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On strictly ergodic models for commuting ergodic transformations

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ABSTRACT. — B. Weiss [W] proved that every ergodic Z^2-action has a strictly ergodic model. We strengthen this result in the following way: If the Z^2-action is ergodic and is generated by two commuting transformations S and T, there exists a strictly ergodic model in which every ergodic Z-action generated by some $S^i T^j$ is itself strictly ergodic.

Key words : Strictly ergodic model — Z^2 action — Uniform partition.

RÉSUMÉ. — B. Weiss [W] a démontré que toute action ergodique de Z^2 possède un modèle strictement ergodique. Nous renforçons ce résultat de la façon suivante : Si l'action de Z^2 est ergodique et est engendrée par deux transformations S et T qui commutent, nous construisons un modèle strictement ergodique dans lequel toute action de Z ergodique, engendrée par une transformation $S^i T^j$, est strictement ergodique.

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I. INTRODUCTION

The action of a group G by homeomorphisms of a compact metric space Y is said to be strictly ergodic, if there is a unique Borel probability measure λ, fixed by the action and $\nu(U)>0$ for every non empty open set $U \subset Y$. In the case where $G=Z$, in 1969, R. Jewett [J] proved that every weakly mixing invertible transformation on a Lebesgue space is measure isomorphic to a strictly ergodic transformation and Krieger [K] proved it for every ergodic invertible transformation in 1970. In 1983, B. Weiss [W] extended this result to any commutative G-action. For $G=Z^2$, B. Weiss asked us the following: Suppose given an ergodic Z^2-action, does there exists a strictly ergodic model for this Z^2-action such that every ergodic element of this action (that generates a Z-action ergodic) is also strictly ergodic?

In this paper, we will give a positive answer to this question.

We point out the following two remarks that motivate our work (both of them are from B. Weiss):

Remark 1. — If $G=Z$ and (X, T) is strictly ergodic, then for every k, if T^k is ergodic, it is strictly ergodic. (The proof of this is easy: if ν is an invariant measure for T^k and λ is the only one for T, if $\nu' = 1/k \sum_{i=0}^{i=k-1} T^i \nu$, ν' is invariant for T so that $\nu'=\lambda$ and this implies that ν is absolutely continuous with respect to λ: $d\nu = f d\lambda$, the fact that ν is T^k invariant then implies that f is T^k invariant, but T^k is ergodic so $f \equiv 1$ and $\nu=\lambda$.

Remark 2. — For $G=Z^2$, the situation is not the same as for Z, in fact B. Weiss has built (oral communication) an example of a Z^2-action (with generators S and T), strictly ergodic such that T is ergodic but not strictly ergodic.

Acknowledgement: Not only did B. Weiss introduce us to the subject, but he also helped us to solve many of the problems we encountered.

Let be given (Y, ρ, ν, S, T) an ergodic Z^2-action with generators S and T. Most of this paper is devoted to a proof of the following theorem:

THEOREM 3. — *If* (Y, ρ, ν, S, T) *is an ergodic* Z^2*-action and the action of* T *alone is ergodic, then there exists a strictly ergodic system* (X, β, λ, S, T) *such that* (X, β, λ, T) *is itself strictly ergodic and* (X, β, λ, S, T) *is measure theoretically isomorphic to* (Y, ρ, ν, S, T).

The proof of this theorem will parallel Weiss's proof for a Z^2-action. In fact, one can reconstruct his proof along ours, with obvious simplifications. In the sequel, we will suppose that the Z^2-action is aperiodic. This will enable us to use Rohlin lemma. Otherwise, for a minimal i, $i>0$ $S^i = T^j$

and we will indicate at the end of part III how to make our proof in that case.

II. CONSTRUCTION OF A UNIFORM TOWER

DEFINITIONS 4. — The M-T-name of x for a partition $P=(p_1, \ldots, p_k)$ is the element of $\{1, 2, \ldots, k\}^M$: (α_i) such that $T^i x \in p_{\alpha_i}$ for $1 \leqq i \leqq M$. By extension, we will also mean the sequence $p_{\alpha_1}, p_{\alpha_2}, \ldots, p_{\alpha_M}$.

A Rohlin tower F with base B is said to be of shape D if $F = \bigcup_{(i,j) \text{ in } D} S^i T^j B$.

Let $D_n = \{(i, j) \in Z^2; \max(|i|, |j|) \leqq n\}$, $C_n = \{i \in Z; |i| \leqq n\}$.

DEFINITION 5. — Let n and M be in N and $\delta > 0$. A set B in ρ is the base of an (n, M, δ, T) uniform Rohlin tower F if:

(i) $B \cap S^i T^j B = 0$ for all (i, j) in $D_n - \{0, 0\}$ and $F = \bigcup_{(i,j) \text{ in } D_n} S^i T^j B$.

(ii) For every p in C_n, for almost every y in Y, if:

$$\beta_p^M(y) = |\{0 \leqq i \leqq M-1;\ T^i y \text{ is in } T^{-n} S^p B\}|$$

then:

$$|\beta_p^M(y)/M - 1/|D_n|| \leqq \delta.$$

We will suppose in the sequel that T is the action that moves points horizontally. Informally, this definition means that for almost every y in Y, in the M successive images of y under T one is most of the time in horizontal level from the tower F, and every such horizontal level is seen with almost the same frequency.

The uniform (n, M, δ, T) Rohlin tower will play a fundamental role in the sequel. Our first goal is to prove:

THEOREM 6. — *For every* n_0 *and every* $\delta > 0$, *if* M *is big enough, there exists a* (n_0, M, δ, T) *uniform Rohlin tower.*

The proof of this theorem depends only on the aperiodicity of the Z^2-action, it is independent of the ergodicity of T.

In order to prove theorem 6, we will first construct a sequence of well-nested (*see* definition below) ordinary Rohlin towers.

If D is in Z^2 and y in Y, by Dy we will mean in the sequel: $\{S^i T^j y; (i, j) \in D\}$.

DEFINITION 7. — Let M and $(h_n)_{n \text{ in N}}$ be in N. A sequence of Rohlin towers $\{F_n\}_{n \text{ in N}}$ with base B_n so that:
$F_n = \bigcup_{(i,j) \text{ in } D_{h_n}} S^i T^j B_n$ is said to be M-well-nested if:

For every p, q, $p<q$, all y in B_p, y' in B_q:
either

(a) $D_{h_p+M}\, y \subset D_{h_q}\, y'$ or

(b) $D_{h_q}\, y' \cap D_{h_p+M}\, y = \varnothing$.

LEMMA 8. — *Given* M *in* N *and* $(h_n)_{n \text{ in N}}$, *if* M/h_1 *is small enough and* h_n/h_{n+1} *decreases sufficiently rapidly with n then:*
There exists a sequence of M *well-nested Rohlin towers* $(F_n)_{n \text{ in N}}$ *with* $F_n = \bigcup_{(i,j) \text{ in } D_{h_n}} S^i T^j B_n$ *and* $\nu(F_n) \to 1$ *as n tends to infinity.*

Proof. — The construction is made by induction. We will suppose M/h_1 very small. The induction will give us a sequence $(F'_i)_{i \text{ in N}}$ with base B_n so that $F'_n = \bigcup_{(i,j) \text{ in } D_{h_n}+M} S^i T^j B_n$ satisfying:

(i) $\nu(F'_n) \to 1$

(ii) For all $p<q$, for all y in B_p, y' in B_q:
either

$$D_{h_q}\, y' \cap D_{h_p+M}\, y = 0 \tag{1}$$

or

$$D_{h_p+M}\, y \subset D_{h_q}\, y'. \tag{2}$$

It is then clear that $F_n = \bigcup_{(i,j) \text{ in } D_{h_n}} S^i T^j B_n$ will satisfy the conditions of the lemma [We replace F'_n by F_n to add to (1) and (2) the case where $p=q$]. We are thus left to build the F'_i by induction:

Let $(\delta_i)_{i \text{ in N}}$ be a given sequence of real decreasing to 0.

Let $F_1^{(1)} = \bigcup_{(i,j) \text{ in } D_{h_1}+M} S^i T^j B_1^{(1)}$ be an ordinary Rohlin tower with $\nu(F_1^{(1)}) > 1 - \delta_1/2$. Let h_2 be big enough relatively to h_1 and let $F_2^{(2)} = \bigcup_{(i,j) \text{ in } D_{h_2}+M} S^i T^j B_2^{(2)}$ be a Rohlin tower chosen with $\nu(F_2^{(2)}) > 1 - \delta_2/2$. Now we change F_1 and B_1 into $F_1^{(2)}$, $B_1^{(2)}$ by erasing from B_1 all the y not satisfying (ii) (1) or (2). If h_1/h_2 was chosen small enough, we get $\nu(F_1^{(2)}) > 1 - \delta_1/2 - \delta_1/4$. This process can be done inductively erasing a small portion from all the $F_i^{(n-1)}$ $i \leq n-1$, at step n to get $F_i^{(n)}$. It is clear that $F'_i = \lim_k F_i^{(k)}$ will satisfy the required conditions and this ends the proof of the lemma.

Proof of Theorem 6. — Let n_0 and δ be fixed and M be such that: $n_0/M \leqq \delta/100$. We use lemma 8 to get a sequence of $100\,M\,n_0 = M'$ well-nested towers $(F_i)_{i \in N}$ such that $F_i = \bigcup_{(k,\,l) \text{ in } D_{h_i}} S^k T^l B_i$.

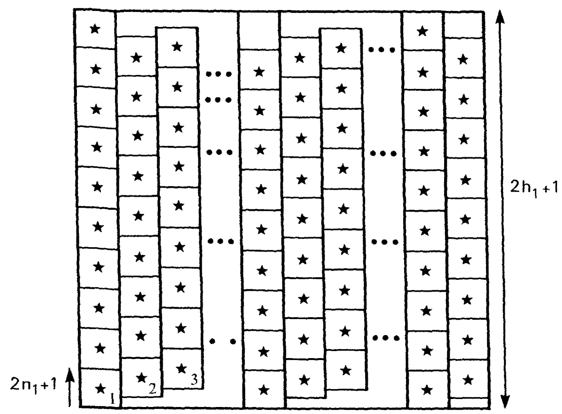


FIG. 1

Let us first consider $F_1 = \bigcup_{(k,\,l) \text{ in } D_{h_1}} S^k T^l B_1$. In the sequel we will use repeatedly the one to one correspondence between (k, l) in D_{h_1} and the level $S^k T^l B_1$ of F_1. We want to define B base of an (n_0, M, δ, T) uniform Rohlin tower. We will pave D_{h_1} as in figure 1 by squares of shape D_{n_0} and put in B, all the levels that correspond (by the above natural correspondence) to centers of the squares that are translate of D_{n_0} (that is the levels marked by a * on the picture). We recall that T is the action that moves points horizontally in the tower. The paving is done by first putting the square in the lower left corner, secondly paving all the column above it consecutively, (we suppose that $2h_1+1$ was chosen to be a multiple of $2n_0+1$), then in the second column, we put the square marked 2 on the figure, that is the translate of square 1 by $(2n_0+1, 1)$, it is one level upward relatively to square 1 and filling what we can of the second column above this square 2, then 3 is one level upward and so on with cycles of length $2n_0+1$. This way in F_1, except near the boundary, in a T-name we see typically: level i of F (defined by its base B), level $i+1$ and so on, and this is what we were looking for.

Let us see, now how to go on this construction:

We want to fill F_2 by towers of shape D_{n_0}. As a first approximation, one paves D_{h_2} by D_{n_0}- squares the same way as we did for F_1.

Let us call base of a F_1-column in F_2, a subset B′ of B_2 (the base of F_2) such that for any (x, y) in B′, any (k, l) in D_{h_2} either ($S^k T^l x \in F_1$ and $S^k T^l y \in F_1$) or ($S^k T^l x \notin F_1$ and $S^k T^l y \notin F_1$). The corresponding column is then $\bigcup_{(k,\,l)\text{ in } D_{h_2}} S^k T^l B'$. Let us fix a F_1-column C in F_2 with base B′. This is a Rohlin tower with shape D_{h_2}. In this F_1-column, by definition, some of the levels ($S^k T^l B'$ for (k, l) in D_{h_2}) are entirely in F_1. We can model this by saying that in D_{h_2}, there are translates of D_{h_1} at some given places, corresponding to levels in $F_1 \cap C$.

The problem we are facing is to match the approximative paving of F_2 (and thus of C) with the already existing paving of F_1-towers. Through the above model this model can be translated into a geometrical problem in D_{h_2}: We want to match the approximative paving of D_{h_2} with the already existing one of the translates of D_{h_1} (corresponding to levels in F_1) that were paved in the first step.

We will first see how to localize the problem around some given image of the F_1-tower. Because the towers are M′ well-nested, around each F_1-tower, we can find a "free zone" such that we thus obtain around each center of F_1-towers a square of size $2h+1$ with $h = h_1 + 10\,M\,n_0$, and in this square there are no other F_1-tower. We now erase in these free zones the paving of D_{h_2} we had (that is we erase all the D_{n_0}-squares intersecting these free zones).

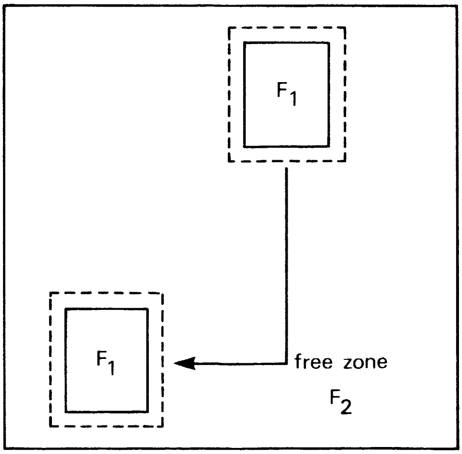


FIG. 2

This way, we localize the problem:

We are now given a free zone around some F_1-tower that was paved in step 1, we want to see how to pave the free zone so as to match the paving both with the existing paving of the F_1-tower and the paving outside the free zone. This is a geometric combinatorial problem, our goal being to keep the uniformity property along every horizontal.

There are, inside the free zone, two matchings to achieve, one for the horizontal coordinate of the squares to be in phase with the already existing paving, the other for the vertical coordinate.

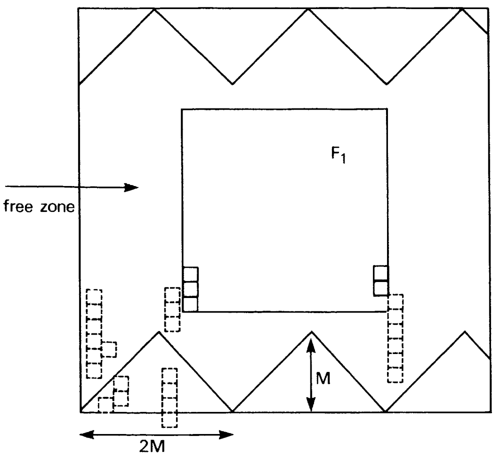


FIG. 3

In the free zone, we put two zigzags with period $2M$, and slope alternatively $+1$ and -1, one at the bottom of the free zone, the other above it (see *Fig.* 3). To be more precise, such a zigzag begins at some point (k, l) on the left vertical boundary of the free zone and ends at (k', l') on the vertical right boundary of it. It is the set of points $(k+p, l+p)$ for $0 \leqq p \leqq M-1$, then $(k+M+p, k+M-p)$ for $0 \leqq p \leqq M-1$ (this will be called a basic zigzag) and then in a periodic way [with period $(2M, 0)$] starting from $(k+2M, l)$ another basic zigzag and so on until (k', l') where we reach the right vertical boundary of the free zone (the last basic zigzag may not be complete). We put two such zigzags in a free zone, one beginning at the lower left corner of it, the other at the point $(k-(M-1), l)$ if (k, l) is the highest left point of the free zone.

These zigzags will play a role of boundaries. We now extend the existing paving of F_1 "naturally", that is the paving of vertical columns is extended by putting under it and above it translates of D_{n_0}, and for a given vertical column, in the next column the paving is the same but moved one level upwards. We so, extend the paving in all the vertical columns until we reach the zigzag boundaries so that there is no square like D_{n_0} intersecting these zigzags, and in the other direction, we stop when we reach the vertical boundary of the free zone.

We similarly "naturally" extend the paving of D_{h_2} from outside the free zone until we reach these zigzag boundaries. We thus obtain a paving of the F_1-column C. We do this successively for all the F_1-columns in F_2.

This way, it is easy to see that in a T−M-name inside F_2, we usually see level i of F then level $i+1$ and so on. The only time this is not true is when we are near the vertical boundaries of a given free zone or near the zigzags inside it.

For the vertical boundaries, the "holes" are at most of length $4n_0$. Because zigzags are of period 2M, in a T−M-name, "holes" because of the zigzags are of length at most $8n_0$. It is easy to deduce that for a point inside F_2 we have [*see* definition 5 (ii)]:

$$|\beta_p^M(y)/M - 1/|D_{n_0}|| \leqq 20\, n_0/M \leqq \delta$$

by the choice of M. It is clear that this same construction can easily be done inductively (because the towers are M′ well-nested) and that almost every y will have a T−M-name inside some F_i [because $\nu(F_i)\to 1$]. This ends the proof of Theorem 6.

III. CONSTRUCTION OF A UNIFORM PARTITION

Before proving the existence of uniform partitions, let us prove a technical lemma; for it we will need the following definition;

DEFINITION 9. — Let k, p, h, $l \in \mathrm{N}$. The (k, p) partition of $[0, h-1]\times[0, h-1]$ is the sequence of sets P_1, K_1, P_2, K_2, . . ., P_l, K_l, P_{l+1} where:

$$P_1=[0,\ h-1]\times[0,\ p-1], \qquad K_1=[0,\ h-1]\times[p,\ p+k-1],$$
$$P_2=[0,\ h-1]\times[p+k,\ p+k+p-1], \qquad K_2=[0,\ h-1]\times[k_2+1, k_2+k]$$

(with $k_2=p+k+p-1$) and so on until

$$P_l=[0,\ h-1]\times[p_l, p_{l+p-1}], \qquad K_l=[0,\ h-1]\times[k_l,\ k_l+k-1],$$
$$P_{l+1}=[0,\ h-1]\times[k_l+k,\ h-1].$$

l was chosen so that for the last set, P_{l+1} we have $[h-1-(k_l+k)]\leqq k+p$. We will say that the P_i are p-bands and the K_1 k-bands. See figure 4.

A special role will be assigned in the sequel to points $(0, s)$ with $0\leqq s\leqq h-1$. We will call them points of the T-boundary. By extension, we will say that a point x in a Rohlin tower $F=\bigcup_{\substack{0\leqq i\leqq h-1\\ 0\leqq j\leqq h-1}} S^i T^j B$, with base B is in the T-boundary of F if $x=S^s y$ for y in B and $0\leqq s\leqq h-1$.

DEFINITION. — Let P be a partition of Y, $\delta>0$, $k\in\mathrm{N}$ and $p\in\mathrm{N}$ satisfy: $p/k\leqq\delta/10$. If F is a Rohlin tower with base B, $F=\bigcup_{(i,j)\in D_n} S^i T^j B$, we can consider the (k, p) partition of D_n. We will say that $x\in Y$ is good

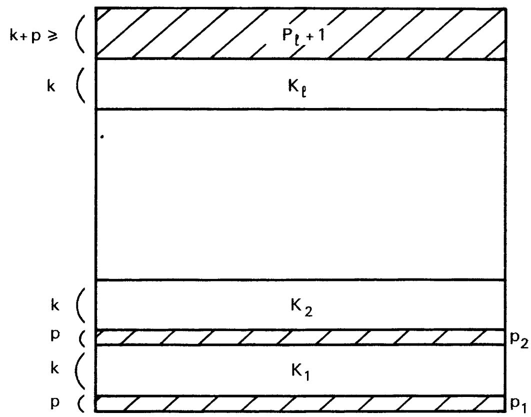


FIG. 4

(for P, δ, k, p, n) if:

(*a*) $x\in F$, $x=S^{i_1}T^{j_1}b$ with $b\in B$ and (i_1, j_1) is in a k-band K_r for some r.

(*b*) If $K_r=[0, h-1]\times[k_r+1, k_r+k]$ for every j_0, $j_0\in[k_r+1, k_r+k]$ so that $(0, j_\theta)$ is in the T-boundary of K_r, we have:

$$\sum_{p_i\in P}\left|1/n\sum_{j=0}^{j=n-1}1_{p_i}(T^j S^{j_0}b)-\nu(p_i)\right|\leqq\delta.$$

LEMMA 10. — *For every partition* P, $\delta>0$, $k\in N$ *and* $p\in N$ *such that* $p/k\leqq\delta/10$, *there exists* n_0 *so that if* $n\geqq n_0$:

If F *a Rohlin tower with* $F=\bigcup_{(i, j)\in D_n}S^iT^jB$ *and* $\nu(F)\geqq 1-\delta/4$, *the set* E *of good points (for* P, δ, k, p, n) *satisfies:* $\nu(E)\geqq 1-\delta$.

Proof. — Let $f_{n, p_i}(x)=1/n\sum_{l=0}^{1=n-1}1_{p_i}(T^l x)$.

Because of the mean ergodic theorem, we can choose n_1 so that there exists D so that:

$$\sum_{p_i\in P}\left|f_{n_1, p_i}(x)-\nu(p_i)\right|\leqq\delta/2$$

for any x in D and $\nu(D)\geqq 1-\delta^2/4k$.

Let us consider now a tower F, $F=\bigcup_{(i, j)\in D_n}S^iT^jB$ with $\nu(F)\geqq 1-\delta/4$ and $n_1/n\leqq\delta/100$. Let us look at a sequence: x, Tx, ..., $T^{n-1}x$ for x in the T-boundary. Let $T^{i_1}x$ the first point in D in this sequence, we thus have a good n_1-block: $T^{i_1}x, T^{i_1+1}x, \ldots, T^{i_1+n_1-1}x$. Starting from $T^{i_1+n_1}x$ we can look at the next point in D, and going on this process, we filled part of our sequence by n_1-blocks, so that if $T^j x$ is the beginning of such an n_1-block, $T^j x \in D$ and the part not filled, is by those j so that $T^j x \notin D$. It is clear that if we filled this sequence in $(1-\delta/2)n$ of the n spaces then:

$$\sum_{p_i \in P} |f_{n, p_i}(x) - \nu(p_i)| \leqq \delta/2 \tag{3}$$

Thus if (3) is not true, more than $\delta n/2$ of the $(T^j x)_{0 \leqq j \leqq n}$ are not in D. It clearly follows that if x is in the set H of points in a k-band (more precisely $x = S^i T^j b$, b is in B and (i, j) is in a k-band) that are not good, at least $\delta n/2$ images of x under T in the k-band are not in D. We point out the fact that for a given x in some k-band (with the same meaning as above) K_r, either x and all its images in the k-band (*see* above) are good or they all are not good, by definition. Thus:

$$\nu(H)\,\delta n/2\,kn \leqq \nu(X - D) \leqq \delta^2/4\,k.$$

From the choice of p and l and using the fact that the points not in k-bands have a measure smaller than $\delta/2$, we conclude that $\nu(E) \geqq 1-\delta$. This ends our proof.

DEFINITION. — Let P be a partition, $P = (p_1, p_2, \ldots, p_a)$. For $D \subset Z^2$, $y \in Y$, the D$-$P-name of y is the sequence $(i_d(x)) \in \{1, 2, \ldots, a\}^D$ such that for any $d \in D$, $d = (i, j)$, $S^i T^j x \in p_{i_d(x)}$.

DEFINITION. — A partition P is said to be T-uniform if:

For avery $n \in N$, for every $\varepsilon > 0$, there exists $N_n \in N$ such that for almost every y in Y:

For any atom $p_i^{(n)}$ in $\bigvee_{(k,\, l) \in D_n} S^k T^l P$:

$$\left| 1/N_n \sum_{k=0}^{k=N_n-1} 1_{p_i^{(n)}}(T^k y) - \nu(p_i^{(n)}) \right| \leqq \varepsilon \tag{4}$$

If (4) is true for a fixed n and for almost every y in Y, we will say that P is (N_n, ε, n) good.

THEOREM 11. — *For every partition* P, *and every* $\delta > 0$, *there exists* $\bar{P}$ T-*uniform such that:* $d(P, \bar{P}) < \delta$.

Proof. — We first fix a sequence $(\varepsilon_n)_{n \in N}$, such that $\sum_{n=0}^{+\infty} \varepsilon_n < \delta$ and we will build $\bar{P}$ as a limit of a sequence of partition P_n such that $P = P_0$ and $d(P_n, P_{n+1}) < \varepsilon_n$. We will build the P_n by induction.

Step 1. — Let us apply Lemma 10, for the partition P, $\delta=\varepsilon_1/3$, $k=1$, $p=0$ (so that there is no p-band) to find a tower T_1 "good" for that lemma and so that T_1 is (n_1, M_1, δ_1, T) uniform with $\delta_1 \leqq \varepsilon_1/3$ and $n_1/M_1 \leqq \varepsilon_1/3$.

A base of a column of T_1 for the partition P is a set of points in B_1 (base of T_1) that have the same $D_{n_1}-P$-name. Because of Lemma 10, most of the horizontal levels in those columns are good (up to $\varepsilon_1/3$) for the ergodic theorem. In such a column, if some horizontal level is not good, we replace the P-name on this level, by the P-name of some good horizontal level (we fix such a good level for all the changes). Doing this for all the columns in T_1, we change P into P_1 so that $d(P, P_1) \leqq \varepsilon_1/3$ and now every horizontal level in T_1 is "good" for P_1. Because of the uniform properties of T_1, for almost every y in Y, in the $T-P_1$-name of y of length M_1, we are at least $(1-\varepsilon_1/3)\,M_1$ times in those "good horizontal levels" of T_1. It is then easy to conclude that P_1 is $(M_1, \varepsilon_1, 1)$ good.

Step 2. — We now choose k_2 with: $6\,M_1/k_2 \leqq \delta/10$ with $\delta=\varepsilon_2/3$ and we apply lemma 10 with:

$P=\bigvee_{(i,\,j)\in D_2} S^i T^j P_1$, $\delta=\varepsilon_2/3$, $k=k_2$ and $p=6\,M_1$ to find a tower T_2 that is (n_2, M_2, δ_2) uniform with $\delta_2 \leqq \varepsilon_2/3$ (from now on we omit the argument T, according to definition 5, T is (n_2, M_2, δ_2, T) uniform). Let $p_2=6\,M_1$. Together with the tower, there is a (k_2, p_2) partition of D_{n_2}.

Let us focus now on a p_2-band.

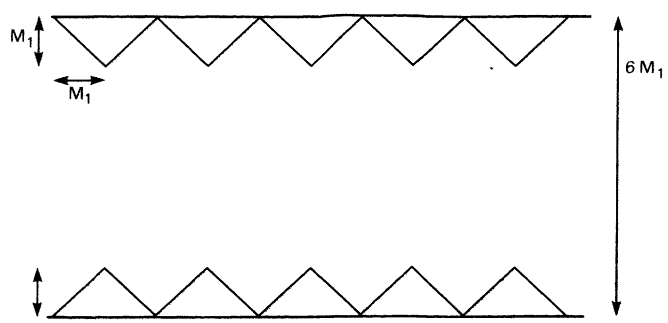


FIG. 5

On these bands, we place 2 zigzags like the one we draw on the picture, the period of which is $2\,M_1$ (the lines are alternatively with slope $+1$ and -1), one at the bottom of the p_2-band, the other one at the top of it. Let us call k_2-p_2 zigzag band, a k_2-band to which we added 2 zigzags, one is the bottom zigzag in the p_2-band above it, the other is the top zigzag in the p_2-band under it.

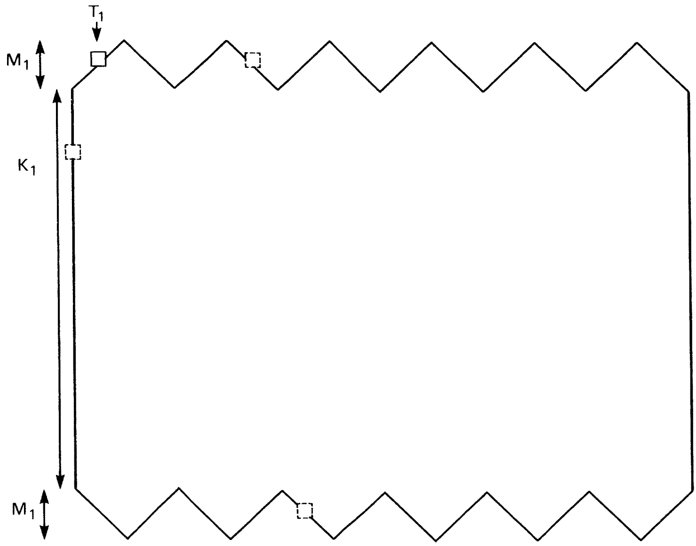


FIG. 6

Let us first consider a fixed $T_1 \vee P_1$ column C with base B′ [T_1 being identified with the partition (T_1, T_1^c)] in the tower T_2. We first delete from T_1, all the towers "like T_1" in C that intersects $S^k T^l B'$ for (k, l) in the boundary of a $k_2 - p_2$ zigzag-band [that is (k, l) is either in one of the 2 zigzags or in the vertical boundaries of the $k_2 - p_2$ zigzag-band]. We change then, the P_1-name in those "deleted towers" and give to the points in it their original P-name. Because of Lemma 10, all the x in C, except a set of measure $\varepsilon_2/3$ (for all the columns), are in a level $S^k T^1 B'$, for (k, l) in a k_2-band and are good. If some x in C is in a k_2-band but is not good, by definition, all the images of x in this k_2-band are not good and we replace the $(P_1 \vee T_1)$-name of x in the entire $k_2 - p_2$ zigzag band, x belongs to, by the $(P_1 \vee T_1)$-name of some y in a good k_2-band, such that all the points in the k_2-band are now good. Doing so, we change T_1. Having done this in all the $P_1 \vee T_1$-column we obtain P_2 with $d(P_1, P_2) \leqq \varepsilon_2/2$ and $T_1^{(2)}$. The $D_{n_1} - P_2$-names of points in $B_1^{(2)}$ (base of $T_1^{(2)}$) are $D_{n_1} - P_2$-names of points in B_1. This crucial fact comes from our construction: we erased towers that were on the boundaries of $k_2 - p_2$ zigzag-bands.

It is now easy to see that, because of the T-uniform property of T_2: For almost every y in Y, if we look at the $T - \bigvee_{(i, j) \in D_2} S^i T^j P_2$ name of y of length M_2, because of property (ii) in definition 5 of T_2, we are $(1 - \varepsilon_2/3) M_2$ times in k_2-bands of T_2. All these k_2-bands are good so that this name is "$(\varepsilon_2, 2)$-good" for P. For almost every y in Y, the $T-P_2$-name of y length M_1 is still "$(2\varepsilon_1, 1)$" good for P_2 because the only change we made in the property of such a name, going from P_1 to P_2 is that we may have erased at most 2 T_1-towers (that were on the boundary of some k_2-p_2 zigzag-band) and because $n_1/M_1 \leqq \varepsilon_1/3$. Another change, but we will absorb it in the other errors, is the difference in the measure of atoms of P_1 and P_2.

Step q. — By induction, we have:

(*a*) a sequence $(T_i^j)_{1 \leqq i \leqq q-1,\ 1 \leqq j \leqq q-1}$ of $(n_i, M_i, \varepsilon_i/3)$ uniform towers, $(p_i)_{1 \leqq i \leqq q-1}$ and $(k_i)_{1 \leqq i \leqq q-1}$ in N.

(*b*) a partition P_{q-1} that is $(M_i, 2\varepsilon_i, i)$ good for $i \leqq q-2$ and $(M_{q-1,}, \varepsilon_{q-1}, q-1)$ good.

(*c*) in fact P_{q-1} is $(M_i, 2\varepsilon_i, i)$ good because at step i for almost every y in Y, the M_i-T-name of y is at least $(1-\varepsilon_i/3)M_i$ times in good k_i-bands that are in T_i (*see* step 2) and in further steps $j>i$, when obtaining P_j, the M_i-T-name of y (for P_j) has almost the same property as the M_i-T-name of y (for P_i) except that we may have erased at most 2 T_i-towers (*see* again step 2) because they were on the boundary of zigzag-bands from some tower $T_l : i<l<j$, for a given l. The fact that l is unique is fondamental and (*c*) is made clearer in our construction below.

Let us construct P_q:

Choose k_q much bigger than $\sup_{i \leqq q-1} M_i$ so that we can apply lemma 10 with:

$$P = \bigvee_{(i,\,j) \in D_q} S^i T^j P_q, \qquad \delta = \varepsilon_q/3, \qquad k = k_q, \qquad p = 6M_{q-1}$$

to find a tower T_q that is (n_q, M_q, δ_q) uniform with $\delta_q \leqq \varepsilon_q/3$. Together with this tower we have the k_q-p_q partition, for $p_q = 6M_{q-1}$. As in step 2, we construct k_q-p_q zigzag-bands.

We first consider a given $\bigvee_{i \leqq q-1} P_i \vee T_i^{q-1}$ column in the tower T_q, where the partition T_i^{q-1} is in fact $(T_i^{q-1}, T_i^{q-1})^c$. We will delete from T_{q-1}, the part that intersects or is at most $2M_{q-2}$ apart from the boundary of some k_q-p_q zigzag-band (*see* step 2). In the following construction, we want to keep our uniform properties from prior steps. To do so, we have to be sure that property (*c*) of the induction remains true. For that, we will do the following (by "picture" of the given column, we will mean a picture of D_{n_q} covered in some part by squares $D_{n_j} j \leqq q-1$, the places of these squares corresponding to T_j^{q-1}-towers in the given column): We give to the parts we delete their previous P_{q-2}-name. Now, by erasing T_{q-1}-towers, we may see again in the picture, towers like T_j^{q-2} for $j \leqq q-2$ that were erased at step $q-1$. We now consider T_{q-2}-towers, both the ones existing before and the ones that "came back" above.

We erase from them all the T_{q-2}-towers that are at most $2M_{q-3}$-apart from the boundary of a $k_q - p_q$ zigzag-band. We give to the part we delete its previous P_{q-3}-name. We thus, may see again in our picture towers like T_j^{q-3} for $j \leqq q-3$ that were erased at step $q-2$. We go on until we look at the T_1-towers in our picture after all this process of "putting on and off towers" and erase these towers that intersect the boundary of a $k_q - p_q$ zigzag-band. We give to the deleted part its previous P-name. We do this for all the columns. Now as in step 2, $1-\varepsilon_q/3$ of the x in k_q-bands are good.

If some x in a k_q-band is not good, we replace its P_{q-1}-name (or more precisely its $\tilde{P}_{q-1}$-name, $\tilde{P}_{q-1}$ being the partition that we obtained above after the above deletion and rebirth process) in the $k_q - p_q$ zigzag-band by one so that the names in the k_q-band are now good.

We thus obtain P_q with $d(P_{q-1}, P_q) < \varepsilon_q$.

As in step 2, it is easy to see that P_q is (M_q, ε_q, q) good. Let us check that we can go on the induction:

Let us fix $j<q$ and y in Y. The M_j-name of y, for P_j, was $(1-\varepsilon_j/3)$ times in good k_j-bands. Suppose we made some change in this property. This means that we are at most $2M_{j-1}$ apart from the boundary of a $k_1 - p_1$ zigzag-band (in a tower T_l, for $l>j$). Let us suppose, now, that we made another change because we were closer than $2M_{j-1}$ from the boundary of a $k_n - p_n$ zigzag-band (in a tower T_n), for $j<l<m \leqq q$.

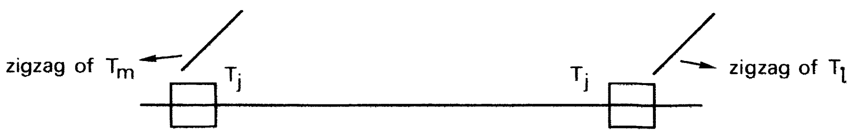


FIG. 7

This would mean that some part of a tower T_l would be closer than $2M_{j-1} + 2M_{j-1} + M_j < 2M_{l-1}$ from the boundary of a $k_n - p_n$ zigzag-band of T_n. This cannot be, by construction. Thus, if we "erased" towers, this happens at most twice in some $T-M_j$-name and this proves that the induction can be pursued. Now, because $\sum_{i=1}^{+\infty} \varepsilon_i < +\infty$, if the k_i were chosen so that $\sum_{i=0}^{+\infty} 10M_{i-1}/k_i < +\infty$, for almost every y in Y and any i, because of the Borel-Cantelli lemma, there is an index j so that we do not change the labelling of the $T-M_i$-name of y after step j. This proves then, that the limiting $\bar{P}$ is $(M_i, 2\varepsilon_i, i)$ good for any i and finishes our proof.

COROLLARY 12. — *There exists a sequence* $\bar{Q}_1 \subset \bar{Q}_2 \subset \ldots \subset \bar{Q}_n \subset \ldots$ *such that* $\bigvee_{n \in N} \bar{Q}_n = \rho$ (*the entire* σ-*algebra*) *and the* $\bar{Q}_n$ *are all* T-*uniform.*

Proof. – Let $Q_1 \subset Q_2 \ldots \subset Q_n \ldots$ be a sequence of partitions such that $\bigvee_{n \in N} Q_n = \rho$. Let $(\varepsilon_i)_{i \text{ in } N}$ be given with $\sum_{i=1}^{+\infty} \varepsilon_i < +\infty$. Using step 1 of Theorem 11, we can find $Q_1^{(1)}$ such that $d(Q_1, Q_1^{(1)}) < \varepsilon_1$ and $Q_1^{(1)}$ is $(M_1, \varepsilon_1, 1)$ good for some M_1.

We can replace Q_2 by $Q_2 \vee Q_1^{(1)}$ to have $Q_1^{(1)} \subset Q_2$. Using now step 2 of Theorem 11 we find $Q_2^{(2)}$ such that $d(Q_2, Q_2^{(2)}) < \varepsilon_2$ and $Q_2^{(2)}$ is $(M_2, \varepsilon_2, 2)$ good. Furthermore, because $Q_1^{(1)} \subset Q_2$, to every atom of Q_2 is corresponding an atom of $Q_1^{(1)}$ and every atom of $Q_1^{(1)}$ is a union of atoms of Q_2. Now using this correspondence $Q_2^{(2)}$ defines $Q_1^{(2)}$ so that $Q_1^{(2)} \subset Q_2^{(2)}$ (For instance if the first atom of $Q_1^{(1)}$ was the union of the second and fourth atom of Q_2, the first atom of $Q_1^{(2)}$ will be the union of the second and fourth atom of $Q_2^{(2)}$. $Q_1^{(2)}$ satisfies:

$$d(Q_1^{(1)}, Q_1^{(2)}) < \varepsilon_2 \text{ and } Q_1^{(2)}$$

is $(M_2, \varepsilon_2, 2)$ good as well as $(M_1, 2\varepsilon_1, 1)$ good (*see* step 2 of Theorem 11). Continuing this process inductively gives us at step n:

$$Q_j^{(n)} \text{ for } j \leqq n \text{ such that: } \quad Q_1^{(n)} \subset Q_2^{(n)} \subset \ldots \subset Q_n^{(n)}.$$

$d(Q_j^{(n)}, Q_j^{(n-1)}) < \varepsilon_n$ for $j < n$ and $d(Q_n, Q_n^{(n)}) < \varepsilon_n$. $Q_j^{(n)}$ is $(M_k, 2\varepsilon_k, k)$ good for any k, $j \leqq k \leqq n$ and $Q_j^{(n)}$ is (M_n, ε_n, n) good for $j \leqq n$. Now defining, $\bar{Q}_j = \lim_{n \to +\infty} Q_j^{(n)}$, as in Theorem 11, we prove that $\bar{Q}_j$ is T-uniform ., $\bar{Q}_1 \subset \bar{Q}_2 \subset \ldots \subset \bar{Q}_j \subset \ldots$, finally because $d(\bar{Q}_j, Q_j) < \sum_{i=j}^{+\infty} \varepsilon_i$, we get $\bigvee_{n \in N} \bar{Q}_n = \rho$.

Sketch of the proof in the non aperiodic case. – In the non aperiodic case it is easy to see that, by if necessary changing the generators of the action, we can suppose that $\sigma^n = \mathrm{Id}$, (σ, τ) being the generators. Then if we have a uniquely ergodic action for (σ, τ), it has to be uniquely ergodic for any $\sigma^i \tau^j$ (if the action of $\sigma^i \tau^j$ is ergodic). This comes as in Remark 1 (*see* introduction) considering ν such that $\sigma^i \tau^j \nu = \nu$ and

$$\nu' = 1/nj\,(\nu + \sigma\nu + \ldots \sigma^{n-1}\nu) + (\tau\nu + \sigma\tau\nu + \ldots \sigma^{n-1}\tau\nu) + \ldots$$
$$(\tau^{j-1}\nu + \sigma\tau^{j-1}\nu + \ldots \sigma^{n-1}\tau^{j-1}\nu),$$

ν' is invariant under σ and τ and the rest of the proof is similar to that of Remark 1.

IV. PROOF OF THEOREM 3

This part is completely due to G. Hansel and J. P. Raoult [H-R], it is just a translation of Corollary 12:

If $P_1 = (p_1, p_2, \ldots, p_a)$ is T-uniform, one can associate to it $\Omega(P_1) \subset \{1, 2, \ldots, a\}^{Z^2}$, with the shifts S_1, T_1 and a measure given by the measure of the cylinder sets in P_1. It is clear that P_1 T-uniform is equivalent to $(\Omega(P_1), S_1, T_1)$ uniquely ergodic and also the action of T_1 alone is uniquely ergodic, with unique invariant measure λ_1. Because of Corollary 12 we can construct, this way:

$$(\Omega(\bar{Q}_1), S_1, T_1, \lambda_1) \overset{\pi_1}{\leftarrow} (\Omega(\bar{Q}_2), S_2, T_2, \lambda_2) \ldots \overset{\pi_n}{\leftarrow} (\Omega(\bar{Q}_{n+1}), S_{n+1}, T_{n+1}, \lambda_{n+1}).$$

The π_i being the projections coming from $\bar{Q}_i \subset \bar{Q}_{i+1}$, λ_i being the invariant measure of $\bar{Q}_i$.

We also have:

$$S_n \pi_{n+1} = \pi_{n+1} S_{n+1} \qquad \text{and} \qquad T_n \pi_{n+1} = \pi_{n+1} T_{n+1}.$$

Let us consider the inverse limit of this diagram $\Omega_\infty = \{(x_n)_{n \in N};\ x_n \in \Omega(\bar{Q}_n)$ and $\pi_n(x_{n+1}) = x_n\}$. This Ω_∞ is compact. Let λ be an ergodic invariant measure for the transformation $T: (x_n) \to (T_n x_n)$. It is clear that projecting λ on the first n components of Ω_∞, that this projection must be λ_n. Because of the definition of the topology of Ω_∞, this shows that λ is unique and finally shows that (Ω_∞, T) is uniquely ergodic. The fact that $\bigvee_{i \in N} \bar{Q}_i = \rho$ implies that if, β is the Borel σ-algebra and S is: $(x_n) \to (S_n x_n)$, $(\Omega_\infty, \beta, \lambda, S, T)$ is isomorphic to (Y, ρ, ν, S, T) and this ends the proof of Theorem 3.

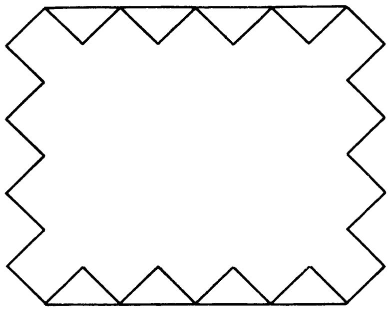
V. GENERALIZATION OF THEOREM 3

THEOREM 13. — *Let* (X, μ, S, T) *an ergodic* Z^2*-action, there exists a strictly ergodic model for this action that satisfies:*

For any (i, j) *such that the Z-action generated by* $S^i T^j$ *on* X *is ergodic, the Z-action generated by the transformation corresponding to* $S^i T^j$ *in the model, is strictly ergodic.*

Remark. — We will suppose that the action is aperiodic, otherwise, *see* above, the proof is trivial.

Proof. — Let $(S_i)_{i \in J}$ (J is finite or countable) be a sequence of all the ergodic Z-actions of the Z^2-action. We will first prove the theorem in the



a k-p zigzag band for S.

FIG. 8

special case where $|J|=2$, $S_1=T$, $S_2=S$:

The proof of theorem 3 can easily be adapted to this case. Theorem 6 remains unchanged. The uniformity for both S and T is obtained, as in Theorem 11 apart from the fact that the proof is done step by step, one step of the induction being to improve the uniformity for S, the next step being then, to improve the uniformity with respect to T. The only real difference is that a zigzag-band for S (or for T) has both an horizontal and vertical boundary that are zigzags (see *Fig.*).

Now for the general case:

Let us see, given a list $S=S^{k(i)}T^{l(i)}$, for $i\in J$, of ergodic Z-actions, how to adapt the proof of Theorem 11 to get from P, a partition $\bar{P}$, close to P and so that $\bar{P}$ is uniform for every S_i, $i\in J$. Doing afterwards, the same kind of proof as in Corollary 12 finishes then, the proof of theorem 13. The proof is, as usual, done by induction. Suppose $J=N$. The first step uses S_1-uniform towers G_1 (we will indicate how to obtain them below), we get from $P_0=P$, a new partition P_1 that is good (we will also indicate below what this means) in G_1 for P_1 and S_1. Then, the same way as in theorem 11, we find M_1 so that for S_1, P_1 is $(M_1, \varepsilon_1, 1)$ good.

In step 2, we construct P_2 such that: P_2 is $(M_1, 2\varepsilon_1, 1)$ and $(M_2, \varepsilon_2, 2)$ good for S_1.

In step 3, we construct P_3 such that: P_3 is $(M_1, 2\varepsilon_1, 1)$ and $(M_2, 2\varepsilon_2, 2)$ good for S_1 and P_3 is also $(M_3, \varepsilon_3, 3)$ good for S_2.

We can then, in the same way, go on the induction and obtain $\bar{P}$.

We have to make two things precise:

(*a*) How to build S_1-uniform towers (and what does S_1-uniform tower mean).

(*b*) What does it mean that P_1 is good for the tower G_1 and the action S_1. We will first explain (*b*), because this will indicate the property we need in (*a*). By (*b*), we will mean as in the proof of Theorem 11: Suppose that we have a Rohlin tower G_1, whose shape is a rectangle, suppose also

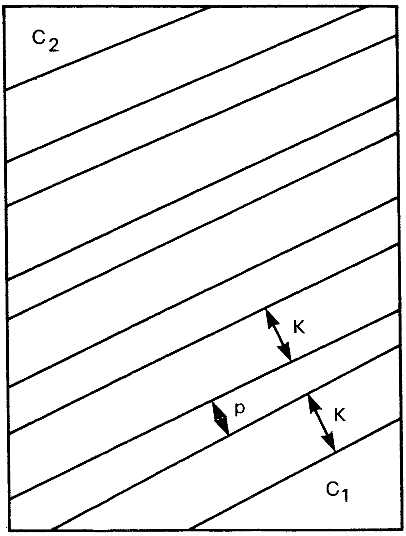


FIG. 9

that we are given a $k-p$ partition in the following sense (see *Fig.*):

Except the two bands on the corner, C_1 and C_2, that are a very small portion of the tower, we divide the points of the lattice corresponding to the tower into successive k_1-bands and p_1-bands. The direction of these bands is parallel to S_1, and in each band we include the points of the lattice that belongs to it.

We also want that in a k_1-band, if we look at the orbit of a point under S_1, we stay longer than a given n_1 in this band (n_1 is choosen to enable us to apply the ergodic theorem to S_1 in those bands). Now P_1-good for S_1 and this partition of G_1, means as before:

All the S_1-names for P_1 along a k_1-band are good for the ergodic theorem (for the atoms of P_1). In the transition from P_0 to P_1 (or from P_n to P_{n+1}), we have to change entire "zigzag-bands". If $S_k = S^{i_k} T^{j_k}$, suppose at step n:

$I_n = \operatorname{Max}_{k \leq n} |i_k|$, $J_n = \operatorname{Max}_{k \leq n} |j_k|$. Our zigzag-bands have thus, zigzags with slope bigger than $\operatorname{Max}(2 I_n, 2 J_n) = K_n$, that is a zigzag-band looks like see figure 10:

The period of these zigzags being bigger than M_{n-1}. The slope of these zigzags will ensure that we can do the induction:

If P_{n-1} was (M_i, ε_i, i) good for the action S_k, P_n will remain good because in a name of length M_i $(i \leq n-1)$ for S_k, we see at most twice, towers near the boundary of a zigzag.

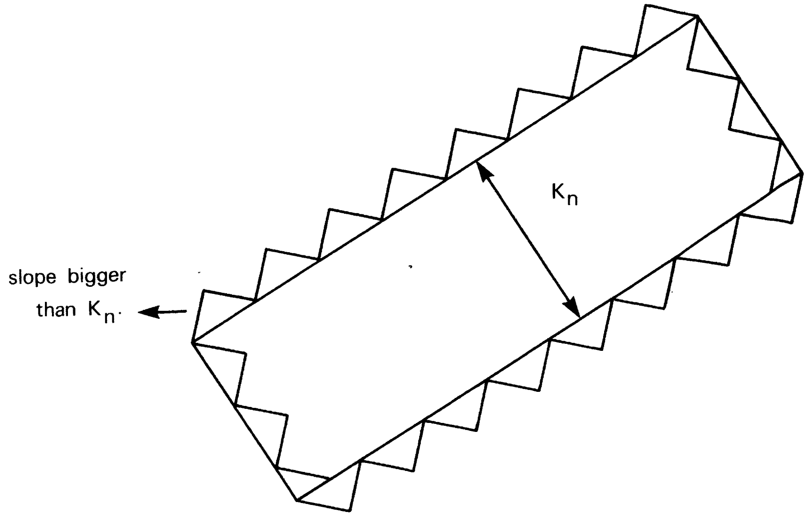


FIG. 10

Now, to ensure that P_1 is $(M_1, \varepsilon_1, 1)$ good for S_1, we have to obtain our towers in such a way that:

For almost every y in Y, if we look at a P_1-name of length M_1, for the action of S_1, we are most of the time in k_1-bands. In the general case, this is what we meant before by a S_1-uniform tower. Let us see now, how to obtain them, that is, we will explain what are the modifications necessary in the proof of Theorem 6 to obtain these towers:

In the proof of Theorem 6, in the first step, we paved D_{h_1} by squares like D_{n_0}. Now, we will pave D_{h_1} by rectangles (thus the uniform Rohlin towers will have a rectangular shape, we suppose $S_1 \neq S$, $S_1 \neq T$). Every column in the paving will now look the same (there is no moving upwards of the "next" column as in the case where $S_1 = T$, see *Fig.* 1). The width p and the length q of the rectangle will be chosen to be prime together and both of them are prime with respect to i_0 and to j_0, if $S_1 = S_0^i T_0^j$. Now, inside F_1, we can look at the orbit of a point x under S_1. x is in some position in one of the rectangles. For $S_1^k x$ to be in the same position in another rectangle we have to have: For the horizontal coordinate: $i_0 k = pk'$ for some k', so that k is a multiple of p, the same way, k is a multiple of q so that the minimal k is $k = pq$. This way the pq images of x under S_1^j: $S_1^j x$ for $j \leqq pq$ are going through all the levels in the rectangle and this exactly once. We are thus in a similar situation as in Theorem 6, for step 1. To go then from the paving of F_1 to one of F_2, we simply pave F_2, by little rectangles as we did in step 1 and remove the ones that intersect F_1 (we do this successively for all the different F_1-column in F_2). It is easy to see (because $i_0 \neq 0$ and $j_0 \neq 0$) that the uniform properties can be obtained this way.

This ends the proof of Theorem 13.

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