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The lifetimes of conditioned diffusion processes

by

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ABSTRACT. – Let $X(t)$ be a diffusion on a smooth bounded domain $D \subset R^d$ and let $X^h(t)$ be a corresponding h-process where h is positive and L-harmonic. Also, let τ_D denote the first exit time from D. DeBlassie proved that for all such h, $\lim_{t \to \infty} \frac{1}{t} \log P_x^h(\tau_D > t) = -\lambda_D$, the lead eigenvalue of L with the Dirichlet boundary condition on ∂D. In this paper, we give a simpler proof of a stronger result. We prove that

$$c_1(L, D, x) P_x(\tau_D > t) \leqq P_x^h(\tau_D > t) \leqq c_2(L, D, x) P_x(\tau_D > t),$$

where the constants $c_1(L, D, x)$ and $c_2(L, D, x)$ are independent of h. From the proof, one can readily see that the upper and lower bounds are essentially consequences of the Hopf maximum principle and Harnack's inequality respectively. We also present a conditional gauge theorem.

RÉSUMÉ. – Soit $X(t)$ une diffusion sur un domaine borné et régulier $D \subset R^d$ et soit $X^h(t)$ un des correspondant h-processus où h est positive et L-harmonique. Soit τ_D le temps de sortie de D. DeBlassie a prouvé que pour une telle h, $\lim_{t \to \infty} \log P_x^h(\tau_D > t) = -\lambda_D$, la première valeur propre de L avec la condition de frontière de Dirichlet sur ∂D. Dans cet article, nous donnons une plus simple preuve d'un résultat plus fort. Nous montrons que

$$c_1(L, D, x) P_x(\tau_D > t) \leqq P_x^h(\tau_D > t) \leqq c_2(L, D, x) P_x(\tau_D > t),$$

où les constantes $c_1(L, D, x)$ et $c_2(L, D, x)$ sont indépendantes de h. La preuve montre que les bornes supérieure et inférieure sont essentiellement

des conséquences respectivement du principe de maximum de Hopf et de l'inégalité de Harnack.

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1. INTRODUCTION

Let $X(t)$ denote the diffusion process on the bounded region $D \subset R^d$ which is generated by

$$L = \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i}$$

and which is killed upon reaching ∂D. We will assume that $a_{ij}, b_i \in C^{1,\alpha}(R^d)$ and that the boundary, ∂D, is $C^{2,\alpha}$. Denote the (substochastic) transition density by $p_D(t, x, y)$ and the corresponding semigroup by T_t. It is well known that given any strictly positive function h on D satisfying $Lh = 0$ in D (henceforth L-harmonic), one may construct a new diffusion $X^h(t)$ on D whose transition probability density is given by $P_D^h(t, x, y) = h(x)^{-1} p(t, x, y) h(y)$. The generator of the new process is $L_h f = \frac{1}{h} L(hf)$.

In fact, following Doob, such a process, called an h-process, may be considered as a conditioned diffusion process. We elaborate on this a bit. As the boundary is assumed smooth, it follows that the Martin boundary coincides with the Euclidean boundary. (Indeed, a Lipschitz boundary is enough to guarantee this [12].) Fix $x_0 \in D$. Let $K(x, y)$ for $x \in D$ and $y \in \partial D$ be the minimal harmonic function with reference point x_0 corresponding to the boundary point y, that is, $K(x, y)$ is L-harmonic in D, $K(x_0, y) = 1$ and $\lim_{D \ni z \to w} K(z, y) = 0$ for $y \neq w \in \partial D$. Then $X^{k(\cdot, y)}$ may be thought of as the original diffusion, $X(t)$, conditioned to exit D at y [8]. By the Martin representation [15], any positive harmonic function $h(x)$ may be represented as

$$h(x) = \int_{\partial D} K(x, y) \mu(dy), \text{ for some measure } \mu \text{ on } \partial D \text{ satisfying}$$
$$\mu(\partial D) = h(x_0). \quad (1.1)$$

Denote by P_x the probability measure on path space induced by the diffusion $X(t)$ starting from $x \in D$ and denote by P_x^h the corresponding probability for $X^h(t)$ starting from $x \in D$. It follows from Doob [8], p. 440, that

$$P_x^h = \int_{\partial D} P_x^{k(\cdot,\, y)} \nu_x(dy), \tag{1.2}$$

where $\nu_x(\partial y) = \dfrac{k(x, y)}{h(x)} \mu(dy)$ and μ is as in (1.1).

Thus, an h-process corresponds to a measure on path space which is a weighted average (with respect to point of exit) of the original process conditioned on its exit point. Of course, the original process $X(t)$ may also be thought of as an h-process; it is the h-process corresponding to $h \equiv 1$.

Let $\tau_D = \inf\{t \geqq 0 : X^h(t) \notin D\}$. It follows from the Donsker-Varadhan theory of large deviations that the original process $X(t)$ satisfies

$$\lim_{t \to \infty} \frac{1}{t} \log P_x(\tau_D > t) = -\lambda_D,$$

where $\lambda_D = \inf \operatorname{Re}(\operatorname{spec}(-L))$ [7]. DeBlassie studied the corresponding asymptotics for h-processes and proved that

$$\lim_{t \to \infty} \frac{1}{t} \log P_x^h(\tau_D > t) = -\lambda_D, \tag{1.3}$$

for every strictly positive harmonic h [5]. In light of Doob's representation in (1.2), this result asserts that the asymptotic rate of decay of the probability of not exiting from D for a long time does not depend on where the process finally exists. Indeed, if one considers Falkner's conditional gauge theorem [9], the intuition behind the above fact becomes quite transparent. See the last paragraph of the introduction in [5] for an explanation. In the special case of Brownian motion, DeBlassie was able to do better than (1.3); he obtained the following eigenfunction expansion.

$$P_x^h(\tau_D > t) = h(x)^{-1} \sum_{k=0}^{\infty} e^{-\lambda_k t} \varphi_k(x) \int_D h(y) \varphi_k(y)\, dy, \tag{1.4}$$

where the convergence is absolute and uniform for $t \geqq T > 0$ [6]. (Also, see [1] and [13] which trent more general reversible diffusions.) Here $\{\lambda_k, \varphi_k\}_{k=0}^{\infty}$ is a complete set of orthonormal eigenvalue-eigenfunction pairs for $\frac{1}{2}\Delta$ on D with the Dirichlet boundary condition on ∂D.

DeBlassie's proof of (1.3) is a nice application of I-function calculating; he evaluated the Donsker-Varadhan I-function for the h-process in a sufficiently explicit manner. In the present paper, we shall improve upon DeBlassie's result using an entirely different approach which appears to us to be somewhat simpler. Our method also has an intuitive appeal; it will be clear from the proof that our upper and lower bounds are essentially consequences of the Hopf maximum principle and Harnack's inequality respectively. We prove

THEOREM 1. — *Let* P_x^h *be the measure corresponding to the h-process* $X^h(t)$ *obtained from the process* $X(t)$ *whose corresponding measure is* P_x *and which is generated by* L *as given above. Assume that the domain* D *possesses a* $C^{2,\alpha}$*-boundary. Then there exist positive constants* $c_1(L, D, x)$ *and* $c_2(L, D, x)$ *which are independent of h such that*

$$c_1(L, D, x)\, P_x(\tau_D > t) \leqq P_x^h(\tau_D > t) \leqq c_2(L, D, x)\, P_x(\tau_D > t). \tag{1.5}$$

Remark. — The constants c_1 and c_2 must necessarily depend on x. To see this, consider the case of Brownian motion. Now $P_x(\tau_D > t)$ solves $u_t = \frac{1}{2}\Delta u$ and vanishes at ∂D. Recall that $P_x^h(\tau_D > t)$ is given by (1.4). By the Hopf maximum principle for parabolic and elliptic operators respectively [17], it follows that $P_x(\tau_D > t)$ and $\varphi_0(x)$ have first order zeroes at ∂D. If we take $h(x) = k(x, y)$ for $y \in \partial D$, then $\lim_{x \to y} h(x) = \infty$ and $\lim_{x \to \partial D - \{y\}} h(x) = 0$. Thus $\frac{P_x^h(\tau_D > t)}{P_x(\tau_D > t)}$ is neither bounded from above nor bounded away from zero on D.

From Theorem 1, we can obtain a conditional gauge theorem. Let

$$u^h(q, D; x) = E_x^h \exp\left(\int_0^{\tau_D} q(X^h(s))\, ds\right),$$

where q is a Borel measurable function. If $h \equiv 1$, u^h is called the gauge of D; for general h it is called the conditional gauge. A number of authors have considered the finiteness or infiniteness of the gauge and of the conditional gauge with various conditions on q, D and L (*See* [3], [4], [9], [10], [16] and [19]). From Theorem 1, we can obtain the following conditional gauge theorem.

THEOREM 2. — *Let* $q \in C_b^2(R^d)$ *and assume that* L *and* D *are as in Theorem* 1. *Let* $\lambda_q = \operatorname{Sup} \operatorname{Re}(\operatorname{Spec}(L + q))$.

(i) *If* $\lambda_q < 0$, *then for each* $x \in D$, $\sup_h u^h(q, D; x) < \infty$.

(ii) *If* $\lambda_q > 0$, *then* $u^h(q, D; x) \equiv \infty$ *for all h.*

(iii) *If* $\lambda_q = 0$, *and* $q \geqq 0$, *then* $u^h(q, D; x) \equiv \infty$ *for all h.*

Proof. — In [16] we used the fact that $\lim_{t \to \infty} \log P_x(\tau_D > t) = -\lambda_D$ to prove a version of the above theorem for the unconditional gauge, that is, the case $h \equiv 1$. Virtually the same proof works here if one uses the above asymptotic result in conjunction with (1.5). The proof is a quite straightforward application of the Donsker and Varadhan large deviations estimates.

Remark. — [4] treats reversible diffusions while the other papers cited above treat Brownian motion. These papers allow for either unbounded q, nonsmooth region D or both of these possibilities. The statement of the theorem in these papers is that if $u^h(q, D; x)$ is finite for some h and some x, then it is bounded for all h and x. Our result, though restricted to bounded q and smooth D, is new in that it treats the general diffusion and that it gives an explicit spectral condition under which $u^h(q, D; x)$ will be finite. Note though that Theorem 2 only gives $\sup_h u^h(q, D; x) < \infty$ for each x and not $\sup_x \sup_h u^h(q, D; x) < \infty$.

In the following section, we will present the proof of Theorem 1, relying on Theorems 3 and 4 appearing below, the proofs of which will be postponed till section three.

In order to state Theorem 3, we must recall the following fact. By the Krein-Rutman theory of positive operators [14], it follows that the infimum of the real part of the spectrum of $-L$ and of its adjoint $-\tilde{L}$ with the Dirichlet boundary condition coincide at a real simple eigenvalue $\lambda_D > 0$ and that the corresponding eigenfunctions φ_0 and $\tilde{\varphi}_0$ are strictly positive in D. From here on, we assume that φ_0 and $\tilde{\varphi}_0$ are normalized so that $\int_D \varphi_0 \tilde{\varphi}_0 \, dx = 1$. Note that, of course, $T_t \varphi_0 = e^{-\lambda_D t} \varphi_0$ and $T_t^* \tilde{\varphi}_0 = e^{-\lambda_D t} \tilde{\varphi}_0$, where T_t^*, the adjoint of T_t, satisfies $T_t^* f(x) = \int_D p(t, y, x) f(y) \, dy$.

We also note here for later use that as a consequence of the regularity assumptions, it follows that $p(t, x, y)$ is a classical solution of $p_t = L_x p$ for each $y \in D$ and of $p_t = \tilde{L}_y p$ for each $x \in D$. Furthermore, $p(t, x, y)$ is continuous on $(0, \infty) \times \bar{D} \times \bar{D}$ and $p(t, x, y) = 0$ for $(x, y) \in (\bar{D} \times \partial D) \cup (\partial D \times \bar{D})$ and $t > 0$.

We now state

THEOREM 3. — $\lim_{t \to \infty} \sup_{x, y \in D} |e^{\lambda_D t} P_D(t, x, y) - \varphi_0(x) \tilde{\varphi}_0(y)| = 0.$

COROLLARY. — *The conditional density*

$$P_x(X(t) \in dy \mid \tau_D > t) = \frac{P_D(t, x, y)}{\int_D p_D(t, x, z) \, dz}$$

satisfies

$$\lim_{t \to \infty} \sup_{x, y \in D} \left| \frac{p_D(t,x,y)}{\int_D p_D(t,x,z)\,dz} - \frac{\tilde{\varphi}_0(y)}{\int_D \tilde{\varphi}_0(z)\,dz} \right| = 0.$$

Proof of Corollary. — From the theorem, we have $p_D(t,x,y) = e^{-\lambda_D t}\varphi_0(x)\tilde{\varphi}_0(y) + o(t)$, uniformly for $x, y \in D$.

Remark. — Actually, the proof of Theorem 1, especially the lower bound, does not at all require the full strength of this result.

THEOREM 4. — *Let $z_0 \in \partial D$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of smooth nonnegative nonidentically zero functions defined on ∂D and define $\mu_n \in P(\partial D)$ by*

$$\mu_n(A) = \frac{\int_A f_n\,d\sigma}{\int_{\partial D} f_n\,d\sigma},$$

for $A \subset \partial D$. Assume that $\mu_n(\partial D - G) = 0$ for each relatively open $z_0 \in G \subset \partial D$ and all sufficiently large n (depending on G). Let u_n denote the solution to the Dirichlet problem $Lu_n = 0$ in D and $u_n = f_n$ on ∂D. Define $v_n(x) = \dfrac{u_n(x)}{u_n(x_0)}$, for $x \in D$, where $x_0 \in D$ is the reference point of section one. Then

$$\lim_{n \to \infty} v_n(x) = k(x, z_0), \quad \textit{for all } x \in \bar{D} - \{z_0\}, \tag{1.8}$$

where $k(x, z_0)$ is the minimal harmonic function with reference point x_0 corresponding to the boundary point z_0. Furthermore

$$\lim_{n \to \infty} P_x^{v_n}(\tau_D > t) = P_x^{k(\cdot, z_0)}(\tau_D > t), \quad \textit{for all } x \in D \textit{ and } t > 0. \tag{1.9}$$

2. PROOF OF THEOREM 1

We first note that, by Doob's representation (1.2), we need only consider the case $h(x) = k(x, z_0)$ for $z_0 \in \partial D$. Fix $x \in D$ and let f_n, u_n and v_n be as in Theorem 4. We have

$$\frac{P_x^{v_n}(\tau_D > t)}{P_x(\tau_D > t)} = \frac{E_x(u_n(X(t)); \tau_D > t)}{u_n(x)\,P_x(\tau_D > t)} = \frac{\int_D u_n(z)\,p_D(t,x,z)\,dz}{u_n(x)\int_D p(t,x,z)\,dz}. \tag{2.1}$$

Now by Theorem 4, the left hand side of (2.1) converges to $\frac{P_x^{k(\cdot, z_0)}(\tau_D > t)}{P_x(\tau_D > t)}$ as $n \to \infty$. Thus, to complete the proof of Theorem 1, it suffices to show that the right hand side of (2.1) is bounded and bounded away from zero independent of n, large t, $z_0 \in \partial D$ and the particular sequence $\{f_n\}_{n=1}^{\infty}$. (It suffices to consider large t since, as $t \downarrow 0$, the left hand side of (2.1) converges to one.)

Lower bound. – From the corollary to Theorem 3, it follows that for $\varepsilon > 0$

$$\frac{p_D(t, x, y)}{\int_D p_D(t, x, z)\, dz} \geqq \frac{\tilde{\varphi}_0(y)}{\int_D \tilde{\varphi}_0(z)\, dz} - \varepsilon \tag{2.2}$$

for all sufficiently large t and all $y \in D$. Let U be open and satisfy $x \in U \subset \bar{U} \subset D$. Then by strict positivity of $\tilde{\varphi}_0$ in U, it follows from (2.2) that for some $\delta > 0$,

$$\frac{p_D(t, x, y)}{\int_D p_D(t, x, z)\, dz} \geqq \delta, \quad \text{for } y \in U \text{ and } t \text{ sufficiently large.} \tag{2.3}$$

By Harnack's inequality, $u_n(x) \leqq \hat{c}\, u_n(y)$ for all $y \in U$, where $\hat{c}$ is independent of n, $\{f_n\}_{n=1}^{\infty}$ and $z_0 \in \partial D$. Thus

$$u_n(x) \leqq c \int_U u_n(y)\, dy, \tag{2.4}$$

for $c = \frac{\hat{c}}{|U|}$. Using (2.3) and (2.4) in the right hand side of (2.1), we conclude that for all t sufficiently large,

$$\frac{\int_D u_n(y)\, p_D(t, x, y)\, dy}{u_n(x) \int_D p(t, x, y)\, dy} \geqq \frac{\delta}{c}. \tag{2.5}$$

Upper bound. – Note that u_n is excessive for T_t:

$$u_n(x) = E_x(u_n(X(t)), \tau_D > t) + E_x(u_n(X(\tau_D)), \tau_D \leqq t)$$
$$\geqq E_x(u_n(X(t)), \tau_D > t) = \int_D p_D(t, x, y)\, u_n(y)\, dy.$$

Thus, we have the following upper bound on the righthand side of (2.1):

$$\frac{\int_D u_n(y)p(t,x,y)\,dy}{u_n(x)\int_D p(t,x,y)\,dy} \leqq \frac{\int_D u_n(y)p(t,x,y)\,dy}{\int_D u_n(y)p(1,x,y)\,dy \int_D p(t,x,y)\,dy}$$

It then follows that to obtain the upper bound, it suffices to show that for each $x \in D$, there exists a $c(x)$ such that

$$\frac{p_D(t,x,y)}{\int_D p_D(t,x,z)\,dz} \leqq c(x)\,p_D(1,x,y), \tag{2.6}$$

for all $y \in D$ and all large t.

Now, as functions of y,

$$\frac{p_D(t,x,y)}{\int_D p_D(t,x,z)\,dz}, \frac{\tilde{\varphi}_0(y)}{\int \tilde{\varphi}_0(z)\,dz}$$

and $p_D(1,x,y)$ are all strictly positive in D and all have first order zeroes on ∂D. Furthermore, by the corollary to Theorem 3, $\dfrac{p_D(t,x,y)}{\int_D p_D(t,x,z)\,dz}$ converges to $\dfrac{\tilde{\varphi}_0(y)}{\int_D \tilde{\varphi}_0(z)\,dz}$ as $t \to \infty$, uniformly for x, $y \in D$. From these facts, it is easy to deduce that, in order to show (2.6), it suffices to show that the unit inward normal derivative of $\dfrac{p_D(t,x,y)}{\int_D p_D(t,x,z)\,dz}$ at ∂D is bounded from above as $t \to \infty$. For $x \in D$ and $y \in \bar{D}$, we write

$$\frac{p_D(t,x,y)}{\int_D p_D(t,x,z)\,dz} = \frac{\int_D p_D(t-1,x,z)\,p_D(1,z,y)\,dz}{\int_D p_D(t,x,z)\,dz}.$$

Thus, letting n_y denote the inward unit normal to ∂D at $y \in \partial D$, we have

$$\frac{\partial}{\partial n_y} \frac{p_D(t,x,y)}{\int_D p_D(t,x,z)\,dz} = \frac{\int_D p_D(t-1,x,z)\,(\partial p_D(1,z,y)/\partial n_y)\,dz}{\int_D p_D(t,x,z)\,dz}. \tag{2.7}$$

The right hand side of (2.7) is bounded as $t \to \infty$ since by Theorem 3, $\dfrac{p_D(t-1,x,.)}{\int_D p_D(t,x,z)\,dz}$ is bounded as $t \to \infty$. This completes the proof of Theorem 1.

3. PROOFS OF THEOREMS 3 AND 4

We need the following lemma to prove Theorem 3.

LEMMA. — *T_t^* is a compact operator on $C(\bar{D})$ for each $t>0$.*

Proof. — We must show that $(T_t^* f : \|f\| \leqq 1\}$ is bounded and equicontinuous. The boundedness follows from the fact that $p(t,x,y)$ is bounded for each $t>0$. For $\|f\| \leqq 1$, we have

$$\sup_{\substack{x,\,y \in \bar{D}\\ |x-y|<\delta}} |T_t^* f(x) - T_t^* f(y)| \leqq \int_D |p(t,z,x)-p(t,z,y)|\,dz$$

$$\leqq |D| \sup_{\substack{x,\,y,\, \in \bar{D}\\ |x-y|<\delta\\ z \in D}} |p(t,z,x)-p(t,z,y)|.$$

By the joint continuity of $p(t,x,y)$ it follows that the righthand side may be made arbitrarily small by picking δ sufficiently small. This gives equicontinuity.

Proof of Theorem 3. — By the lemma, T_t^* is a compact operator on $C(\bar{D})$. Thus, by the Riesz-Schauder theory of compact operators on Banach spaces, it follows that $C(\bar{D}) = N \otimes R$ where $N = \{ c\tilde{\varphi}_0, c \in R \}$ and T_t^* leaves N and R invariant ([2], Chapter 6, section 6). Since $e^{-\lambda_D t} = \sup \operatorname{Re}(\operatorname{spec}(T_t^*))$, and since the nonzero spectrum of a compact operator is isolated, it follows that $e^{\lambda_D t} \| T_t^* |_R \| \leqq e^{-\nu t}$, for large t and some $\nu > 0$. Let $f \in C(\bar{D})$. By the above direct sum decomposition of $C(\bar{D})$, it follows that $f = c_f \tilde{\varphi}_0 + \psi_f$, where $\psi_f \in R$. Thus for large t,

$$\| e^{\lambda_D t} T_t^* f - c_f \tilde{\varphi}_0 \| = e^{\lambda_D t} \| T_t^* \psi_f \| \leqq e^{-\nu t} \| \psi_f \|. \tag{3.1}$$

We now identify c_f. Recalling that $\int_D \varphi_0 \tilde{\varphi}_0 \, dx = 1$, we have

$$0 = \lim_{t \to \infty} \int_D (e^{\lambda_D t} T_t^* f - c_f \tilde{\varphi}_0) \varphi_0 \, dx = \lim_{t \to \infty} \int_D e^{\lambda_D t} f T_t \varphi_0 - c_f = \int f \varphi_0 \, dx - c_f.$$

Thus $c_f = \int f \varphi_0 \, dx$ and $|c_f| \leqq \|f\| \int_D \varphi_0 \, dx$. Furthermore then

$$\|\psi_f\| \leqq \|f\| + |c_f| \, \|\tilde{\varphi}_0\| \leqq \|f\| [1 + \|\tilde{\varphi}_0\| \int_D \varphi_0 \, dx] \equiv k \|f\|.$$

Thus it follows from (3.1) that for all large t,

$$\|e^{\lambda_D t} T_t^* f - c_f \tilde{\varphi}_0\| \leqq k \|f\| e^{-\nu t}. \tag{3.2}$$

We now write

$$e^{\lambda_D t} p(t, x, y) = e^{\lambda_D t} \int_D p(1, x, z) p(t-1, z, y) \, dz = e^{\lambda_D t} T_{t-1}^* f_x(y), \tag{3.3}$$

where

$$f_x(z) = p(1, x, z).$$

We calculate

$$c_{f_x} = \int_D p(1, x, z) \varphi_0(z) \, dz = e^{-\lambda_D} \varphi_0(x). \tag{3.4}$$

Let $M = \sup_{x, z \in D} p(1, x, z)$. From (3.2), (3.3) and (3.4), we have for large t

$$\begin{aligned} \sup_{x, y \in D} &|e^{\lambda_D t} p(t, x, y) - \varphi_0(x) \tilde{\varphi}_0(y)| \\ &= \sup_{x, y \in D} |e^{\lambda_D t} T_{t-1}^* f_x(y) - e^{\lambda_D} c_{f_x} \tilde{\varphi}_0(y)| \\ &\leqq \sup_{\|f\| \leqq M} \|e^{\lambda_D t} T_{t-1}^* f - e^{\lambda_D} c_f \tilde{\varphi}_0\| \\ &= \sup_{\|f\| \leqq M} e^{\lambda_D} \|e^{\lambda_D (t-1)} T_{t-1}^* f - c_f \tilde{\varphi}_0\| \leqq e^{\lambda_D} M k e^{-\nu (t-1)}. \end{aligned}$$

This proves the theorem.

Proof of Theorem 4. – Let $G \subset \bar{G} \subset D$ be an open set. We claim that $\{v_n\}_{n=1}^{\infty}$ is bounded and equicontinuous on G. The boundedness follows from Harnack's inequality. From the Schauder interior estimates (*see* [11], Theorem 6.2), it follows that for open sets G and G_1 satisfying $G \subset \bar{G} \subset G_1 \subset \bar{G}_1 \subset D$, there exists a c independent of n such that

$$|u_n(x) - u_n(y)| \leqq c (\sup_{z \in G_1} u_n(z)) |x - y|, \qquad \text{for} \quad x, y \in \bar{G}.$$

Thus, from Harnack's inequality again, for $x, y \in \bar{G}$,

$$|v_n(x) - v_n(y)| \leqq c \left(\sup_{z \in G_1} \frac{u_n(z)}{u_n(x_0)} \right) |x-y| \leqq c_1 |x-y|,$$

where c_1 is independent of n. This gives equicontinuity. Now let G_m be an increasing sequence of open sets whose union is D. Then we can find a subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ of $\{v_n\}$ which converges on D to a continuous function which we denote by g. From the representation $v_n(x) = E_x v_n(X(\tau_A))$ for any open A satisfying $x \in A \subset \bar{A} \subset D$, it follows that $g(x) = E_x g(X(\tau_A))$, for such A. Thus g is L-harmonic in D. Furthermore $g(x_0) = 1$. If we can show that $\lim_{x \to z} g(x) = 0$ for all $z_0 \neq z \in \partial D$, then it will follow that $g(x)$ is in fact $k(x, z_0)$, the minimal harmonic function with reference point x_0 corresponding to the boundary point z_0. From this it then follows that in fact the original sequence $\{v_n\}_{n=1}^{\infty}$ converges to $k(.,z_0)$.

By construction, for each $z_0 \neq z \in \partial D$, $v_n(z) = 0$ for n sufficiently large. Thus, to show that $\lim_{x \to z} g(x) = 0$ for $z_0 \neq z \in \partial D$, it suffices to show that the inward unit normal derivative $\frac{\partial v_n}{\partial n_x}(x)$ of v_n at $z_0 \neq x \in \partial D$ is bounded independent of n. Fix such an x. From the Martin representation, we have

$$v_n(x) = \frac{u_n(x)}{u_n(x_0)} = \frac{\int_{\partial D} k(x,z)\, \mu_n(dz)}{\int_{\partial D} k(x_0,z)\, \mu_n(dz)},$$

for some finite measure μ_n on ∂D. The inward unit normal derivative to v_n at x is

$$\frac{\partial v_n(x)}{\partial n_x} = \frac{\int_{\partial D} (\partial k/\partial n_x)(x,z)\, \mu_n(dz)}{\int_{\partial D} k(x_0,z)\, \mu_n(dz)}.$$

Now $\frac{\partial k}{\partial n_x}(x,z)$ is bounded for z bounded away from x. Since the measure μ_n is supported on the support of f_n which, by construction, is bounded away from x for sufficiently large n, it follows that $\frac{\partial v_n}{\partial n_x}(x)$ is bounded. This completes the proof of (1.8).

We now consider (1.9). Let G_m be an increasing sequence of open sets whose union is D. Let $P_x^{v_n, m}$ denote the measure corresponding to the process $X^{v_n}(t)$ restricted to the interval $0 \leqq t \leqq \tau_{G_m}$. The corresponding transition probability, $p^{n,m}(t,x,y) \equiv v_n^{-1}(x) p_D(t,x,y) v_n(y)$, satisfies $\lim_{t \to 0} \frac{1}{t} \sup_n \sup_{\substack{|x-y|<\delta \\ x, y \in G_m}} p^{n,m}(t,x,y) = 0$, for all $\delta > 0$, since the same is true of $p_D(t,x,y)$. It then follows that $\{P_x^{v_n,m}\}_{n=1}^{\infty}$ is tight [19], and since $\lim_{n \to \infty} p^{n,m}(t,x,y) = k(x,z_0)^{-1} p_D(t,x,y) k(y,z_0)$, we conclude that in fact $P_x^{v_n,m} \Rightarrow P_x^{k(\cdot,z_0),m}$ as $n \to \infty$, where $P_x^{k(\cdot,z_0),m}$ is the measure corresponding to the process $X^{k(\cdot,z)}$ restricted to the interval $0 \leqq t \leqq \tau_{G_m}$. Invoking Lemma 11.1.1 in [18], it follows that $P_x^{v_n} \Rightarrow P_x^{k(\cdot,z_0)}$ as $n \to \infty$. (1.9) now follows.

REFERENCES

[1] R. BAÑUELOS and B. DAVIS, Heat kernel, eigenfunctions, and conditioned Brownian motion in planar domains, *J. Funct. Anal.*, Vol. **84**, 1989, pp. 188-200.

[2] A. L. BROWN and A. PAGE, *Elements of Functional Analysis*, Van Nostrand Reinhold Co., London, 1970.

[3] K. L. CHUNG, The gauge and conditional gauge theorem. Séminaire de Probabilités XIX, 1983/1984, *Lect. Notes Math.*, **1123**, pp. 496-503, Springer, Berlin.

[4] M. CRANSTON, E. FABES and Z. ZHAO, Conditional gauge and potential theory for the Schrödinger operator, *Trans. Am. Math. Soc.*, Vol. **307**, 1988, pp. 171-194.

[5] R. D. DEBLASSIE, Doob's conditioned diffusions and their lifetimes, *Ann. Probab.*, July 1988 (to appear).

[6] R. D. DEBLASSIE, The lifetime of conditioned Brownian motion on certain Lipschitz domains, *Probab. Th. Rel. Fields*, Vol. **75**, 1987, pp. 55-65.

[7] M. D. DONSKER and S. R. S. VARADHAN, On the principal eigenvalue of second-order elliptic differential operators. *Comm. Pure Appl. Math.*, Vol. **29**, 1976, pp. 595-621.

[8] J. L. DOOB, Conditioned Brownian motion and the boundary limits of harmonic functions, *Bull. Soc. Math. France*, Vol. **85**, 1957, pp. 431-458.

[9] N. FALKNER, Feynman-Kac functionals and positive solutions of $\frac{1}{2}\Delta u + qu = 0$, *Z. Wahrsch. Verw. Gebiete*, Vol. **65**, 1983, pp. 19-33.

[10] N. FALKNER, Conditional Brownian motion in rapidly exhaustible domains, *Ann. Probab.*, Vol. **15**, 1987, pp. 1501-1514.

[11] D. GILBARG and N. S. TRUDINGER, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin, 1977.

[12] R. A. HUNT and R. L. WHEEDEN, Positive harmonic functions on Lipschitz domains, *Trans. Am. Math. Soc.*, Vol. **143**, 1968, pp. 307-322.

[13] C. E. KENIG and J. PIPHER, The *h*-path distribution of the lifetime of conditioned Brownian motion for nonsmooth domains, *Probab. Theory Related Fields*, Vol. **82**, 1989, pp. 615-623.

[14] M. G. KREIN and M. A. RUTMAN, Linear operators leaving invariant a cone in a Banach space, *A.M.S. Translations series*, Vol. **1**, 10, 1962, pp. 199-324.

[15] R. S. MARTIN, Minimal positive harmonic functions, *Trans. Am. Math. Soc.*, Vol. **49**, 1941, pp. 137-172.

[16] R. PINSKY, A spectral criterion for the finiteness or infiniteness of stopped Feynman-Kac functionals of diffusion processes, *Ann. Probab.*, Vol. **14**, 1986, pp. 1180-1187.

[17] M. H. PROTTER and H. F. WEINBERGER, *Maximum Principles in Differential Equations*, Springer-Verlag, New York, 1984.

[18] D. W. STROOCK and S. R. S. VARADHAN, *Multidimensional Diffusion Processes*, Springer-Verlag, Berlin, 1979.

[19] S. R. S. VARADHAN, *Stochastic processes*, Lecture Notes, Courant Institute of Mathematical Sciences, 1968.

[20] Z. ZHAO, Conditional gauge with unbounded potential, *Z. Wahrsch. Verw. Gebiete*, Vol. **65**, 1983, pp. 13-18.

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