ANNALES DE L'I. H. P., SECTION B

J. MICHAEL STEELE

Kingman's subadditive ergodic theorem

Annales de l'I. H. P., section B, tome 25, nº 1 (1989), p. 93-98

<http://www.numdam.org/item?id=AIHPB_1989__25_1_93_0>

Ann. Inst. Henri Poincaré,

Vol. 25, n° 1, 1989, p. 93-98. *Probabilités et Statistiques*

Kingman's subadditive ergodic theorem

by

J. Michael STEELE

School of Engineering and Applied Science, Engineering Quadrangle E-220,
Princeton University, Princeton, NJ 08544

ABSTRACT. — A simple proof of Kingman's subadditive ergodic theorem is developed from a point of view which is conceptually algorithmic and which does not rely on either a maximal inequality or a combinatorial Riesz lemma.

Key words : Ergodic theorem, subadditive ergodic theorem.

RÉSUMÉ. — Une preuve simple du théorème ergodique sous-additif de Kingman est développée d'un point de vue qui est conceptuellement algorithmique et qui ni se base ni sur une inégalité maximale ni sur un lemme combinatoire de Riesz.

1. INTRODUCTION

This note concerns the proof and application of the following theorem of Kingman (1968, 1973, 1976):

THEOREM. — *If* T *is a measure preserving transformation of the probability space* $(\Omega, \mathbf{F}, \mu)$ *and* $\{g_n, 1 \leqq n < \infty\}$ *is a sequence of integrable functions*

Classification A.M.S. : 60 G 10, 60 F 15.

satisfying

$$g_{n+m}(x) \leqq g_n(x) + g_m(T^n x), \tag{1.1}$$

then, with probability one, we have

$$\lim_{n \to \infty} g_n(x)/n = g(x) \geqq -\infty,$$

where $g(x)$ is an invariant function.

This result has been given elegant proofs in Burkholder (1973), Derriennic (1975), Katznelson and Weiss (1982), Neveu (1983), and Smeltzer (1977). Also, proofs which relax the original requirements or sharpen the original conclusions have been given by Ackoglu and Sucheston (1978), Ackoglu and Krengle (1981), Ghoussoub and Steele (1980), Del Junco (1977), Derriennic (1983), Ligget (1985), Moulin-Ollagnier (1983), and Schürger (1986, 1988). Finally, there have been applications of Kingman's theorem in a variety of areas, including, for example, Deken (1979), Derriennic (1980), Kingman (1976), and Steele (1978).

The next section gives a proof of Kingman's theorem which is possibly the simplest — in conception and in calculation — of the available proofs. The third section then examines some evolutionary relationships between approaches to the theory of subadditive processes. Finally, to give an example of its application, Kingman's theorem is used to prove the ergodic theorem for vector valued stationary processes.

2. PROOF OF KINGMAN'S THEOREM

If we define a new process g'_m by

$$g'_m(x) = g_m(x) - \sum_{i=1}^{m-1} g_1(T^i x),$$

then $g'_m(x) \leqq 0$ for all x, and g'_m again satisfies (1.1). Since Birkhoff's ergodic theorem can be applied to the second term of g'_m, the almost sure convergence of g'_m/m implies the almost sure convergence of g_m/m. Thus, without loss of generality, we can assume $g_m(x) \leqq 0$.

Next, we check that $g(x) = \liminf_{n \to \infty} g_n(x)/n$ is an invariant function. By (1.1) we have $g_{n+1}(x)/n \leqq g_1(x)/n + g_n(Tx)/n$, so taking the limit inferior we see $g(x) \leqq g(Tx)$. Thus, one finds $\{g(x) > \alpha\} \subset T^{-1}\{x : g(x) > \alpha\}$, and since T is measure preserving, the sets $\{g(x) > \alpha\}$ and $T^{-1}\{g(x) > \alpha\}$ differ by at most a set of measure zero. Consequently, $g(Tx) = g(x)$ almost surely, and g is measurable with respect to the invariant σ-field A. Thus, we can also assume without loss of generality that for all $x \in \Omega$ we have $g(T^k x) = g(x)$ for all integers k.

Now, given $\varepsilon>0$, $1<N<\infty$, and $0<M<\infty$, we let

$$G_M(x)=\max\{-M, g(x)\}$$

and consider the set

$$B(N, M)=\{x: g_l(x)>l(G_M(x)+\varepsilon) \text{ for all } 1\leqq l\leqq N\},$$

and its complement $A(N, M)=B(N, M)^c$. For any $x\in\Omega$ and $n\geqq N$, we then decompose the integer set $[1, n)$ into a union of three classes of intervals by the following algorithm:

Begin with $k=1$. If k is the least integer in $[1, n)$ which is not in an interval already constructed, then consider $T^k x$. If $T^k x\in A(N, M)$, then there is an $l\leqq N$ so that $g_l(T^k x)\leqq l(G_M(T^k x)+\varepsilon)=l(G_M(x)+\varepsilon)$, and if $k+l\leqq n$ we take $[k, k+l)$ as an element of our decomposition. If $k+l>n$, we just take the singleton interval $[k, k+1)$, and finally if $T^k x\in B(N, M)$ we also take the singleton $[k, k+1)$.

Thus, for any x we have a decomposition of $[1, n)$ into a set of u intervals $[\tau_i, \tau_i+l_i)$ where $g_{l_i}(T^{\tau_i}x)\leqq l_i(G_M(x)+\varepsilon)$ with $1\leqq l_i\leqq N$, and two sets of singletons: one set of v singletons $[\sigma_i, \sigma_i+1)$ for which $1_{B(N, M)}(T^{\sigma_i}x)=1$ and a second set of w singletons contained in $(n-N, n)$.

By the fundamental subadditive inequality (1.1), our decomposition of $[1, n)$, the invariance of g, and the assumption that $g_m(x)\leqq 0$ one has the bounds:

$$g_n(x)\leqq \sum_{i=l}^{u} g_{l_i}(T^{\tau_i}x)+\sum_{i=1}^{v} g_1(T^{\sigma_i}x)+\sum_{i=1}^{w} g_1(T^{n-i}x) \tag{2.1}$$
$$\leqq (G_M(x)+\varepsilon)\sum_{i=1}^{u} l_i\leqq G_M(x)\sum_{i=1}^{n} l_i+n\varepsilon.$$

Also, by the construction of the covering intervals we have

$$n-\sum_{k=1}^{n} 1_{B(N, M)}(T^k x)-N\leqq \sum_{i=1}^{u} l_i, \tag{2.2}$$

so, by Birkhoff's theorem we find

$$\liminf_{n\to\infty} n^{-1}\sum_{i=1}^{n} l_i\geqq \lim_{n\to\infty} n^{-1}\sum_{k=1}^{n} 1_{A(N, M)}(T^k x)$$
$$=1-E(1_{B(N, M)}\mid A)\quad \text{a. s.} \tag{2.3}$$

By (2.1) we then conclude for any $K\leqq M$ that

$$\limsup_{n\to\infty} g_n(x)/n\leqq G_M(x)(1-E(1_{B(N, M)}\mid A))+\varepsilon\quad \text{a. s.} \tag{2.4}$$
$$\leqq G_K(x)(1-E(1_{B(N, M)}\mid A))+\varepsilon\quad \text{a. s.}$$

Now, the definition of B(N, M) guarantees that $1_{B(N,M)} \to 0$ a. s. as $N \to \infty$ for any fixed K, so letting $N \to \infty$ in (2.4) implies

$$\limsup_{n \to \infty} g_n(x)/n \leqq G_K(x) + \varepsilon \quad \text{a. s.} \tag{2.5}$$

Since (2.5) holds for all $K \geqq 0$ and $\varepsilon > 0$, we see

$$\limsup_{n \to \infty} g_n(x)/n \leqq \liminf_{n \to \infty} g_n(x)/n \quad \text{a. s.}$$

and the proof of Kingman's theorem is complete.

3. RELATIONSHIPS AND AN APPLICATION

Kingman's subadditive ergodic theorem has inspired many proofs, possibly even more than the fundamental ergodic theorem of Birkhoff. The preceding approach was motivated by the proof of Birkhoff's ergodic theorem of Shields (1987), which in turn, owes a debt to Katznelson and Weiss (1982) and Ornstein and Weiss (1983). Further, the proofs of the Birkhoff and Kingman theorems given in Katznelson and Weiss (1982) have roots in the intriguing proof of Birkhoff's theorem given by Kamae (1982) which drew on techniques of non-standard analysis, specifically the non-standard measure theory of Loeb (1975). The reduction to negative processes by means of g'_m developed independently of this evolutionary chain and was introduced in Ackoglu and Sucheston (1978).

As a simple but typical application of Kingman's subadditive ergodic theorem, we will give a brief proof of the Banach space ergodic theorem which extends the law of large numbers of Mourier (1953). If $\{X_i\}$ denotes an ergodic stationary sequence of Bochner integrable random variables with values in the Banach space F with norm $\|.\|$, we first note that for the purpose of proving an ergodic theorem there is no loss in assuming that $E(X_1)=0$. Next, for any $\varepsilon>0$, the Bochner integrability of the $\{X_i\}$ implies there is a linear operator θ on F with finite dimensional range such that $E\|X_i - \theta X_i\| \leqq \varepsilon$ for each $1 \leqq i < \infty$. The usual Birkhoff's ergodic theorem applied to linear functionals of $\theta(X_i)$ then shows that $n^{-1} \sum_{i=1}^{n} \theta(X_i)$ converges a. s. and in L_F^1 to $E\theta(X_1)$. The L_F^1 convergence then guarantees $\overline{\lim}\, E\|S_n/n - E\theta(X_1)\| \leqq \varepsilon$ from which we see $\lim_{n \to \infty} E\|S_n/n\| = 0$. Since $\|S_n\|$ is a subadditive process, the real sequence $\|S_n/n\|$ converges almost surely, and, by the preceding identification, it must now converge almost surely to zero, exactly as was to be proved.

ACKNOWLEDGEMENT

Thanks are due Paul Shields for making available a preliminary version of Shields (1987). Also, David Aldous, Andrés del Junco, Ulrich Krengle, Thomas Ligget, Joseph Marhoul, David Pollard, and Klaus Schürger kindly provided comments on a (much) earlier draft. In particular, the proof of Mourier's theorem was provided by David Aldous for inclusion here. This research was supported in part by NSF Grant # DMS-8414069.

REFERENCES

M. A. Ackoglu and L. Sucheston, A Ratio Ergodic Theorem for Superadditive Processes, *Z. Wahr. Verw. Gebiete*, Vol. **44**, 1978, pp. 269-278.

M. A. Ackoglu and U. Krengle, Ergodic Theorems for Superadditive Processes, *J. für die Reine und Angewante Math.*, Vol. **323**, 1981, pp. 53-67.

D. L. Burkholder, Discussion of Prof. Kingman's paper, *Ann. Probab.*, Vol. **1**, 1973, pp. 900-902.

A. Del Junco, On the Decomposition of a Subadditive Stochastic Process, *Ann. Probability*, Vol. **5**, 1977, pp. 298-302.

J. G. Deken, Some Limit Result for Longest Common Subsequences, *Discrete Math.*, Vol. **26**, 1979, pp. 17-31.

Y. Derriennic, Sur le théorème ergodique sous-additif, *C.R. Acad. Sci. Paris*, T. **281**, Series A, 1975, pp. 985-988.

Y. Derriennic, Quelques applications du théorème ergodique sous-additif, *Astérisque*, Vol. **74**, 1980, pp. 183-201.

Y. Derriennic, Un théorème ergodique presque sous-additif, *Ann. Probability*, Vol. **11**, 1983, pp. 669-677.

N. Ghoussoub and J. M. Steele, Vector Valued Subadditive Processes and Applications in Probability, *Ann. Probability*, Vol. **8**, 1980, pp. 83-95.

T. Kamae, A Simple Proof of the Ergodic Theorem Using Non-Standard Analysis, *Israel J. Math.*, Vol. **42**, 1982, pp. 284-290.

Y. Katznelson and B. Weiss, A Simple Proof of Some Ergodic Theorems, *Israel J. Math.*, Vol. **42**, 1982, pp. 291-296.

J. F. C. Kingman, The Ergodic Theory of Subadditive Stochastic Processes, *J. Roy. Statist. Soc. Ser.*, Vol. **B 30**, 1968, pp. 499-510.

J. F. C. Kingman, Subadditive Ergodic Theory, *Ann. Probability*, Vol. **1**, 1973, pp. 883-909.

J. F. C. Kingman, Subadditive Processes, *Lecture Notes in Math.*, Vol. **539**, 1976, pp. 167-223.

T. Ligget, An Improved Subadditive Ergodic Theorem, *Ann. Probability*, Vol. **13**, 1985, pp. 1279-1285.

P. Loeb, Conversion from Non-Standard Measure Theory and Applications to Probability Theory, *Transactions of the American Math. Soc.*, Vol. **211**, 1975.

E. Mourier, Éléments aléatoire dans un espace de Banach, *Ann. Inst. H. Poincaré*, 1953, pp. 161-244.

J. Moulin-Ollagnier, Théorème ergodique presque sous-additif et convergence en moyenne de l'information, *Ann. Inst. Henri Poincaré*, Vol. **B 19**, 1983, pp. 257-266.

J. Neveu, Courte démonstration du théorème ergodique sur-additif, *Ann. Inst. Henri Poincaré*, Vol. **B 19**, 1983, pp. 87-90.

D. ORNSTEIN and B. WEISS, The Shannon-McMillan-Breiman Theorem for a Class of Amenable Groups, *Israel J. Math.*, Vol. **44**, 1983, pp. 53-60.

K. SCHÜRGER, A limit theorem for almost monotone sequences of random variables, *Stochastic Processes and their Applications*, Vol. **21**, 1986, pp. 327-338.

K. SCHÜRGER, On Derrienic's almost subadditive ergodic theorem, *Acta Math. Hungarica*, Vol. **52**, 1988 (to appear).

P. SHIELDS, The Ergodic and Entropy Theorems Revisited, *I.E.E.E. Transactions in Information Theory*, IT-33(2), 1987, pp. 263-265.

M. D. SMELTZER, Subadditive Stochastic Processes, *Bull. Amer. Math. Soc.*, Vol. **83**, 1977, pp. 1054-1055.

J. M. STEELE, Empirical Processes and Subadditive Processes, *Ann. Probability*, Vol. **6**, 1978, pp. 118-127.

(*Manuscrit reçu le* 1er *mars* 1988.)