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On subsets of L^p and p-stable processes

by

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ABSTRACT. — Each subset of L^p defines a p-stable process. We study for which sets this process is bounded. We show that sample boundedness occurs as the result of two phenomenon. One, well understood, is the existence of a suitable majorizing measure. In the other, more delicate, all aspects of cancellation have disappeared. Examples are given showing that these subsets can be very complicated.

Key words : p-stable processes, sample continuity.

RÉSUMÉ. — Chaque sous-ensemble de L^p défini un processus p-stable. On étudie quels sous-ensembles définissant un processus p-stable borné. On montre que c'est le cas exactement sous la conjonction de deux conditions. L'une, bien comprise, est du type mesure majorante, l'autre condition est plus délicate, mais n'est pas basée sur des phénomènes d'annulation.

Mots clés : Processus p-stable, bornitude, mesure majorante.

Classification A.M.S. : 60 G 17, 60 E 07.

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1. INTRODUCTION

For $1 \leqq p \leqq 2$, a real valued random variable X (r. v.) will be called symmetric p-stable of parameter σ if

$$\forall u \in \mathbb{R}, \quad \mathrm{E} \exp iu\mathrm{X} = \exp - \sigma^p |u|^p.$$

Given an index set T, a stochastic process $(\mathrm{X}_t)_{t \in \mathrm{T}}$ will be called p-stable if each finite linear combination $\Sigma \alpha_t \mathrm{X}_t$ is a p-stable r. v. We say that $(\mathrm{X}_t)_{t \in \mathrm{T}}$ is sample bounded if $\sup_{\mathrm{T}} |\mathrm{Y}_t| < \infty$ a. s., where $(\mathrm{Y}_t)_{t \in \mathrm{T}}$ is a separable version of $(\mathrm{X}_t)_{t \in \mathrm{T}}$. It is known [2] that when $(\mathrm{X}_t)_{t \in \mathrm{T}}$ is a sample bounded p-stable process there exists a positive measure m on $[-1, 1]^{\mathrm{T}}$ such that whenever $(\alpha_t)_{t \in \mathrm{T}} \in \mathbb{R}^{(\mathrm{T})}$, we have

$$\mathrm{E} \exp iu \Sigma \alpha_t \mathrm{X}_t = \exp - \int |\Sigma \alpha_t \beta_t|^p \, dm(\beta).$$

To each $t \in \mathrm{T}$ associate the element $\varphi_t \in \mathrm{L}^p(m)$ given by $\varphi_t(\beta) = \beta_t$ for $\beta \in [-1, 1]^{\mathrm{T}}$. Then, the r. v. $\Sigma \alpha_t \mathrm{X}_t$ is p-stable of parameter $\| \Sigma \alpha_t \varphi_t \|_p$. It is shown in [3] that for any finite measure m, one can define a p-stable process $(\mathrm{Y}_t)_{t \in \mathrm{L}^p(m)}$ such that Y_t has parameter $\| t \|_p$. As we have seen any sample-bounded p-stable process can be considered as equal in distribution to the restriction of (Y_t) to a subset T of $\mathrm{L}^p(m)$. So, the question of understanding which p-stable processes are sample-bounded amounts to understand for which subsets T of $\mathrm{L}^p(m)$, the restriction of $(\mathrm{Y}_t)_{t \in \mathrm{T}}$ to T is sample bounded. Such a subset of $\mathrm{L}^p(m)$ will be called a pB-set. The Gaussian case ($p=2$) is completely understood [14], but, as well known, the case $1 \leqq p < 2$ is much more delicate. We suppose now on $p<2$, and without loss of generality that T is countable. From general considerations concerning p-stable Banach space valued random variables, it is known [2] that the appropriate measure of the size of a pB-set T is the quantity

$$\mathrm{S}_p(\mathrm{T}) = (\sup_{c>0} c^p \, \mathrm{P}(\sup_{t \in \mathrm{T}} |Y_t| > c))^{1/p}.$$

For $p > 1$, we have

$$\mathrm{K}^{-1} \mathrm{S}_p(\mathrm{T}) \leqq \mathrm{E} \sup_{t \in \mathrm{T}} |\mathrm{Y}_t| \leqq \mathrm{K} \mathrm{S}_p(\mathrm{T}).$$

Here, as well as in all the paper, K denotes a constant that depends on p only, but that may vary from line to line.

We can and do assume without loss of generality that m is a probability on [0, 1]. Much of the recent progress on p-stable processes has been permitted by a remarkable representation of these processes that was brought to light in the fundamental work of M. Marcus and G. Pisier [6], (following ideas of [5]).

We denote once and for all by $(X_i)_{i \geqq 1}$ an independent sequence valued in [0, 1] and distributed like m, and by $(\varepsilon_i)_{i \geqq 1}$ a Bernoulli sequence independent of (X_i).

THEOREM 1.1 [6]. — *For $p \geqq 1$, and any countable subset* T *of* $L^p(m)$, *we have*

$$(1.1) \qquad K^{-1} S_p(T) \leqq E \sup_{f \in T} \Big| \sum_{i \geqq 1} \varepsilon_i f(X_i) i^{-1/p} \Big| \leqq K S_p(T).$$

Theorem 1.1 is very useful since it allows to replace a rather abstract question by the concrete problem of estimating $E \| \sum_{i \geqq 1} \varepsilon_i f(X_i) i^{-1/p} \|_T$, where we set

$$\sup_{f \in T} \Big| \sum_{i \geqq 1} \varepsilon_i f(X_i) i^{-1/p} \Big| = \Big\| \sum_{i \geqq 1} \varepsilon_i f(X_i) i^{-1/p} \Big\|_T.$$

We now need to recall definitions about majorizing measures. Given a metric space (T, d), a probability measure μ on T, and a function h: $[0, 1] \to \mathbb{R}^+$, we set

$$\gamma(T, d, h, \mu) = \sup_{x \in T} \int_0^D h\left(\frac{1}{\mu(\{ y \in T;\ d(x, y) \leqq \varepsilon \})} \right) d\varepsilon$$

where D is the diameter of T. We set $\gamma(T, d, h) = \inf \gamma(T, d, h, \mu)$, where the inf is taken over all probability measures μ on T. We refer the reader to [11] for a systematic exposition of the theory of majorizing measures. We recall that for a sequence of numbers $(a_i)_{i \geqq 1}$, we set

$$\| (a_i) \|_{p, \infty} = \sup_{t > 0} (t \operatorname{card} \{ i;\ |a_i| \geqq t^{1/p} \})^{1/p}.$$

We will later use the fact that $\| (a_i) \|_{p, \infty} = \sup_{i \geqq 1} i^{1/p} a_i^*$ where (a_i^*) is the non-decreasing rearrangement of $(|a_i|)_{i \geqq 1}$.

We will now state and explain our main theorem. For $p > 1$, define q by $1/p + 1/q = 1$.

THEOREM 1.2. — *Let $p > 1$. Consider a contable pB-seet* T, *and let* $\bar{f}(x) = \sup_{f \in T} |f(x)|$. *Then for $f \in$* T, *and $i \geqq 1$, we can define $u(f, i)$, $v(f, i) \in L^p$ with the following properties:*

$$(1.2) \qquad f = u(f, i) + v(f, i)$$

$$(1.3) \qquad |u(f, i)|,\ |v(f, i)| \leqq 2 \bar{f}$$

$$(1.4) \qquad E \Big\| \sum_{i \geqq 1} |v(f, i)(X_i)| i^{-1/p} \Big\|_T \leqq K S_p(T).$$

(1.5) *Consider the random distance $\delta_X(f, g)$ on* T *given by*

$$\delta_X(f, g) = \| ((u(f, i)(X_i) - u(g, i)(X_i)) i^{-1/p})_{i \geqq 1} \|_{p, \infty}.$$

Then there is a metric δ *on* T *such that* $\gamma(\mathrm{T}, \delta, \log^{1/q})) \leqq \mathrm{KS}_p(\mathrm{T})$, *and such that the identity map* $(\mathrm{T}, \delta) \to (\mathrm{T}, \delta_{\mathrm{X}})$ *is Z-Lipschitz, where* $(\mathrm{EZ}^2)^{1/2} \leqq \mathrm{K}$.
(1.6) *Consider the random distance* $\delta'_{\mathrm{X}}(f, g)$ *on* T *given by*

$$\delta'_{\mathrm{X}}(f, g) = \Big(\sum_{i \geqq 1} (u(f, i) - u(g, i))^2\, i^{-2/p}\Big)^{1/2}.$$

Then there is a metric δ' *on* T *such that* $\gamma(\mathrm{T}, \delta', \log^{1/2}) \leqq \mathrm{KS}_p(\mathrm{T})$, *and such that the identity map* $(\mathrm{T}, \delta') \to (\mathrm{T}, \delta'_{\mathrm{X}})$ *is* Z'*-Lipschitz, with* $(\mathrm{E}(\mathrm{Z}')^2)^{1/2} \leqq \mathrm{K}$.

This theorem is close in spirit to the results of [10]. It roughly speaking states that T can be decomposed in two parts, one for which we have control of the absolute values, and the other for which we have control thanks to a majorizing measure. More precisely, to see that Theorem 2 obviates the fact that T is a p B-set, we write

$$\mathrm{E}\Big\|\sum_{i \geqq 1} \varepsilon_i f(\mathrm{X}_i)\, i^{-1/p}\Big\|_{\mathrm{T}} \leqq \mathrm{E}\Big\|\sum_{i \geqq 1} \varepsilon_i u(f, i)(\mathrm{X}_i)\, i^{-1/p}\Big\|_{\mathrm{T}} + \mathrm{E}\Big\|\sum_{i \geqq 1} \varepsilon_i v(f, i)(\mathrm{X}_i)\, i^{-1/p}\Big\|_{\mathrm{T}}.$$

We have

$$\mathrm{E}\Big\|\sum_{i \geqq 1} \varepsilon_i v(f, i)(\mathrm{X}_i)\, i^{-1/p}\Big\|_{\mathrm{T}} \leqq \mathrm{E}\Big\|\sum_{i \geqq 1} |v(f, i)(\mathrm{X}_i)|\, i^{-1/p}\Big\|_{\mathrm{T}}$$

which is controlled by (1.4).

On the other hand, we have $\gamma(\mathrm{T}, \delta', \log^{1/2}) \leqq \mathrm{Z}'\, \gamma(\mathrm{T}, \delta', \log^{1/2})$. Denoting by E_ε and P_ε the conditional expectation and conditional probability when (X_i) is fixed, we have, from the classical subgaussian inequality

$$\mathrm{P}_\varepsilon\Big(\Big|\sum_{i \geqq 1} \varepsilon_i a_i\Big| \geqq t\Big) \leqq 2 \exp - \frac{t^2}{2 \sum_{i \geqq 1} a_i^2}$$

and the majorizing measure bound, that

$$\mathrm{E}_\varepsilon\Big\|\sum_{i \geqq 1} \varepsilon_i u(f, i)\, i^{-1/p}\Big\|_{\mathrm{T}} \leqq \mathrm{K}\, \gamma(\mathrm{T}, \delta'_{\mathrm{X}}, \log^{1/2})$$

and this implies by (1.6) that

$$(1.7) \qquad \mathrm{E}\Big\|\sum_{i \geqq 1} \varepsilon_i u(f, i)\, i^{-1/p}\Big\|_{\mathrm{T}} \leqq \mathrm{KS}_p(\mathrm{T}).$$

An intriguing feature of [6] is that, while the necessary conditions are obtained via the random distance $d'(f, g) = \big(\sum_{i \geqq 1} (f(\mathrm{X}_i) - g(\mathrm{X}_i))^2\, i^{-2/p}\big)^{1/2}$ on T, the sufficient conditions are obtained via the random distance $d(f, g) = \|((f(\mathrm{X}_i))\, i^{-1/p})_{i \geqq 1}\|_{p, \infty}$. (The desire to understand this phenomenon was at the origin of the present work.) Theorem 2 points toward a close relationship in general between these two distances. Indeed, the inequality (1.7) also follows from the inequality ([6], lemma 3.1)

$$P_\varepsilon(|\sum_{i\geqq 1}\varepsilon_i a_i|>t)\leqq 2\exp-\frac{t^q}{K\|(a_i)_{i\geqq 1}\|_{p,\infty}},$$

the fact $\gamma(T, \delta_X, \log^{1/q})\leqq Z\gamma(T, \delta, \log^{1/q})$, and the corresponding majorizing measure bound "for tails in e^{-t^q}". (It would indeed be also possible to use many other distances interpolating between δ_X and δ'_X.)

Consider the random distance d_X on T given by $d_X(f, g)=\|(i^{-1/p}(f(X_i)-g(X_i))_{i\geqq 1}\|_{p,\infty}$.

In the class of processes studied by Marcus and Pisier, the condition $S_p(T)<\infty$ is proved by showing that $E\gamma(T, d_X, \log^{1/q})<\infty$. So, it is natural to ask whether it is true in general that $E\gamma(T, d_X, \log^{1/q})<\infty$ whenever $S_p(T)<\infty$. We conjecture that this is always the case. (We have been able to show that it is the case under the hypothesis of Theorem 1.4.)

We now turn to the question of finding sufficient conditions for $S_p(T)<\infty$. We recall that for a r. v. Y, we set

$$\|Y\|_{p,\infty}=(\sup_{t>0} t^p P(|Y|>t))^{1/p}.$$

THEOREM 1.3. — *Let $p>1$. Consider a metric space* (T, *d*) *and a process* $(Y_t)_{t\in T}$ *such that*

$$\|Y_t-Y_u\|_{p,\infty}\leqq d(t, u)$$

whenever $t, u\in T$. *Then*

$$\|\sup_{t,u\in T}(Y_t-Y_u)\|_{p,\infty}\leqq K\gamma(T, d, |\,.\,|^p).$$

This theorem is not suprising. The corresponding weaker statement, where $\gamma(T, d, |\,.\,|^p)$ is replaced by the corresponding entropy condition is due to G. Pisier [7], while the author has proved an anologous result for $\|\,.\,\|_p$ instead of $\|\,.\,\|_{p,\infty}$ [11]. Theorem 1.3 can be applied to a subset T of L^p provided with its natural distance.

We now mention a "bracketing theorem" that is actually a rather immediate consequence of the results of [1]. We will give a simple proof that is a small variation of the argument of Theorem 1.2.

THEOREM 1.4. — *Let $p>1$. Suppose that on a subset* T *of* L^p *we are given a distance* ρ *finer than the natural distance. Assume that for* $\varepsilon>0$ *and* $f\in T$, *we have*

$$\|\operatorname{ess\,sup}\{|f-g|;\ \rho(f, g)\leqq\varepsilon\}\|_p\leqq\varepsilon.$$

Then we have $S_p(T)\leqq K\gamma(T, \rho, \log^{1/q})$.

By suitable choices of ρ, this allows to prove statements when we have some control over $A(f, \varepsilon)\overset{d}{=}:\|\operatorname{ess\,sup}\{|f-g|;\|f-g\|_p\leqq\varepsilon\}\|_p$. It would be nice to have a statement that contains Theorem 1.4 and which, when specialized to the case where we have no control on $A(f, \varepsilon)$ coincide with Theorem 1.3.

The paper is organized as follows. Theorem 1.2 is proved in Section 2. Theorems 1.3 and 1.4 are proved in Section 3. Various examples of classes of sets that are (are not) p B-classes are provided in Section 4.

2. STRUCTURE THEOREMS

Our starting point to prove Theorem 1.2 is the following

PROPOSITION 2.1 [15]. — $\gamma(T, \|\cdot\|_p, \log^{1/q}) \leqq KS_p(T)$.

In order to unify the proofs of Theorems 1.2 and 1.4, we assume now that $\gamma(T, \rho, \log^{1/q}) = M$ for some metric ρ finer than the distance induced by $\|\cdot\|_p$. From a (simple) discretization procedure (*see* [1], [12], [11]) one obtains the following.

PROPOSITION 2.2. — *Denote by* l_0 *the largest integer with* $2^{-l_0} \leqq \operatorname{diam} T$, *where the diameter is taken for the distance* ρ. *For* $l \geqq l_0$, *one can find finite subsets* T_l *of* T, *maps* π_l: $T \to T_l$, *and a probability measure* m *on* T *such that is we set* $\alpha_{l,f} = m(\{\pi_l f\})$, *the following properties hold.*

(2.1) $$\rho(f, \pi_l f) \leqq 2^{-l}$$

(2.2) $$\pi_l \circ \pi_{l+1} = \pi_l$$

(2.3) $\pi_{l_0} f$ *is independent of* f *(and will be denoted by* π_0*)*

(2.4) *For each* $f \in T$, *there is a nondecreasing sequence* $\beta_{l,f} \geqq \log \dfrac{e^2}{\alpha_{l,f}}$ *depending only on* $\pi_l f$ *such that*

$$\sum_{l \geqq l_0} 2^{-l} \beta_{l,f}^{1/q} \leqq KM.$$

We define the distance δ on T by $\delta(f, g) = 2^{-r}$, where $r = \max[l;\ l \geqq l_0,\ \pi_l f = \pi_l g\}$. The fact that it is a distance follows from (2.2). Clearly (2.4) implies that $\gamma(T, \delta, \log^{1/q}) \leqq KM$.

Let $\alpha = 1/2 - 1/q = 1/p - 1/2$. We set $\delta'(f, g) = 2^{-l} \beta_{l,f}^{-\alpha}$ where $\delta(f, g) = 2^{-l}$. This definition makes sense, since, as $\beta_{l,f}$ depends only on $\pi_l f$, we have $\beta_{l,f} = \beta_{l,g}$.

PROPOSITION 2.3. — δ' *is a distance, and* $\gamma(T, \delta', \log^{1/2}) \leqq KM$.

Proof. — We check first that δ' is a distance. Only the triangle inequality needs proof. Let $f, g, h \in T$. If $\delta(f, g) = \delta(g, h) = 2^{-l}$ then $\pi_l f = \pi_l g = \pi_l h$, so that $\delta(f, h) = 2^{-l'}$ with $l' \geqq l$, and $\delta'(f, h) = 2^{-l'} \beta_{l',f}^{-\alpha} \leqq \delta'(f, g)$. If $\delta(f, g) = 2^{-l} < \delta(g, h) = 2^{-l'}$ then

$\pi_{l'} f = \pi_{l'} g = \pi_{l'} h$, while $\pi_{l'+1} f = \pi_{l'+1} g \neq \pi_{l'+1} h$, so $\delta(f, h) = 2^{-l'}$, and $\delta'(f, h) = 2^{-l'} \beta_{l', g}^{-\alpha} = \delta'(g, h)$.

We now prove the second assertion. Fix $f \in T$. For $l \geqq l_0$, let $r_l = 2^{-l} \beta_{l, f}^{-\alpha}$. We have $\delta'(f, \pi_l f) \leqq r_l$, so that

$$m(B(r_l)) \geqq m(\{\pi_l f\}) = \alpha_{l, f}$$

where $B(\varepsilon) = \{g \in T;\ \delta'(f, g) \leqq \varepsilon\}$. Since $1/2 - \alpha = 1/q$, we have

$$\sum_{l \geqq l_0} r_l (\log(1/m(B(r_l))))^{1/2} \leqq \sum_{l \geqq l_0} r_l \beta_{l, f}^{1/2} = \sum_{l \geqq l_0} 2^{-l} \beta_{l, f}^{1/q} \leqq KM.$$

We note that for $g \in T$, $\delta(f, g)$ is necessarily of the form r_l. Since $r_{l+1} \leqq r_l/2$, this implies easily that

$$\int_0^{\operatorname{diam} T} (\log(1/m(B(\varepsilon))))^{1/2} d\varepsilon \leqq 2 \sum_{l \geqq l_0} r_l (\log(1/m(B(r_l))))^{1/2}$$

and this concludes the proof.

For $f \in T$, $i \geqq 1$, we define $l(f, i)(x)$ as the smallest integer $l \geqq l_0$ for which $|\pi_{l+1} f(x) - \pi_l f(x)| > 2^{-l-1} i^{1/p} \beta_{l+1, f}^{-1/p}$, and we set $u(f, i)(x) = \pi_{l(f, i)(x)} f(x)$ (if no such integer exists, we set $u(f, i)(x) = f(x)$). We set $v(f, i) = f - u(f, i)$, so that (1.3) holds.

PROPOSITION 2.4. — $\forall f \in T$, $\sum_{i \geqq 1} i^{-1/p} E|v(f, i)| \leqq KM.$

We observe that

$$|v(f, i)| \leqq \sum_{l \geqq l_0} |f - \pi_l f| 1_{\{|\pi_{l+1} f - \pi_l f| > 2^{-l} i^{1/p} \beta_{l+1, f}^{-1/p}\}},$$

so that Proposition 2.4 follows from (2.4) and the following.

LEMMA 2.5. — *Let $g, h \in L^p$, $g, h \geqq 0$ and $a > 0$. Then*

$$\sum_{i \geqq 1} i^{-1/p} E(g 1_{\{h \geqq i^{1/p} a^{-1/p}\}}) \leqq K a^{1/q} \|g\|_p \|h\|_p^{p/q}.$$

Proof. — We have $h \geqq i^{1/p} a^{-1/p}$ only if $i \leqq a h^p$. Since $\sum_{i \leqq a h^p} i^{-1/p} \leqq K(a h^p)^{1/q}$, the left hand side is $\leqq K a^{1/q} E(g h^{p/q})$, and the result follows by Holder's inequality.

For $f \in T$, $l \geqq l_0$, set $h_{l, f} = (\pi_l f - \pi_{l-1} f)$. Denote by $g_{l, f, X}^{(r)}$ the r-th largest term of the sequence $(i^{-1/p} h_{l, f}(X_i))_{i \geqq 1}$. For $u > 1$, consider the following event Ω_u:

$$\forall f \in T, \quad \forall l \geqq l_0, \quad \forall r \geqq \beta_{l, f}, \qquad g_{l, f, g, X}^{(r)} \leqq 2^{-l+5} r^{-1/p} u.$$

LEMMA 2.6. — *For $u \geqq e$, $P(\Omega_u) \geqq 1 - u^{-2p}$.*

Proof. — If $g^{(r)}$ is the r-th largest term of the sequence $i^{-1/p} h(X_i)$, a very useful computation of J. Zinn ([8], p. 37) shows that

$$\text{(2.5)} \qquad P(\forall r \geqq m,\ g^{(r)} \geqq (2e)^{1/p} r^{-1/p} \| h \|_p) \leqq u^{-mp}.$$

Applying (2.5) to the functions $h_{l,f}$ that satisfy $\| h_{l,f} \|_p \leqq 3.2^{-l}$, it follows that $P(\Omega_u) \geqq 1 - \sum_{f \in T, l \geqq l_0} u^{-p\beta_{l,f}}$. Now

$$u^{-p\beta_{l,f}} = (e^{-\beta_{l,f}})^{p \log u} \leqq (\alpha_{l,f}/e^2)^{p \log u}$$

and the result follows since $\sum \alpha_{l,f} \leqq 1$.

LEMMA 2.7. — *On Ω_u, we have*

$$\text{(2.6)} \quad \begin{cases} \forall f \in T, \quad \forall l \geqq l_0, \\ \| (i^{-1/p} h_{l,f}(X_i)\, 1_{\{h_{l,f} < 2^{-l} i^{1/p} \beta_{l,f}^{-1/p}\}}(X_i))_{i \geqq 1} \|_{p,\infty} \leqq 2^{-l+5} u. \end{cases}$$

Proof. — Consider the r-th largest term $\theta^{(r)}$ of that sequence. If $\theta^{(r)} > 2^{-l+5} r^{-1/p} u$, by definition of Ω_u, we must have $r \leqq \beta_{l,f}$, so that $\theta^{(r)} > 2^{-l+5} u \beta_{l,f}^{-1/p}$ which is impossible since each term of the sequence is $\leqq 2^{-l} \beta_{l,f}^{-1/p}$. Thus $\theta^{(r)} \leqq 2^{-l+5} r^{-1/p} u$, which implies the result.

PROPOSITION 2.8. — *On Ω_u, the following holds.*

(1) *The identity map* $(T, \delta) \to (T, \delta_X)$ *is* $K u$*-Lipschitz*

(2) *The identity map* $(T, \delta') \to (T, \delta'_X)$ *is* $K u$*-Lipschitz.*

Proof. — (1) Let f, $g \in T$ and let $\delta(f, g) = 2^{-r}$. We recall that $u(f, i)(x) = \pi_{l(f,i)(x)} f(x)$, where $l(f, i)(x)$ is the smallest integer such that $|\pi_{l+1} f(x) - \pi_l f(x)| > 2^{-l-1} i^{1/p} \beta_{l+1,f}^{-1/p}$. It follows that if $l(f, i)(x) < r$, then, since $\pi_r f = \pi_r g$, we have $l(f, i)(x) = l(g, i)(x)$ and $u(f, i)(x) = u(g, i)(x)$. So we have

$$|u(f, i)(x) - u(g, i)(x)| \leqq |u(f, i)(x) - \pi_r f(x)|\, 1_{\{l(f,i)(x) \geqq r\}} + |u(g, i)(x) - \pi_r g(x)|\, 1_{\{l(f,i)(x) \geqq r\}}.$$

Let $h_{l,f} = \pi_l f - \pi_{l-1} f$, $\bar{h}_{l,f,i} = h_{l,f}\, 1_{\{h_{l,f} < 2^{-l} i^{1/p} \beta_{l,f}^{-1/p}\}}$. If $l(f, i)(x) \geqq r$, we have

$$\text{(2.8)} \quad |u(f, i)(x) - \pi_r f(x)| \leqq \sum_{r < l < l(f,i)(x)} h_{l,f}(x) \leqq \sum_{r < l \leqq l(f,i)(x)} \bar{h}_{l,f,i}(x)$$

by definition of $l(f, i)(x)$. By definition of Ω_u, we have $\| (i^{-1/p} \bar{h}_{l,f,i}(X_i))_{i \geqq 1} \|_{p,\infty} \leqq K u 2^{-l}$, so that

$$\| (i^{-1/p} (u(f, i) - \pi_r f)\, 1_{\{l(f,i) \geqq r\}}(X_i))_{i \geqq 1} \|_{p,\infty} \leqq K u \sum_{l \geqq r} 2^{-l} \leqq K u 2^{-r}.$$

The same inequality holds for f in place of g; this proves the result.

(2) We use the same notations as in (1). Fixing l, let $a_i = i^{-1/p} \bar{h}_{l,f,i}(X_i)$, so $\| (a_i)_{i \geqq 1} \|_{p,\infty} < K u 2^{-l}$. Also $i^{1/p} a_i \leqq 2^{-l} \beta_{l,f}^{-1/p}$. So, the i-th largest term

$b^{(i)}$ of the sequence $i^{-1/p} a_i$ is $\leqq 2^{-l} \min(K u i^{-1/p}, \beta_{l,f}^{-1/p})$. A short computation shows that $(\sum_{i \geqq 1} i^{-2/p} a_i^2)^{1/2} \leqq 2^{-l} K \beta_{l,f}^{-\alpha} u^{p/2}$. Since the sequence $(\beta_{l,f}^{-\alpha})$ decreases, we see that $\sum_{l \geqq r} 2^{-l} K \beta_{l,f}^{-\alpha} u^{p/2} \leqq K 2^{-r} \beta_{r,f}^{-\alpha} u^{p/2}$. It follows that $\delta_X(f, g) \leqq K 2^{-r} \beta_{r,f}^{-\alpha} u^{p/2} \leqq K u \delta'(f, g)$, and this concludes the proof.

We observe that if the r. v. Z satisfies $|Z| \leqq K u$ on Ω_u, we have $(EZ^2)^{1/2} \leqq K u$ from Lemma 2.6. Thus (1.5) follows from Proposition 2.8. Only (1.4) remains to prove. As observed in the discussion of Theorem 2, (1.5) implies that $E\| \sum_{i \geqq 1} \varepsilon_i i^{-1/p} u(f, i)(X_i) \|_T \leqq K$ so that $E\| \sum_{i \geqq 1} \varepsilon_i i^{-1/p} v(f, i)(X_i) \|_T \leqq K S_p(T)$ since $v(f, i) = f - u(f, i)$. It follows from [4], Proposition 1 [applied conditionally with respect to (X_i)] that

$$E\| \sum_{i \geqq 1} \varepsilon_i i^{-1/p} | v(f, i)(X_i) | \|_T \leqq K S_p(T).$$

Now, by a standard symmetrization argument

$$E\| \sum_{i \geqq 1} i^{-1/p} (| v(f, i)(X_i) | - E | v(f, i) |) \|_T \leqq 2 E\| \sum_{i \geqq 1} \varepsilon_i i^{-1/p} | v(f, i)(X_i) | \|_T$$

so

$$E\| \sum_{i \geqq 1} i^{-1/p} | v(f, i)(X_i) \|_T \leqq KM + \sup_{f \in T} \sum_{i \geqq 1} i^{-1/p} E | v(f, i)(X_i) |$$

$$\leqq KM + KS_p(T)$$

by Proposition 2.4. The proof of Theorem (1.2) is complete.

Remark. – The families $v(f, i)$ can be defined whenever we have a metric ρ on T finer than $\|.\|_p$ with $M = \gamma(T, \rho, \log^{1/q}) < \infty$. Then our argument, together with the comments of the introduction show that T is a pB-set if and only it $E\| \sum_{i \geqq 1} i^{-1} | v(f, i)(X_i) | \|_T < \infty$.

3. SUFFICIENT CONDITIONS

It is enough to prove Theorem 1.3 when T is finite. The main step is to reduce to the case where T is ultrametric, that is where the distance d satisfies $d(f, h) \leqq \max(d(f, g), d(g, h))$ whenever $f, g, h \in T$. This is done *via* the following lemma.

LEMMA 3.1 ([11], Theorem 4.6). – *Consider a finite metric space* (T, d). *Then* (T, d) *is image by a contraction of a finite ultrametric space* (U, δ) *such that* $\gamma(U, \delta, |.|^p) \leqq K \gamma(T, d, |.|^p)$.

We also need the following, that is certainly known.

LEMMA 3.2. — *$\|.\|_{p,\infty}$ is equivalent to a norm* N *that has the property that* $N(\max_{i\leq n}|Y_i|)\leq(\sum_{i\leq n}N(Y_i)^p)^{1/p}$ *for all n, all* $Y_1,\ldots,Y_n\in L_{p,\infty}$.

Proof. — Define

$$N(Y)=\sup\left\{P(A)^{-1/q}\int_A |Y|\,dP : P(A)>0\right\}$$

which is well known to be equivalent to $\|.\|_{p,\infty}$. Given A with $P(A)>0$, and $Y_1,\ldots,Y_n$, consider a partition $A_1,\ldots,A_n$ of A such that $Y_i\geq Y_j$ on A_i whenever $j\neq i$. Then

$$\int_A \max_{i\leq n}|Y_i|\,dP=\sum_{i\leq n}\int_{A_i}|Y_i|\,dP\leq\sum_{i\leq n}P(A_i)^{1/q}N(Y_i)\leq P(A)^{1/q}(\sum_{i\leq n}N(Y_i)^p)^{1/p}$$

by Holder's inequality. Since A is arbitrary, this concludes the proof.

We denote by k_T the largest integer with $2^{-k_T}\geq \operatorname{diam} T$. Let us call the *depth* of T the smallest integer k for which each ball of T of radius 2^{-k-k_T} contains at most one point. (If T contains only one point, we define its depth as zero.) We prove by induction over k that whenever we have a process $(Y_t)_{t\in T}$, such that $N(Y_t-Y_u)\leq d(t,u)$ for all $t, u\in T$, where (T, d) is ultrametric of depth $\leq k$, for each probability m on T, we have

$$N(\sup_{t\in T}|Y_t-Y_{t_0}|)\leq S(m)$$

where we set

$$S(m)=\sup_{x\in T}\sum_{l\geq k_T+1}2^{-l+1}(m(B(x,2^{-l})))^{-1/p}.$$

Here, t_0 is arbitrary in T and as usual $B(x,\varepsilon)$ denotes the closed ball of radius ε centered at x. For $k=0$, T contains only one point, so there is nothing to prove. We now perform the induction step from k to $k+1$. Suppose that T is of depth $k+1$. Denote by $T_1,\ldots,T_n$ the closed balls of T of radius 2^{-k_T-1}, noting that any two such balls are either equal or disjoint, by ultrametricity. Denote by m_i the restriction of m to T_i for $i\leq n$. Pick $t_i\in T_i$, and let $Z_i=\sup_{t\in T_i}(Y_t-Y_{t_i})$. If we apply the induction hypothesis to T_i and the measure $m_i/m(T_i)$ we have

$$(3.1)\quad N(Z_i)\leq(m(T_i))^{1/p}\sup_{x\in T_i}\sum_{l\geq k_T+2}2^{-l+1}(m(B(x,2^{-l})))^{-1/p}$$
$$\leq(m(T_i))^{1/p}(S(m)-2^{-k_T}m(T_i)^{-1/p})\leq m(T_i)^{1/p}S(m)-2^{k_T}.$$

Let $U_i=|Y_{t_i}-Y_{t_0}|$, where t_0 is arbitrary in T, so $N(U_i)\leq 2^{-k_T}$. Let $Z=\sup_{t\in T}|Y_t-Y_{t_0}|$. We have clearly $Z\leq\max_{i\leq n}(U_i+Z_i)$, so, by Lemma 3.2, we have $N(Z) \leqq (\sum_{i \leqq n} N(U_i + Z_i)^p)^{1/p}$. Now, by (3.1)

$$N(U_i + Z_i) \leqq 2^{-k_T} + N(Z_i) \leqq m(T_i)^{1/p} S(m)$$

so that $N(Z) \leqq S(m)$. This completes the proof.

We now prove Theorem 4.1. The proof procedes as the proof of Theorem 1.1. We can surely assume that $\beta_{l,f} \leqq 2\beta_{l-1,f}$. In the definition of Ω_u, we replace $h_{l,f}$ by $h'_{l,f}$ defined as follows

$$h'_{l,f} = \operatorname{ess\,sup}\{|g - \pi_{l-1} f| : \rho(g, \pi_{l-1} f) \leqq 2^{-l+2}\}$$

(so that $\|h'_{l,f}\|_p \leqq 2^{-l+2}$ by hypothesis). We observe that $h'_{l+1,f} \leqq h'_{l,f}$. We denote by $l'(f, i)(x)$ the smallest integer l such that $h'_{l+1,f}(x) > 2^{-l-1} i^{1/p} \beta_{l+1,f}^{-1}$, and we replace $l(f, i)(x)$ by $l'(f, i)(x)$ in the definition of $u(f, i)$ and $v(f, i)$. Since $l'(f, i) \leqq l(f, i)$, Proposition 2.8 still hold. To conclude it is enough to prove that on Ω_u we have

$$(3.2) \qquad \forall f \in T, \quad \sum_{i \geqq 1} i^{1/p} |v(f, i)(X_i)| \leqq KM u^p.$$

If we set $l = l'(f, i)(x)$, we have $v(f, i) = f - \pi_l f$ so that $|v(f, i)| \leqq h'_{l,f+1} \leqq h'_{l,f}$. Also the definition of $l'(f, i)(x)$ shows that $|v(f, i)| \leqq h'_{l,f} 1_C$, where

$$C = \{2^{-l-1} \beta_{l+1,f}^{-1/p} \leqq i^{-1/p} h'_{l,f} \leqq 2^{-l} \beta_{l,f}^{-1/p}\}.$$

Now, on Ω_u, whenever $i^{-1/p} h'_{l,f}(X_i) \geqq 2^{-l-1} \beta_{l+1,f}^{-1/p}$, if the left hand side is the r-th largest terms of the sequence $(i^{1/p} h'_{l,f}(X_i))_{i \geqq 1}$, then when $r > \beta_{l,f}$, we have

$$2^{-l+5} r^{-1/p} u \geqq 2^{-l-1} \beta_{l+1,f}^{-1/p}$$

so that $r \leqq K u^p \beta_{l+1,f}$. In particular there are at most $K u^p \beta_{l+1,f}$ such terms. The sum of the values of those terms that are $\leqq 2^{-l} \beta_{l,f}^{-1/p}$ is hence $\leqq K 2^{-l} u^p \beta_{l,f}^{1/q}$. This proves (3.2) and finishes the proof.

4. EXAMPLES

A class T of subsets of a probability space is called a pB-class of sets if the class of functions of the type 1_A, $A \in T$ is a pB-set. We denote Lebesgue's measure on [0, 1] by λ.

Example 4.1. — There exists a pB-class of sets T in [0, 1] that cannot be covered by a finite number of brackets $B_i = \{A : A_i \subset A \subset C_i\}$ where $\lambda(C_i \setminus A_i) < 1$.

Proof. — Consider a finite sequence $s = (r_1, \ldots, r_n)$ of rationals $0 < r_1 < \ldots < r_n < 1$. For $u > 0$, consider the class $T_{s,u}$ of sets of the type

$\bigcup_{i\leqq r}[t+r_i, t+r_i+u]$ (mod 1) for all $0\leqq t\leqq 1$, and let $T_s=\bigcup_{0\leqq u\leqq 1} T_{s,u}$. It follows from Theorem 1.3 that T_s is a pB-class of sets. It follows from Lebesgue's theorem that

$$\lim_{u\to 0}\int_1^{\infty} P(\sup_{A\in T_{s,u}} |\Sigma i^{-1/p}\varepsilon_i 1_A(X_i)|>r)\,dr=0.$$

It follows that if (s_m) is an enumeration of all the finite sequences of rationals $r_1<\ldots<r_n$, we can find a sequence $u_m>0$ with $\lim_{m\to\infty} u_m k_m=0$, where k_m is the length of s_m, and such that $T=\bigcup T_{s_m, u_m}$ is a pB-class. Suppose for contradiction, that it could be covered by a finite sequence $(B_i)_{i\leqq k}$ of brackets $\{A;\ A_i\subset A\subset C_i\}$ with $\lambda(C_i\setminus A_i)<\delta<1$. Fix m_0 large enough that $\lambda(A)<\delta/2$ whenever $A\in T'=\bigcup_{m\geqq m_0} T_{s_m, u_m}$. $(B_i)_{i\leqq k}$ covers T', and we disregard those brackets that do not meet T'. Then $\lambda(A_i)<\delta/2$ for $i\leqq k$, so $\lambda(C_i)<1$. It follows that for some l large enough $\sum_{i\leqq k}(\lambda(C_i))^l<1$. It is then a routine exercise to see that there exists $t>0$ and a sequence $r_1<\ldots<r_l$ of rationals such that the sequence $t+r_1, \ldots, t+r_l$ (mod 1) cannot be covered by any C_i, $i\leqq k$. This contradiction completes the proof.

We recall that one says that a class of sets T shatters a finite set $\{x_1, \ldots, x_n\}$ if for any subset I of $\{1, \ldots, n\}$, there exists $A\in T$ such that $x_i\in A \Leftrightarrow i\in I$. For a class of sets T, we denote $V_n(T)$ the set of all sequences $(x_1, \ldots, x_n)$ such that T shatters $\{x_1, \ldots, x_n\}$. In [13], it is shown that the sets $V_n(T)$ essentially determine whether a class of sets is a Donsker class. This is not true for pB-classes.

Example 4.2. — There exists two classes of sets T, T' on [0, 1] with the following properties.

(4.1) T' is pB-class, T fails to be a pB-class.

(4.2) $T'\subset T$; $\gamma(T, \|.\|_p, \log^{1/q})<\infty$.

(4.3) $\forall n$, $V_n(T')=V_n(T)$.

Proof. — Consider a sequence (k_n), to be determined later. For $n\geqq 3$, we consider disjoint subintervals I_n of [0, 1], of lenth $2^{-n/q}$, that we partition in k_n equal intervals. We set $T'=\bigcup_{n\geqq 3} T_n$, where T_n consists of the unions of any collection of less than $r_n=[2^{-1+n/p}]$ subintervals of I_n, for any n. We claim that if the sequence (k_n) increases fast enough, T is a pB-class of sets. One first shows, by routine computations, that if $S_{n,X}$ denotes the sum of the r_n longest terms of the sequence $i^{-1/p}1_{I_n}(X_i)$, $E\sup_n S_{n,X}<\infty$, and one uses a limit procedure as in example 1.

Consider now a one to one map φ from $\bigcup_{n \geqq 1} T_n$ to $\mathbb{N}$. We define T as the union of T′ and of the sets of the type $\bigcup_{l \geqq 1} A_l$, where for some sequence $n(k)$, we have $A_l \in T_{n(l)}$ and $n(l+1) = \varphi(A_l)$ for each $l \geqq 1$. Call a sequence $(n(l))_{l \geqq 1}$ admissible if it arises this way. Clearly, we can choose φ with the following additional property: given two admissible sequences $(n(k))$, $(m(k))$ if $n(r) \neq m(r)$ for some r, then for l, $l' > r$ we have $n(l) \neq m(l')$. It is an easy, but tedious task to show that $\gamma(T, \|\cdot\|_p, \log^{1/q}) < \infty$ provided (k_n) increases fast enough. We show now that T is not a pB-class. If the sequence (k_n) increases fast enough, with probability one, for each sequence $(X_i)_{i \geqq 1}$ and large enough n, we can find at least r_n terms of this sequence, of index $\leqq 2^n$, that are located in disjoint subinterals of I_n. Given (ε_i) and (X_i), with probability one, one can then construct by induction over l a sequence $A_l \in T_{n(l)}$ where $n(l) = \varphi(A_{l-1})$, such that $\sum i^{-1/p} \varepsilon_i 1_{A_l}(X_i) \geqq 1/4$. This shows that T is not a pB-class. We now prove that (4.3) holds. Let $(x_1, \ldots, x_n) \in V_n(T')$. Suppose that $Z = \{x_1, \ldots, x_n\}$ meets two different intervals I_r, I_s with, say, $r < s$. Then we can find $A \in T$ such that $A \supset (I_r \cup I_s) \cap Z$. By definition $A = \bigcup_{l \geqq 1} A_l$, where $A_{l+1} \in T_{n(l)}$, for an admissible sequence $(n(l))$. We can also find $A' \in T$ with $A' \cap I_r \cap Z = \varnothing$, $A' \supset I_s \cap Z$. Then $A' = \bigcup_{l \geqq 1} A'_l$, where $A_{l+1} \in T_{m(l)}$ for some admissible sequence $(m(l))$. We observe that s is both of the type $n(l)$ and $m(l')$. This implies that $l = l'$ and that $n(i) = m(i)$ for $i \leqq l$; but then $A \cap I_r = A' \cap I_r$, which is impossible. Then we have shown that Z is contained is a single interval I_r. Since the trace of T and T′ coincide on that interval, we have $Z \in V_n(T)$. This concludes the proof.

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