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The bootstrap of the mean with arbitrary bootstrap sample size

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ABSTRACT. — We study the bootstrap central limit theorem a. s. and in probability, for random variables X with infinite second moment, in the domain of attraction of the normal law, or of the other stable laws, or in the partial domain of attraction of an infinitely divisible law, and with the size m_n of the bootstrap sample not necessarily equal to (and in some cases necessarily different from) the size n of the original sample.

Main results:

(1) If X is in the domain of attraction of the normal law then it satisfies the bootstrap CLT in probability for all $m_n \to \infty$;

(2) If $\{m_n\}$ is a regular sequence such that $m_n (\log\log n)/n \to 0$, and X is in the domain of attraction of a stable law then the bootstrap CLT holds a. s.; but it does not hold a. s. if $EX^2 = \infty$ and $\inf m_n (\log\log n)/n > 0$.

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RÉSUMÉ. — Nous étudions le théorème central limite « bootstrap » presque sûr et en probabilité, pour des variables aléatoires X de second moment infini, dans le domaine d'attraction de la loi normale ou des autres lois stables, ou dans le domaine partiel d'attraction d'une loi infiniment divisible. La taille m_n de l'échantillon « bootstrap » n'est pas nécessairement égale (et même dans certains cas elle est nécessairement différente) à la taille n de l'échantillon original.

Les résultats principaux sont :

(i) Si X est dans le domaine d'attraction de la loi normale, il vérifie le théorème central limite « bootstrap » en probabilité, quelle que soit m_n tendant vers ∞.

(2) Si $\{m_n\}$ est une suite régulière telle que $m_n(\log\log n)/n \to 0$, et si X est dans le domaine d'attraction d'une loi stable, alors le théorème central limite « bootstrap » a lieu presque sûrement. Mais il n'est pas exact presque sûrement si $EX^2=\infty$ et $\inf m_n(\log\log n)/n > 0$.

1. INTRODUCTION

In this article we study the bootstrapped central limit theorem (the bootstrap of the mean) for sizes m_n of the bootstrapped sample $\{X_{nj}^{\omega}\}_{j=1}^{m_n}$ different from the size n of the original sample $\{X_i\}_{i=1}^{n}$. Our work is motivated by results of Bickel and Freedman [5], Athreya ([3], [4]), and Csörgö and Mason [6]. To describe their results as well as ours we require some definitions and notation that will be in force for the rest of the paper. Let $X, X_1, X_2, \ldots$ be independent identically distributed (i. i. d.) real random variables and, for each $n\in\mathbb{N}$ and $\omega\in\Omega$, let $X_{nj}^{\omega}(\omega')$, $j\in\mathbb{N}$, be i. i. d. random variables with law $P_n(\omega)=n^{-1}\sum_{i=1}^{n}\delta_{X_i(\omega)}$, defined in another probability space Ω'. The bootstrap variables $X_{nj}^{\omega}(\omega')$ can be realized on the product probability space $\Omega\times\Omega'$ as follows: let $\{A_{1j}, \ldots, A_{nj}\}$ be disjoint independent partitions of Ω' with $P' A_{ij}=1/n$ for all i, j; then $X_{nj}^{\omega}(\omega')=\sum_{i=1}^{n} X_i(\omega)\, I_{A_{ij}(\omega')}$. In what follows we drop reference to ω' and write $\hat{P}$ for P' and $\hat{\mathscr{L}}$ for conditional distribution given $\{X_i\}$. Let d be any distance metrizing weak convergence in $\mathbb{R}$. We write

$$\hat{\mathscr{L}}\left[\sum_{j=1}^{m_n} X_{nj}^{\omega}/a_n - c_n(\omega)\right] \to_w \mu \quad \text{in probability} \tag{1.1}$$

to mean

$$d\left[\hat{\mathscr{L}}\left[\sum_{j=1}^{m_n} X_{nj}^{\omega}/a_n - c_n(\omega)\right], \mu\right] \to 0 \quad \text{in probability.} \tag{1.1$'$}$$

If (1.1)′ holds for some such d, it holds for all, and even for the sup norm of distribution functions if μ is continuous [12]. The notation

$$\hat{\mathscr{L}}\left[\sum_{j=1}^{m_n} X_{nj}^{\omega}/a_n - c_n(\omega)\right] \to_w \mu \quad \text{a.s.} \tag{1.2}$$

is self explanatory, and is obviously equivalent to

$$d\left[\hat{\mathscr{L}}\left[\sum_{j=1}^{m_n} X_{nj}^{\omega}/a_n - c_n(\omega)\right], \mu\right] \to 0 \quad \text{a.s.} \tag{1.2$'$}$$

so that in particular the "a.s. boostrap CLT" (1.2) implies the "bootstrap CLT in probability" (1.1). Ideally a_n is not a constant, but a function of the sample $X_1, \ldots, X_n$: this will end up being the case here whenever μ is the normal law $N(0, \sigma^2)$. If either (1.1) or (1.2) hold and also

$$\mathscr{L}\left[\sum_{i=1}^{n} X_i/b_n - c_n\right] \to_w \mu \tag{1.3}$$

then, as is well known, the bootstrap limit theorems (1.1) and (1.2) can be used to estimate asymptotic probabilities of sets defined through $\sum_{i=1}^{u} X_i/b_n - c_n$ (e.g. [7], [5]; *see* also [11]). It is customary to take $m_n = n$ in (1.1) and (1.2), but this is not necessary and, if μ in (1.3) is not normal, then it is even impossible ([3], [12]).

Bickel and Freedman [5] show that if $EX^2 < \infty$ then the bootstrap CLT (1.2) holds a.s. as long as $m_n \to \infty$. But for $EX^2 = \infty$ the situation is different: Athreya [4] and Csörgö and Mason [6] prove that that if $EX^2 = \infty$, μ is normal and X satisfies (1.3) then the bootstrap CLT in probability (1.1) holds for $0 < C_1 < m_n/n < C_2 < \infty$ (it is proved in [12] that the a.s. bootstrap CLT does not hold if $EX^2 = \infty$ and $m_n = n$). Athreya also shows in [4] that if X is in the domain of attraction of a p-stable law μ, $0 < p \leqq 2$, with norming constants b_n then (1.1) holds for $a_n = b_{m_n}$ and $m_n/n \to 0$ (it does not hold if $m_n = n$ [12]). These results obviously suggest the following questions: Can $m_n \to \infty$ be taken to be arbitrary if $EX^2 = \infty$ and μ is normal? Under what conditions on $\{m_n\}$ does the a.s.

bootstrap CLT hold if μ is p-stable, $0 < p < 2$? If (1.3) holds only along a subsequence $\{n'\} \subset \mathbb{N}$ (*i.e.* X is in the domain of partial attraction of μ) does (1.1) or (1.2) also hold along $\{n'\}$? These are the questions we consider in this article.

In Section 2, we determine essentially all the situations for which the bootstrap CLT in probability (1.1) holds. We show in particular that if μ is normal, $EX^2 = \infty$ and (1.3) holds, then (1.1) holds too with $a_n = b_{m_n}$ if $m_n \leqq n$, and $a_n = b_n (m_n/n)^{1/2}$ if $m_n > n$. In the normal case, using the fact that $\sum_{i=1}^{n} X_i^2/b_n^2 \to \sigma^2$ in probability, a_n can be estimated from the sample without changing the limit in (1.1). Results for X in the domain of partial attraction of an infinitely divisible law are also given; these imply, in particular, Athreya's result for μ p-stable and $m_n/n \to 0$.

In Section 3, we consider the a. s. bootstrap CLT. We prove that if $EX^2 = \infty$ and X is in the domain of attraction of a p-stable law μ then the a. s. bootstrap CLT does not hold for $\inf_n m_n (\log\log n)/n > 0$. On the other hand, at least under regularity ($\{m_n\}$ non-decreasing and $m_n/m_{2n} \geqq c > 0$ for all $n \in \mathbb{N}$) if $m_n (\log\log n)/n \to 0$ then the bootstrap CLT does hold a. s.

The methods of proof are not new: we repeatedly check the usual conditions for the central limit theorem in $\mathbb{R}$, very much as in [12]. We can do this because, as observed in [12], if $a_n \to \infty$ then the system $\{X_{nj}^{\omega}/a_n : j = 1, \ldots, m_n\}_{n=1}^{\infty}$ is a. s. infinitesimal by the law of large numbers: just note

$$\max_{j \leqq m_n} \hat{P}\{|X_{nj}^{\omega}| > \lambda a_n\} = n^{-1} \sum_{i=1}^{n} I(|X_i| > \lambda a_n) \to 0 \quad \text{a. s.}$$

The results on the a. s. bootstrap rely on the usual techniques for the law of the iterated logarithm, suitably modified. This work owes much to [12] on technique, particularly for Section 2.

2. The Bootstrap CLT in Probability

The following theorem gives necessary conditions for the bootstrap in probability. Its proof closely follows [12]. We use the notation c_τ Pois π for a generalized Poisson measure with Lévy measure π, as in [2]. If a sequence $\{c_n\}$ is non-decreasing and $c_n \to \infty$, we write $c_n \nearrow \infty$.

2.1. THEOREM. – *Let* X *be a random variable for which there exist a sequence of positive integers* $m_n \nearrow \infty$, *a sequence of positive real numbers*

$a_n \nearrow \infty$, random variables $c_n(\omega)$, $n \in \mathbb{N}$, and a random probability measure $\mu(\omega)$, non-degenerate with positive probability, such that

$$\hat{\mathscr{L}}\left[\sum_{j=1}^{m_n} X^{\omega}_{nj}/a_n - c_n(\omega)\right] \to_w \mu(\omega) \quad \textit{in probability.} \tag{2.1}$$

Then:

(*a*) *There are a Lévy measure π and a real number $\sigma^2 \geqq 0$ such that for all τ with $\pi\{-\tau, \tau\}=0$,*

$$\hat{\mathscr{L}}\left[\sum_{j=1}^{m_n}\left[X^{\omega}_{nj} - \frac{1}{n}\sum_{i=1}^{n} X_i(\omega) I(|X_i(\omega)| \leqq \tau a_n)\right]/a_n\right]$$
$$\to_w N(0, \sigma^2) * c_\tau \operatorname{Pois} \pi \quad \textit{in probability.} \tag{2.1)'$$

(*b*) *If c is a positive number and*

$$b_{n,c} = \begin{cases} a_n & \textit{for } m_n \leqq cn \\ a_n (n/m_n)^{1/2} & \textit{for } m_n > cn, \end{cases} \qquad r_{n,c} = \begin{cases} m_n & \textit{for } m_n \leqq cn \\ n & \textit{for } m_n > cn \end{cases} \tag{2.2}$$

then, for all τ with $\pi\{-\tau, \tau\}=0$,

$$\mathscr{L}\left[\sum_{i=1}^{r_{n,c}} (X_i - EXI(|X| \leqq \tau b_{n,c})/b_{n,c}\right] \to_w N(0, \sigma^2) * c_\tau \operatorname{Pois} \pi. \tag{2.3}$$

(*c*) *If $\limsup_{n\to\infty} m_n/n > 0$ then $\pi = 0$ and $\sigma^2 \neq 0$ in* (*a*) *and* (*b*).

(*d*) *If $\liminf_{n\to\infty} m_n/n > 0$ then X is in the domain of attraction of the normal law with norming constants $b_n = a_n (n/m_n)^{1/2}$, that is*

$$\mathscr{L}\left[\sum_{i=1}^{n} (X_i - EX)/b_n\right] \to_w N(0, \sigma^2) \tag{2.4}$$

(*e*) *If* (2.1) *holds only along a subsequence $\{n'\} \subset \mathbb{N}$ with $\{m_{n'}\} \subset \{n'\}$, then* (*a*)-(*d*) *hold but with n' instead of n* [*in* (*d*) (2.4) *holds for n' but X is not necessarily in the domain of attraction of* $N(0, \sigma^2)$].

Proof. — If $EX^2 < \infty$ this theorem is a consequence of Theorem 2.1 in [5]. So, we assume $EX^2 = \infty$. As shown in the introduction, $\{X^{\omega}_{nj}/a_n\}$ is an infinitesimal system, a fact that will be used throughout without further mention. (2.1) holds if and only if every subsequence has a further subsequence along which the limit in (2.1) holds a. s. Hence, $\mu(\omega)$ is a. s. infinitely divisible. Let $\{n'\}$ be such a subsequence. The fact that $a_{n'} \to \infty$ readily gives, by the corresponding argument in [12], that the Lévy measure $\pi(\omega)$ of $\mu(\omega)$ is a. s. a fixed Lévy measure π. Moreover, if D is a countable set of points δ dense in $\mathbb{R}^+$ such that $\pi\{-\delta, \delta\}=0$, then the following limits hold for all $\delta \in D$ almost surely (by the general CLT in $\mathbb{R}$, e. g. [2]

Chapter 2):

$$\frac{m_{n'}}{n'}\sum_{i=1}^{n'} I(X_i > \delta a_{n'}) \to \pi(\delta, \infty)$$
$$\frac{m_{n'}}{n'}\sum_{i=1}^{n'} I(X_i < -\delta a_{n'}) \to \pi(-\infty, -\delta) \tag{2.5}$$

and

$$\frac{m_{n'}}{n' a_{n'}^2}\sum_{i=1}^{n'} X_i^2 I(|X_i| \leqq \delta a_{n'}) - \frac{m_{n'}}{n'^2 a_{n'}^2}\left[\sum_{i=1}^{n'} X_i I(|X_i| \leqq \delta a_{n'})\right]^2 \to \sigma^2(\omega) + \int_{-\delta}^{\delta} x^2 \, d\pi(x) \tag{2.6}$$

where $\sigma^2(\omega)$ is the variance of the normal component of $\mu(\omega)$. Since $EX^2 = \infty$, the law of large numbers implies $\sum_{i=1}^{n} X_i^2 I(|X_i| \leqq \delta a_n)/n \to \infty$ a.s. and this is all that is needed to prove, exactly as in Lemma 3 of [12], that

$$\frac{\left[\sum_{i=1}^{n} X_i I(|X_i| \leqq \delta a_n)/n\right]^2}{\sum_{i=1}^{n} X_i^2 I(|X_i| \leqq \delta a_n)/n} \to 0 \quad \text{a.s.} \tag{2.7}$$

Then (2.6) becomes

$$\frac{m_{n'}}{n' a_{n'}^2}\sum_{i=1}^{n'} X_i^2 I(|X_i| \leqq \delta a_{n'}) \to \sigma^2(\omega) + \int_{-\delta}^{\delta} x^2 \, d\pi(x) \quad \text{a.s.} \tag{2.8}$$

In particular, $m_{n'}/n' a_{n'}^2 \to 0$ (even $m_{n'}/a_{n'}^2 \to 0$ by the law of large numbers) and therefore $\sigma^2(\omega)$ is a tail random variable, hence a.s. constant, say σ^2. We have thus shown that $\mu(\omega)$ is a.s. a (possibly random) shift of $\mu = N(0, \sigma^2) * c_\tau \text{Pois}\, \pi$. Moreover (2.5) and (2.6) show, by the general CLT in $\mathbb{R}$, that (2.1)′ holds a.s. along n'. Therefore (2.1)′ holds.

We will prove (*b*) for $c=1$, the case of general c being entirely similar. Set $b_n = b_{n,1}$, and $r_n = r_{n,1} = m_n \wedge n$. Consider a subsequence for which (2.1) converges a.s. Then, either $m_n/n \leqq 1$ infinitely often along this subsequence, or $m_n/n > 1$ i.o. or both. So, we may specialize to two types of subsequences n' for which (2.1) holds a.s.: those satisfying $m_{n'}/n' \leqq 1$, and those for which $m_{n'}/n' > 1$. In the first case the summands in (2.5) and (2.8) are bounded and therefore their expected values converge to the expected values of their respective limits (by e.g. [1], Theorem 3.2 or exercises 13 and 14, p. 69, 70 in [2]). Thus, we obtain for all $\delta \in D$,

$$m_{n'} P\{X > \delta a_{n'}\} \to \pi(\delta, \infty), \qquad m_{n'} P\{X < -\delta a_{n'}\} \to \pi(-\infty, -\delta)$$

and

$$\frac{m_{n'}}{a_{n'}^2} EX^2 I(|X| \leq \delta a_{n'}) \to \sigma^2 + \int_{-\delta}^{\delta} x^2 \, d\pi(x).$$

These three limits imply [recall $EX^2 = \infty$ and (2.7)] by the general CLT in $\mathbb{R}$ (e. g. [2], Chapter 2) that the limit (2.3) holds along $\{n'\}$ ($b_{n'} = a_{n'}$ in this case). Suppose now $m_{n'}/n' > 1$ and that (2.1) holds a. s. along n'. Then, by taking a further subsequence if necessary, we may assume $m_{n'}/n' \to c \in [1, \infty]$. If $c = \infty$ then (2.5) implies $\sum_{i=1}^{n'} I(|X_i| > \delta a_{n'}) \to 0$ a. s. If $c \in (0, \infty)$ then the corresponding argument (on binomial limits) in the proof of Theorem 1 in [12] gives also that $\sum_{i=1}^{n'} I(|X_i| > \delta a_{n'}) \to 0$ a. s. Then $\sum_{i=1}^{n'} I(|X_i| > \delta a_{n'}) = 0$ eventually a. s. (since this sequence is integer valued and tends to 0), which implies that the limits in (2.5) are 0 a. s. that is, $\pi = 0$. This argument already proves (*c*). Then, since $\sum_{i=1}^{n'} X_i^2 I(|X_i| > \delta a_{n'}) = 0$ eventually a. s., (2.8) becomes

$$\frac{m_{n'}}{n' a_{n'}^2} \sum_{i=1}^{n'} X_i^2 \to \sigma^2 \quad \text{a. s.} \tag{2.9}$$

We can apply the converse CLT (e. g. [2], p. 61) to (2.9) and obtain [recall $b_{n'} = a_{n'} (n'/m_{n'})^{1/2}$ in this case] that for all $\delta > 0$,

$$n' P\{|X| > \delta b_{n'}\} \to 0, \qquad \frac{n'}{b_{n'}^2} EX^2 I(|X| \leq \delta b_{n'}) \to \sigma^2. \tag{2.10}$$

But (2.10) implies by the CLT in $\mathbb{R}$ (recall $EX^2 = \infty$),

$$\mathcal{L}\left[\sum_{i=1}^{n'} (X_i - EX)/b_{n'}\right] \to N(0, \sigma^2) \tag{2.11}$$

[where EX can be replaced by $EXI(|X| \leq \delta b_{n'})$] *i.e.* (2.3) for the sequence $\{n'\}$. This argument already proves (*d*) since it gives (2.4) along a subsequence of every subsequence. We have thus proved that every subsequence has a further subsequence along which the limit (2.3) holds. Hence, (2.3) holds. (*a*)-(*d*) are proved. The proof of (*e*) follows exactly along the same lines since nothing in the above arguments depends on the sequence $\{n\}$ being all of $\mathbb{N}$. [ ]

Before proving the converse of Theorem 2.1 in the general case, we will consider the case of X in the domain of attraction of the normal law. There are two reasons for this: one is its importance, and the other is that in this case we have regular variation as an additional convenient tool. Theorem 2.2 was obtained by Athreya [4] in the special case $m_n = n$ and by Csörgö and Mason [6] for $0 < c_1 \leqq m_n/n \leqq c_2 < \infty$. We extend their results to arbitrary $\{m_n\}$.

2.2 THEOREM. – *Let* X *be in the domain of attraction of the normal law. Concretely, assume there are constants* $b_n \nearrow \infty$ *such that*

$$\mathscr{L}\left[\sum_{i=1}^{n} (X_i - EX)/b_n\right] \to_w N(0, 1). \tag{2.12}$$

Let $m_n \to \infty$. *For* $c \in (0, \infty)$ *fixed, let*

$$a_n = \begin{cases} b_{m_n} & \text{if } m_n \leqq cn \\ b_n (m_n/n)^{1/2} & \text{if } m_n > cn \end{cases}$$

$$\tilde{X}_n^{\omega} = \begin{cases} \bar{X}_n(\omega) := n^{-1} \sum_{i=1}^{n} X_i(\omega) & \text{if } m_n \leqq cn \\ n^{-1} \sum_{i=1}^{n} X_i(\omega) I(|X_i(\omega)| \leqq a_n) & \text{if } m_n > cn \end{cases}$$

Then,

$$\hat{\mathscr{L}}\left[\sum_{j=1}^{m_n} (X_{nj}^{\omega} - \tilde{X}_n^{\omega})/a_n\right] \to_w N(0, 1) \quad \textit{in probability} \tag{2.13}$$

Proof. – The case $EX^2 < \infty$ is proved in [5]; see the beginning of Section 3 below for an alternative simple proof that in this case (2.13) holds a.s. So, we can (and do) assume $EX^2 = \infty$. To dispense with the centering for the rest of the proof note that once shift tightness of the law of $\sum_{j=1}^{m_{n'}} X_{nj}^{\omega}/a_{n'}$ has been established for a subsequence $\{n'\}$ of $\mathbb{N}$ then the centering variables prescribed by the usual theory are

$$a_{n'}^{-1} m_{n'} \hat{E} X_{n'1}^{\omega} I(|X_{n'1}^{\omega}| \leqq a_{n'}) = \frac{m_{n'}}{n' a_{n'}} \sum_{i=1}^{n'} X_i I(|X_i| \leqq a_{n'});$$

hence, $a_{n'}^{-1} m_{n'} \bar{X}_{n'}(\omega)$ is allowed as centering if

$$\frac{m_{n'}}{n' a_{n'}} \sum_{i=1}^{n'} X_i I(|X_i| > a_{n'}) \to 0 \quad \text{in probability} \tag{2.14}$$

for $m_{n'}/n' \leqq c$. Define $U(t)=EX^2 I(|X| \leqq t)$ and note that $E|X| I(|X|>t)=\int_t^\infty dU(s)/s$ $\left(\text{since } Ef(|X|)=\int_0^\infty \frac{f(s)}{s^2}\,dU(s) \text{ in general}\right)$. Integrating by parts and using Theorem 1, p. 281 in [8] we obtain

$$\frac{m_{n'}}{n' a_{n'}} E\left|\sum_{i=1}^{n'} X_i I(|X_i|>a_{n'})\right| \leqq \frac{m_{n'}}{b_{m_{n'}}} E|X| I(|X|>b_{m_{n'}}) = \frac{m_{n'}}{b_{m_{n'}}^2} U(b_{m_{n'}})\,\varepsilon_{n'}$$

where $\varepsilon_{n'} \to 0$. Then, since $\frac{m_n}{b_{m_n}} U(b_{m_n}) \to 1$ (by (2.12) and the classical domains of attraction theorem – e. g. [2], p. 86), (2.14) follows.

For every subsequence there is a futher subsequence $\{n'\}$ such that either $m_{n'}/n' \to 0$ or $m_{n'}/n' \to \lambda \in (0, \infty]$. In the first case we may assume $a_{n'}=b_{m_{n'}}$. By (2.12) and the domains of attraction theorem, normal case,

$$m_{n'} P\{|X|>\delta b_{m_{n'}}\} \to 0 \quad \text{for all } \delta>0$$

and (2.15)

$$\frac{m_{n'}}{b_{m_{n'}}^2} U(b_{m_{n'}}) \to 1.$$

Therefore,

$$m_{n'} E(\hat{P}\{|X_{n' 1}^{\omega}|>\delta b_{m_{n'}}\}) = \frac{m_{n'}}{n'} E\left[\sum_{i=1}^{n'} I(|X_i|>\delta b_{m_{n'}})\right] = m_{n'} P\{|X|>\delta b_{m_{n'}}\} \to 0,$$

$$E\left[\frac{m_{n'}}{b_{m_{n'}}^2} \hat{E}(X_{n' 1}^{\omega})^2 I(|X_{n' 1}^{\omega}| \leqq b_{m_{n'}})\right] = \frac{m_{n'}}{n' b_{m_{n'}}^2} E\left[\sum_{i=1}^{n'} X_i^2 I(|X_i| \leqq b_{m_{n'}})\right] = \frac{m_{n'}}{b_{m_{n'}}^2} U(b_{m_{n'}}) \to 1$$

and

$$\operatorname{Var}\left(\frac{m_{n'}}{b_{m_{n'}}^2} \hat{E}(X_{n' 1}^{\omega})^2 I(|X_{n' 1}^{\omega}| \leqq b_{m_{n'}})\right) \leqq \frac{m_{n'}^2}{n' b_{m_{n'}}^4} EX^4 I(|X| \leqq b_{m_{n'}}) \leqq \frac{m_{n'}}{n'}\left(\frac{m_{n'}}{b_{m_{n'}}^2} U(b_{m_{n'}})\right) \to 0.$$

Therefore,

$$m_{n'} \hat{P}\{|X_{n' 1}^{\omega}|>\delta b_{m_{n'}}\} \to 0 \quad \text{in probability}$$

and

$$\frac{m_{n'}}{b_{m_{n'}}^2}\,\hat{\mathrm{E}}\,(\mathrm{X}_{n'\,1}^{\omega})^2\,\mathrm{I}\,(|\mathrm{X}_{n'\,1}^{\omega}|\leqq b_{m_{n'}})\to 1 \quad \text{in probability.}$$

Then, for a subsequence $\{n''\}$ of $\{n'\}$ these limits hold a. s. for all $\delta\in\mathrm{D}$, and the CLT, normal convergence case (e. g. [2], p. 63), together with (2.7), gives

$$\hat{\mathscr{L}}\left[\sum_{j=1}^{m_{n''}}(\mathrm{X}_{n''\,j}^{\omega}-\tilde{\mathrm{X}}_{n''}^{\omega})/a_{n''}\right]\to_w \mathrm{N}(0,\,1) \quad \text{a. s.} \tag{2.16}$$

Suppose now $m_{n'}/n'\to\lambda\in(0,\,\infty]$. We can assume $a_{n'}=b_{n'}(m_{n'}/n')^{1/2}$ since this holds at least from some n' on if $\lambda=\infty$, and if $\lambda<\infty$ then $b_{m_{n'}}/b_{n'}\to\lambda^{1/2}$ by regular variation. We observe first that, by (2.12),

$$\sum_{i=1}^{n}\mathrm{X}_i^2/b_n^2\to 1 \quad \text{in probability.}$$

This is just Raikov's theorem (e. g. [9]), but it follows trivially from

$n\mathrm{P}\{|\mathrm{X}|>b_n\}\to 0,$

$$\frac{n}{b_n^2}\,\mathrm{U}\,(b_n)\to 1 \qquad \text{and} \qquad \frac{n}{b_n^4}\,\mathrm{Var}\,[\mathrm{X}^2\,\mathrm{I}\,(|\mathrm{X}|\leqq b_n)]\to 0;$$

the first two limits are necessary (and sufficient) for (2.12) to hold, and the last one is a consequence of

$$\mathrm{EX}^4\,\mathrm{I}\,(|\mathrm{X}|\leqq a)=\int_0^a t^2\,d\mathrm{U}\,(t)=a^2\,\mathrm{U}\,(a)-2\int_0^a t\,\mathrm{U}\,(t)\,\mathrm{d}t$$

and theorem 1, p. 281 in [8], which give

$$\frac{n}{b_n^4}\,\mathrm{Var}\,(\mathrm{X}^2\,\mathrm{I}\,(|\mathrm{X}|\leqq b_n)\leqq\frac{n}{b_n^4}\,\mathrm{EX}^4\,\mathrm{I}\,(|\mathrm{X}|\leqq b_n)$$

$$=\frac{n}{b_n^2}\,\mathrm{U}\,(b_n)\left[1-\frac{2\int_0^{b_n}t\,\mathrm{U}\,(t)\,dt}{b_n^2\,\mathrm{U}\,(b_n)}\right]\to 0.$$

Hence we have, by the definition of $a_{n'}$,

$$\frac{m_{n'}}{n'\,a_{n'}^2}\sum_{i=1}^{n'}\mathrm{X}_i^2\to 1 \quad \text{in probability.} \tag{2.17}$$

Since in the case we are considering $a_{n'}>db_{n'}$ for some constant $d>0$, we have $n'\,\mathrm{P}\{|\mathrm{X}|>\delta a_{n'}\}\leqq n'\,\mathrm{P}\{|\mathrm{X}|>\delta db_{n'}\}\to 0$ for all $\delta>0$. This implies

$\sum_{i=1}^{n'} I(|X_i| > \delta a_{n'}) \to 0$ in probability, hence

$$P\left\{\sum_{i=1}^{n'} I(|X_i| > \delta a_n) > 0\right\} \to 0;$$

hence, even if $\lambda = \infty$,

$$\frac{m_{n'}}{n'} \sum_{i=1}^{n'} I(|X_i| > \delta a_{n'}) \to 0 \quad \text{in probability.}$$

Moreover, this limit and (2.17) give

$$\frac{m_{n'}}{n' a_{n'}^2} \sum_{i=1}^{n'} X_i^2 I(|X_i| \leqq a_{n'}) \to 1 \quad \text{in probability.}$$

So, these two limits hold a. s. along some subsequence $\{n''\}$ of $\{n'\}$ and all $\delta \in D$. For this subsequence the usual CLT and (2.7) implies (2.16). We have thus proved that (2.16) holds along some subsequence of $\{n''\}$ of any subsequence of $\mathbb{N}$. Hence (2.13) holds. [ ]

2.3. *Remark* (estimating a_n from the sample). – In Theorem 2.2, let

$$Y_n^{\omega} = \sum_{j=1}^{m_n} [X_{nj}^{\omega} - \bar{X}_n^{\omega}]/a_n \qquad \text{and} \qquad Z_n^{\omega} = \sum_{j=1}^{m_n} [X_{nj}^{\omega} - \bar{X}_n^{\omega}]/\hat{a}_n^{\omega}$$

with $\hat{a}_n^{\omega}(\omega')/a_n \to 1$ in probability (as a random variable defined on $\Omega \times \Omega'$). Then it is easy to prove (by e. g. passing to subsequences) that

$$\mathscr{L}(Z_n^{\omega}) \to_w N(0, 1) \quad \text{in probability.} \tag{2.18}$$

Since $\sum_{i=1}^{n} X_i^2/b_n^2 \to 1$ in probability (2.18) shows that a_n can be replaced in (2.13) by the following function a_n^{ω} of the sample:

$$\hat{a}_n^{\omega} = \begin{cases} \left[\dfrac{m_n}{n} \sum_{i=1}^{n} X_i^2(\omega)\right]^{1/2} & \text{if } m_n \geqq n \\ \text{average over all the } \dbinom{n}{m_n} \text{ combinations} & \\ 0 < j_1 < \ldots < j_{m_n} \leqq n \text{ of } \left[\sum_{i=1}^{m_n} X_{j_i}^2\right]^{1/2} & \text{if } m_n < n. \end{cases} \tag{2.19}$$

[To check that (2.19) satisfies $|\hat{a}_n^{\omega}/a_n - 1| \to 0$ in pr. just note that $\sum_{i=1}^{n} X_i^2/b_n^2 \to 1$ in L_p for every $p < 1$ (e. g. Theorem 3.2 in [1]); then the case

$m_n \geqq n$ is obvious and the case $m_n < n$ follows by Chebyshev's inequality.] The choice (2.19) of $\hat{a}_n^\omega$ may not be too practical for $m_n < n$. Another possible choice, for all n, is

$$\hat{a}_n^\omega(\omega') = \left[\sum_{j=1}^{m_n} (X_{nj}^\omega(\omega'))^2\right]^{1/2} \tag{2.20}$$

[Actually, by Raykov's theorem for every subsequence there is another subsequence $\{n'\}$ such that $\hat{a}_{n'}^\omega / a_{n'} \to 1$ in ω'-probability; then dominated convergence on $\hat{P}\left\{\left|\frac{\hat{a}_{n'}^\omega}{a_{n'}} - 1\right| > \varepsilon\right\} \to 0$ shows that $\hat{a}_{n'}^\omega(\omega')/a_{n'} \to 1$ in (ω, ω')-probability.]

2.4. *Remark* (On the centering in Theorem 2.2.). – The centering $\tilde{X}_n^\omega$ in (2.13) cannot in general be replaced by $\bar{X}_n^\omega$. For instance, if the law of X has density $c(\log x)/x^3$, $x \geqq 1$ and $m_n = 2^n$, then $b_n \simeq n^{1/2} \log n$ and

$$E\left[\frac{m_n}{n a_n} \sum_{i=1}^{n} X_i I(|X_i| > a_n)\right] \simeq n/(\log n)^4 \to \infty.$$

Finally we consider the general case of X in the domain of *partial* attraction of an infinitely divisible law. Without regular variation for $U(t)$ we do not know how to treat the case $m_{n'}/n' \to \infty$ in the following theorem.

2.5. THEOREM. – *Let* X *satisfy*

$$\mathscr{L}\left[\sum_{i=1}^{m_{n'}} (X_i - EXI(|X| \leqq \tau a_{n'}))/a_{n'}\right] \to_w N(0, \sigma^2) * c_\tau \operatorname{Pois} \pi \tag{2.21}$$

for some $n' \nearrow \infty$, $a_{n'} \nearrow \infty$, $\sigma^2 \geqq 0$ *and Lévy measure* π (*possibly* 0), *with* τ *such that* $\pi\{-\tau, \tau\} = 0$. *Let* $m_{n'} \nearrow \infty$, $\{m_{n'}\} \subset \{n'\}$. *Then, if* $m_{n'}/n' \to 0$ *or if* $\pi = 0$ *and* $\sup_{n'} m_{n'}/n' < \infty$,

$$\mathscr{L}\left[\sum_{j=1}^{m_{n'}} \left[X_{n'j}^\omega - \frac{1}{n'} \sum_{i=1}^{n'} X_i(\omega) I(|X_i(\omega)| \leqq \tau a_{m_{n'}})\right] / a_{m_{n'}}\right] \to_\omega N(0, \sigma^2) * c_\tau \operatorname{Pois} \pi \tag{2.22}$$

in probability.

Proof. – Here again, we can assume $EX^2 = \infty$. Let us first consider the case $m_{n'}/n' \to 0$. By the usual arguments (with subsequences and the

CLT) it suffices to show that for every δ in D,

$$m_{n'}\hat{P}\{X^{\omega}_{n'1} > \delta a_{m_{n'}}\} = \frac{m_{n'}}{n'}\sum_{i=1}^{n'} I(X_i > \delta a_{m_{n'}}) \to \pi(\delta, \infty) \quad \text{in probability,} \tag{2.23}$$

$$m_{n'}\hat{P}\{X^{\omega}_{n'1} < -\delta a_{m_{n'}}\} = \frac{m_{n'}}{n'}\sum_{i=1}^{n'} I(X_i < -\delta a_{m_{n'}}) \to \pi(-\infty, -\delta) \quad \text{in probability,} \tag{2.24}$$

and

$$\frac{m_{n'}}{a^2_{m_{n'}}}\hat{V}\mathrm{ar}(X^{\omega}_{n'} I(|X^{\omega}_{n'1}| \leqq \delta a_{m_{n'}})) = (1+o_P)\frac{m_n}{n' a^2_{m_{n'}}}\sum_{i=1}^{n'} X_i^2 I(|X| \leqq \delta a_{m_{n'}}) \to \sigma^2 + \int_{-\delta}^{\delta} x^2\, d\pi(x) \quad \text{in probability} \tag{2.25}$$

where o_P denotes a sequence of random variables tending to zero in probability. [The equal sign in (2.25) follows from (2.7).] (2.23) follows from the following two limits which hold by the converse CLT applied to (2.21) and $m_{n'}/n' \to 0$:

$$E\left[\frac{m_{n'}}{n'}\sum_{i=1}^{n'} I(X_i > \delta a_{m_{n'}})\right] = m_{n'} P\{X > \delta a_{m_{n'}}\} \to \pi(\delta, \infty)$$

and

$$\mathrm{Var}\left[\frac{m_{n'}}{n'}\sum_{i=1}^{n'} I(X_i \geqq \delta a_{m_{n'}})\right] \leqq \frac{m^2_{n'}}{n'} P\{X > \delta a_{m_{n'}}\} = \frac{m_{n'}}{n'}[m_{n'} P\{X > \delta a_{m_{n'}}\}] \to 0.$$

(2.24) is proved in exactly the same way. As for (2.25) we similarly note that [by (2.7)]

$$E\left[\frac{m_{n'}}{n' a^2_{m_{n'}}}\sum_{i=1}^{n'} X_i^2 (I(|X_i| \leqq \delta a_{m_{n'}})\right] = \frac{m_{n'}}{a^2_{m_{n'}}} U(\delta a_{m_{n'}}) \to \sigma^2 + \int_{-\delta}^{\delta} x^2\, d\pi(x)$$

and

$$\operatorname{Var}\left[\frac{m_{n'}}{n' a_{m_{n'}}^2} \sum_{i=1}^{n} X_i^2 I(|X_i| \leqq \delta a_{m_{n'}})\right] \leqq \frac{m_{n'}^2}{n' a_{m_{n'}}^4} EX^4 I[|X| \leqq \delta a_{m_{n'}}] \leqq \left[\frac{m_{n'}}{n'}\right] \frac{\delta^2 m_{n'}}{a_{m_{n'}}^2} U(\delta a_{m_{n'}}) \to 0.$$

Let us now assume $\sup m_{n'}/n' \leqq c < \infty$ and $\pi = 0$. By passing to subsequences and applying the first part of the proof if necessary, we assume $\inf m_{n'}/n' > 0$. In this case we require a relatively more refined method for proving (2.22). Let $d_3(\mu, \nu) = \sup\left\{\left|\int f\, d(\mu - \nu)\right| : \sum_{i=0}^{3} \|f^{(i)}\|_\infty \leqq 1\right\}$ where $f^{(0)} = f$, $f^{(i)}$ is the i-th derivative of f and $\|f\|_\infty$ is the sup norm of $f: \mathbb{R} \to \mathbb{R}$. Then d_3 metrizes weak convergence ([2], Chapter 1). The Lindeberg method of proof of the CLT (e. g. [2], Theorem 2.1.3) and the triangle inequality together with a few trivial estimates, as in [2], Theorem 2.3.2, give

$$\begin{aligned} M_{n'}(\omega) := d_3\Bigg[\mathscr{L}\Bigg[\sum_{j=1}^{m_{n'}}\Bigg[X_{n'j}^{\omega} - \frac{1}{n}\sum_{i=1}^{n'} X_i(\omega) I(|X_i(\omega)| \leqq \tau a_{m_{n'}})/a_{m_{n'}}\Bigg]\Bigg], N(0, \sigma^2)\Bigg] \\ \leqq 2 \wedge \left[\frac{m_{n'}}{n' a_{m_{n'}}}\left|\sum_{i=1}^{n'} X_i(\omega) I(\delta a_{m_{n'}} < |X_i(\omega)| \leqq \tau a_{m_{n'}})\right|\right] \\ + \hat{E}\left[2 \wedge \left|\sum_{j=1}^{m_{n'}} X_{n'j}^{\omega} I(|X_{n'j}^{\omega}| > \delta a_{m_{n'}})/a_{m_{n'}}\right|\right] \\ + 2K\delta\hat{\sigma}_{n'\delta}^2(\omega) + (2/\pi)^2 |\hat{\sigma}_{n',\delta}(\omega) - \sigma| \\ := \sum_{i=1}^{4} (I_i^{n',\delta}), \quad \text{for all } \delta > 0 \qquad (2.26) \end{aligned}$$

where $K = (1 + (\delta/\pi)^{1/2})/6$ and

$$\hat{\sigma}_{n'\delta}(\omega) = \hat{\operatorname{Var}}\left[\sum_{j=1}^{m_{n'}} X_{n'j}^{\omega} I(|X_{n'j}^{\omega}| \leqq \delta a_{m_{n'}})/a_{m_{n'}}\right].$$

Since $\pi = 0$ and $\inf m_{n'}/n' > 0$, it follows that for all $\lambda > 0$, $\varepsilon > 0$,

$$P\left\{\frac{m_{n'}}{n' a_{m_{n'}}} \sum_{i=1}^{n'} |X_i| I(|X_i| > \lambda a_{m_{n'}}) > \varepsilon\right\} \leqq \left[\frac{n'}{m_{n'}}\right] m_{n'} P\{|X| > \lambda a_{m_{n'}}\} \to 0$$

and therefore,

$$\left.\begin{aligned}\lim_{n' \to \infty} (I_1^{n', \delta}) = 0, \\ \lim_{n' \to \infty} (I_2^{n', \delta}) = 0 \quad \text{in probability.}\end{aligned}\right\} \tag{2.27}$$

By (2.27), $\hat{\sigma}_{n', \delta} = (1 + o_P)\, \bar{\sigma}_{n', \delta}$ where

$$\bar{\sigma}^2_{n', \delta} = \frac{m_{n'}}{n' a^2_{m_{n'}}} \sum_{i=1}^{n'} X_i^2\, I(|X_i| \leqq \delta a_{m_{n'}}).$$

Hence, as long as $\bar{\sigma}_{n, \delta}$ is bounded in probability [as is the case: see (2.28) below] $\hat{\sigma}_{n, \delta}$ can be replaced by $\bar{\sigma}_{n, \delta}$ in $(I_3^{n', \delta})$ and $(I_4^{n', \delta})$. By the converse CLT in the infinite variance case, (2.21) implies

$$\lim_{n' \to \infty} E\,\bar{\sigma}^2_{n' \delta} = \lim_{n' \to \infty} \frac{m_{n'}}{a^2_{m_{n'}}} U(\delta a_{m_{n'}}) = \sigma^2. \tag{2.28}$$

This and the previous observation show that, for all $\varepsilon > 0$,

$$\lim_{\delta \to 0} \limsup_{n' \to \infty} P\{(I_3^{n', \delta}) > \varepsilon\} = 0. \tag{2.29}$$

Analogously, since $\sup_{n'} m_{n'}/n' \leqq c < \infty$,

$$\begin{aligned}\limsup_{n' \to \infty} \operatorname{Var}(\bar{\sigma}^2_{n', \delta}) &\leqq \limsup_{n' \to \infty} \frac{m^2_{n'}}{n' a^4_{m_{n'}}} EX^4\, I(|X| \leqq \delta a_{m_{n'}}) \\ &\leqq \delta^2 \limsup_{n' \to \infty} \left[\frac{m_{n'}}{n'}\right] \frac{m_{n'}}{a^2_{m_{n'}}} U(\delta a_{m_{n'}}) \leqq c\,\delta^2. \end{aligned} \tag{2.30}$$

(2.28) and (2.30) then give, by the previous observation,

$$\lim_{\delta \to 0} \limsup_{n'} P\{(I_4^{n', \delta}) > \varepsilon\} = 0. \tag{2.31}$$

Now, (2.27), (2.29) and (2.31) yield

$$\limsup_{n' \to \infty} P\{M_{n'}(\omega) > \varepsilon\} \leqq \lim_{\delta \to 0} \limsup_{n' \to \infty} \sum_{i=1}^{4} P\{(I_i^{n', \delta}) > \varepsilon/4\} = 0$$

for all $\varepsilon > 0$. This is just (2.22). [ ]

This previous theorem (together with easy considerations on centering that we omit) immediately gives:

2.6. Corollary (Athreya, [4]). — *Let θ be a non-degenerate p-stable random variable, $0 < p < 2$, and let X be in its domain of attraction, with norming constants b_n that is*

$$\mathscr{L}\left[\sum_{i=1}^{n} (X_i - EXI(|X| \leqq \tau b_n))/b_n\right] \to_w \mathscr{L}(\theta) \tag{2.32}$$

where τ can be zero if $p < 1$ and $+\infty$ if $p > 1$. Then if $m_n/n \to 0$,

$$\hat{\mathscr{L}}\left[\sum_{i=1}^{m_n}\left(X_{nj}^{\omega}-\frac{1}{n}\sum_{i=1}^{n}X_i(\omega)I(|X_i(\omega)|\leqq \tau b_{m_n})\right)/b_{m_n}\right]\to_w \mathscr{L}(\theta)$$

in probability. (2.33)

2.7. *Remark.* — It is easy to check, with the methods used in this section, that in Theorems 2.5 and 2.6 the centers in the bootstrap limit results (2.22) and (2.33) are stochastically equivalent to the centers in the respective limits (2.21) and (2.32) for the original sample.

3. THE a.s. BOOTSTRAP CLT

Bickel and Freedman [5] showed that if $EX^2 < \infty$ then the bootstrap CLT holds a. s. for any bootstrap sample size $m_n \to \infty$. Their proof uses distances in a way somewhat similar to the last part of the proof of Theorem 2.5 above. Here is a somewhat more natural proof of their theorem:

3.1. *Short proof of Theorem* 2.1 *in* [5]. — In order to show

$$\hat{\mathscr{L}}\left[\sum_{j=1}^{m_n}(X_{nj}^{\omega}-\bar{X}_n^{\omega})/m_n^{1/2}\right]\to_w N(0, \mathrm{Var}(X)) \quad \text{a. s.,} \tag{3.1}$$

by the usual CLT (e. g. [2], Cor. 2.4.8, p. 63) it suffices to prove

$$m_n\hat{P}\{|X_{n1}^{\omega}| > \delta m_n^{1/2}\}\to 0 \quad \text{a. s.} \tag{3.2}$$

for all $\delta > 0$ (then one makes the set of measure one where convergence takes place independent of δ just by taking a countable set of δ's, as usual),

$$\hat{\mathrm{V}}\mathrm{ar}(X_{n1}^{\omega} I(|X_{n1}^{\omega}|\leqq m_n^{1/2}))\to \mathrm{Var}(X) \quad \text{a. s.} \tag{3.3}$$

and

$$\frac{m_n^{1/2}}{n}\sum_{i=1}^{n}X_i I(|X_i| > m_n^{1/2})\to 0 \quad \text{a. s.} \tag{3.4}$$

Now, since $EX^2 < \infty$, we have that for any $p \leqq 2$,

$$\frac{1}{n}\sum_{i=1}^{n}|X_i|^p I(|X_i)| > \delta m_n^{1/2})\to 0 \quad \text{a. s.}$$

by the strong law of large numbers (replace $\delta m_n^{1/2}$ by c rational and take $\lim_{c\to 0}\lim_{n\to\infty}$). Then (3.4) follows immediately, and so do (3.2) and (3.3):

$$m_n \hat{\mathrm{P}}\{|X_{n1}^{\omega}|>\delta m_n^{1/2}\}=\frac{m_n}{n}\sum_{i=1}^{n} \mathrm{I}(|X_i|>\delta m_n^{1/2})$$
$$\leqq \frac{\delta^{-2}}{n}\sum_{i=1}^{n} X_i^2\,\mathrm{I}(|X_i|>\delta m_n^{1/2})\to 0 \quad \text{a. s.}$$

and

$$\hat{\mathrm{V}}\mathrm{ar}(X_{n1}^{\omega}\,\mathrm{I}(|X_{n1}^{\omega}|\leqq m_n^{1/2})=\frac{1}{n}\sum_{i=1}^{n} X_i^2\,\mathrm{I}(|X_i|\leqq m_n^{1/2})$$
$$-\left[\frac{1}{n}\sum_{i=1}^{n} X_i\,\mathrm{I}(|X_i|\leqq m_n^{1/2})\right]^2$$
$$\simeq \frac{1}{n}\sum_{i=1}^{n} X_i^2-\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right]^2\to \mathrm{Var}(X) \quad \text{a. s.}$$

The a. s. bootstrap CLT for $EX^2=\infty$ is somewhat more complicated. We know from [12] that if $EX^2=\infty$ and $m_n=n$ the CLT cannot be bootstrapped a. s. The same is true if $m_n \geqq cn$ for some $c>0$:

3.2. Proposition. — *If for some sequence $a_n\to\infty$, random variables $c_n(\omega)$ and random measure $\mu(\omega)$ non-degenerate with positive probability there exist $m_n \nearrow \infty$ such that* $\inf m_n/n>0$ *for which*

$$\hat{\mathscr{L}}\left[\sum_{j=1}^{m_n} X_{nj}^{\omega}/a_n-c_n(\omega)\right]\to_w \mu(\omega) \quad a.s. \tag{3.5}$$

then $EX^2<\infty$.

Proof. — By Theorem 2.1 we can take

$$c_n(\omega)=\frac{m_n}{na_n}\sum_{i=1}^{n} X_i(\omega)\,\mathrm{I}(|X_i(\omega)|\leqq a_n)$$

and then $\mu(\omega)=N(0,\sigma^2)$ a.s. for some $\sigma\in(0,\infty)$. Then (3.5) implies (converse CLT)

$$\frac{m_n}{n}\sum_{i=1}^{n} \mathrm{I}(|X_i|>\delta a_n)\to 0 \quad a.s.$$

for all $\delta>0$. This and $\inf m_n/n>0$ gives

$$\sum_{i=1}^{n} \mathrm{I}(|X_i|>\delta a_n)=0 \quad \text{eventually a. s.} \tag{3.6}$$

hence, also

$$\sum_{i=1}^{n} |X_i|^p I(|X_i| > \delta a_n) = 0 \quad \text{eventually a. s.} \tag{3.6)'}$$

for all p. So, if $EX^2 = \infty$ the truncated variance condition of the CLT for X_{nj}^{ω} becomes [recall (2.7)]

$$\frac{m_n}{n a_n^2} \sum_{i=1}^{n} X_i^2 \to \sigma^2 \quad \text{a. s.} \tag{3.7}$$

If we let $b_n = a_n (n/m_n)^{1/2}$, then (3.7) gives, as in the last part of the proof of Theorem 2.1, that X is in the domain of attraction of the normal law with norming constants b_n. Since $EX^2 = \infty$ this implies in particular that $b_n \simeq d_n$ (in the sense that $b_n/d_n \to 1$) with $d_n = n^{1/2} L(n)$, where $L(n) \nearrow \infty$. Hence $d_n^2/n \nearrow \infty$, so that we can apply a result of Feller (e. g. [15], Theorem 3.2.5, p. 132) to conclude that the limit in (3.7) is either 0 or $+\infty$, a contradiction. Therefore, $EX^2 < \infty$. [ ]

We believe that Proposition 3.2 is not best possible. In fact, in view of the following result, it is possible that Proposition 3.2 holds true for all sequences $m_n \nearrow \infty$ such that $\liminf_{n \to \infty} \frac{m_n \, LL\, n}{n} > 0$, where $L r = \log(r \vee e)$, and $LL r = L(L r)$.

3.3 THEOREM. — *If* $EX^2 = \infty$, *if* X *is in the domain of attraction of a non-degenerate p-stable law,* $0 < p \leqq 2$, *and if* $A := \inf m_n (LL\, n)/n > 0$, *then the a. s. bootstrap* CLT *does not hold for* X.

Proof. — We can assume $m_n/n \to 0$ by Proposition 3.2. If θ is the p-stable limit and $\{b_n\}$ are the norming constants then the results of Section 2 give

$$\hat{\mathscr{L}}\left[\sum_{j=1}^{m_n}\left[X_{nj}^{\omega} - \frac{1}{n}\sum_{i=1}^{n} X_i(\omega) I(|X_i(\omega)| \leqq b_{m_n})\right] / b_{m_n}\right] \to_w \mathscr{L}(\tilde{\theta}) \text{ in pr.} \tag{3.8}$$

where $\tilde{\theta}$ is a shift of θ. Let us assume $0 < p < 2$. If this limit held a. s. then we would necessarily have

$$\frac{m_n}{n} \sum_{i=1}^{n} I(|X_i| > \delta b_{m_n}) \to \pi_\delta \quad \text{a. s.} \tag{3.9}$$

for all $\delta > 0$, where $\pi_\delta = \pi\{(-\delta, \delta)^c\}$, π being the Lévy measure of $\mathscr{L}(\theta)$. Hence $\pi_\delta \neq 0$, $\delta > 0$. We will show that (3.9) does not hold under the hypotheses of the theorem. If suffices to consider $\delta = 1$. Let

$$p_n = P\{|X| > b_{m_n}\}.$$

Since $m_n p_n \to \pi_1$, (3.9) with $\delta=1$ is equivalent to

$$\frac{m_n}{n} \sum_{i=1}^{n} (\mathrm{I}(|\mathrm{X}_i| > b_{m_n}) - p_n) \to 0 \quad \text{a. s.} \tag{3.9$'$}$$

Let $n_k = k^k$ and let $\bar{m}_k = m_{n_k}$, $\bar{a}_k = b_{\bar{m}_k}$ and $\bar{p}_k = p_{n_k}$. We claim

$$\sum_{k=1}^{\infty} \mathrm{P}\left\{\frac{\bar{m}_k}{n_k}\left|\sum_{i=1}^{n_{k-1}} (\mathrm{I}(|\mathrm{X}_i| > \bar{a}_k) - \bar{p}_k\right| > \mathrm{K}\right\} < \infty \tag{3.10}$$

for all $\mathrm{K} > 0$. To estimate the probabilities in (3.10) we use Prokorov's exponential inequality (e. g. [15], Theorem 5.2.2, p. 262) which states that for ξ_i independent centered, $|\xi_i| \leqq c s_n$ where $s_n^2 = \sum_{i=1}^{n} \mathrm{E}\xi_i^2$,

$$\mathrm{P}\left\{\left|\sum_{i=1}^{n} \xi_i/s_n\right| \geqq \varepsilon\right\} \leqq 2\exp\left\{-\frac{\varepsilon}{2c}\operatorname{arcsinh}\left[\frac{\varepsilon c}{2}\right]\right\}.$$

We take $\xi_i = \mathrm{I}(|\mathrm{X}_i| > \bar{a}_n) - \bar{p}_n$, $\varepsilon = \dfrac{\mathrm{K} n_k}{\bar{m}_k s_k}$ where $s_k^2 = n_{k-1}\bar{p}_k(1-\bar{p}_k)$, and $c = 1/s_k$. Since $\bar{m}_k \bar{p}_k \to \pi_1$ and $n_k/k n_{k-1} \to e$ we have for $\delta > 0$ and k large enough,

$$\frac{(1-\delta)e\mathrm{K}k}{2\pi_1} \leqq \varepsilon c/2 \leqq \frac{(1+\delta)e\mathrm{K}k}{2\pi_1}.$$

Hence we can replace arcsinh by $(1-\delta')\log$ in Prohorov's inequality. We obtain

$$\mathrm{P}\left\{\frac{\bar{m}_k}{n_k}\left|\sum_{i=1}^{n_{k-1}} (\mathrm{I}(|\mathrm{X}_i| > \bar{a}_n) - \bar{p}_n)\right| > \mathrm{K}\right\}$$
$$< 2\exp\left\{-\frac{\mathrm{K} n_k}{2\bar{m}_k}(1-\delta')\log k\right\} \leqq 2\exp(-2\log k) = 2k^{-2}$$

for k large enough, since $n_k/\bar{m}_k \to \infty$. (3.10) is proved. And (3.10) implies that for all $\mathrm{K} > 0$,

$$\frac{\bar{m}_k}{n_k}\left|\sum_{i=1}^{n_{k-1}} (\mathrm{I}(|\mathrm{X}_i| > \bar{a}_k) - \bar{p}_k)\right| \leqq \mathrm{K} \quad \text{eventually a. s.} \tag{3.11}$$

We show next that there exists $\mathrm{L} > 0$ such that

$$\sum_{k=1}^{\infty} \mathrm{P}\left\{\frac{\bar{m}_k}{n_k}\left|\sum_{i=n_{k-1}+1}^{n_k} (\mathrm{I}(|\mathrm{X}_i| > \bar{a}_k) - \bar{p}_k)\right| > \mathrm{L}\right\} = \infty. \tag{3.12}$$

To prove (3.12) we will invoke Kolmogorov's exponential minorization ([15], p. 262): for ξ_i as above there are, for all $\gamma > 0$, $\varepsilon(\gamma)$ and $\pi(\gamma)$ such that if $\varepsilon \geqq \varepsilon(\gamma)$ and $\varepsilon c \leqq \pi(\gamma)$ then

$$P\left\{\sum_{i=1}^{n} \xi_i/s_n > \varepsilon\right\} \geqq \exp\{-\varepsilon^2(1+\gamma)/2\}.$$

Now the variance of the sum is $\bar{s}_k^2 := (n_k - n_{k-1})\bar{p}_k(1-\bar{p}_k)$ so that, with $c = 1/\bar{s}_k$ and $\varepsilon = \frac{n_k L}{\bar{m}_k \bar{s}_k}$ we have, for k large enough, $\varepsilon c \leqq 2L/\pi_1$ and $\varepsilon \geqq \frac{L}{2\pi_1}(n_k/\bar{m}_k)^{1/2} \to \infty$. Hence, taking $\gamma = 1$ we can apply Kolmogorov's inequality for

$$L \leqq \pi(1)\pi_1/2.$$

We then have, for k large enough,

$$P\left\{\frac{\bar{m}_k}{n_k}\left|\sum_{i=n_{k-1}+1}^{n_k}(I(|X_i| > \bar{a}_k) - \bar{p}_n)\right| > L\right\} \geqq \exp\left[-\frac{2L^2}{\pi_1 A}\log k\right],$$

which is the general term of a divergent series if

$$L^2 \leqq A\pi_1/2.$$

Hence (3.12) holds for all $0 < L \leqq \frac{\pi(1)\pi_1}{2} \wedge \left[\frac{A\pi_1}{2}\right]^{1/2}$. But (3.12) implies (by disjointness of the intervals $(n_{k-1}, n_k]$ and Borel-Cantelli) that

$$\frac{\bar{m}_k}{n_k}\left|\sum_{i=n_{k-1}+1}^{n_k}(I(|X_i| > \bar{a}_k) - \bar{p}_k)\right| > L \quad \text{a. s. infinitely often.} \tag{3.13}$$

(3.11) for $K = L/2$ and (3.13) show that a. s.

$\frac{\bar{m}_k}{n_k}\left|\sum_{i=1}^{n_k}(I(|X_i| > \bar{a}_k) - \bar{p}_k)\right| > L/2 > 0$ infinitely often, and therefore (3.9)′ does not hold.

If $p = 2$ and the bootstrap CLT holds a. s. then, since $EX^2 = \infty$, by the converse CLT and (2.7) the limit

$$\frac{\bar{m}_k}{n_k \bar{a}_k^2}\sum_{i=1}^{n_k} X_i^2 I(|X_i| \leqq \bar{a}_k) \to \sigma^2 \text{ a. s.} \tag{3.14}$$

must hold. A proof completely analogous to the above one shows that for some $L > 0$,

$$\frac{\bar{m}_k}{n_k \bar{a}_n^2} \left| \sum_{i=1}^{n_k} (X_i^2 I(|X_i| \leqq \bar{a}_k) - U(\bar{a}_k)) \right| > L/2 > 0 \text{ a. s.}$$

infinitely often, which contradicts (3.14) because $\frac{\bar{m}_k}{\bar{a}_k} U(\bar{a}_k) \to \sigma^2$. [ ]

Finally we show that the a. s. bootstrap CLT for X in the domain of attraction of any stable law always holds if $m_n (\mathrm{LL}\, n)/n \to 0$, at least under regularity of $\{m_n\}$ ($m_n/m_{2n} \geqq c$ for some $c > 0$ and $m_n \nearrow \infty$). Theorem 3.3 shows that the result is sharp. Thus, the next theorem improves Athreya's result (Corollary 2.6 above) for these sequences.

3.4. THEOREM. — *Let θ be a non-degenerate p-stable random variable, $0 < p \leqq 2$, and let X be a random variable in its domain of attraction, concretely, let X satisfy*

$$\mathscr{L}\left[\sum_{i=1}^{n} (X_i - \mathrm{EXI}(|X| \leqq \tau b_n))/b_n\right] \to_w \mathscr{L}(\theta) \tag{3.15}$$

with $b_n \nearrow \infty$. Let $\{m_n\}$ be a sequence of positive integers regular in the sense that $m_n \nearrow \infty$ and $m_n/m_{2n} \geqq c$ for some $c \geqq 0$ and all $n \in \mathbb{N}$, and such that

$$m_n (\mathrm{LL}\, n)/n \to 0. \tag{3.16}$$

Then

$$\hat{\mathscr{L}}\left[\sum_{j=1}^{m_n} \left(X_{nj}^{\omega} - \frac{1}{n} \sum_{i=1}^{n} X_i I(|X_i| \leqq b_{m_n})\right)/b_{m_n}\right] \to_w \mathscr{L}(\theta) \quad \text{a. s.} \tag{3.17}$$

Proof. — By Bickel and Freedman's theorem, only the case $EX^2 = \infty$ requires proof. Let, for $\lambda > 0$, $\delta > 0$,

$$\pi_\lambda = \lim_{n \to \infty} n P\{X > \lambda b_n\}, \qquad \pi_{-\lambda} = \lim_{n \to \infty} n P\{X < -\lambda b_n\},$$

$$\sigma_\delta^2 = \lim_{n \to \infty} \frac{n}{b_n^2} U(\delta b_n) \qquad \text{and} \qquad \sigma^2 = \lim_{\delta \to 0} \sigma_\delta^2.$$

As usual, it suffices to show that

$$\frac{m_n}{n} \sum_{i=1}^{n} I(X_i > \lambda b_{m_n}) \to \pi_\lambda \quad \text{a. s.,}$$

$$\frac{m_n}{n} \sum_{i=1}^{n} I(X_i < -\lambda b_{m_n}) \to \pi_{-\lambda} \quad \text{a. s.,}$$

$$\lim_{\delta \to 0} \lim_{n \to \infty} \frac{m_n}{n b_{m_n}^2} \sum_{i=1}^{n} X_i^2 I(|X_i| \leqq \delta b_{m_n}) = \sigma^2 \quad \text{a. s.}$$

(π_λ, $\pi_{-\lambda}=0$ if $p=2$; $\sigma^2=0$ if $p<2$; for $p=2$, $\lim_{\delta\to 0}$ is redundant). Letting $p_{n,\lambda}=P\{X>\lambda b_{m_n}\}$ and $p_{n,-\lambda}=P\{X<-\lambda b_{m_n}\}$, these limits are equivalent to

$$\frac{m_n}{n}\sum_{i=1}^{n}(I(X_i>\lambda b_{m_n})-p_{n,\lambda})\to 0 \quad \text{a.s.} \tag{3.18}$$

$$\frac{m_n}{n}\sum_{i=1}^{n}(I(X_i<-\lambda b_{m_n})-p_{n,-\lambda})\to 0 \quad \text{a.s.} \tag{3.19}$$

$$\lim_{\delta\to 0}\lim_{n\to\infty}\frac{m_n}{nb_{m_n}^2}\sum_{i=1}^{n}[X_i^2 I(|X_i|\leqq\delta b_{m_n})-U(\delta b_{m_n})]=0 \quad \text{a.s.} \tag{3.20}$$

The three limits can be proved using exactly the same technique. So, we give the details of the proof only for (3.18). By Borel Cantelli (3.18) will follow if we show

$$\sum_{n=1}^{\infty}P\left\{\max_{2^{n-1}<k\leqq 2^n}\frac{m_k}{k}\left|\sum_{i=1}^{k}(I(X_i>\lambda b_{m_k})-p_{k,\lambda})\right|>\varepsilon\right\}<\infty \tag{3.21}$$

for all $\varepsilon>0$. To prove this we symmetrize, apply Lévy's maximal inequality, and then use an exponential inequality to estimate the resulting probability. For the symmetrization we use an idea of Hoffmann-Jorgensen ([13], proof of Corollary 3.4). Let us consider the following $l^{\infty}_{2^{n-1}}$ valued vectors

$$v_i=\left(\frac{m_k}{k}I(X_i>\lambda b_{m_k}):2^{n-1}<k\leqq 2^n\right),\qquad i=1,\ \ldots,\ 2^{n-1}+1$$

$$v_i=(\underbrace{0,\ \ldots,\ 0}_{i-1-2^{n-1}};\ \frac{m_k}{k}I(X_i>\lambda b_{m_k}):i\leqq k\leqq 2^n)$$

$$i=2^{n-1}+2,\ \ldots,\ 2^n.$$

Then

$$\left\|\sum_{i=1}^{2^n}(v_i-E\,v_i)\right\|=\max_{2^{n-1}<k\leqq 2^n}\frac{m_k}{k}\left|\sum_{i=1}^{k}(I(X_i>\lambda b_{m_k})-p_{k,\lambda})\right|. \tag{3.22}$$

If $(v)^r$ denotes the r-th coordinate of $v \in l_{2^{n-1}}$, then

$$\max_r \mathrm{P}\left\{\left|\sum_{i=1}^{2^n} (v_i - \mathrm{E}\, v_i)^r\right| > \varepsilon\right\}$$
$$\leqq \varepsilon^{-1} \max_{2^{n-1} < k \leqq 2^n} \frac{m_k}{k} \mathrm{E}\left|\sum_{i=1}^{k} (\mathrm{I}(\mathrm{X}_i > \lambda\, b_{m_k}) - p_{k,\lambda}\right|$$
$$\leqq \varepsilon^{-1} \max_{2^{n-1} < k \leqq 2^n} \frac{m_k}{k} (k p_{k,\lambda})^{1/2}$$
$$= \varepsilon^{-1} \max_{2^{n-1} < k \leqq 2^n} \left[\frac{m_k}{k}\right]^{1/2} (m_k p_{k,\lambda})^{1/2} \to 0$$

since $m_k p_{k,\lambda} \to \pi_\lambda$ and $m_k/k \to 0$. Therefore we can apply to $\sum (v_i - \mathrm{E}\, v_i)$ e. g. the symmetrization lemmas in [10] (Lemma 2.5) to obtain that for each $\varepsilon > 0$ there is $n(\varepsilon) < \infty$ such that for $n > n(\varepsilon)$,

$$\mathrm{P}\left\{\left\|\sum_{i=1}^{2^n} (v_i - \mathrm{E}\, v_i)\right\| > \varepsilon\right\} \leqq 2\mathrm{P}\left\{\left\|\sum_{i=1}^{2^n} \varepsilon_i\, v_i\right\| > \varepsilon/3\right\} \tag{3.23}$$

where $\{\varepsilon_i\}$ is an independent sequence of i. i. d. random variables with $\mathrm{P}\{\varepsilon_i = 1\} = \mathrm{P}\{\varepsilon_i = -1\} = 1/2$, independent of $\{v_i\}$. By (3.22) and (3.23), the proof of (3.21) reduces to showing

$$\sum_{n=1}^{\infty} \mathrm{P}\left\{\max_{2^{n-1} < k \leqq 2^n} \frac{m_k}{k}\left|\sum_{i=1}^{k} \varepsilon_i \mathrm{I}(\mathrm{X}_i > \lambda\, b_{m_k})\right| > \varepsilon\right\} < \infty \tag{3.24}$$

for all $\varepsilon > 0$. In order to apply P. Lévy's maximal inequality we write

$$\mathrm{P}\left\{\max_{2^{n-1} < k \leqq 2^n} \frac{m_k}{k}\left|\sum_{i=1}^{k} \varepsilon_i \mathrm{I}(\mathrm{X}_i > \lambda\, b_{m_k})\right| > \varepsilon\right\}$$
$$\leqq \mathrm{P}\left\{\frac{m_{2^{n-1}}}{2^{n-1}} \max_{2^{n-1} < k \leqq 2^n}\left|\sum_{i=1}^{2^n} \varepsilon_i \mathrm{I}(\mathrm{X}_i > \lambda\, b_{m_k})\right| > c\,\varepsilon/2\right\}$$
$$+ \mathrm{P}\left\{\frac{m_{2^{n-1}}}{2^{n-1}} \max_{2^{n-1} < k \leqq 2^n}\left|\sum_{i=k+1}^{2^n} \varepsilon_i \mathrm{I}(\mathrm{X}_i > \lambda\, b_{m_k})\right| > c\,\varepsilon/2\right\}$$

and notice that the sets of indices

$$\mathrm{A}_k = \{i \leqq 2^n : \mathrm{X}_i > \lambda\, b_{m_k}\}, \qquad k = 2^{n-1}, \ldots, 2^n$$

and

$$\mathrm{B}_k = \{i : k+1 \leqq i \leqq 2^n, \mathrm{X}_i > \lambda\, b_{m_k}\}, \qquad k = 2^{n-1}, \ldots, 2^n,$$

are both decreasing as k increases since b_{m_k} increases with k. Therefore, for $\mathrm{X}_1, \ldots, \mathrm{X}_{2^n}$ fixed, the above maxima are actually maxima of partial sums respectively of $\sum_{i \in \mathrm{A}_{2^{n-1}}} \varepsilon_i$ and $\sum_{i \in \mathrm{B}_{2^{n-1}}} \varepsilon_i$, suitably ordered. So we can apply Lévy's maximal inequality conditionally on the X_i's and then integrate with respect to the X_i's. Taking into account that $B_{2^{n-1}} \subset A_{2^{n-1}}$, so that we can apply Lévy's inequality twice to the second probability, we obtain

$$P\left\{\max_{2^{n-1}<k\leqq 2^n}\frac{m_k}{k}\left|\sum_{i=1}^{k}\varepsilon_i I(X_i>\lambda b_{m_k})\right|>\varepsilon\right\} \leqq 6P\left\{\frac{m_{2^{n-1}}}{2^{n-1}}\left|\sum_{i=1}^{2^n}\varepsilon_i I(X_i>\lambda b_{m_{2^{n-1}}})\right|>c\varepsilon/2\right\}. \quad (3.25)$$

Since $m_{2^{n-1}} p_{2^{n-1},\lambda} \to \pi_\lambda$ and since $2^{n-1}/m_{2^{n-1}} \geqq K_n \log n$, $K_n \to \infty$, Prohorov's inequality (stated in the proof of Theorem 3.3) applied to the last probability in (3.25) shows that, for n large,

$$P\left\{\max_{2^{n-1}<k\leqq 2^n}\frac{m_k}{k}\left|\sum_{i=1}^{k}\varepsilon_i I(|X_i|>\lambda b_{m_k})\right|>\varepsilon\right\} \leqq 12\exp\left\{-\frac{c\varepsilon 2^{n-1}}{4m_{2^{n-1}}}\operatorname{arcsinh}\left[\frac{c\varepsilon}{4m_{2^{n-1}}p_{2^{n-1},\lambda}}\right]\right\} \leqq 12\exp\{-dK_n\log n\} \quad (3.26)$$

where d is a fixed constant. (3.26) is the general term of a convergent series and therefore (3.24), hence (3.21) holds. This proves (3.18). [ ]

3.5. *Remark.* – The centering in (3.15) can be taken to be EX for $p>1$ and zero for $p<1$. We may ask if the centering in (3.17) can analogously be taken to be $\bar{X}_n(\omega)$ for $p>1$ and zero for $p<1$. In the case $p>1$ the answer is affirmative if and only if $\frac{m_n}{nb_{m_n}}\sum_{i=1}^{n}(X_i-EX)\to 0$ a.s. Theorem 3.2.5 [15] (Feller's theorem) shows that, under regularity conditions, this is equivalent to $\sum P\left\{|X|>\frac{nb_{m_n}}{m_n}\right\}<\infty$, but (3.16) does not imply that this series converges. This holds if e.g. $m_n(\log n)^{\alpha/(p-1)}/n \leqq$ constant for some $\alpha>1$. A similar observation can be made for $p<1$.

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