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Factorising Brownian motion at two boundaries; an example

by

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ABSTRACT. — Time-changing Brownian motion by the inverse of a fluctuating additive functional produces a factorisation, analogous to the Wiener-Hopf factorisation of a Lévy process. In only a few cases are these factors known explicitly. The purpose of this note is to solve a two boundary example in detail, using a complex martingale construction.

Key words : Brownian motion, fluctuating additive functional, time change, factorisation.

RÉSUMÉ. — Le mouvement brownien changé de temps par l'inverse d'une fonctionnelle additive oscillante nous donne une factorisation, analogue à la factorisation Wiener-Hopf d'un processus de Lévy. Les facteurs ne sont connus que dans très peu de cas. On donne ici la solution détaillée d'un problème avec deux frontières, à partir d'une martingale conforme.

Our example arises in the following general setting. Let B_t be a real Brownian motion and A_t an *increasing* adapted process. If $\tau_t = \inf\{s : A_s > t\}$ we define $X_t = B_{\tau_t}$, noting that in general τ_t and X_t can be discontinuous. The special case $A_t = \int L(a, t)\,\mu(da)$, where $L(a, t)$ is the local time of B_t and $\mu \geqq 0$ a measure, has been extensively studied and is of particular importance in the theory of one-dimensional diffusions [6]. Then X_t is a gap diffusion, X_t lives on the support of μ, and roughly speaking every gap diffusion in natural scale is obtained from B_t by such a time-change.

In contrast, if we replace μ by a signed measure $m=m^+-m^-$, the situation is not so transparent. For a start A_t is no longer increasing so we now have two processes

$$\tau_t^+=\inf\{s:A_s>t\},\qquad \tau_t^-=\inf\{s:A_s<-t\},$$

and, writing $B_t^+=B_{\tau_t^+}$ and $B_t^-=B_{\tau_t^-}$, the problem is to describe these new "factors" probabilistically. In particular we want an explicit description of their jumps, the first jump computation being crucial.

All this has a formal (and practical) analogy with the so-called Wiener-Hopf factorisation of Lévy processes. If Y_t is a Lévy process then for certain purposes it is easier to work with its maximum Y_t^+. The Wiener-Hopf factorisation of Y_t is the pair $(Y_t^+, Y_t^+ - Y_t)$, the second part being known as the reflection functional. This is a powerful concept in the analysis of Lévy processes but unfortunately, with the exception of certain simple cases, explicitly computing these factors is extremely difficult.

Going back to our problem one can appreciate the analogy; we are asking for an explicit description of the factorisation (B_t^+, B_t^-) generated by our fluctuating additive functional A_t. Though the question now seems more complicated it also has extra structure, and there are non-trivial examples where the computation is not only feasible but quite straightforward and elegant. This paper examines one such case in detail.

0. PROBLEM AND METHOD

Suppose B_t is a Brownian motion started at the point $x\in(-p,p)$, and define the additive functional $A_t=\int_0^t 1_{(|B_s|>p}\,ds-\delta^2\int_0^t 1_{(|B_s|<p)}\,ds$. Then for $\tau=\inf\{t>0:A_t=0\}$ we want to compute the kernel $\mathbf{P}_x[B_\tau\in dy]=\Pi(x,dy)$. Notice that there are two critical points here (namely $\pm p$) where A_t changes direction, which is why this is called a two boundary problem [1] (the examples in [9] all have one boundary). The kernel $\Pi(x,dy)$ has already been computed in [1] but here we want to derive the answer by using a two boundary version of the method used in [9]. Why? Well, our proof is easier, it leads to an interesting probabilistic interpretation, and the idea has a quite remarkable implication for the dual problem.

First we recall the general tactic from [10], which is to choose a bounded function f_θ such that the process $f_\theta(B_t)\,e^{A_t\theta^2/2}$ is a local martingale. The values of θ for which this is possible are called the *eigenvalues* of the problem. Because the process is uniformly integrable up to time τ the

Doob stopping theorem gives

$$\int f_\theta(y)\,\Pi(x, dy) = f_\theta(x) \tag{0.1}$$

noting how $\Pi(x, .)$ is supported on $R\setminus(-p, p)$ since $x\in(-p, p)$. The problem is to find $\Pi(x, dy)$ from this rather cryptic equation.

We do more; we will construct a complex analogue of the martingale $f_\theta(B_t)\,e^{A_t\theta^2/2}$ and use it to rederive 0.1.

In fact we pursue the following "method".

Step 1. – Using complex Brownian motion we construct a conformal martingale so closely related to $f_\theta(B_t)\,e^{A_t\theta^2/2}$ that a suitable application of Doob's stopping theorem gives us 0.1.

Step 2. – We prove that 0.1 uniquely determines the kernel $\Pi(x, dy)$.

The point is that step 1 provides us with $\Pi(x, dy)$ as long as we know the stopping distribution of our complex Brownian motion.

The crucial difference with the one boundary case is that step 2 needs to be proved separately; in [9] we could rely on the result of [7]. Nevertheless the difficult part of the solution was to carry out step 1, obvious though the answer may seem in retrospect.

The example studied here is important for several reasons. It seems to be the first direct use of the above "eigenvalue check" to solve a two boundary problem, and the method used to prove uniqueness may be of interest also. Another novelty is the connection with the dual problem, where we use essentially the same harmonic function to solve when $B_0>p$. But the really intriguing aspect of our approach is how closely it mimicks the derivation of 0.1. Perhaps we are missing some simpler interpretation.

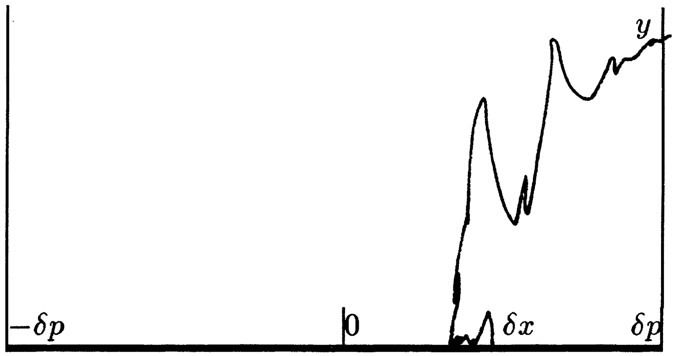
Notation. – We adopt the convention that $x\in(-p, p)$, while y corresponds to real points outside this range.

1. COMPUTING $\Pi(x, dy)$

When $A_t = \int_0^t 1_{(|B_s|>p)}\,ds - \delta^2\int_0^t 1_{(|B_s|<p)}\,ds$ Ito's formula shows that $f_\theta(B_t)\,e^{A_t\,\theta^2/2}$ is a local martingale if

$$f_\theta(x) = \begin{cases} e^{\theta\delta x} & \text{if } |x|\leqq p; \\ e^{\theta\delta p}[\cos\theta(x-p)+\delta\sin\theta(x-p)] & \text{if } x>p; \\ e^{-\theta\delta p}[\cos\theta(x+p)+\delta\sin\theta(x+p)] & \text{if } x<-p \end{cases} \tag{1.1}$$

with $\theta>0$. For the computation of $\Pi(x, dy)$ we take a complex Brownian motion Z_t reflected upwards from the real axis, so that our basic diagram is



the reflecting boundary being represented by the thicker line. We consider $M_t = \Re\left[e^{\theta Z_t}\cos^\rho \frac{\pi Z_t}{2\delta p}\right]$ with $\rho \in (0,1)$ to be chosen later. The appearance of $e^{\theta z}$ is motivated from the explicit form of f_θ at 1.1, but the function $cos^\rho \frac{\pi z}{2\delta p}$ is called the *integrating factor* and its choice is determined by the region. For the branch of $\cos^\rho \frac{\pi z}{2\delta p}$ we take the one which is real on $(-\delta p, \delta p)$, defining it elsewhere in the region by analytic continuation. Of course Z_t never hits the points $\pm\delta p$ so we can safely ignore them. Since the process $e^{\theta Z_t}\cos^\rho \frac{\pi Z_t}{2\delta p}$ is real on the real axis one sees that M_t, as the real part of a conformal martingale, must be a martingale. We will start at $Z_0 = \delta x > 0$ and we write ξ for the first hitting time of $\Re z = \pm\delta p$ by Z_t. It is not too hard to check that the process $M_{t \wedge \xi}$ is uniformly integrable. We also need (*see* [6], p. 29 and [5], p. 31) the respective hitting distributions

$$\mathbf{P}[Z_\xi \in -\delta p + i\,dy] = \frac{1}{2p\delta}\,\frac{\cos(\pi x/2p)\,dy}{\cosh(\pi y/2\delta p) + \sin(\pi x/2p)};$$

$$\mathbf{P}[Z_\xi \in \delta p + i\,dy] = \frac{1}{2p\delta}\,\frac{\cos(\pi x/2p)\,dy}{\cosh(\pi y/2\delta p) - \sin(\pi x/2p)}$$

Applying the Doob theorem to the martingale $M_{t \wedge \xi}$ we compute that $\mathbf{E}[M_\xi] = M_0$ is just

$$\frac{1}{2p\delta}\int_0^\infty \sinh^\rho \frac{\pi y}{2\delta p}\left[\frac{\Re[i^\rho e^{\theta(iy-\delta p)}]\cos(\pi x/2p)\,dy}{\cosh(\pi y/2\delta p) + \sin(\pi x/2p)} + \frac{\Re[(-i)^\rho e^{\theta(iy+\delta p)}]\cos(\pi x/2p)\,dy}{\cosh(\pi y/2\delta p) - \sin(\pi x/2p)}\right] = e^{\theta\delta x}\cos^\rho \frac{\pi x}{2p}$$

So if we make the choice $\delta=\tan\frac{\pi\rho}{2}$, then we find 0.1 is satisfied by the kernel

$$\Pi(x, p+dy)=\frac{1}{p\delta}\cos\frac{\pi\rho}{2}\left(\frac{\sinh(\pi y/2\delta p)}{\cos(\pi x/2p)}\right)^{\rho}\frac{\cos(\pi x/2p)\,dy}{\cosh(\pi y/2\delta p)-\sin(\pi x/2p)}$$

$$\Pi(x, -p-dy)=\frac{1}{p\delta}\cos\frac{\pi\rho}{2}\left(\frac{\sinh(x|y|/2\delta p)}{\cos(\pi x/2p)}\right)^{\rho}\frac{\cos(\pi x/2p)\,dy}{\cosh(\pi y/2\delta p)+\sin(\pi x/2p)}$$

This is just the answer obtained by Baker [1] but we wish to emphasie the quite remarkable linkage between the problem and the solution. In fact the real part of Z_t behaves like B_t when the latter is in $(-p, p)$, while the imaginary part behaves like B_t outside this interval. The left/right splitting of $\Pi(x, dy)$ highlights this point.

To complete the proof we must show uniqueness. This is in two parts. First we show that $\Pi^e(x, dy)=\Pi(x, dy)+\Pi(x, -dy)$, the "even" part of $\Pi(x, dy)$, is zero. For this notice that from 0.1 and 1.1 we get

$$\int_p^{\infty}\Pi^e(x, dy)\,[\cosh\theta\delta p\cos\theta(y-p)+\delta\sinh\theta\delta p\sin\theta(y-p)]=e^{\theta\delta x}$$

Our hypothesis is therefore that

$$\int_0^{\infty}\mu(dy)\,[\cosh\theta\,\delta p\cos\theta\, y+\delta\sinh\theta\,\delta p\sin\theta\, y]=0$$

for a bounded signed measure μ of total mass zero, from which we want to deduce $\mu\equiv 0$. However this is the same uniqueness condition as in the single boundary example obtained by placing a reflecting barrier at zero, see [9], and so [7], p. 62 applies here. The proof that 0.1 uniquely determines the "odd" part $\Pi^0(x, dy)=\Pi(x, dy)-\Pi(x, dy)$ is similar, we relate it to the single boundary problem with an absorbing barrier at zero.

2. DUAL PROBLEM

We now look at the dual case, by which we mean the calculation of $\tilde{\Pi}(y, dx)=\mathbf{P}_y[B_\tau\in dx;\ \tau<+\infty]$ when $B_0=y>p$. We show it suffices to use almost the same harmonic function as before: the other difference being that we need to interchange absorbing and reflecting boundaries for the driving complex Brownian motion, and this involves adjusting the integrating factor accordingly.

To begin the solution we first establish an analogue of 1.1. Suppose we retain the notation of the previous section and consider the local martingale $f_\theta(B_t)\,e^{A_t\theta^2/2}$. As it stands this is no longer uniformly bounded.

Note that if the exponential is to remain bounded we must switch $\theta^2 \to -\theta^2$, but then the corresponding function is unbounded. The way out of this difficulty is to consider the real part only, restricting θ to get what we want. So the analogue of 1.1 turns out to be

$$\tilde{f}_\theta(x) = \begin{cases} \cos\theta\,\delta x & \text{if } |x| \leq p; \\ e^{-\theta(x-p)}\cos\theta\,\delta p & \text{if } x > p; \\ e^{\theta(x+p)}\cos\theta\,\delta p & \text{if } x < -p \end{cases} \tag{2.1}$$

with $\theta > 0$ chosen so that the derivatives match on the boundary, namely $\delta = \cot\theta\,\delta p$. The equation 0.1 then gives us

$$\int_{-p}^{p} \tilde{\Pi}(y, dx)\cos\theta\,\delta x = e^{-\theta(y-p)}\cos\theta\,\delta p \tag{2.2}$$

when $y > 0$, and we want to solve this eigenvalue relation for $\tilde{\Pi}(y, dx)$.

The basic idea is just like before. We first find a candidate for $\tilde{\Pi}(y, dx)$ by constructing and stopping a conformal martingale, and then hope that 2.2 uniquely specifies the kernel. So we use the conformal local martingale $e^{i\theta Z_t}\cos^{1-\rho}\frac{\pi Z_t}{2\delta p}$ where $\tilde{Z}_t$ is a complex Brownian motion having reflecting boundaries on $\Re z = \pm\delta p$, and we stop at the first hitting time ζ of the interval $(-\delta p, \delta p)$ by $\tilde{Z}_t$. Here we have $\delta = \tan\frac{\pi\rho}{2}$ as before, and notice that the choice of integrating factor makes the function real on the boundary.

For the computation note that the eigenvalue condition $\cot\theta\,\delta p = \delta$ means $N_t = \Re\left[e^{i\theta\tilde{Z}_t}\cos^{1-\rho}\frac{\pi\tilde{Z}_t}{2\delta p}\right]$ remains a martingale at the boundaries $\Re z = \pm\delta p$, because the conformal martingale is real-valued there. Moreover a check on the lowest eigenvalue verifies uniform boundedness. We will start the process at $\tilde{Z}_0 = \delta p + iy$ and hence must compute the hitting distribution $\mathbf{P}_{\delta p + iy}[\tilde{Z}_\zeta \in dx]$. For this we use Paul Lévy's theorem on conformal invariance, mapping our region to the positive quadrant via the conformal function $z \to \tan\frac{\pi(\delta p - z)}{4\delta p}$ and computing the hitting distribution as the conformal image of the Cauchy law. The answer comes out to be

$$\mathbf{P}_{\delta p + iy}[\tilde{Z}_\zeta \in \delta\,dx] = \frac{2}{\pi}\,\frac{\tanh(\pi y/4\,\delta p)\,d(\tan(\pi x'/4\,\delta p))}{\tanh^2(\pi y/4\,\delta p) + \tan^2(\pi x'/4\,\delta p)}$$
$$= \frac{1}{2p}\,\frac{\sinh(\pi y/2\,\delta p)\,dx}{\cosh(\pi y/2\,\delta p) - \sin(\pi x/2p)}$$

where $x' = \delta p - \delta x$. Using Doob's theorem we know that $\mathbf{E}[N_\zeta] = N_0$, which when we write it out gives us

$$\frac{1}{2p}\int \frac{\sinh(\pi y/2\delta p)\,dx}{\cosh(\pi y/2\delta p) - \sin(\pi x/2p)} \cos\theta\,\delta x \cos^{1-\rho}\frac{\pi x}{2p} = e^{-\theta y}\operatorname{cosec}\frac{\pi\rho}{2}\cos\theta\,\delta p \sinh^{1-\rho}\frac{\pi y}{2\delta p}$$

If we compare with 2.2 we find that the kernel should be

$$\tilde{\Pi}(p+y, dx) = \frac{1}{2p}\sin\frac{\pi\rho}{2}\left(\frac{\cos(\pi x/2p)}{\sinh(\pi y/2\delta p)}\right)^{1-\rho} \times \left[\frac{\sinh(\pi y/2\delta p)\,dx}{\cosh(\pi y/2\delta p) - \sin(\pi x/2p)}\right] \quad (2.3)$$

where $\delta = \tan\frac{\pi\rho}{2}$.

Finally we must show that $\tilde{\Pi}(y, dx)$ is uniquely determined. We begin with 2.2 and suppose μ is a bounded signed measure on $[-p, p]$ satisfying $\int_{-p}^{p} \mu(dx)\cos\theta_n\,\delta x = 0$ for all the eigenvalues $\theta_n > 0$, defined by $\cot\theta_n\,\delta p = \delta$. We claim that μ must be odd.

To see why consider the function $h(z) = \int_{-p}^{p} \mu(dx)\cos z\,\delta x$ which is analytic on the complex plane and has zeros at the points $(\pm\theta_n, n \geq 1)$. First remark that $\theta_n \sim \pi n/\delta p$, so in particular the product $u(z) = \prod_{n\geq 1}(1 - z^2\theta_n^{-2})$ converges absolutely and defines an analytic function on the complex plane. Comparison with $\cos z\,\delta p$ shows that $\lim_{z\to\pm i\infty} e^{-\delta p|z|}u(z) \geq 1$ and since by 2.2 $\tilde{\Pi}(y, dx)$ does not charge $\pm p$ it follows that $\lim_{z\to\pm i\infty} h/u(z) = 0$, something which remains true also on rays close to the imaginary axis. Next remark that by [3], p. 20 the ratio of two entire functions of finite type is again of finite type provided their ratio is entire, so if we apply the Phragmen-Lindelöf theorem to h/u on the sector $\operatorname{Arg}(z) < \frac{\pi}{2} - \varepsilon$ we find it is bounded on the sector by a constant independent of ε. Hence it is bounded on the right half plane and, being even, must be constant. We see that h vanishes by noting the behaviour of h/u on the imaginary axis. This implies that μ must be odd.

It remains to remove this possibility. Consider the harmonic conjugate of $\tilde{f}_\theta$, namely the function

$$\tilde{g}_\theta(x) = \begin{cases} \sin\theta\,\delta x & \text{if } |x| < p; \\ e^{-\theta(x-p)}\sin\theta\,\delta p & \text{if } x > p; \\ -e^{\theta(x+p)}\sin\theta\,\delta p & \text{if } x < -p, \end{cases}$$

with $\theta > 0$ chosen so that $\tan\theta\,\delta p = -\delta$. It is easy to see that $\tilde{g}_\theta(B_t)\,e^{-A_t\theta^2/2}$ is a uniformly bounded martingale until time τ and by Doob we have

$$\int_{-p}^{p} \sin\theta\,\delta x\,\tilde{\Pi}(y, dx) = e^{-\theta(y-p)}\sin\theta\,\delta p$$

Moreover this relation can be written as $\mathbf{E}[U_\zeta] = U_0$, if we take $U_t = \Im\left[e^{i\theta\tilde{Z}_t}\cos^{1-p}\dfrac{\pi\tilde{Z}_t}{2\,\delta p}\right]$, so it holds for our kernel at 2.3. Then we start from the assumption that μ is a bounded signed measure on $(-p, p)$ such that $\int_{-p}^{p}\sin\theta\,\delta x\,\mu(dx) = 0$, and by an argument as before (using $\sin z\,\delta p/z\,\delta p$ in place of $\cos z\,\delta p$) we deduce that μ must be even. Our uniqueness proof is finished.

We ought to comment on why this is termed the dual problem (terminology from [1]) since duality in probability theory implies a connection with time reversal. Notice that the solution of the dual problem is a time reversal of the original; we just switch absorbing and reflecting boundaries, and then, using essentially the same harmonic function, we run the process back again.

3. FACTORISATION

Now we tackle the original problem posed in [10], namely we want to give an explicit description of our Brownian motion B_t when it is time-changed by the inverse of our fluctuating additive functional $A_t = \int_0^t 1_{(|B_s| > p)}\,ds - \delta^2 \int_0^t 1_{(|B_s| < p)}\,ds$. By this we mean the following. Suppose we define the time changes

$$\tau_t^+ = \inf\{s : A_s > t\}; \qquad \tau_t^- = \inf\{s : A_s < -t\},$$

and write $B_t^+ = B_{\tau_t^+}$, $B_t^- = B_{\tau_t^-}$, noting that by definition both of these processes are right continuous. The problem is to describe (B_t^+, B_t^-), which we call *the Wiener-Hopf factorisation of* B_t *by the additive functional* A_t, in as much detail as possible.

On an informal level the description is clear. Suppose for example that $B_0 = x \in (-p, p)$. In this case the process B_t^+ starts off with initial distribution given by $\Pi(x, dy)$, running as a Brownian motion until it hits the boundary $\pm p$. It then jumps out at a rate governed by the boundary local time before continuing once more as a Brownian motion, and so on.

We now want to prove this rigorously, and in particular we want to give a clearer description of the jumps. For this we need certain facts from the "even problem" which is the one boundary case obtained by placing a reflecting barrier at zero. Details can be found in [9]. Remark too that since $\tau < +\infty$ the strong Markov property gives us $\tau_t^+ < +\infty$ so the process B_t^+ runs for all time.

It will be convenient to write $\bar{A}_t = \sup\{A_s : 0 < s < t\}$, and to give the description in four stages.

(1) The set $H = \{t : B_t^+ = p \text{ or } B_{t-}^+ = p\}$ is a closed, nowhere dense, perfect set.

The set is closed since B_t^+ is right continuous. To see that it is nowhere dense suppose $[T_1, T_2]$ is a non-trivial stochastic interval defined by B_t^+ stopping times throughout which $B_t^+ = 0$. But this would mean that B_t itself spends positive time at zero, something which is known to be false [6]. To prove that H is perfect, suppose not. Then there are stopping times $T_1 < \zeta < T_2$ such that $B_\zeta^+ = 0$ and such that the process has no other zeros on $[T_1, T_2]$. However if B_t^+ starts at zero then it hits zero again instantly as we see by taking limits in 0.1, so applying the strong Markov property of B_t at ζ gives a contradiction.

Our argument shows that H has in fact zero Lebesgue measure. In any case it follows from [8] that H has a continuous local time $L^+(p, t)$, defined as the unique (up to multiplicative constant) continuous increasing additive functional of B_t^+ whose closed support is precisely H.

(2) The jumps $\sum_0^t 1_{(B_{s-}^+ = p)} \Delta B_s^+$ occur as a Lévy process of bounded variation in the $L^+(p, t)$ time scale.

First of all, by reference to the "even problem" where all jumps have the same sign, this process is of finite variation. Moreover, if $\sigma^+(p, t)$ is the right continuous inverse of $L^+(p, t)$ then by the strong Markov property of B_t the process $\sum_0^{\sigma^+(p, t)} 1_{(B_{s-}^+ = p)} \Delta B_s^+$ has independent increments. That it must be a Lévy process in this time scale follows since it clearly has no fixed jumps.

This takes care of the jumps, next we remove the martingale part. Note that the process $\beta_t^+ = \int_0^{\tau_t^+} 1_{(A_s = A_s)} dB_s$ is a martingale with quadratic variation t, and so it must be a Brownian motion in the B_t^+ filtration by Paul Lévy's theorem. We can therefore introduce the following "phantom" process.

(3) The continuous process $C_t^+ = B_t^+ - \beta_t^+ - \sum_0^t \Delta B_s^+$ is supported on the boundary set H and has zero quadratic variation.

Since B_t^+ runs as a Brownian motion on any stochastic interval of B_t^+ stopping times disjoint from the boundary set we see that H supports C_t. Next, writing $B_t^+ = B_0^+ + \beta_t^+ + C_t^+ + \sum_0^t \Delta B_s^+$ we remark that by (1) the lhs has quadratic variation $t + \sum_0^t (\Delta B_s^+)^2$. On the other hand the processes β_t^+ and C_t^+ are supported on disjoint sets, so their joint quadratic variation is zero. It now follows that C_t^+ has zero quadratic variation.

Here we are using the standard probabilistic convention where one computes the quadratic variation pathwise along a refinement limit of (say) dyadic partitions.

Now the process $f_\theta(B_t)\, e^{A_t \theta^2/2}$ used in 0.1 is a local martingale and is uniformly bounded up to time τ_t^+. Time-changing we find $f_\theta(B_t^+)\, e^{t\theta^2/2}$ is a martingale. To see what martingale it is, note that by [2], p. 237, B_t^+ is a semimartingale and then use Ito's formula to get

$$f_\theta(B_t^+)\, e^{t\theta^2/2} = f_\theta(B_0^+) + \int_0^t e^{s\theta^2/2} f_\theta'(B_s^+)\, d\beta_s^+ + \sum_{0 < s \leq t} e^{s\theta^2/2}\, \Delta f_\theta(B_s^+) + \int_0^t e^{s\theta^2/2} f_\theta'(B_s^+)\, dC_s^+ \quad (3.1)$$

However this expression is simpler than it looks, as we shall now see.

(4) The process C_t^+ is identically zero.

The rhs of the above equation is a martingale so the bounded variation component is itself a totally discontinuous martingale. We have two possibilities. The first is that $C_t^+ = L^+(p, t) + L^+(-p, t)$. In which case, taking account of (2) above, we have the relation

$$\int [f_\theta(y) - f_\theta(p)]\, \nu_p^+(dy) = f_\theta'(p)$$

where ν_p^+ is the Lévy measure governing the jumps of B_t^+ out from the point p. However this is impossible, as we can see by writing f_θ explicitly since by (2) $\int (|x| \wedge 1)\, \nu_p^+ (dx) < +\infty$ and hence

$$\lim_{\theta \uparrow \infty} \theta^{-1} e^{-\theta\delta p} \int [f_\theta(y) - f_\theta(p)]\, \nu_p^+ (dy) = 0 \neq \lim_{\theta \uparrow \infty} \theta^{-1} e_\theta^{-\theta\delta p} f'(p) = \delta.$$

We have estimated the integral by splitting at $y = \theta^{-1}$ and noting that $\lim \theta^{-1} \nu_p^+ [\theta^{-1}, \infty) = 0$ $\left(\text{use } \varepsilon \nu_p^+ [\varepsilon, 1) \leqq \int_\varepsilon^1 x\, \nu_p^+ (dx) \right)$. This contradiction shows us that C_t^+ vanishes.

It now follows from 3.1 that $\sum_{0 < s \leqq t} \Delta f_\theta (B_s^+)$ is a (totally discontinuous) martingale so we are forced to conclude then that its compensator vanishes, to wit $\int [f_\theta(y) - f_\theta(p)]\, \nu_p^+ (dy) = 0$. The proof that this determines the measure uniquely is the same as in section 1 and we only need a candidate for ν_p^+. Observe however that from 0.1 and $\mathbf{P}[\tau < +\infty] = 1$ we get $\int [f_\theta(y) - f_\theta(x)]\, \Pi(x, dy) = 0$, so by renormalising and letting $x \uparrow p$ we obtain

$$\nu_p^+ (p + dy) = \frac{1}{p\delta} \cos \frac{\pi\rho}{2} \sinh^\rho \frac{\pi y}{2\delta p} \frac{dy}{\cosh(\pi y / 2\delta p) - 1}$$

Obviously the computation for the other boundary is the same, yielding

$$\nu_{-p}^+ (-p - dy) = \frac{1}{p\delta} \cos \frac{\pi\rho}{2} \sinh^\rho \frac{\pi |y|}{2\delta p} \frac{dy}{\cosh(\pi y / 2\delta p) - 1}$$

We formalise these results in the following.

THEOREM. — *The process* B_t^+ *can be written as*

$$B_t^+ = B_0^+ + \beta_t^+ + X_t^p + X_t^{-p}$$

where β_t^+ *is a Brownian motion, and* X_t^p, X_t^{-p} *are Lévy processes run in the time scales* $L^+(p, t)$, $L^+(-p, t)$ *with respective jump measures* ν_p^+, ν_{-p}^+.

There is an analogous description of the process B_t^- which lives on the interval $[-p, p]$. The proof is carried out in a similar fashion, but $\mathbf{P}[\tau < +\infty] < 1$ so we must take account of the process dying. In this case the Lévy measures are given by

$$\nu_p^- (dx) = \frac{1}{2p} \sin \frac{\pi\rho}{2} \cos^{1-\rho} \frac{\pi x}{2p} \frac{dx}{1 - \sin(\pi x / 2p)};$$

$$\nu_{-p}^- (dx) = \frac{1}{2p} \sin \frac{\pi\rho}{2} \cos^{1-\rho} \frac{\pi x}{2p} \frac{dx}{1 + \sin(\pi x / 2p)}$$

with each one having a mass

$$\gamma=\frac{2}{\pi\rho}\sin\frac{\pi\rho}{2}\mathrm{B}\left(\frac{1}{2},1-\frac{\rho}{2}\right)\frac{3-\rho}{2-\rho}$$

at infinity (these are the same by symmetry). Then the description of the process B_t^- can be written as follows.

THEOREM. – *The process* B_t^- *lives on the interval* $[-p, p]$ *and can be written as*

$$\mathrm{B}_t^- = \mathrm{B}_0^- + \beta_t^- + \mathrm{Y}_t^p + \mathrm{Y}_t^{-p}$$

where β_t^- *is a killed Brownian motion,* Y_t^p *and* Y_t^{-p} *are independent Lévy processes having jump measures* ν_p^- *and* ν_{-p}^- *and run in the respective time scales* $\mathrm{L}^-(p, t)$ *and* $\mathrm{L}^-(-p, t)$.

Proof. – The only difference from the previous case is computing the Lévy measures. An argument like before leads to the equation $\int [\tilde{f}_\theta(x)-\tilde{f}_\theta(p)]\,\nu_p^-(dx)=\gamma\,\tilde{f}_\theta(p)$ where θ is any eigenvalue. However from 0.1

$$\int [\tilde{f}_\theta(x)-\tilde{f}_\theta(p)]\,\tilde{\Pi}(y, dx)=\tilde{f}_\theta(y)-\mathbf{P}_y[\tau<+\infty]\,\tilde{f}_\theta(p)$$

so from 2.3 we deduce that

$$\nu_p^-(dx)=\frac{1}{2p}\sin\frac{\pi\rho}{2}\cos^{1-\rho}\frac{\pi x}{2p}\,\frac{dx}{1-\sin(\pi x/2p)}$$

while

$$\gamma=\lim_{y\downarrow p}\frac{1-\mathbf{P}_y[\tau<+\infty]}{\sinh^\rho(\pi y/2\delta p)}=\frac{2}{\pi\rho}\sin\frac{\pi\rho}{2}\mathrm{B}\left(\frac{1}{2},1-\frac{\rho}{2}\right)\frac{3-\rho}{2-\rho}\qquad(3.2)$$

Details of the latter computation are to be found in the appendix.

This then completes the study of the simplest two boundary example. There are other problems which fit the same pattern and some which do not. For instance the "circle case" discussed by Baker in his thesis is presumably solved by running a Brownian motion on the torus. But so far I have been unable to formulate a general theory. This "proof by conformal mapping" may be a step in the right direction.

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APPENDIX

Here we give details of how we calculated the killing constant γ in the dual problem. The computation uses certain identities for the hypergeometric function ${}_2F_1$, our source for these being [4]. The starting point is our expression for $\tilde{\Pi}(p+y,\, dx)$ derived at 2.3, which on integration gives us

$$\begin{aligned}
&\mathbf{P}_{p+y}[\tau < +\infty] \\
&\quad = \frac{1}{2p}\sin\frac{\pi\rho}{2}\int_{-p}^{p}\left(\frac{\cos(\pi x/2p)}{\sinh(\pi y/2\delta p)}\right)^{1-\rho}\left[\frac{\sinh(\pi y/2\delta p)\,dx}{\cosh(\pi y/2\delta p)-\sin(\pi x/2p)}\right] \\
&\quad = \frac{1}{\pi}\sin\frac{\pi\rho}{2}\sinh^{\rho}\frac{\pi y}{2\delta p}\int_{-1}^{1}(1-u^2)^{-\rho/2}\frac{du}{\cosh(\pi y/2\delta p)-u} \\
&\quad = \frac{1}{\pi}\sin\frac{\pi\rho}{2}\sinh^{\rho}\frac{\pi y}{2\delta p}\int_{0}^{1}\frac{1}{\sqrt{u}}(1-u)^{-\rho/2}\frac{\cosh(\pi y/2\delta p)\,du}{\cosh^2(\pi y/2\delta p)-u}
\end{aligned}$$

$$= \frac{1}{\pi} \sin \frac{\pi \rho}{2} \sinh^{\rho} \frac{\pi y}{2 \delta p} \operatorname{sech} \frac{\pi y}{2 \delta p} \mathrm{B}\left(\frac{1}{2}, 1-\frac{\rho}{2}\right) \times {}_2\mathrm{F}_1\left(1, \frac{1}{2}; \frac{3-\rho}{2}; \operatorname{sech}^2 \frac{\pi y}{2 \delta p}\right).$$

In general the hypergeometric function has singularities at the points $0, 1, \infty$. Here we are interested in the singularity at 1 which we can examine by using the identities

$${}_2\mathrm{F}_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)};$$
$${}_2\mathrm{F}_1(a, b; c; z) = (1-z)^{c-a-b}\, {}_2\mathrm{F}_1(c-a, c-b; c; z)$$

the first one being valid for $c \neq 0, -1, -2, \ldots -n, \ldots$ and $\Re(c) > \Re(a+b)$. This shows that $\mathbf{P}_{p+y}[\tau < +\infty]$ is given by

$$\frac{1}{\pi} \sin \frac{\pi \rho}{2} \cosh^{\rho-1} \frac{\pi y}{2 \delta p} \mathrm{B}\left(\frac{1}{2}, 1-\frac{\rho}{2}\right) {}_2\mathrm{F}_1\left(\frac{1-\rho}{2}, 1-\frac{\rho}{2}; \frac{3-\rho}{2}; \operatorname{sech}^2 \frac{\pi y}{2 \delta p}\right),$$

a more convenient alternative expression. For example notice how, by using the identity $\Gamma(1-\alpha)\Gamma(\alpha) = \pi \operatorname{cosec} \pi\alpha$, we find that as $y \downarrow 0$ this has limit one.

Then to get γ we use 3.2 and compute the limit

$$\lim_{y \downarrow 0} \frac{1 - \mathbf{P}_{p+y}[\tau < +\infty]}{\sinh^{\rho}(\pi y / 2 \delta p)}$$

by L'Hôpital's rule. The appropriate differentiation formulae are

$$\frac{d^n}{dx^n} {}_2\mathrm{F}_1(a, b; c; z) = \frac{(a)_n (b)_n}{(c)_n} {}_2\mathrm{F}_1(a+n, b+n, ; c+n; z)$$

with the notation $(a)_0 = 1$, $(a)_n = a(a+1) \ldots (a+n-1)$. So the limit is the same as

$$\frac{2}{\pi \rho} \sin \frac{\pi \rho}{2} \mathrm{B}\left(\frac{1}{2}, 1-\frac{\rho}{2}\right) \times \lim_{y \downarrow 0} \left[{}_2\mathrm{F}_1\left(\frac{3-\rho}{2}, 2-\frac{\rho}{2}; \frac{5-\rho}{2}; \operatorname{sech}^2 \frac{\pi y}{2 \delta p}\right) \sinh^{2-\rho} \frac{\pi y}{2 \delta p} \right]$$
$$= \frac{2}{\pi \rho} \sin \frac{\pi \rho}{2} \mathrm{B}\left(\frac{1}{2}, 1-\frac{\rho}{2}\right) {}_2\mathrm{F}_1\left(1, \frac{1}{2}; \frac{5-\rho}{2}; 1\right)$$
$$= \frac{2}{\pi \rho} \sin \frac{\pi \rho}{2} \mathrm{B}\left(\frac{1}{2}, 1-\frac{\rho}{2}\right) \frac{3-\rho}{2-\rho}$$

again using the above hypergeometric identities. And this gives what we want.

Remark. — Using the relation

$$ {}_2F_1(a, b, ; c ; z) = (1-z)^{-a} {}_2F_1(a, c-b, ; c ; z/(z-1)) $$

we see that $\mathbf{P}_{p+y}[\tau < +\infty] \downarrow 0$ exponentially fast with rate parameter $\pi(1-\rho)/\delta p$ as $y \uparrow +\infty$.

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