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# On the asymptotic equidistribution of sums of independent identically distributed random variables

by

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ABSTRACT. — For a sum $S_n$ of $n$ I.I.D. random variables the idea of approximate equidistribution is made precise by introducing a notion of asymptotic translation invariance. The distribution of $S_n$ is shown to be asymptotically translation invariant in this sense iff $S_1$ is nonlattice. Some ramifications of this result are given.

*Key words :* Sums of I.I.D. random variables, asymptotic equidistribution, asymptotic translation invariance.

RÉSUMÉ. — On introduit, pour une somme $S_n$ de $n$ variables aléatoires indépendantes équidistribuées, une notion d'invariance asymptotique par translation, qui permet de rendre précise l'idée d'équidistribution approximative. On montre que la loi de $S_n$ est asymptotiquement invariante par translation en ce sens si, et seulement si, la loi de $S_1$ est non arithmétique. On donne quelques extensions de ce résultat.

## 1. INTRODUCTION

Let $T_1$, $T_2$, ... be a sequence of independent random variables with a common distribution $P^{T_n} = P^{T_1}$ and let $S_n = T_1 + \ldots + T_n$. Intuitively, if

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$E(T_1)=0$ and $T_1 \neq 0$, the mass of the probability measure $P^{S_n}$ is expected to be approximately "equidistributed", as $n$ becomes large. If $E(T_1)>0$, one is inclined to think of something like an "approach to uniformity at infinity". An old result of this kind is due to Robbins (1953). If $T_1$ is not concentrated on a lattice,

$$n^{-1} \sum_{i=1}^{n} h(S_i) \rightarrow \lim_{T \rightarrow \infty} (2T)^{-1} \int_{-T}^{T} h(x)\, dx, \qquad \text{as} \quad n \rightarrow \infty \tag{1.1}$$

for all almost periodic functions $h$ (*i. e.* if $h$ is the uniform limit of trigonometric polynomials). For a partial sharpening of this result *see* Theorem 3 of Stadje (1985). In the case when $T_1$ is not concentrated on a lattice, $E(T_1)=0$ and $0<\sigma^2:=\operatorname{Var}(T_1)<\infty$, the expectation of asymptotic equidistribution can also be justified by the limiting relation

$$\sigma(2\pi n)^{1/2} P(S_n \in I) \rightarrow \lambda(I), \qquad \text{as} \quad n \rightarrow \infty \tag{1.2}$$

which is valid for all bounded intervals $I \subset \mathbb{R}$, where $\lambda$ denotes the Lebesgue measure (Shepp (1964), Stone (1965, 1967), Breiman (1968), chapt. 10).

One might try to interpret approximate uniformity of $P^{S_n}$ by stating that $P(S_n \in I)$ asymptotically only depends on the length of I. Since $\lim_{n \rightarrow \infty} P(S_n \in I)=0$ for every bounded interval I, this idea should be made reasonable by examining the speed of convergence of $P(S_n \in I)$. This is done in (1.2) stating $a_n P^{S_n}$ approaches the Lebesgue measure, where $a_n=\sigma(2\pi n)^{1/2}$.

In this paper another approach to the idea of equidistribution of $P^{S_n}$ is developed. The essential property of an "equidistribution" is the invariance under translations. To measure the degree of translation invariance of a probability measure Q on $\mathbb{R}$, we introduce, for $a \in \mathbb{R}$ and $t>s>0$, the quantities

$$d(a, t, Q):=\sum_{i=-\infty}^{\infty} \left| Q((a+it, a+(i+1)t]) - Q((a+(i-1)t, a+it]) \right| \tag{1.3}$$

$$D(t, Q):=\sup_{a \in \mathbb{R}} d(a, t, Q) \tag{1.4}$$

$$\tilde{D}(s, t, Q):=\sup_{s \leqq u \leqq t} D(u, Q). \tag{1.5}$$

We call a sequence $(Q_n)_{n \geqq 1}$ of probability measures *asymptotically translation invariant* (ATI), if $\lim_{n \rightarrow \infty} \tilde{D}(s, t, Q_n)=0$ for all $t>s>0$. The main theorem of this paper states that $(P^{S_n})_{n \geqq 1}$ is ATI if, and only if, $P^{T_1}$ is not concentrated on a lattice. No moment conditions are needed for this equivalence. Let $D_n(t):=\tilde{D}(t^{-1}, t, P^{S_n})$, $t>1$. Regarding the speed of convergence of $D_n(t)$ we remark that

$$\liminf_{n \rightarrow \infty} n^{1/2} D_n(t)>0 \quad \text{for all } t>1, \qquad \text{if} \quad E(T_1^2)<\infty. \tag{1.6}$$

To see (1.6), let without loss of generality $E(T_1)=0$ and $E(T_1^2)=1$. Then, by Chebyshev's inequality,

$$P(|S_n|<n^{1/2})\geqq 1-n^{-1}. \tag{1.7}$$

The interval $(-n^{1/2}, n^{1/2})$ can be covered by $[2n^{1/2}/t]+1$ half-open intervals of length $t$. One of these intervals, say I, obviously satisfies

$$P(S_n\in I)\geqq(1-n^{-1})/([2n^{1/2}/t]+1)\geqq\frac{1}{2}\frac{1}{2n^{1/2}t+1}, \quad \text{if} \quad n\geqq 2. \tag{1.8}$$

Choose $a\in[0, t]$ and $i_0\in\mathbb{Z}$ such that $I=(a+i_0 t, a+(i_0+1)t]$.

Then

$$\begin{aligned} d(a, t, P^{S_n})\geqq \sum_{i=-\infty}^{i_0} [P(S_n\in(a+it, a+(i+1)t]) \\ -P(S_n\in(a+(i-1)t, a+it])] \\ =P(S_n\in(a+i_0 t, a+(i_0+1)t])\geqq\frac{1}{4t+2}n^{-1/2}, \quad n\geqq 2 \end{aligned} \tag{1.9}$$

(1.6) follows from (1.9).

In order to derive a converse result to (1.6), we need a further notion. A distribution Q on $\mathbb{R}$ is called strongly nonlattice, if its characteristic function $\varphi$ satisfies

$$\limsup_{|\zeta|\to\infty}|\varphi(\zeta)|<1. \tag{1.10}$$

The second main result of this note is that if $P^{T_1}$ is strongly nonlattice,

$$\limsup_{n\to\infty} n^{1/2} D_n(t)<\infty \quad \text{for all } t>1. \tag{1.11}$$

## 2. THE MAIN THEOREM

We shall prove

THEOREM 1. — *The following two statements are equivalent.*

(*a*) $P^{T_1}$ *is nonlattice.*

(*b*) $(P^{S_n})_{n\geqq 1}$ *is ATI.*

*Proof.* — (*a*) $\Rightarrow$ (*b*). Assume first that $E|T_1|^3<\infty$ and $E(T_1)=0$. Let $\sigma^2=E(T_1^2)$, $\mu_3=E(T_1^3)$ and denote the distribution function of $S_n$ by $F_n$. Then

$$\begin{aligned} d(a, t, P^{S_n})=\sum_{i=-\infty}^{\infty} |F_n(a+(i+1)t)-2F_n(a+it) \\ +F_n(a+(i-1)t)|. \end{aligned} \tag{2.1}$$

Since $P^{T_1}$ is nonlattice, a well-known expansion for distribution functions yields

$$F_n(n^{1/2}\sigma x) = \Phi(x) + \frac{\mu_3}{6\sigma^3 n^{1/2}}(1-x^2)\varphi(x) + \varepsilon_n(x) n^{-1/2} \tag{2.2}$$

for all $x \in \mathbb{R}$, where

$$\varepsilon_n := \sup_{x \in \mathbb{R}} |\varepsilon_n(x)| \to 0, \qquad \text{as} \quad n \to \infty \tag{2.3}$$

and $\Phi$ and $\varphi$ are the distribution function and density of $N(0, 1)$ (*see* e. g. Feller (1971), p. 539). It is easy to check that for each $j \in \mathbb{N}$ and $a \in [0, t]$

$$\sum_{i>j} |F_n(a+(i+1)t) - 2F_n(a+it) + F_n(a+(i-1)t)| \\ \leqq 1 - F_n(a+(j+1)t) + 1 - F_n(a+jt) \leqq 2P(S_n > jt) \tag{2.4}$$

and

$$\sum_{i<-j} |F_n(a+(i+1)t) - 2F_n(a+it) + F_n(a+(i-1)t)| \\ \leqq F_n(a-jt) + F_n(a-(j+1)t) \leqq 2P(S_n \leqq -(j-1)t). \tag{2.5}$$

Inserting (2.2)-(2.5) into (2.1) we obtain, for $a \in [0, t]$,

$$d(a, t, P^{S_n}) \leqq 2P(|S_n| \geqq (j-1)t) + (2j+1)\varepsilon_n n^{-1/2} \\ + \sum_{i=-\infty}^{\infty} \left| \Phi\left(\frac{a+(i+1)t}{\sigma n^{1/2}}\right) - 2\Phi\left(\frac{a+it}{\sigma n^{1/2}}\right) + \Phi\left(\frac{a+(i-1)t}{\sigma n^{1/2}}\right) \right| \\ + \frac{|\mu_3|}{6\sigma^3 n^{1/2}} \sum_{i=-\infty}^{\infty} |(1-x_{i+1}^2)\varphi(x_{i+1}) \\ - 2(1-x_i^2)\varphi(x_i) + (1-x_{i-1}^2)\varphi(x_{i-1})| \tag{2.6}$$

where $x_i := (a+it)/\sigma n^{1/2}$. By Chebyshev's inequality,

$$2P(|S_n| \geqq (j-1)t) + (2j+1)\varepsilon_n n^{-1/2} \leqq \frac{2\sigma^2 n}{t^2(j-1)^2} + (2j+1)\varepsilon_n n^{-1/2}. \tag{2.7}$$

The smallest order of magnitude of the righthand side of (2.7) is attained for $j = j_n$ being equal to the integer part of $n^{1/2}\varepsilon_n^{-1/3}$; in this case

$$2P(|S_n| \geqq (j_n-1)t) + (2j_n+1)\varepsilon_n n^{-1/2} \\ = (1+t^{-2})O(\varepsilon_n^{2/3}), \qquad \text{as} \quad n \to \infty. \tag{2.8}$$

Next we estimate the two series in (2.6). Let X be a standard normal random variable. Then the first sum at the righthand side of (2.6) is equal to

$$\sum_{i=-\infty}^{\infty} \left| P(\sigma n^{1/2} X \in (a+it, a+(i+1)t]) - P(\sigma n^{1/2} X - t \in (a+it, a+(i+1)t)] \right|$$
$$\leqq (\sigma n^{1/2})^{-1} \int_{-\infty}^{\infty} \left| \varphi(x/\sigma n^{1/2}) - \varphi((x+t)/\sigma n^{1/2}) \right| dx$$
$$= 2(\sigma^{1/2})^{-1} \left[ \int_{-\infty}^{-t/2} \varphi((x+t)/\sigma n^{1/2})\, dx - \int_{-\infty}^{-t/2} \varphi(x/\sigma n^{1/2})\, dx \right]$$
$$= 2 \int_{-t/2\sigma n^{1/2}}^{t/2\sigma n^{1/2}} \varphi(u)\, du = t\, O(n^{-1/2}), \qquad \text{as} \quad n \to \infty. \tag{2.9}$$

To estimate the last sum at the right side of (2.6), note that the function $(1-x^2)\exp(-x^2/2)$ has four points of inflexion. Regarding the sequence

$$a_i := (1-x_{i+1}^2)\, \varphi(x_{i+1}) - (1-x_i^2)\, \varphi(x_i), \qquad i \in \mathbb{Z}$$

this implies that $(a_i - a_{i-1})_{i \in \mathbb{Z}}$ changes signs at most four times. Using its telescoping form the sum in question can be bounded from above as follows:

$$\sum_{i=-\infty}^{\infty} |a_i - a_{i-1}| \leqq 8 \sup_{-\infty < i < \infty} |a_i|. \tag{2.10}$$

Further, by the mean value theorem,

$$|a_i| \leqq |x_{i+1} - x_i| \sup_{x \in \mathbb{R}} \left| \frac{d}{dx} (1-x^2)\, \varphi(x) \right| = K\, t/\sigma n^{1/2} \tag{2.11}$$

for some constant K. Inserting (2.8)-(2.11) into (2.6) we arrive at

$$d(a, t, P^{S_n}) = (1+t^{-2})\, O(\varepsilon_n^{2/3}) + t\, O(n^{-1/2}), \qquad \text{as} \quad n \to \infty, \tag{2.12}$$

so that

$$D(t, P^{S_n}) = (1+t^{-2})\, O(\varepsilon_n^{2/3}) + t\, O(n^{-1/2}), \qquad \text{as} \quad n \to \infty. \tag{2.13}$$

To establish the assertion without moment conditions we first remark that $D(t, Q)$ is translation invariant in the sense that

$$D(t, Q) = D(t, Q * \varepsilon_x) \quad \text{for all} \ x \in \mathbb{R}, \quad t > 0, \tag{2.14}$$

where $*$ denotes convolution and $\varepsilon_x$ is the point mass at $x$. Thus, (2.13) holds, if $E|T_1|^3 < \infty$ (without the assumption $E(T_1) = 0$). Further, for probability measures Q and R we have

$$D(t, Q * R) \leqq D(t, Q). \tag{2.15}$$

(2.15) is proved as follows:

$$D(t, Q*R)=\sup_a \sum_{i=-\infty}^{\infty} |(Q*R)((a+it, a+(i+1)t]) - (Q*R)((a+(i-1)t, a+it])|$$

$$\leqq \sup_a \sum_{i=-\infty}^{\infty} \int_{-\infty}^{\infty} |Q((a+it-x, a+(i+1)t-x]) - Q((a+(i-1)t-x, a+it-x)])|\, dR(x)$$

$$\leqq \int_{-\infty}^{\infty} \sup_a \sum_{i=-\infty}^{\infty} |Q((a+it-x, a+(i+1)t-x]) - Q((a+(i-1)t-x, a+it-x])|\, dR(x)$$

$$= \int_{-\infty}^{\infty} D(t, Q)\, dR(x) = D(t, Q). \quad (2.16)$$

Next suppose that $P^{T_1}=\alpha Q+(1-\alpha)R$ for some $\alpha\in(0, 1]$ and probability measures Q and R such that Q satisfies, for some constants $K_1$, $K_2$,

$$D(t, Q^{*n})\leqq(1+t^{-2})K_1\varepsilon_n^{2/3}+tK_2n^{-1/2}, \qquad \text{as} \quad n\to\infty \quad (2.17)$$

($Q^{*n}$ is the $n$-fold convolution of Q with itself). Then

$$D(t, P^{S_n})=\sup_a \sum_{i=-\infty}^{\infty} \left| \sum_{l=0}^{n} \binom{n}{l} \alpha^l (l-\alpha)^{n-l} \{ Q^{*l} * R^{*(n-l)}((a+it, a+(i+1)t]) - Q^{*l} * R^{*(n-l)}((a+(i-1)t, a+it]) \} \right|$$

$$\leqq \sum_{l=0}^{n} \binom{n}{l} \alpha^l (1-\alpha)^{n-l} D(t, Q^{*l} * R^{*(n-l)})$$

$$\leqq \sum_{l=0}^{n} \binom{n}{l} \alpha^l (1-\alpha)^{n-l} D(t, Q^{*l}). \quad (2.18)$$

Let $\delta_n := \sup_{l>n} \varepsilon_l$. Then $\delta_n \downarrow 0$ and, by Chebyshev's inequality for the binomal distribution and (2.17), it follows that, for arbitrary $\varepsilon\in(0, \alpha)$,

$$\sum_{l=0}^{n} \binom{n}{l} \alpha^l (1-\alpha)^{n-l} D(t, Q^{*l}) \leqq 2 \sum_{l\leqq(\alpha-\varepsilon)n} \binom{n}{l} \alpha^l (1-\alpha)^{n-l} + (1+t^{-2})K_1 \sup_{l>(\alpha-\varepsilon)n} \varepsilon_l + tK_2((\alpha-\varepsilon)n)^{-1/2}$$

$$\leqq 2\alpha(1-\alpha)\varepsilon^{-2}n^{-1} + K[(1+t^{-2})\delta_{(\alpha-\varepsilon)n} + tn^{-1/2}] \quad (2.19)$$

where $K=\max(K_1, K_2)$. (2.19) implies that $D_n(t)\to 0$, as $n\to\infty$, for all $t>1$.

Now we choose a function $f$ on $\mathbb{R}$ such that $0<f(x)<1$ for all $x\in\mathbb{R}$ and

$$\int_{-\infty}^{\infty} |x|^3 f(x)\, dP^{T_1}(x)<\infty.$$

Let $\alpha := \int_{-\infty}^{\infty} f(x)\, dP^{T_1}(x)$ and define the probability measures Q and R by $dQ := \alpha^{-1} f\, dP^{T_1}$, $dR=(1-\alpha)^{-1}(1-f)\, dP^{T_1}$. Then the third moment of Q is finite and Q is nonlattice so that Q satisfies (2.17). Since $P^{T_1}=\alpha Q+(1-\alpha) R$, it follows that $D_n(t)\to 0$.

$(b)\Rightarrow(a)$. Let $P^{T_1}$ have span $\lambda>0$. If $t\in(0, \lambda/2)$, at most one of the successive intervals $(a+(i-1)t, a+it]$ and $(a+it, a+(i+1)t]$ contains a multiple of $\lambda$. Therefore it is obvious that $d(a, t, P^{S_n})=2$ for all $a\in\mathbb{R}$, $t\in(0, \lambda/2)$ and $n\in\mathbb{N}$.

## 3. THE STRONGLY NONLATTICE CASE

Concerning the speed of convergence of $D_n(t)$ we shall now prove

THEOREM 2. — *If* $P^{T_1}$ *is strongly nonlattice,*

$$\limsup_{n\to\infty} n^{1/2} D_n(t)<\infty \quad \text{for all } t>1. \tag{3.1}$$

*Proof.* — Let $\eta := \limsup_{|\zeta|\to\infty} |\varphi(\zeta)|<1$. We can decompose $P^{T_1}=\alpha Q+(1-\alpha) R$, where $\alpha\in(0, 1]$ and Q and R are probability measures such that Q is strongly nonlattice and concentrated on a bounded interval. If $P^{T_1}$ itself is concentrated on a bounded interval, this is trivial. Otherwise let $\alpha_N := P(T_1\in[-N, N])$, where N is large enough to ensure $0<\alpha_N<1$. Define, for Borel sets B,

$$Q_N(B) := \alpha_N^{-1} P(T_1\in B\cap[-N, N])$$
$$R_N(B) := (1-\alpha_N)^{-1} P(T_1\in B\setminus[-N, N]).$$

Then the characteristic functions $\tilde{\varphi}_N$ and $\tilde{\tilde{\varphi}}_N$ of $Q_N$ and $R_N$ satisfy $\tilde{\varphi}_N=\alpha_N^{-1}(\varphi-(1-\alpha_N)\tilde{\tilde{\varphi}}_N)$ so that

$$\limsup_{|\zeta|\to\infty} |\tilde{\varphi}_N(\zeta)|\leqq \alpha_N^{-1}(\eta+1-\alpha_N), \tag{3.2}$$

and the righthand side of (3.2) is smaller than 1 for sufficiently large N, because $\alpha_N\uparrow 1$.

We proceed by proving the assertion for Q instead of $P^{T_1}$. Obviously we may assume that $\int x\, dQ(x)=0$. Let $F_n$ be the distribution function of

$Q^{*n}$ and $\sigma^2 := \int x^2 \, dQ(x)$. Since Q is strongly nonlattice and possesses moments of all orders, a well-known expansion yields, for every $r \geqq 3$,

$$F_n(n^{1/2}\sigma x) - \Phi(x) - \varphi(x) \sum_{k=3}^{r} n^{-(k/2)+1} R_k(x) = o(n^{-(r/2)+1}) \quad (3.3)$$

uniformly in $x$, where $R_k$ is a polynomial depending only on the first $r$ moments of Q(*see*, e. g., Feller (1971), p. 541). Letting $r=5$ and proceeding as in (2.4)-(2.6) we obtain, for arbitrary $j$,

$$\begin{aligned} d(a, t, Q^{*n}) \leqq\ & 2\, Q^{*n}(\mathbb{R} \setminus [-(j-1)t, (j-1)t]) \\ & + (2j+1)\, o(n^{-3/2}) + \sum_{i=-j}^{j} \left| \Phi\left(\frac{a+(i+1)t}{\sigma n^{1/2}}\right) - 2\Phi\left(\frac{a+it}{\sigma n^{1/2}}\right) + \Phi\left(\frac{a+(i-1)t}{\sigma n^{1/2}}\right) \right| \\ & + \sum_{k=3}^{5} n^{-(k/2)+1} \sum_{i=-j}^{j} \left| \varphi(x_{i+1}) R_k(x_{i+1}) - 2\varphi(x_i) R_k(x_i) + \varphi(x_{i-1}) R_k(x_{i-1}) \right|. \quad (3.4) \end{aligned}$$

Here again $x_i = (a+it)/\sigma n^{1/2}$. Since each function $\varphi(x) R_k(x)$ has a bounded derivative and only a finity number of points of inflexion, the same reasoning as in the proof of Theorem 1 (for $R(x) = 1 - x^2$) shows that, for $k = 3, 4, 5$,

$$\begin{aligned} \sum_{i=-\infty}^{\infty} & \left| \varphi(x_{i+1}) R_k(x_{i+1}) - \varphi(x_i) R_k(x_i) - [\varphi(x_i) R_k(x_i) - \varphi(x_{i-1}) R_k(x_{i-1})] \right| \\ & \leqq L \sup_{-\infty < i < \infty} \left| \varphi(x_{i+1}) R_k(x_{i+1}) - \varphi(x_i) R_k(x_i) \right| \\ & \leqq \tilde{L} \sup_{-\infty < i < \infty} |x_{i+1} - x_i| = \tilde{L}\, t/\sigma n^{1/2}, \quad (3.5) \end{aligned}$$

where L and $\tilde{L}$ are appropriate constants. Thus the last term at the right side of (3.4) is $t\,O(n^{-1/2})$. Further using (2.9) for the remaining sum in (3.4) and Chebyshev's inequality we arrive at

$$\begin{aligned} d(a, t, Q^{*n}) & \leqq 2\, Q^{*n}(\mathbb{R} \setminus [-(j-1)t, (j-1)t]) \\ & \quad + (2j+1)\, o(n^{-3/2}) + t\, O(n^{-1/2}) \\ & = t^{-2} O(n/j^2) + (2j+1)\, o(n^{-3/2}) + t\, O(n^{-1/2}). \quad (3.6) \end{aligned}$$

Choosing $j = j_n = n^{5/6}$, (3.6) implies that

$$d(a, t, Q^{*n}) = t^{-2} O(n^{-2/3}) + t\, O(n^{-1/2}). \quad (3.7)$$

Now arguing similarly as in (2.18) and (2.19),

$$\mathrm{D}(t, \mathrm{P}^{S_n}) \leqq \sum_{l=0}^{n} \binom{n}{l} a^l (1-\alpha)^{n-l} \mathrm{D}(t, \mathrm{Q}^{*l})$$
$$\leqq 2\alpha(1-\alpha)\varepsilon^{-2} n^{-1} + \mathrm{K}[t^{-2} O(n^{-2/3}) + t\, O(n^{-1/2})], \quad (3.8)$$

where $\varepsilon > 0$ and K are constants. It follows that $\mathrm{D}_n(t) = O(n^{-1/2})$ for each $t > 1$, as claimed.

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