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The energy functional, balayage, and capacity

by

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ABSTRACT. — The relationships among the energy functional, the balayage operators, and capacities for a Markov process and the same objects for its q-subprocesses and u-transforms are investigated. Special emphasis is on the behavior of these objects as q and u vary.

Key words : Markov process, energy, balayage, capacity, probabilistic potential theory.

RÉSUMÉ. — La relation entre la fonctionnelle d'énergie, les opérateurs de balayage, et les capacités d'un processus de Markov et les mêmes objets pour ses q-sous-processus et u-transformés sont étudiées. Le comportement de ces objets quand q et u varient est étudié en détails.

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1. INTRODUCTION

In this paper we investigate the relationships between the energy functional (defined in section 3) and the balayage operators and apply the results to the discussion of capacities and cocapacities for Markov processes as defined in [16]. In particular we are interested in the study of q-capacities (i. e. the capacities associated with the q-subprocess) and their properties as a function of q.

The notion of balayage of excessive funtions or excessive measures is due to Hunt [17]. He showed that the balayage of an excessive function f on a Borel set B is given by $P_B f$ where P_B is the hitting operator associated with the underlying Markov process. Perhaps because of this, for many years most attention was devoted to the study of the potential theoretical properties of excessive functions and their generalizations. Recently there has been a renewed interest in the potential theory of excessive measures. *See* e. g. [7], [8], and [13]. In particular in [8], Fitzsimmons and Maisonneuve gave a direct probabilistic expression for Hunt's balayage operation $R_B m$ of an excessive measure m on a set B in terms of the stationary process (Y, Q_m). *See* (2.7).

The energy functional was introduced explicitly by Meyer in [20], but may be traced back to Hunt as well who used similar techniques in discussing balayage. *See* sections 7 and 8 of [17]. Our attention was drawn to the energy functional in connection with capacities by a remark of C. Dellacherie on our previous paper [16]. It has already been pointed out in a paper [23] by the second author that using the energy functional yields in connection with the balayage operations a method of defining capacities which is more general than that in [16]. Moreover, it provides a useful tool in the study of the properties of q-capacities.

We assume given a Borel right process X with semigroup (P_t). If m is an excessive measure and u an excessive function $L(m, u)$ denotes the energy functional evaluated at m and u. *See* (3.2) and (3.9) for the definition of L. The reason L is called the energy functional is explained in (3.15). It is shown in (3.16) that $L(m, P_B u) = L(R_B m, u)$; that is, $L(.,.)$ makes P_B and R_B dual objects. Now fix an excessive measure m and let $C(B)$ and $\hat{C}(B)$ be the "capacity" and "cocapacity" of B as defined in [16]. One main observation (3.22) is that

$$\Gamma(B) \equiv L(R_B m, 1) = \hat{C}(B) + L((R_B m)_i, 1) = C(B) + L(m, (P_B 1)_i),$$

where the subscript i denotes the invariant part. This implies that in case B is transient (resp. cotransient) as defined in [16], one obtains $\Gamma(B) = C(B)$

[resp. $\Gamma(B)=\hat{C}(B)$]. Actually Γ turns out [*see* (7.12)] to be the proper extension (at least for dissipative m) of C and $\hat{C}$ as an outer capacity.

More generally, considering the q-subprocess associated with X (for $q \geqq 0$) there are defined P^q_B, R^q_B, C^q, and $\hat{C}^q$ and as well L^q and Γ^q. For $0 \leqq r < q$ we obtain the following relations for these objects (the numbers refer to the places where the formulas appear in the later sections):

$$(3.26) \qquad L^q(m, u) = L^r(m, u) + (q-r)\, m(u);$$

$$(6.17) \qquad R^q_B m = R^r_B m + (q-r)\, R^r_B m\, V_q$$

where (V_q) denotes the resolvent of X killed when hitting B;

$$(7.4) \qquad \Gamma^q(B) = \Gamma^r(B) + (q-r)\, R^q_B m\, P^r_B 1;$$

$$(6.19) \qquad R^q_B m\, P^r_B = R^r_B m\, P^q_B.$$

Since for $q>0$ any Borel set B is both transient and cotransient for the q-subprocess one has that $C^q(B)=\Gamma^q(B)=\hat{C}^q(B)$ for $q>0$. The behavior of $\Gamma^q(B)$ as a function of q is described in section 8. For $r=0$ there is a formula similar to (7.4) above for $q>0$

$$(7.9) \qquad C^q(B) = \hat{C}^q(B) = \hat{C}(B) + q\, R_B m\, P^q_B 1 = C(B) + q\, R^q_B m\, P_B 1.$$

However, it seems to be difficult to deduce the behavior of $\Gamma^q(B)$ as q approaches zero from either (7.4) or (7.9). Nevertheless it is shown in (8.1) and (8.3) that $q \to \Gamma^q(B)$ is increasing and continuous on $]0, \infty[$, and that if $(R_B m)\, P^q_B 1$ is finite for some $q>0$, then $\Gamma^q(B)$ decreases to $\Gamma(B)=C(B)=\hat{C}(B)$ as $q \downarrow 0$. This generalizes a result of the first author [11] in the context of weak duality. (Un)fortunately we have not yet found a proof for this last result purely in terms of the energy functional and related notions. In fact, our current proof uses exit systems as in [8].

More generally than explained so far and more generally than in [16], we define capacity and cocapacity of a set B with respect to an excessive measure m and an excessive function u (instead of 1) satisfying $m(u=\infty)=0$ by means of the Kuznetsov measure Q^u_m associated with m, u, and the semigroup (P_t). *See* (2.3) and (5.1). However, Q^u_m is the same as the Kuznetsov measure associated with um, 1, and the semigroup $(P^{(u)}_t)$-the h-transform of (P_t) by the excessive function $h=u$ [*see* (4.1)]. Therefore the study of these capacities for a general u may be reduced to the case $u=1$ for the u-transform of X. The capacities studied by Hunt in section 19 of [17] and the conditional capacities discussed in section 7 of [11] are special cases of these general capacities. Associated with the

h-transform are the corresponding energy functional and balayage operators. Their properties are studied in sections 4 and 5. These results not only are of interest in themselves but also provide the tools for reducing the case of a general u to the case $u=1$.

NOTATION. – Our notation is for the most part standard. However, the following special notation will be used without comment. The symbol "$\equiv$" means "is defined to be". If $(H, \mathscr{H})$ is a measurable space, $h \in \mathscr{H}$ means that h is an extended real valued measurable function on H, while $h \in p\mathscr{H}$ (resp. $b\mathscr{H}$) means $h \geqq 0$ (resp. bounded) in addition. $\mathscr{H}^*$ denotes the σ-algebra of universally measurable sets over $\mathscr{H}$. If $(G, \mathscr{G})$ is another measurable space, $\varphi \in \mathscr{H} | \mathscr{G}$ means that φ is a measurable map from $(H, \mathscr{H})$ to $(G, \mathscr{G})$. If μ is a measure on $\mathscr{H}$ and $f \in \mathscr{H}$ we use both $\mu(f)$ and $\langle \mu, f \rangle$ to denote $\int f d\mu$ whenever the integral exists, and sometimes just μf for $\mu(f)$. On the other hand $f\mu$ or $f.\mu$ always denotes the measure $f(x)\,\mu(dx)$. The infimum (resp. supremum) of the empty set is $+\infty$ (resp. $-\infty$). As usual $\mathbb{R}$ denotes the reals and $\mathbb{Q}$ the rationals.

2. PRELIMINARIES

Let E be a Borel subset of a compact metric space and $\mathscr{E}$ the σ-algebra of Borel subsets of E. Let Δ be a point not in E and let $E_\Delta = E \cup \{\Delta\}$ where Δ is adjoined to E as an isolated point. Let $\mathscr{E}_\Delta = \sigma(\mathscr{E} \cup \{\Delta\})$. A function f on E is automatically extended to E_Δ by $f(\Delta)=0$.

Let Ω be the set of all right continuous trajectories $\omega : \mathbb{R}^+ \to E_\Delta$ with Δ as cemetery. As usual $X_t(\omega) = \omega(t)$, $\theta_t \omega(s) = \omega(t+s)$, $\mathscr{F}^0 = \sigma(X_t, t \geqq 0)$ and $\mathscr{F}_t^0 = \sigma(X_s, 0 \leqq s \leqq t)$. We assume given a Borel right process $X = (\Omega, \mathscr{F}^0, \mathscr{F}_t^0, X_t, \theta_t, P^x)$ in the sense of [9]. Let $(P_t)_{t \geqq 0}$ and $(U_q)_{q \geqq 0}$ denote the transition semigroup and resolvent of X respectively. Here $P_0 = I$ and we write $U = U_0$. Let $\zeta = \inf\{t : X_t = \Delta\}$ be the lifetime of X and P^Δ denote unit mass at $[\Delta]$ - the trajectory that is identically equal to Δ.

Denote by W the set of all maps $w : \mathbb{R} \to E_\Delta$ such that there exists an open interval $]\alpha(w), \beta(w)[$ on which w is E-valued and right continuous and with $w(t) = \Delta$ for t not in $]\alpha(w), \beta(w)[$. Note that $]\alpha(w), \beta(w)[$ empty corresponds to $w = [\Delta]$ the constant map identically equal to Δ. Observe that $[\Delta]$ is used in two senses: $[\Delta] \in \Omega$ (resp. $[\Delta] \in W$) is the constant map defined for $t \geqq 0$ (resp. $t \in \mathbb{R}$). (In [16] we used two distinct points a and b for the pre-birth and death points, but here it seems more convenient to take $a=b=\Delta$ as in [8].) Let $Y_t(w)=w(t)$ be the coordinate maps on W and $\theta_t w(s)=w(s+t)$ for $t \in \mathbb{R}$. Note that θ_t is used for the shift in W and in Ω. Let $\mathscr{G}^0=\sigma(Y_t; t \in \mathbb{R})$ and $\mathscr{G}_t^0=\sigma(Y_s; s \leqq t)$. Set $\alpha([\Delta])=+\infty$ and $\beta([\Delta])=-\infty$. As usual we sometimes write $Y(t)$ for Y_t and $X(t)$ for X_t.

The spaces Ω and W are related by the mappings $\gamma_t: W \to \Omega$ defined for $t \in \mathbb{R}$ as follows:

$$(2.1) \qquad \begin{aligned} \gamma_t w(s) &= w(t+s) \quad \text{for } s \geqq 0 \text{ if } \alpha(w)<t \\ &= \Delta \quad \text{for } s \geqq 0 \text{ if } \alpha(w) \geqq t. \end{aligned}$$

Clearly if $t \in \mathbb{R}$, $\gamma_t=\gamma_0 \circ \theta_t$. If $\alpha<t$, $X_s \circ \gamma_t = Y_{s+t}$, and if $\alpha<t<\beta$, $\zeta \circ \gamma_t = \beta \circ \theta_t$. Note that γ_t is $\mathscr{G}_{t+s}^0/\mathscr{F}_s^0$ measurable for each $s \geqq 0$ and $t \in \mathbb{R}$. One easily checks the following useful identities:

$$(2.2) \qquad \begin{aligned} &\text{(i)} \quad \gamma_t \circ \theta_s = \gamma_{t+s} \quad \textit{on } W \quad \text{for } \textit{all } s, t \in \mathbb{R} \\ &\text{(ii)} \quad \theta_s \circ \gamma_t = \gamma_{t+s} \quad \textit{on } \{\alpha<t\} \quad \text{for } s \geqq 0,\ t \in \mathbb{R}. \end{aligned}$$

A family $\nu=\{\nu_t; t \in \mathbb{R}\}$ of σ-finite measures on $(E, \mathscr{E})$ is an *entrance rule* (for X or P_t) provided $\nu_s P_{t-s} \uparrow \nu_t$ as $s \uparrow t$. An *entrance law* ν at t_0, $-\infty \leqq t_0 < \infty$ is an entrance rule such that $\nu_t=0$ for $t \leqq t_0$ and $\nu_s P_{t-s}=\nu_t$ for $t_0<s<t$. An entrance law at zero is simply called an entrance law. An *excessive measure* is an entrance rule that is independent of t; that is, a σ-finite measure m such that $m P_t \uparrow m$ as $t \downarrow 0$. Arguments similar to those in the proof of Lemma 5.1 in [5] show that $t \to \nu_t(B)$ is Borel measurable for each $B \in \mathscr{E}$.

Let $\nu=(\nu_t)$ be an entrance rule and u an excessive function with $\nu_t(u=\infty)=0$ for each $t \in \mathbb{R}$. Then it follows from a theorem of Kuznetsov [18] (*see* also [19] or [14]) that there exists a unique measure Q_ν^u on $(W, \mathscr{G}^0)$ not charging $[\Delta]$ such that if $t_1 < \ldots < t_n$,

$$(2.3) \quad Q_\nu^u(\alpha<t_1, Y_{t_1} \in dx_1, \ldots, Y_{t_n} \in dx_n, t_n<\beta) = \nu_{t_1}(dx_1) P_{t_2-t_1}(x_1, dx_2) \ldots P_{t_n-t_{n-1}}(x_{n-1}, dx_n) u(x_n)$$

and Q_ν^u is σ-finite. We shall call Q_ν^u the *Kuznetsov measure* corresponding to ν, u, and (P_t). Strictly speaking the theorem in [18] would require u to be Borel measurable, but the extension to arbitrary excessive u poses no difficulty. *See* remark (3.14) of [14]. (Another approach is to observe that $\bar{\nu} \equiv \int \nu_t \, dt$ is a countable sum of finite measures, and so one may use (6.11) of [15] to find a Borel excessive function $v \leqq u$ such that $\nu_t (v < u) = 0$ for Lebesgue almost all t, and hence for every t since $\nu_t P_s \leqq \nu_{t+s}$. Then (2.3) is unchanged if u is replaced by v and so one may define Q^u_ν to be Q^v_ν.)

If m is an excessive measure and u is an excessive function with $m(u = \infty) = 0$ we write Q^u_m for Q^u_ν where $\nu_t = m$ for each t. In this case Q^u_m is *invariant* in the sense that $\theta_t (Q^u_m) = Q^u_m$. Finally we write simply Q_m or Q_ν when $u = 1$. It is immediate from the uniqueness assertion that the Kuznetsov measure Q^u_m corresponding to m, u, and (P_t) is the same as the Kuznetsov measure corresponding to um, 1, and the h-transform semigroup $(P^{(u)}_t)$. *See* section 4 for a discussion of h-transforms. Moreover $Y = (Y_t)$ under Q^u_m is strong Markov with semigroup $(P^{(u)}_t)$. *See* [21] or [19].

Let Exc denote the class of excessive measures and **E** the class of excessive functions for X. We recall two decompositions of an excessive measure from [8]. *See* also [16]. Firstly each $m \in \text{Exc}$ has a unique decomposition $m = m_i + m_p$ where m_i is *invariant* (i.e. $m_i P_s = m_i$ for each $s \geqq 0$) and m_p is *purely excessive* [i.e. $f \in p\mathcal{E}$ with $m_p(f) < \infty$ implies $m_p P_t (f) \to 0$ as $t \to \infty$]. If $u \in \mathbf{E}$ with $m(u = \infty) = 0$, then $Q^u_m = Q^u_{m_i} + Q^u_{m_p}$ and by checking finite dimensional distributions one finds

$$(2.4) \qquad \begin{cases} Q^u_{m_i}(\,.\,) = Q^u_m(\,.\,; \alpha = -\infty) \\ Q^u_{m_p}(\,.\,) = Q^u_m(\,.\,; \alpha > -\infty). \end{cases}$$

This is proved in [8] when $u = 1$. We let Inv and Pur denote the classes of invariant and purely excessive measures respectively.

Secondly each $m \in \text{Exc}$ may be written uniquely as $m = m_c + m_d$ where m_c is *conservative* and m_d is *dissipative*. Recall that $m \in \text{Exc}$ of the form $m = \mu U - \mu$ is necessarily σ-finite—is called a *potential*, and that $m \in \text{Exc}$ is dissipative provided there exists a sequence of potentials $(\mu_n U)$ increasing to m while m is conservative provided $\mu U \leqq m$ implies $\mu U = 0$. If $g > 0$ with $m(g) < \infty$, then by (4.3) of [8](, m_c(resp. m_d) is the restriction of m to $\{Ug = \infty\}$ (resp. $\{Ug < \infty\}$). An elementary proof of these facts is given in [1]. We let Pot, Dis, and Con denote the class of potentials, dissipative, and conservative excessive measures respectively. It is shown in [8] that Pot $\subset$ Pur $\subset$ Dis and Con $\subset$ Inv.

The decomposition $m=m_i+m_p$ has an analogue for excessive functions. Given $u \in \mathrm{E}$, let $\bar{u}_i \equiv \lim_{t\to\infty} \mathrm{P}_t u$. Then $\mathrm{P}_t \bar{u}_i(x)=\bar{u}_i(x)$ for each $t>0$ if $\bar{u}_i(x)<\infty$. It follows that $\bar{u}_i$ is *supermedian* (i.e. $\mathrm{P}_t \bar{u}_i \leqq \bar{u}_i$). Let $u_i \equiv \lim_{t\downarrow 0} \mathrm{P}_t \bar{u}_i$ be the excessive regularization of $\bar{u}_i$. One readily checks that on $\{\bar{u}_i<\infty\}=\{\inf_{t>0} \mathrm{P}_t u<\infty\}$ one has $\mathrm{P}_t u_i=u_i$ for each t and $\lim_{t\to\infty} \mathrm{P}_t u=u_i$. Next define $\bar{u}_p(x)=u(x)-u_i(x)$ if $\bar{u}_i(x)<\infty$, $\bar{u}_p(x)=\infty$ if $\bar{u}_i(x)=\infty$. Again $\bar{u}_p$ is supermedian and we set $u_p \equiv \lim_{t\downarrow 0} \mathrm{P}_t \bar{u}_p$. One verifies that on $\{\inf_{t>0} \mathrm{P}_t u<\infty\}$ one has

$$\text{(2.5)}\quad \begin{array}{ll} \text{(i)} & u=u_i+u_p; \\ \text{(ii)} & \mathrm{P}_t u_p \to 0 \quad \textit{and} \quad \mathrm{P}_t u \to u_i \quad \textit{as } t\to\infty, \\ \text{(iii)} & \mathrm{P}_t u_i=u_i \quad \textit{for each } t>0. \end{array}$$

We call u_i (resp. u_p) the invariant (resp. purely excessive) part of u. We say that u is *invariant* if $u=u_i$ on $\{u<\infty\}$ which is equivalent to $\mathrm{P}_t u=u$ for each t on $\{u<\infty\}$, and that u is *purely excessive* if $u=u_p$ on $\{u<\infty\}$ which is equivalent to $\mathrm{P}_t u \to 0$ as $t\to\infty$ on $\{u<\infty\}$. If $m \in \mathrm{Exc}$ we say that u is m-invariant (resp. m-purely escessive) if $u=u_i$ (resp. $u=u_p$) a.e. m on $\{u<\infty\}$ which is equivalent to $\mathrm{P}_t u=u$ for each t (resp. $\mathrm{P}_t u\to 0$ as $t\to\infty$) a.e. m on $\{u<\infty\}$. By checking finite dimensional distributions one has the following dual of (2.4) for $m \in \mathrm{Exc}$ and $u \in \mathrm{E}$ with $m(u=\infty)=0$,

$$\text{(2.6)}\quad \begin{aligned} \mathrm{Q}_m^{u_i}(\,.\,) &= \mathrm{Q}_m^u(\,.\,;\beta=\infty), \\ \mathrm{Q}_m^{u_p}(\,.\,) &= \mathrm{Q}_m^u(\,.\,;\beta<\infty). \end{aligned}$$

It is immediate from (2.5) and (2.6) that $\mathrm{Q}_m^u(\beta<\infty)=0$ if and only if u is m-invariant.

If $q>0$ we let Exc^q and E^q denote the q-excessive measures and functions respectively; that is, excessive relative to the semigroup $\mathrm{P}_t^q \equiv e^{-qt}\mathrm{P}_t$. Using the obvious notation one has for $q>0$, $\mathrm{Exc}^q=\mathrm{Dis}^q$, but there may exist non-zero elements in Inv^q.

Next we recall Hunt's balayage operation on Exc as extended in [8]. Let $\mathrm{B} \in \mathscr{E}$ and define $\tau_\mathrm{B} \equiv \inf\{t: \mathrm{Y}_t \in \mathrm{B}\}$. Then $\alpha \leqq \tau_\mathrm{B} \leqq \infty$ and $\tau_\mathrm{B} \circ \theta_t=\tau_\mathrm{B}-t$. Also $\{\tau_\mathrm{B}<t\}$ is in $\mathscr{G}_t^* \equiv (\mathscr{G}_t^0)^*$. If $m \in \mathrm{Exc}$, define for $f \in p\mathscr{E}$

$$\text{(2.7)}\quad \mathrm{R_B}\, m(f)=\mathrm{Q}_m[f\circ \mathrm{Y}_t;\, \tau_\mathrm{B}<t].$$

This is independent of t. It is shown in [8], that if $m \in \text{Dis}$ and $\mu_n U \uparrow m$, then $\mu_n P_B U \uparrow R_B m$. In particular $R_B(\mu U) = \mu P_B U$. Of course, $R_B^q m$ for $m \in \text{Exc}^q$ is defined similarly relative to the Kuznetsov measure ${}^qQ_m$ corresponding to m and the semigroup (P_t^q). In referring to [8] one should note that Fitzsimmons and Maisonneuve use L_B for what we denote by R_B defined in (2.7).

Finally we record some results about dissipative measures that will be needed later. It is shown in [1], that if $m \in \text{Con}$ and $u \in \mathbf{E}$ then $P_t u = u$ a. e. m. Consequently by the remark below (2.6) and (2.4) — recall Con $\subset$ Inv — one has for $m \in \text{Con}$, $u \in \mathbf{E}$ with $m(u = \infty) = 0$,

$$(2.8) \qquad Q_m^u(\beta < \infty) = 0 \quad \text{and} \quad Q_m^u(\alpha > -\infty) = 0.$$

The proof we give of the following proposition is due to R. M. Blumenthal. It is much simpler than our original proof.

(2.9) Proposition. — *Let $m \in \text{Con}$ and $u \in \mathbf{E}$. Then*

$$P^m[u \circ X_t \neq u \circ X_0 \textit{ for some } t \geqq 0] = 0.$$

Proof. — It suffices to suppose u is bounded. If D is a nearly Borel set let $T_D \equiv \inf\{t > 0 : X_t \in D\}$ and $L_D \equiv \sup\{t : X_t \in D\} \vee 0$. Let $\psi_D(x) \equiv P^x(0 < L_D < \infty)$. Then ψ_D is excessive and because $P_t \psi_D \to 0$ as $t \to \infty$ and $m \in \text{Con}$, $\psi_D = 0$ a. e. m (since $P_t \psi_D = \psi_D$ a. e. m). If $q \in \mathbb{Q}$ let $A_q = \{u > q\}$ and $B_q = \{u < q\}$, and set $F = \bigcap_{r, q \in \mathbb{Q}} \{\psi_{A_r} = 0 = \psi_{B_q}\}$. Then $m(F^c) = 0$. Suppose $x \in F$ and $q < r < u(x)$. Since $P^x(L_{A_r} > 0) = P^x(T_{A_r} < \infty) = 1$, one must have $P^x(L_{A_r} = \infty) = 1$; that is, a. s. P^x, $u \circ X_t > r$ for arbitrarily large t. Suppose $P^x(T_{B_q} < \infty) > 0$. Then $P^x(L_{B_q} = \infty) > 0$ and so $u \circ X_t < q$ for arbitrarily large t with positive P^x probability. But $\lim_{t \to \infty} u \circ X_t$ exists a. s. P^x and so $P^x(T_{B_q} < \infty) = 0$. Since $q < u(x)$ was arbitrary one has a. s. P^x, $u \circ X_t \geqq u(x)$ for all t, and similarly $u \circ X_t \leqq u(x)$ for all t. □

Let $B \in \mathscr{E}$ and $\varphi_B(x) \equiv P^x(T_B < \infty)$. Then using (2.9) one has for m almost all x

$$\begin{aligned} \varphi_B(x) &= P^x(T_B \leqq t) + P^x(t < T_B < \infty) \\ &= P^x(T_B \leqq t) + P^x(\varphi_B \circ X_t; t < T_B) \\ &= P^x(T_B \leqq t) + \varphi_B(x) P^x(t < T_B), \end{aligned}$$

and letting $t \to \infty$ this implies that $\varphi_B(x) = [\varphi_B(x)]^2$. This proves the following:

(2.10) COROLLARY. — *Let* $B \in \mathscr{E}$ *and* $m \in \mathrm{Con}$. *Then for* m *a. e.* x, $\varphi_B(x)$ *is either zero or one.*

3. THE ENERGY FUNCTIONAL

In [20] (*see* also [4]) Meyer associated with each $m \in \mathrm{Exc}$ and $u \in \mathbf{E}$ a number $L(m, u)$ with $0 \leqq L(m, u) \leqq \infty$, which generalizes the notion of "energy" of two measures with respect to a potential kernel in the situation of duality. In [4] and [20] the resolvent is assumed to be transient, but the proofs carry over to our general situation (as described in section 2) with only minor modifications. These are based on the following observations.

Let $m \in \mathrm{Dis}$ and $g \in \mathscr{E}$ with $g > 0$ and $m(g) < \infty$. Then $Ug < \infty$ a. e. m, and the argument on page 402 of [10] shows that there exists an $h \in pb\mathscr{E}$ with $Uh \leqq 1$ on E and $\{Ug < \infty\} \subset \{Uh > 0\}$. It now follows by standard arguments (*see* for example II-2.19 of [2]) that if $u \in \mathbf{E}$ then there exists an increasing sequence of potentials Uf_k such that $u = \lim Uf_k$ on $\{Uh > 0\}$, and hence a. e. m. If, moreover, $m = \mu U$ then $m(g) < \infty$ implies that $Ug < \infty$ a. e. μ, and so Uf_k increases to u a. e. μ as well as a. e. μU.

We shall now sketch the steps in the construction of the functional L referring to [4] or [20] for the proofs. It is first shown that if $m \in \mathrm{Pur}$ and $u = Uf$ with $m(f) < \infty$, then

$$(3.1) \qquad q\langle m - qmU_q, Uf\rangle = qmU_q(f) \uparrow m(f) \quad \text{as } q \to \infty.$$

Obviously, this then extends to arbitrary $f \in p\mathscr{E}^*$. In (3.1), $\eta \equiv m - qmU_q$ is the unique σ-finite measure such that $m = \eta + qmU_q$. Since $\eta \leqq m$ clearly $\langle \eta, u\rangle$ is unchanged if u is changed on a set of m-measure zero. Since according to the above remark, for $u \in \mathbf{E}$ there exist potentials Uf_k increasing to u a. e. m, it then follows from (3.1) that $q\langle m - qmU_q, u\rangle$ increases with q. Hence one defines for $m \in \mathrm{Pur}$ and $u \in \mathbf{E}$

$$(3.2) \qquad L(m, u) \equiv \uparrow \lim_{q \to \infty} q\langle m - qmU_q, u\rangle.$$

(Here and in the sequel $\uparrow \lim$ means that the limit in question is an increasing limit.) From (3.1) and (3.2) the following facts for $m \in \mathrm{Pur}$

and $u \in \mathbf{E}$ can immediately be derived:

(3.3) $$L(m, Uf) = m(f) \quad \text{for} \quad f \in p\mathcal{E}.$$

(3.4) $$L(m, u) = \uparrow \lim_k L(m, Uf_k) = \uparrow \lim_k m(f_k) \quad \text{if} \quad Uf_k \uparrow u \text{ a. e. } m;$$

that is, $Uf_k \leqq Uf_{k+1}$ a. e. m for each k and $\lim_k Uf_k = u$ a. e. m.

(3.5) $$L(m, u) = \uparrow \lim_n L(m, u_n) \quad \text{if } u_n \uparrow u \text{ a. e. } m, \qquad u_n \in \mathbf{E}.$$

(3.6) $$L(m, u) = \uparrow \lim_n L(m_n, u) \quad \text{if } m_n \uparrow m, \qquad m_n \in \text{Pur}.$$

(3.7) $$L(\mu U, u) = \mu(u) \quad \text{if } \mu U \in \text{Exc}; \qquad \text{i. e. if } \mu U \text{ is } \sigma\text{-finite}.$$

To obtain (3.7) recall that one may choose $Uf_k \uparrow u$ a. e. μU and also a. e. μ. Therefore from (3.4) $L(\mu U, u) = \lim_k \mu U(f_k) = \mu(u)$. Furthermore, if $(\mu_t)_{t>0}$ is an entrance law for (P_t) representing m, i. e. $m = \int_0^\infty \mu_t \, dt$, then since $\mu_s U \uparrow m$ as $s \downarrow 0$ one has from (3.6) and (3.7),

(3.8) $$L(m, u) = \uparrow \lim_{s \downarrow 0} \mu_s(u).$$

Finally, for general $m \in \text{Exc}$ and $u \in \mathbf{E}$ one defines

(3.9) $$\begin{aligned} L(m, u) &\equiv \sup \{ L(\eta, u) : \eta \in \text{Pur}, \eta \leqq m \} \\ &= \sup \{ L(\eta, u) : \eta \in \text{Pot}, \eta \leqq m \} \end{aligned}$$

where the second equality follows from (3.6) and the fact that any purely excessive measure is the increasing limit of potentials [as explained preceding (3.8)]. Let $g \in p\mathcal{E}$ with $g > 0$ and $m(g) < \infty$. If $m = m_c + m_d$ and $\eta \in \text{Pur}$, then m_c is carried by $\{ Ug = \infty \}$ while m_d and η are carried by $\{ Ug < \infty \}$. Thus $\eta \leqq m$ if and only if $\eta \leqq m_d$. Consequently one has

(3.10) $$L(m, u) = L(m_d, u) = \uparrow \lim \mu_n(u) \quad \text{if} \quad \mu_n U \uparrow m_d,$$

(3.11) $$L(m, u) = 0 \quad \text{if} \quad m \in \text{Con}.$$

One readily checks that (3.3)-(3.6) remain valid for general $m \in \text{Dis}$. In fact, L is the unique map from $\text{Exc} \times \mathbf{E}$ to $[0, \infty]$ that is bilinear for positive scalars and satisfies (3.11) and (3.3), (3.7), (3.5), (3.6) extended to $m \in \mathrm{Dis}$. For $m \in \mathrm{Exc}$, $\mathrm{L}(m, .)$ is extended to the class of supermedian functions by setting

$$(3.12) \qquad \mathrm{L}(m, s) \equiv \mathrm{L}(m, \hat{s})$$

for s supermedian where $\hat{s} \equiv \lim_{t \downarrow 0} \mathrm{P}_t s$ denotes the excessive regularization of s. Furthermore in using $\mathrm{L}(m, u)$, there is no loss of generality in supposing that u is Borel measurable excessive, because given $u \in \mathbf{E}$ and $m \in \mathrm{Exc}$ there exists a $v \in \mathbf{E} \cap \mathscr{E}$ with $u = v$ a. e. m by (6.11) of [15], and $\mathrm{L}(m, u) = \mathrm{L}(m, v)$. If $m \in \mathrm{Pur}$ and $f \in p\mathscr{E}$ with $m(f) < \infty$ then arguments analagous to those leading to (3.1) and (3.2) yield

$$\uparrow \lim_{q \to \infty} q \langle m - qm \mathrm{U}_q, \mathrm{U}f \rangle = \uparrow \lim_{t \downarrow 0} \frac{1}{t} \langle m - m \mathrm{P}_t, \mathrm{U}f \rangle,$$

where (P_t) denotes the associated semigroup. This implies that for $m \in \mathrm{Pur}$ and $u \in \mathbf{E}$

$$(3.13) \qquad \mathrm{L}(m, u) = \uparrow \lim_{t \downarrow 0} \frac{1}{t} \langle m - m \mathrm{P}_t, u \rangle.$$

If $u \in \mathbf{E}$, then, as is well known—*see* e. g. the argument leading to (4.6) in [16]—the potential of $t^{-1}(u - \mathrm{P}_t u)$ increases to u as $t \downarrow 0$ on $\{u < \infty$ and $\lim_{t \to \infty} \mathrm{P}_t u = 0\}$. Consequently if $m \in \mathrm{Dis}$ with $m(u = \infty) = 0$ and u_p is the purely excessive part of u [*see* (2.5)], then using (3.4) and (2.5) (iii);

$$(3.14) \quad \mathrm{L}(m, u_p) = \uparrow \lim_{t \downarrow 0} t^{-1} \langle m, u_p - \mathrm{P}_t u_p \rangle = \uparrow \lim_{t \downarrow 0} t^{-1} \langle m, u - \mathrm{P}_t u \rangle.$$

Recall that if $m \in \mathrm{Con}$, $u = \mathrm{P}_t u$ a. e. m, so m, so (3.14) is valid for all $m \in \mathrm{Exc}$ with $m(u = \infty) = 0$. All of the preceding statements either may be found in [4] or [20] or are easy consequences of results proved there.

The name "energy functional" for L is motivated by the following:

(3.15) *Remark.* — Suppose X and $\hat{\mathrm{X}}$ are in strong duality with respect to some excessive reference measure and with potential density kernel $u(x, y)$ as in Chapter VI of [2]. If $m = \mu \mathrm{U} \in \mathrm{Exc}$ is the potential of a measure μ and $f = \mathrm{U}\nu \equiv \int u(., y) \nu(dy) \in \mathbf{E}$ is the potential generated by the measure ν, then

$$\mathrm{L}(m, f) = \mu(f) = \iint \mu(dx) u(x, y) \nu(dy).$$

That is $L(\mu U, U\nu)$ is the mutual energy of the measures μ and ν with respect to the kernel $u(.,.)$. Therefore, in general, L should be thought of as the "energy" between an excessive measure and an excessive function.

The next proposition states that the energy functional is the pairing that makes R_B and P_B dual objects.

(3.16) PROPOSITION. — *Let* $B \in \mathscr{E}$, $m \in \mathrm{Exc}$ *and* $u \in \mathbf{E}$. *Then*

$$L(R_B m, u) = L(m, P_B u).$$

Proof. — In (5.8) of [8] it is shown that $(R_B m)_d = R_B(m_d)$ and $(R_B m)_c = R_B(m_c)$. Therefore because of (3.11) it suffices to prove (3.16) for $m \in \mathrm{Dis}$. Choose $(\mu_n U)$ increasing to m. Then according to (5.9) of [8], $(\mu_n P_B U)$ increases to $R_B m$, and using (3.6) and (3.7) one obtains

$$L(R_B m, u) = \lim_n L(\mu_n P_B U, u) = \lim_n \mu_n P_B(u)$$

$$= \lim_n L(\mu_n U, P_B u) = L(m, P_B u). \quad \square$$

Before we come to discuss the relationship of the energy functional with the definitions of capacity and cocapacity as given in a previous paper [16], we state one more property of the functional L.

(3.17) LEMMA. — *Let* $m \in \mathrm{Dis}$ *and* $u \in \mathbf{E}$. *Then* $L(m, u) = 0$ *if and only if* $u = 0$ *a. e. m.*

Proof. — Clearly $L(m, u) = 0$ if $u = 0$ a. e. m. On the other hand, since $m \in \mathrm{Dis}$ there exist potentials $U f_k$ increasing to u a. e. m. Hence from (3.4) one has $L(m, u) = \uparrow \lim_k m(f_k)$, and $L(m, u) = 0$ implies $m(f_k) = 0$ for all k, which gives $m U_q(f_k) \leqq \frac{1}{q} m(f_k) = 0$ for all k and all $q > 0$. Hence $U f_k = 0$ a. e. m for each k and so $u = 0$ a. e. m. $\square$

(3.18) *Remarks.* — In [16], given $m \in \mathrm{Exc}$, we associated two numbers $C(B) \equiv C_m(B)$ and $\hat{C}(B) \equiv \hat{C}_m(B)$ with any set $B \in \mathscr{E}$ by defining

$$(3.19) \qquad \hat{C}(B) \equiv Q_m(0 < \tau_B < 1); \qquad C(B) \equiv Q_m(0 < \lambda_B < 1),$$

where $\tau_B \equiv \inf\{t : Y_t \in B\}$ and $\lambda_B \equiv \sup\{t : Y_t \in B\}$. We defined B to be *transient* (resp. *cotransient*) (relative to m) provided $Q_m(\lambda_B = \infty) = 0$ [resp. $Q_m(\tau_B = -\infty) = 0$], which according to (4.1) [resp. (4.7)] of [16] is equivalent to $P_B 1$ being m-purely excessive (resp. $R_B m \in \mathrm{Pur}$). [Recall the definition of m-purely excesssive below (2.5).] It follows from (5.3) (iii) of [8] that $\rho_t^B(f) \equiv Q_m(f \circ Y_{t+\tau_B};\ 0 < \tau_B \leqq 1)$ – as in (4.9) of [16] – defines an entrance law for (P_t) representing the purely excessive part of $R_B m$, i.e. $(R_B m)_p = \int_0^\infty \rho_t^B\, dt$. And the proof of (4.10) of [16] actually yields

$$\hat{C}(B) \equiv \lim_{t \downarrow 0} \rho_t^B(1) \qquad \text{for} \quad B \in \mathscr{E}. \tag{3.20}$$

On the other hand, in (4.3) of [16] it was proved

$$C(B) = \lim_{t \downarrow 0} \frac{1}{t} m(\varphi_B - P_t \varphi_B) \qquad \text{for} \quad B \in \mathscr{E}, \tag{3.21}$$

where $\varphi_B \equiv P_B 1$.

The following relations of the set functions C and $\hat{C}$ with the energy functional have already been pointed out in (1.4) and (2.3) of [23].

(3.22) Proposition. — *Let* $m \in \mathrm{Exc}$ *and* $B \in \mathscr{E}$. *Then*

$$\hat{C}(B) = L((R_B m)_p,\ 1), \tag{3.23}$$

$$C(B) = L(m,\ (P_B 1)_p). \tag{3.24}$$

Proof. — (3.23) follows from (3.20) and (3.8); (3.24) follows from (3.21) and (3.14). □

(3.25) *Remarks.* — In particular, if B is cotransient one has $\hat{C}(B) = L(R_B m, 1)$, and if B is transient one has $C(B) = L(m, P_B 1)$. Moreover, if B is both transient and cotransient the equality of C(B) and $\hat{C}(B)$ follows from (3.16). In (3.15) of [16] we gave a purely probabilistic proof for that. Also the proof of (3.24) shows that $\frac{1}{t} m(\varphi_B - P_t \varphi_B)$ increases to C(B) as t decreases to zero, this sharpens (4.3) of [16], which merely states that the limit is increasing along the sequence $\{2^{-k}\}$. Furthermore from (3.17) we obtain the following characterization: a set $B \in \mathscr{E}$ is m-polar (i.e. $\varphi_B = 0$ a.e. m) if and only if $L(m, \varphi_B) = 0 = L(R_B m, 1)$ and $m_c(\varphi_B) = 0$. This in fact generalizes the result (3.11) of [16].

We turn now to some additional properties of L. We denote the energy functional relative to the semigroup (P_t^q) by L^q (for $q > 0$). Thus L^q is defined on $\mathrm{Exc}^q \times E^q$. In case $q > 0$ there are no q-conservative measures, therefore $\mathrm{Exc}^q = \mathrm{Dis}^q$. In general, for $q \geqq 0$, one has $\mathrm{Exc}^q = \bigcap_{r > q} \mathrm{Exc}^r$.

(3.26) PROPOSITION. — *Suppose that* $0 \leqq r < q$ *and* $m \in \mathrm{Exc}^r$ *and* $u \in \mathbf{E}^r$. *Then*

$$\mathrm{L}^q(m, u) = \mathrm{L}^r(m, u) + (q - r)\, m(u). \tag{3.27}$$

Proof. — We shall prove this when $r=0$. The general case then follows by taking (P_t^r) to be the basic semigroup. Suppose $m \in \mathrm{Con}$. Then $\mathrm{L}(m, u) = 0$ according to (3.11), and since $\mathrm{Con} \subset \mathrm{Inv}$ one has $m = qm\,\mathrm{U}_q$ and therefore $\mathrm{L}^q(m, u) = \mathrm{L}^q(qm\,\mathrm{U}^q, u) = qm(u)$ [according to (3.7) applied to the q-subprocess], which gives (3.27) for $m \in \mathrm{Con}$. Next suppose $m \in \mathrm{Dis}$. Then there exist potentials $\mu_n \mathrm{U}$ increasing to m. Let $\nu_n \equiv \mu_n(\mathrm{I} + q\,\mathrm{U})$, so that $\nu_n \mathrm{U}_q = \mu_n \mathrm{U}$ increases to m. Then

$$\mathrm{L}^q(m, u) = \uparrow \lim_n \nu_n(u) = \uparrow \lim_n [\mu_n(u) + q\,\mu_n \mathrm{U}(u)].$$

But $(\mu_n(u))$ increases to $\mathrm{L}(m, u)$ and $(\mu_n\,\mathrm{U})$ increases to m, which establishes (3.27) for $m \in \mathrm{Dis}$. □

(3.28) COROLLARY. — (*a*) *If* $m \in \mathrm{Inv}$ *and* $u \in \mathbf{E}$, *then* $m(u) < \infty$ *implies* $\mathrm{L}(m, u) = 0$. (*b*) *If* $0 \leqq r < q$ *and* $m \in \mathrm{Exc}^r$ *and* $u \in \mathbf{E}^r$, *then*

$$\mathrm{L}^q(m, u) = \mathrm{L}^r(m_p^r, u) + (q - r)\, m(u),$$

where m_p^r *denotes the r-purely excessive part of m.*

Proof. — Arguing as in the first part of the proof of (3.26) one finds for $q > 0$, $\mathrm{L}^q(m, u) = qm(u)$. Thus part (*a*) follows from (3.27). To prove (*b*) we take $r = 0$ [as in the proof of (3.27)], then (*b*) follows as well from (3.27) because according to (*a*) $\mathrm{L}(m_i, u) > 0$ only if $m(u) = \infty$. □

4. THE ENERGY FUNCTIONAL AND h-TRANSFORMS

We begin this section with some facts about h-transforms which will be needed later. Most of these facts are well known. Some of them may be found in [24]. There is a complete exposition in the Paris lecture notes of J. B. Walsh [25]. *See* also [22].

As in previous sections X is a Borel right process constructed on the canonical space Ω of right continuous paths, and (P_t) and (U_q) are the semigroup and resolvent of X. Let h be an excessive function and let $\mathrm{E}_h = \{ 0 < h < \infty \}$. Then E_h is nearly Borel (in particular $\mathrm{E}_h \in \mathscr{E}^*$), and Borel if h is Borel. Defines kernels $P_t^{(h)}$ by

$$(4.1)\qquad \begin{aligned} P_t^{(h)}(x, dy) &= h(x)^{-1} P_t(x, dy)\, h(y), \qquad x \in E_h \\ &= \varepsilon_x(dy), \qquad x \in E - E_h. \end{aligned}$$

Then it is well known and easy to check that $(P_t^{(h)})_{t \geqq 0}$ is a subMarkov semigroup on E and each $P_t^{(h)}$ maps Borel functions into nearly Borel functions—in fact into Borel functions if h is Borel. If $x \in E_h$, the measure $P_t^{(h)}(x, .)$ is carried by E_h for each $t \geqq 0$. It is evident that $P_t(x, .)$ does not charge $\{ h = \infty \}$ when $h(x) < \infty$ and does not charge $\{ h > 0 \}$ when $h(x) = 0$. It is known (*see*, e. g. [22], [24], or [25]) that there exist probabilities $P^{x/h}$ for $x \in E$ on Ω so that $X^h \equiv (X, P^{x/h})$ is a right process (Borel right process if h is Borel) with state space $(E, \mathscr{E})$ and semigroup $P_t^{(h)}$. Both E_h and $E - E_h$ are absorbing sets for X^h and, of course, if $x \in E - E_h$, then under $P^{x/h}$ the process sits at x forever. We denote the resolvent of $P_t^{(h)}$ by $(U_q^{(h)})$. From (4.1),

$$\begin{aligned} U_q^{(h)}(x, dy) &= h(x)^{-1} U_q(x, dy)\, h(y) \qquad \text{if} \quad x \in E_h \\ &= q^{-1} \varepsilon_x(dy) \qquad \text{if} \quad x \in E - E_h. \end{aligned}$$

A function or measure is *h-excessive* provided it is excessive relative to the semigroup $(P_t^{(h)})$. We use the notation $\mathbf{E}(h)$ and $\mathrm{Exc}(h)$ for these classes. Note the distinction between $P_t^q = e^{-qt} P_t$ for $q \in \mathbb{R}^+$ and $P_t^{(h)}$ for $h \in \mathbf{E}$. Also note that if $h \in \mathbf{E}$ and $q > 0$, then $h \in \mathbf{E}^q$ and $e^{-qt} P_t^{(h)} = (P_t^q)^{(h)}$; that is taking the h-transform and the q-subprocess commute.

The following result is due to Walsh [24].

(4.2) PROPOSITION. — (i) *If v is h-excessive, then there exists an excessive u with $u = hv$ on $\{ h < \infty \}$. If h and v are Borel one may choose u Borel.*

(ii) *If u is excessive and $v \in p\,\mathscr{E}^*$ satisfies $u = vh$ on $\{ h < \infty \}$, then v is h-excessive.*

The next proposition describes the situation for excessive measures. The notation Pur(h), Dis(h), etc. is self-explanatory.

(4.3) PROPOSITION. — *Let $m \in \mathrm{Exc}$ and $h \in \mathbf{E}$ with $m(h = \infty) = 0$. Then:*

(i) *$hm \in \mathrm{Exc}(h)$.*

(ii) *If $m \in$ Pur [resp. Inv], then $hm \in$ Pur (h) [resp. Inv (h)].*

(iii) *If $m \in$ Dis, then $hm \in$ Dis(h) and one may choose measures μ_n carried by $\{ h \leqq n \}$ so that $\mu_n U \uparrow m$ and $(h \mu_n) U^{(h)} \uparrow hm$.*

(iv) *If $m \in$ Con, then $hm \in$ Con(h).*

Proof. — The proofs of (i) and (ii) are straightforward and left to the reader. For (iii) suppose first that there exist measures μ_n carried by $\{h \leqq n\}$ with $\mu_n U \uparrow m$. Since $U(fh)(x)=0$ if $h(x)=0$, the following steps are easily justified for $f \in p\mathcal{E}$.

$$(h\mu_n)(U^{(h)}f) = \int_{\{0<h\leqq n\}} \mu_n(dx)\, U(fh)(x)$$

$$= \int \mu_n(dx)\, U(fh)(x) = \mu_n U(fh) \uparrow m(fh)$$

$$= (hm)(f) \quad \text{as } n \to \infty.$$

To produce such μ_n, first choose ν_n with $\nu_n U \uparrow m$. Let $B_k = \{h<k\}$. Each B_k is a finely open nearly Borel set and $B_k \uparrow B$ where $B \equiv \{h<\infty\} = E$ a.e. m. Now $\nu_n P_{B_k} U$ increases with both n and k and $\nu_n P_{B_k} U \uparrow R_{B_k} m$ as $n \to \infty$. According to (5.14) (*b*) of [8], $R_{B_k} m \uparrow R_B m$ as $k \to \infty$. We claim that $R_B m = m$. From (2.7), it suffices to show $Q_m(\tau_B > \alpha) = 0$. But for each rational r

$$Q_m(\alpha < r < \tau_B) = Q_m(\alpha < r < \tau_B,\ T_B \circ \gamma_r > 0) \leqq P^m(T_B > 0) = 0,$$

where the last equality follows because B is finely open and $m(E-B)=0$. This shows that $Q_m(\tau_B > \alpha) = 0$, establishing (iii) with $\mu_n = \nu_n P_{B_n}$.
Finally suppose $m \in \text{Con}$. If $g>0$, then because

$$\{U^{(h)} g < \infty\} \cap E_h = \{U(gh) < \infty\} \cap E_h$$

one has

$$hm[0 < U^{(h)} g < \infty] = \int_{E_h} 1_{\{0<U(gh)<\infty\}}\, h\, dm$$

$$= \lim_{n\to\infty} \int_{\{0<h\leqq n\}} 1_{\{0<U(gh)<\infty\}}\, h\, dm.$$

But the integral over $\{0<h\leqq n\}$ is dominated by $n.m[0<U(gh)<\infty]=0$ since $m \in \text{Con}$. *See* [1]. Similarly

$$hm[U^{(h)} g = 0] = hm[U(gh)=0] = 0,$$

because

$$\{U(gh)=0\} = \{Uh=0\} = \{h=0\}$$

a. e. m since $P_t h = h$ a. e. m. *See* [1]. Therefore $hm \in \mathrm{Con}(h)$. □

(4.4) *Remark.* — An immediate consequence of (4.3) is that $(hm)_p = hm_p$, $(hm)_i = hm_i$, $(hm)_d = hm_d$, and $(hm)_c = hm_c$. Of course, $(hm)_p$ denotes the purely excessive part of hm with respect to $(P_t^{(h)})$ and the other expressions are defined analogously.

For $h \in \mathbf{E}$ we want to define the energy functional L_h corresponding to $P_t^{(h)}$. If h is Borel, the semigroup $P_t^{(h)}$ is Borel and the discussion in section 3 applies. In general X^h is only a right process and one can not apply the results of section 3 directly. Of course, in [4] the basic object is a transient resolvent on an abstract measure space subject to certain hypotheses which are satisfied by the resolvent $(U_q^{(h)})$ restricted to E_h provided it is transient. However, the extension in section 3 to dissipative m in the non-transient case uses results which are proved in the literature only when the underlying process is a Borel right process. Although these results are undoubtedly true more generally, we can avoid the difficulty by considering only $\xi \in \mathrm{Exc}(h)$ which are of the form $\xi = hm$ with $m \in \mathrm{Exc}$ and $m(h = \infty) = 0$. Such a ξ is carried by E_h and determines m uniquely on $\{h > 0\}$. By (6.11) of [15] there exists a Borel measurable excessive function g with $g \leq h$ and $m(g < h) = 0$. So $\xi = gm$ also, and for ξ and m almost all x, $P_t^{(h)}(x, .) = P_t^{(g)}(x, .)$ for all t. Since $P_t^{(g)}$ is Borel we may define $L_g(\xi, u)$ for $u \in \mathbf{E}(g)$ relative to the semigroup $(P_t^{(g)})$. If $v \in \mathbf{E}(h)$, then $P_t^{(g)} v = P_t^{(h)} v$ a. e. ξ. Consequently v satisfies the hypotheses of (6.19) in [15] relative to ξ and $P_t^{(g)}$, so there exists $w \in \mathbf{E}(g) \cap \mathscr{E}$ with $w = v$ a. e. ξ. We then define $(\xi = hm = gm)$

$$(4.5) \qquad L_h(hm, v) \equiv L_g(gm, w),$$

and this does not depend on the choice of g or w. If $m \in \mathrm{Dis}$ and $v \in \mathbf{E}(h)$ and u is the excessive function in (4.2) (i), then there exist $U f_k \uparrow u$ a. e. m. Let $f_k^* = h^{-1} f_k$ on E_h and $f_k^* = 0$ off E_h. Then $U^{(h)} f_k^* \uparrow v$ a. e. m on E_h. Whenever $U^{(h)} g_k \uparrow v$ a. e. m on E_h one has $U^{(g)} g_k \uparrow w$ a. e. $\xi = gm$ and so from (4.5)

$$(4.6) \qquad L_h(hm, v) = \uparrow \lim_k hm(g_k).$$

If $m_k \uparrow m \in \mathrm{Dis}$ and $U^{(h)} g_n \uparrow v \in \mathbf{E}(h)$ a. e. m, then

$$(4.7) \quad L_h(hm, v) = \uparrow \lim_n hm(g_n) = \uparrow \lim_n \uparrow \lim_k hm_k(g_n) = \uparrow \lim_k L_h(hm_k, v).$$

Thus (4.6) and (4.7) extend the basic properties of L to L_h as defined in (4.5). The restriction to measures of the form hm is completely analagous to considering the Kuznetsov measures Q^h_m which correspond to hm and $P^{(h)}_t$.

(4.8) PROPOSITION. — *Let* $m \in \text{Exc}$ *and* $h \in \mathbf{E}$ *with* $m(h=\infty)=0$. *Let* $v \in \mathbf{E}(h)$ *and* $u \in \mathbf{E}$ *with* $u=hv$ a. e. m *on* $\{h<\infty\}$. *Then* $L_h(hm, v)=L(m, u)$.

(4.9) *Remarks.* — By (4.2) (i) for a given $v \in \mathbf{E}(h)$ there always exists such a u with the equality holding everywhere on $\{h<\infty\}$. Since $u=hv$ a. e. m because $m(h=\infty)=0$ one may abbreviate the conclusion of (4.8) as $L_h(hm, v)=L(m, hv)$. Note that $s=hv$ on $\{h<\infty\}$ and $s=\infty$ on $\{h=\infty\}$ is supermedian and its excessive regularization $\hat{s}=u$. The most important special case of (4.8) is $v=1$, in which case the conclusion is

$$L(m, h)=L_h(hm, 1). \tag{4.10}$$

Proof. — In view of the definition (4.5) and (4.4) we may suppose $m \in \text{Dis}$ since both $L_h(hm, .)$ and $L(m, .)$ vanish if $m \in \text{Con}$. Also from (4.5), $L_h(hm, v)=L_g(gm, w)$ where $w \in \mathbf{E}(g)$ with $v=w$ a. e. $gm=hm$. Consequently $u=hv=gw$ a. e. m, and so in proving (4.8) we may suppose that $h \in \mathscr{E}$. Then from (4.2) (i) it suffices to consider the case $u=hv$ everywhere on $\{h<\infty\}$. By (4.3) (iii) there exist measures μ_k carried by $\{h<\infty\}$ with $\mu_k U \uparrow m$ and $(h\mu_k) U^{(h)} \uparrow hm$. Therefore

$$\begin{aligned} L(m, u) &= \lim_k L(\mu_k U, u)=\lim_k \mu_k(u) \\ &= \lim_k \mu_k(hv)=\lim_k (h\mu_k)(v) \\ &= \lim_k L_h((h\mu_k) U^{(h)}, v)=L_h(hm, v), \end{aligned}$$

where the third equality follows because μ_k is carried by $\{h<\infty\}$ and $u=hv$ on $\{h<\infty\}$. □

5. THE ENERGY FUNCTIONAL AND KUZNETSOV MEASURES

In this section we shall express $L(m, u)$ in terms of Q^u_m under some conditions on m or u. In the course of our discussion we shall need some extensions of results in [8] and [16] which are of interest in their own right. We fix $m \in \text{Exc}$ and $u \in \mathbf{E}$ with $m(u=\infty)=0$.

We first extend the definition of C and $\hat{C}$ in [16]—*see* (3.19)—to general u.

(5.1) DEFINITION. — *For each* $B \in \mathscr{E}$ *define*

$$C(B) \equiv C_{m,u}(B) \equiv Q_m^u(0<\lambda_B<1),$$
$$\hat{C}(B) \equiv \hat{C}_{m,u}(B) \equiv Q_m^u(0<\tau_B<1).$$

It is then immediate that $Q_m^u(\lambda_B \in dt) = C_{m,u}(B)\,dt$ and $Q_m^u(\tau_B \in dt) = \hat{C}_{m,u}(B)\,dt$. When u and m are fixed we often shall suppress them in our notation if no confusion is possible. In particular we write $C_m = C_{m,1}$ and $\hat{C}_m = \hat{C}_{m,1}$ to agree with the notation in (3.19). The following proposition will often allow us to reduce the case of a general u to the case $u=1$. We need some notation for its statement. If $B \in \mathscr{E}$ let $P_B^{(u)}$ denote the hitting operator of B relative to the u-transform X^u; that is,

$$P_B^{(u)} f(x) = P^{x/u}[f \circ X_{T_B};\ T_B < \infty]. \tag{5.2}$$

Similarly $R_B^{(u)}$ denotes the balayage operator on Exc(u) relative to X^u. Since Q_m^u is the Kuznetsov measure corresponding to um and the semigroup $P_t^{(u)}$, from (2.7) we have

$$R_B^{(u)}(um)(f) = Q_m^u[f \circ Y_t;\ \tau_B < t]. \tag{5.3}$$

(5.4) PROPOSITION. — *Let* $B \in \mathscr{E}$, $u \in \mathbf{E}$, *and* $m \in \text{Exc}$ *with* $m(u=\infty)=0$. *Then* (i) $P_B u = u\, P_B^{(u)} 1$ *on* $\{u<\infty\}$ *and* (ii) $R_B^{(u)}(um) = u \,.\, R_B m$.

Proof. — Assertion (i) is just Proposition 1.4 in [24]. For (ii) one first checks that if $Z \in p\mathscr{G}_t^*$, then

$$Q_m^u(Z;\ \alpha<t<\beta) = Q_m(Z\, u \circ Y_t). \tag{5.5}$$

Therefore from (5.3)

$$R_B^{(u)}(um)(f) = Q_m[(fu) \circ Y_t;\ \tau_B < t] = R_B m(fu),$$

proving (ii). □

We have the following generalization of (3.22).

(5.6) PROPOSITION. — *Let* $B \in \mathscr{E}$. *Then*

$$C(B) = C_{m,u}(B) = L(m, (P_B u)_p)$$

and

$$\hat{C}(B) = \hat{C}_{m,u}(B) = L((R_B m)_p, u).$$

Proof. — If $u=1$, this reduces to (3.22). It follows readily from (5.4) (i) that $(P_B u)_p = u(P_B^{(u)} 1)_p$ on $\{u<\infty\}$. (Of course, $(P_B u)_p$ [resp. $(P_B^{(u)} 1)_p$] is the purely excessive part of $P_B u$ [resp. $P_B^{(u)} 1$] relative to the semigroup (P_t) [resp. $(P_t^{(u)})$].) Consequently in light of (3.24) for the process X^u and (4.8)

$$C_{m,u}(B) = Q_m^u(0<\lambda_B<1) = L_u(um, (P_B^{(u)} 1)_p) = L(m, (P_B u)_p).$$

Using (4.4), (4.10), and (5.4) (ii) one has

$$L((R_B m)_p, u) = L_u(u(R_B m)_p, 1) = L_u((u R_B m)_p, 1) = L_u((R_B^{(u)}(um))_p, 1),$$

and so from (3.23) applied to X^u

$$\hat{C}_{m,u}(B) = Q_m^u(0<\tau_B<1) = L_u((R_B^{(u)}(um))_p, 1) = L((R_B m)_p, u). \quad \square$$

Remark. — The argument reducing (5.6) to (3.22) is the prototype of an argument that will be used several times in the sequel.

We shall say that $B \in \mathscr{E}$ is *u-m-transient* (resp. *u-m-cotransient*) provided $Q_m^u(\lambda_B=\infty)=0$ [resp. $Q_m^u(\tau_B=-\infty)=0$]. These agree with the definitions in [16] when $u=1$. Proposition 4.1 of [16] applied to the u-transform of X states that B is u-m-transient if and only if $(P_B^{(u)} 1)_i = 0$ a. e. um, and, since $m(u=\infty)=0$, in view of (5.4) (i), this is equivalent to $(P_B u)_i = 0$ a. e. m. Similarly by (4.7) of [16], B *is* u-m-cotransient if and only if $(R_B^{(u)}(um))_i = 0$, or by (5.4) (ii) and (4.4), $u(R_B m)_i = 0$. This immediately implies the following statement.

(5.7) Proposition. — *Let* $B \in \mathscr{E}$. *If* B *is m-cotransient, then it is u-m-cotransient. If* $u>0$ a. e. $(R_B m)_i$, *in particular if* $u>0$ a. e. m, *and* B *is u-m-cotransient, then it is m-cotransient.*

It follows from (5.6) that $C_{m,u}(B) = L(m, P_B u)$ if B is u-m-transient and that $\hat{C}_{m,u}(B) = L(R_B m, u)$ if B is u-m-cotransient. If B is both u-m-transient and u-m-cotransient, one has $C_{m,u}(B) = \hat{C}_{m,u}(B)$ because of (3.16).

(5.8) *Remarks.* — Both assertions in (5.7) are *false* if cotransience is replaced by transience. Let X be translation to the right on $\mathbb{R}$ at unit speed killed exponentially with parameter one so that $P_t f(x) = e^{-t} f(x+t)$. Let m be Lebesgue measure. Since $P_t 1 \to 0$ as $t \to \infty$, each $B \in \mathscr{E}$ is m-transient. Let $u(x) = e^x$. Then $P_t u = u$. If $B =]0, \infty[$, one checks that $P_B u = u$ and so by the remarks above (5.7), B is not u-m-transient. Similarly if X is translation to the right at unit speed on $\mathbb{R}$ and $u(x)=e^{-x}$, then $B=]0, \infty[$ is u-m-transient but not m-transient. Hence there is an essential difference between transience and cotransience as far as (5.7) is concerned.

We now introduce the birthing operators b_r, $-\infty \leqq r < \infty$ and the killing operators k_s, $-\infty < s \leqq \infty$ on W as follows:

$$(5.9) \qquad \begin{cases} b_r w(t)=w(t) \text{ if } t>r, & b_r w(t)=\Delta \text{ if } t \leqq r; \\ k_s w(t)=w(t) \text{ if } t<s, & k_s w(t)=\Delta \text{ if } t \geqq s. \end{cases}$$

The next result is an extension of (5.3)(ii) in [8].

(5.10) Proposition. — *Let* $B \in \mathscr{E}$. *Then*

(i) $Q^u_{R_B m}=b_{\tau_B}[Q^u_m(\,.\,;\tau_B<\infty)]$;

(ii) $Q^{P_B u}_m=k_{\lambda_B}[Q^u_m(\,.\,;\lambda_B>-\infty)]$.

Proof. — If $u=1$, (i) is just (5.3)(ii) in [8]. Let Q^*_{um} denote the Kuznetsov measure corresponding to um, 1, and $(P^{(u)}_t)$ so $Q^*_{um}=Q^u_m$. Then

$$b_{\tau_B}[Q^u_m(\,.\,;\tau_B<\infty)]=b_{\tau_B}[Q^*_{um}(\,.\,;\tau_B<\infty)]=Q^*_{R_B^{(u)}(um)}=Q^*_{uR_B m}=Q^u_{R_B m},$$

where the second equality follows from the case $u=1$ and the third from (5.4)(ii). Similarly using (5.4)(i) it will suffice to prove (5.10)(ii) when $u=1$. Denote the measure on the right side of (5.10)(ii) with $u=1$ by Q. Then

$$Q(f\circ Y_t)=Q_m(f\circ Y_t\circ k_{\lambda_B};\lambda_B>-\infty)=Q_m(f\circ Y_t;\,t<\lambda_B).$$

But $\{t<\lambda_B\}=\{L_B\circ\gamma_t>0\}$ where $L_B\equiv\sup\{t:X_t\in B\}\vee 0$ is the last exit time from B for X. Now $P^x(L_B>0)=\varphi_B(x)\equiv P_B 1(x)$, and so

$$Q(f\circ Y_t)=Q_m(f\circ Y_t\,\varphi_B\circ Y_t)=Q^{\varphi_B}_m(f\circ Y_t).$$

A similar calculation shows that Q and $Q^{\varphi_B}_m$ have the same finite dimensional distributions, and hence (5.10)(ii) with $u=1$ follows by the uniqueness property of Kuznetsov measures. □

(5.11) *Remark.* — As in [8], (5.10)(i) extends immediately to the class of intrinsic stopping times τ defined there. There is a similar extension of (5.10)(ii) to "stationary times λ corresponding to coterminal times". We shall not pursue this here. However, note that (5.10)(ii) gives the following formula which is analagous to the definition (2.7) of $R_B m$,

(5.12) $$m(f\,P_B u)=Q_m^u(f\circ Y_t;\ \lambda_B>t)$$

for each $t\in\mathbb{R}$.

(5.13) COROLLARY. — *Let* $B\in\mathscr{E}$. *Then for* $Z\in p\,\mathscr{G}^*$.

(i) $Q_{R_B m}^u(\alpha\in dt)=Q_m^u(\tau_B\in dt)=\hat{C}_{m,u}(B)\,dt$;

(ii) $Q_{(R_B m)_i}^u(Z)=Q_m^u(Z;\ \tau_B=-\infty)$;

(iii) $Q_m^{P_B u}(\beta\in dt)=Q_m^u(\lambda_B\in dt)=C_{m,u}(B)\,dt$;

(iv) $Q_m^{(P_B u)_i}(Z)=Q_m^u(Z;\ \lambda_B=\infty)$.

Proof. — Since $\alpha\leqq\tau_B<\beta$ on $\{\tau_B<\infty\}$, it follows that $\alpha\circ b_{\tau_B}=\tau_B$ on $\{\tau_B<\infty\}$, and hence (i) is an immediate consequence of (5.10)(i). Combining the above remark with (2.4) and (5.10)(i) we have

$$Q_{(R_B m)_i}^u(Z)=Q_{R_B m}^u(Z;\ \alpha=-\infty)$$
$$=Q_m^u(Z\circ b_{\tau_B};\ \alpha\circ b_{\tau_B}=-\infty,\ \tau_B<\infty)=Q_m^u(Z;\ \tau_B=-\infty)$$

where for the last equality we also use $Z\circ b_{-\infty}=Z$. Similarly $\beta\circ k_{\lambda_B}=\lambda_B$ on $\{\lambda_B>-\infty\}$ and $Z\circ k_\infty=Z$, and so one obtains (iii) and (iv) from (2.6) and (5.10)(ii). □

Here is the relationship between the energy functional and the Kuznetsov measure promised in the first sentence of this section.

(5.14) THEOREM. — *Let* $m\in\text{Exc}$ *and* $u\in E$ *with* $m(u=\infty)=0$. *Then* $Q_m^u(0<\alpha<1)=L(m_p,\ u)$ *and* $Q_m^u(0<\beta<1)=L(m,\ u_p)$. *In particular if* $m\in\text{Pur}$, $L(m,\ u)=Q_m^u(0<\alpha<1)$ *and if* u *is* m-*purely excessive* $L(m,\ u)=Q_m^u(0<\beta<1)$.

Proof. — Applying (5.13) (i) with $B=E$ yields $Q_m^u(0<\alpha<1)=\hat{C}_{m,u}(E)$ since $R_E m=m$. But from (5.6), $\hat{C}_{m,u}(E)=L(m_p,\ u)$ proving the first assertion in (5.14). The second is established in a similar manner using (5.13)(iii). □

Remarks. — There is another approach to (5.14). If $m_p=\int_0^\infty \nu_t\,dt$ where $\nu=(\nu_t)_{t>0}$ is an entrance law, then $Q_{m_p}^u=\int_{-\infty}^\infty \theta_t(Q_\nu^u)\,dt$. Using this and (2.4) a direct calculation shows that

$$Q_m^u(0<\alpha<1)=Q_{m_p}^u(0<\alpha<1)=Q_\nu^u(\alpha\in\mathbb{R})=\lim_{t\downarrow 0}\nu_t(u)=L(m_p,\ u)$$

where the last equality follows from (3.8). If u_p is the integral of an "exit law" – *see* [6] – then a similar argument gives $Q^u_m(0<\beta<1)=L(m, u_p)$. However, u_p need not be the integral of an exit law and so here one obtains a weaker result than (5.14).

6. SOME BALAYAGE IDENTITIES

In this section the balayage operators R^q_B will be investigated and particularly their dependence on q. We first need to establish some auxiliary relations between the resolvent (U_q) and the hitting operators (P^q_B) of X. Recall that $(T_B \equiv \inf\{t>0: X_t \in B\})$

(6.1) $$P^q_B f \equiv P^{\cdot}[e^{-qT_B} f \circ X_{T_B}].$$

We shall let (V_q) denote the resolvent of X killed when it first hits B; that is

(6.2) $$V_q f \equiv P^{\cdot}\left[\int_0^{T_B} e^{-qt} f \circ X_t \, dt\right].$$

It is well known and easily checked that (V_q) satisfies the resolvent equation and that

(6.3) $$U_q = V_q + P^q_B U_q \quad \text{for} \quad q \geqq 0.$$

(6.4) LEMMA. – *Let $q, r \geqq 0$. Then*

(i) $P^r_B U_r P^q_B + V_r P^q_B = U_r P^q_B$;

(ii) $(q-r) V_r P^q_B = P^r_B - P^q_B$.

Proof. – Let $f \in pb\mathscr{E}$. Then

$$U_r P^q_B f = P^{\cdot}\left[\int_0^{\infty} e^{-rt} P^q_B f \circ X_t \, dt\right]$$

and splitting the integral into $\int_0^{T_B} \ldots$ and $\int_{T_B}^{\infty} \ldots$ yields (i). Moreover, since $T_B \circ \theta_t = T_B - t$ on $\{t < T_B\}$

$$(q-r) V_r P^q_B f = (q-r) P^{\cdot}\left[\int_0^{T_B} e^{-rt} e^{-qT_B \circ \theta_t} f \circ X_{t+T_B \circ \theta_t} \, dt\right]$$

$$= P^{\cdot}[e^{-qT_B} f \circ X_{T_B}(e^{(q-r)T_B} - 1)] = P^r_B f - P^q_B f,$$

which is (ii). □

(6.5) COROLLARY. — *Let* $q, r \geqq 0$. *Then*

(6.6) $$(q-r) U_r P_B^q + P_B^q = P_B^r + (q-r) P_B^r U_r P_B^q.$$

Proof. — This is an immediate consequence of (6.4). □

(6.7) COROLLARY. — *Let* $0 \leqq r < q$. *Then*

(6.8) $$[I+(q-r) U_r] P_B^q U_q + (q-r) P_B^r U_r U_q = P_B^r U_r + (q-r) P_B^r U_r P_B^q U_q.$$

In particular in case $r > 0$ *this implies*

(6.9) $$[I+(q-r) U_r] P_B^q U_q = P_B^r U_r - (q-r) P_B^r U_r V_q;$$

furthermore for $r \geqq 0$

(6.10) $$[I+(q-r) U_r] P_B^q U_q P_B^r = P_B^r U_r P_B^q.$$

Proof. — Applying (6.6) to U_q and adding the term $(q-r) P_B^r U_r U_q$ to both sides of the equality yields (6.8) because of the resolvent equation. In case $r > 0$ one obtains (6.9) from (6.8) by subtracting the second term on the right hand side and using (6.3). To see (6.10) assume first that $0 < r < q$. Applying (6.9) to P_B^r yields (6.10) because

$$P_B^r U_r P_B^r - (q-r) P_B^r U_r V_q P_B^r = P_B^r U_r (I-(q-r) V_q) P_B^r = P_B^r U_r P_B^q,$$

by (6.4) (ii). Letting r decrease to zero gives (6.10) for $r=0$ as well since the convergence is monotone. □

In discussing the balayage identities we shall need some properties of conservative excessive measures.

(6.11) LEMMA. — *Let* $m \in \mathrm{Con}$ *and* $B \in \mathscr{E}$. *Then* $Q_m(-\infty < \tau_B < \infty) = 0$ *and* $Q_m(-\infty < \lambda_B < \infty) = 0$.

Proof. — $m \in \mathrm{Con}$ implies $R_B m \in \mathrm{Con} \subset \mathrm{Inv}$, hence $(R_B m) P_t = R_B m$ for any $t > 0$. Let $f > 0$ with $m(f) < \infty$. Then

$$\begin{aligned} 0 = R_B m(f) - R_B m(P_t f) &= Q_m[f \circ Y_0;\ \tau_B < 0] - Q_m[f \circ Y_t;\ \tau_B < 0] \\ &= Q_m[f \circ Y_0;\ -t < \tau_B < 0]. \end{aligned}$$

Letting $t \to \infty$ and using (2.8) we obtain $Q_m(-\infty < \tau_B < 0) = 0$. But $\tau_B \circ \theta_t = \tau_B - t$, and so by the stationarity of Q_m, $Q_m(-\infty < \tau_B < t) = 0$.

Letting $t \to \infty$ establishes the first claim in (6.11). A similar argument starting from $m(f\varphi_B) = m(f P_t \varphi_B)$ establishes the second. □

For the next result recall that $\varphi_B \equiv P_B 1 = P^{\cdot}(T_B < \infty)$ and the definition of R_B^q below (2.7).

(6.12) PROPOSITION. — *Let* $m \in \mathrm{Con}$, $B \in \mathscr{E}$. *Then*

(i) $R_B m = \varphi_B m$;

(ii) $(R_B m) P_B^q = m P_B^q$ *for* $q \geqq 0$;

(iii) $R_B^q m = qm P_B^q U_q = q(R_B m) P_B^q U_q$ *for* $q > 0$.

Proof. — To prove the first assertion let $f > 0$ with $m(f) < \infty$. Then using (6.11) one obtains

$$\{T_B \circ \gamma_0 = \infty\} = \{\lambda_B \leqq 0\} = \{\lambda_B = -\infty\} = \{\tau_B = \infty\}$$

a. e. Q_m and so

$$\begin{aligned} R_B m(f) = Q_m[f \circ Y_0;\ \tau_B < 0] &= Q_m[f \circ Y_0;\ \tau_B = -\infty] \\ &= Q_m[f \circ Y_0] - Q_m[f \circ Y_0;\ \tau_B = \infty] \\ &= m(f) - Q_m[f \circ Y_0 P^{Y_0}[T_B = \infty]] \\ &= m(f) - m[f(1-\varphi_B)] = m(f.\varphi_B), \end{aligned}$$

which establishes (i). Now from (2.10) we know that a. e. m, φ_B is either zero or one, and on $\{\varphi_B = 0\}$, $P_B^q 1$ will be zero for $q \geqq 0$. Therefore $(R_B m) P_B^q = (\varphi_B m) P_B^q = m P_B^q$. To see (iii) observe that since m is invariant, for $q > 0$, $m = qm U_q$, hence by the remarks below (2.7) applied to the q-subprocess, $R_B^q m = qm P_B^q U_q = q(R_B m) P_B^q U_q$ by (ii). □

(6.13) *Remark.* — Since $\mathrm{Pur} \subset \mathrm{Dis}$, it is clear that B cotransient implies $R_B m \in \mathrm{Dis}$. But from (6.12) (i) it follows as well that B transient implies $R_B m \in \mathrm{Dis}$. To see this observe that $(R_B m)_d = R_B m_d$ by (5.8) of [8], and so $R_B m \in \mathrm{Dis}$ is equivalent to $R_B m_c = 0$. But B transient implies $m_c(\varphi_B) = 0$ and therefore according to (6.12) (i) $R_B m_c = \varphi_B . m_c = 0$.

We shall now state a relationship between $R_B^q m$ and $R_B^r m$ for general m.

(6.13) THEOREM. — *Let* $0 \leqq r < q$. *Then for* $m \in \mathrm{Exc}^r$

$$R_B^q m + (q-r) R_B^r m U_q = R_B^r m + (q-r) R_B^r m P_B^q U_q. \tag{6.14}$$

Proof. — We shall prove this for $r = 0$; the general case then follows by taking (P_t^r) to be the basic semigroup. First let $m \in \mathrm{Con}$. Then since $R_B m \in \mathrm{Con} \subset \mathrm{Inv}$ (6.14) is obtained from (6.12) (iii). If $m \in \mathrm{Dis}$ then m is the increasing limit of potentials $\mu_n U$, and so $R_B m = \uparrow \lim_n \mu_n P_B U$ and also $m = \uparrow \lim_n \nu_n U_q$ where $\nu_n \equiv \mu_n (I + q U)$. Consequently $R_B^q m = \uparrow \lim_n \nu_n P_B^q U_q$. Hence from (6.8)

$$R_B^q m + q R_B m U_q = \uparrow \lim_n \mu_n [(I + q U) P_B^q U_q + q P_B UU_q]$$

$$= \uparrow \lim \mu_n [P_B U + q P_B UP_B^q U_q] = R_B m + q R_B m P_B^q U_q,$$

which is (6.14) for $r=0$. □

(6.15) *Remark.* — Since (6.14) is an identity between σ-finite measures even when $r=0$ (both terms on either side are dominated by m) one may write it as

$$(6.16) \qquad R_B^q m = [R_B^r m - (q-r) R_B^r m U_q] + (q-r) R_B^r m P_B^q U_q$$

where the expression in brackets is a positive σ-finite measure. Also, using (6.3), $R_B^r m U_q - R_B^r m P_B^q U_q = R_B^r m V_q$ as σ-finite measures. Therefore (6.14) may as well be written

$$(6.17) \qquad R_B^q m = R_B^r m - (q-r) R_B^r m V_q.$$

(6.18) PROPOSITION. — *Let* $m \in \mathrm{Exc}$, $r \geqq 0$, *and* $B \in \mathscr{E}$. *Then* $R_B^q m$ *increases to* $R_B^r m$ *as* q *decreases to* r.

Proof. — As in the proof of (6.13) it suffices to prove this for $r=0$. Let (K_t) denote the semigroup of X killed when it first hits B so that (V_q) defined in (6.2) is the resolvent of (K_t). In the course of the proof of (7.1) in [16] (*see* the paragraph below (7.16) in [16]) it was shown that $R_B m K_t$ decreases to zero as $t \to \infty$, or in terms of the resolvent, $q R_B m V_q$ decreases to zero as $q \downarrow 0$ [i. e. $q R_B m V_q(f) \downarrow 0$ as $q \downarrow 0$ provided $R_B m(f) < \infty$ or only $R_B m V_q(f) < \infty$ for some $q > 0$]. Consequently because of (6.17), $R_B^q m(f)$ increases to $R_B m(f)$ whenever $R_B m(f)$ is finite. Since $R_B m$ is σ-finite this implies that $R_B^q m \uparrow R_B m$ as q decreases to zero. □

Remark. — Letting q decrease to zero in the first equality in (9.3) gives an alternate proof of (6.18).

The following important identity will be used in section 7.

(6.19) THEOREM. — *Let* $0 \leqq r < q$. *Then for* $m \in \mathrm{Exc}^r$

$$(6.20) \qquad (R_B^q m) P_B^r = (R_B^r m) P_B^q.$$

Proof. — Since as r decreases to zero P_B^r increases to P_B and $R_B^r m$ increases to $R_B m$ according to (6.18), it suffices to prove (6.20) for $r > 0$. But for $r > 0$, $\mathrm{Exc}^r = \mathrm{Dis}^r$ so that there exists a sequence of r-potentials $\mu_n U_r$ increasing to m, which implies $\mu_n P_B^r U_r \uparrow R_B^r m$. Let $\nu_n \equiv \mu_n [I + (q-r) U_r]$. Then $\nu_n U_q = \mu_n U_r \uparrow m$, and thus $\nu_n P_B^q U_q \uparrow R_B^q m$. Hence using (6.10),

$$(R_B^q m) P_B^r = \lim_n \mu_n [I + (q-r) U_r] P_B^q U_q P_B^r = \lim_n \mu_n P_B^r U_r P_B^q = (R_B^r m) P_B^q,$$

proving (6.20). □

7. CAPACITIES AND q-CAPACITIES

In this section we fix $m \in \mathrm{Exc}$ and $u \in E$ with $m(u=\infty)=0$. For each $q \geqq 0$ and $B \in \mathscr{E}$ we define

(7.1) $$\Gamma^q(B) \equiv \Gamma^q_{m,u}(B) \equiv L^q(R_B^q m, u) = L^q(m, P_B^q u)$$

where the last equality follows from (3.16) (applied to the q-subprocess). If $q=0$, then because of (5.6)

(7.2) $$\Gamma(B) \equiv \Gamma^0(B) = C(B) + L(m, (P_B u)_i) = \hat{C}(B) + L((R_B m)_i, u).$$

Also, if $q=0$, by (3.10), $\Gamma_{m,u} = \Gamma_{m_d, u}$. On the other hand, if $q > 0$, then $R_B^q m \in \mathrm{Dis}^q$ always. Our first result shows that Γ^q behaves like a capacity. In (7.12) it will be shown that, at least for dissipative m, Γ^q is the proper extension of the notion of capacity and cocapacity as defined in [16] to arbitrary Borel sets.

(7.3) THEOREM. — *Let* A, $B \in \mathscr{E}$, *and* $q \geqq 0$. *Then*

(i) $\Gamma^q(A) \leqq \Gamma^q(B)$, *if* $A \subset B$;

(ii) $\Gamma^q(A \cup B) + \Gamma^q(A \cap B) \leqq \Gamma^q(A) + \Gamma^q(B)$;

(iii) $\Gamma^q(B_n) \uparrow \Gamma^q(B)$, *if* $B_n \in \mathscr{E}$ and $(B_n) \uparrow B$.

Proof. — We prove (7.3) only in case $q=0$, the general result then follows by taking (P_t^q) as the basic semigroup. The argument is the same as part of the proof of (4.5) in [16]. If $A \subset B$, then $\varphi_A \equiv P_A 1 \leqq P_B 1 \equiv \varphi_B$, and $L(m, .)$ is monotone, which yields (i). For (ii) observe that

$$\begin{aligned} P^\cdot(T_{A\cup B} < \infty) - P^\cdot(T_A < \infty) &= P^\cdot(T_{A\cup B} < \infty,\ T_A = \infty) \\ &\leqq P^\cdot(T_B < \infty,\ T_{A\cap B} = \infty) = P^\cdot(T_B < \infty) - P^\cdot(T_{A\cap B} < \infty), \end{aligned}$$

and so $\varphi_{A \cup B} + \varphi_{A \cap B} \leqq \varphi_A + \varphi_B$, which yields (ii). If $(B_n) \uparrow B$, then $T_{B_n} \downarrow T_B$ and $\varphi_{B_n} \uparrow \varphi_B$, which implies (iii) according to (3.5). □

The next result deals with the dependence of Γ^q on q.

(7.4) THEOREM. – *Let* $0 \leqq r < q$ *and* $B \in \mathscr{E}$. *Then*

$$\Gamma^q(B) = \Gamma^r(B) + (q-r) R_B^r m (P_B^q u) = \Gamma^r(B) + (q-r) R_B^q m (P_B^r u).$$

Proof. – Note that the second equality follows from (6.20). As before it suffices to prove (7.4) in case $r=0$. Suppose first $m \in \text{Con}$. Then $\Gamma(B) = L(m, P_B u) = 0$, hence since m is invariant and because of (6.12) (ii) we obtain

$$\Gamma^q(B) = L^q(m, P_B^q u) = qm P_B^q u = q R_B m (P_B^q u) = \Gamma(B) + q R_B m (P_B^q u).$$

Now, if $m \in \text{Dis}$ then there exists a sequence of potentials $\mu_n U$ increasing to m. Let $\nu_n \equiv \mu_n (I + qU)$. Then $\nu_n U_q = \mu_n U \uparrow m$. Therefore

$$\Gamma^q(B) = L^q(m, P_B^q u) = \lim_n \nu_n (P_B^q u)$$

$$= \lim_n \mu_n [P_B^q + q\, U P_B^q](u) = \lim_n \mu_n [P_B + q P_B U P_B^q](u),$$

where the last equality is because of (6.6). Since

$$\mu_n P_B u \uparrow L(m, P_B u) = \Gamma(B)$$

by (3.6) and (3.7), and $\mu_n P_B U P_B^q u \uparrow R_B m (P_B^q u)$, this yields (7.4). □

In the following we investigate the relationship of the set functions C^q and $\hat{C}^q$, i. e. the function C and $\hat{C}$ relative to the q-subprocess, with Γ^q. Let ${}^q Q_m^u$ denote the Kuznetsov measure corresponding to m, u, and the semigroup (P_t^q), which is the same as the Kuznetsov measure corresponding to um, 1 and the semigroup $(e^{-qt} P_t^{(u)})$. Recall that $\tau_B \equiv \inf\{t : Y_t \in B\}$ and $\lambda_B \equiv \sup\{t : Y_t \in B\}$ and define

$$(7.5) \quad \begin{cases} C^q(B) \equiv C_{m,u}^q(B) \equiv {}^q Q_m^u (0 < \lambda_B < 1), \\ \hat{C}^q(B) \equiv \hat{C}_{m,u}^q(B) \equiv {}^q Q_m^u (0 < \tau_B < 1). \end{cases}$$

If $q > 0$ then $m P_t^q(f) \to 0$ as $t \to \infty$, when $f \in p\mathscr{E}$ with $m(f) < \infty$. Thus $m \in \text{Pur}^q$ and so $R_B m \in \text{Pur}^q$. Therefore by (5.7) applied to the q-subprocess each $B \in \mathscr{E}$ is q-u-m-cotransient. Also $P_t^q P_B^q u(x) \leqq e^{-qt} u(x)$ and hence tends to zero as $t \to \infty$ if $u(x) < \infty$. Consequently by the remarks above (5.7) each $B \in \mathscr{E}$ is q-u-m-transient. Therefore by (5.6) and the definitions

(7.5) we have

$$C^q(B) = \Gamma^q(B) = \hat{C}^q(B), \qquad \text{if} \quad q > 0. \tag{7.6}$$

In case $q=0$ the corresponding relation is (7.2). From (7.6) it is clear that Theorem 7.4 can be stated as well with C or $\hat{C}$ in place of Γ provided $r > 0$. What we are going to prove next is that this is true even for $r=0$. For that we first prove two auxiliary results.

(7.7) LEMMA. – *Let* $B \in \mathscr{E}$. *Then* (i) $R_B[(R_B m)_i] = (R_B m)_i$ *and* (ii) $P_B[(P_B u)_i] = (P_B u)_i$ *a. e. m.*

Proof. – Let $f \in p\mathscr{E}$. Using (2.7) and (5.13) (ii) twice one obtains

$$R_B[(R_B m)_i](f) = Q_{(R_B m)_i}(f \circ Y_0;\ \tau_B < 0)$$
$$= Q_m(f \circ Y_0;\ \tau_B = -\infty) = (R_B m)_i(f),$$

proving (i). On the other hand using (5.12) and (5.13) (iv) twice,

$$m[f P_B(P_B u)_i] = Q_m^{(P_B u)_i}(f \circ Y_0 : \lambda_B > 0)$$
$$= Q_m^u(f \circ Y_0;\ \lambda_B = \infty) = m[f(P_B u)_i].$$

Since $f \in p\mathscr{E}$ is arbitrary this yields (ii). □

(7.8) LEMMA. – *Let* $B \in \mathscr{E}$ *and* $q > 0$. *If* $(R_B m)_i P_B^q u < \infty$, *then* $L((R_B m)_i, u) = 0$; *and if* $R_B^q m (P_B u)_i < \infty$, *then* $L(m, (P_B u)_i) = 0$.

Proof. – Since $(R_B m)_i = q(R_B m)_i U_q$ one obtains by applying (3.7), (7.1), (7.4), and (7.7),

$$q(R_B m)_i P_B^q u = L^q((R_B m)_i, P_B^q u) = \Gamma^q_{(R_B m)_i, u}(B)$$
$$= \Gamma_{(R_B m)_i, u}(B) + q R_B (R_B m)_i P_B^q u$$
$$= L(R_B(R_B m)_i, u) + q R_B (R_B m)_i P_B^q u$$
$$= L((R_B m)_i, u) + q(R_B m)_i P_B^q u.$$

Hence, if $(R_B m)_i P_B^q u$ is finite, then $L((R_B m)_i, u)$ must be zero. On the other hand, analogously, since $(P_B u)_i = q U_q (P_B u)_i$ a. e. m,

$$q R_B^q m (P_B u)_i = L^q(R_B^q m, (P_B u)_i) = \Gamma^q_{m, (P_B u)_i}(B)$$
$$= \Gamma_{m, (P_B u)_i}(B) + q R_B^q m [P_B(P_B u)_i]$$
$$= L(m, P_B(P_B u)_i) + q R_B^q m [P_B(P_B u)_i]$$
$$= L(m, (P_B u)_i) + q R_B^q m (P_B u)_i,$$

which leads to the second statement accordingly. □

(7.9) THEOREM. — *Let* $B \in \mathcal{E}$ *and* $q > 0$. *Then*

$$C^q(B) = \hat{C}(B) + q\,R_B\,m(P_B^q\,u) = C(B) + q\,R_B^q\,m(P_B\,u).$$

Proof. — We know from (7.4), (7.6) and (5.6) that

$$\begin{aligned} C^q(B) = \Gamma^q(B) = \Gamma(B) + q\,R_B\,m(P_B^q\,u) \\ = L(R_B\,m,\ u) + q\,R_B\,m(P_B^q\,u) \\ = \hat{C}(B) + q\,R_B\,m(P_B^q\,u) + L((R_B\,m)_i,\ u) \end{aligned}$$

and

$$\begin{aligned} C^q(B) = \Gamma(B) + q\,R_B^q\,m(P_B\,u) \\ = L(m,\ P_B\,u) + q\,R_B^q\,m(P_B\,u) \\ = C(B) + q\,R_B^q\,m(P_B\,u) + L(m,\ (P_B\,u)_i). \end{aligned}$$

Now, Lemma (7.8) implies that if $(R_B\,m)_i\,P_B^q\,u < \infty$ then $L((R_B\,m)_i,\ u) = 0$, and if $R_B^q\,m(P_B\,u)_i < \infty$ then $L(m,\ (P_B\,u)_i) = 0$. If either of them is infinite, however, then $C^q(B) = \infty = q\,R_B\,m(P_B^q\,u) = q\,R_B^q\,m(P_B\,u)$. Therefore in any case the equalities claimed in (7.9) are valid. □

The subsequent corollary states some characterizations of *u*-*m*-(co-) transience. The proof is immediate from the discussion preceding (5.7), from (7.8), (3.17) and (6.18).

(7.10) COROLLARY. — *Let* $m \in \mathrm{Dis}$ *and* $B \in \mathcal{E}$. *Then*

(1) *the following are equivalent:*

(i) $R_B^q\,m((P_B\,u)_i) < \infty$ *for some* $q > 0$;

(ii) B *is u-m-transient;*

(iii) $R_B\,m((P_B\,u)_i) = 0$;

(iv) $R_B^q\,m((P_B\,u)_i) = 0$ *for all* $q \geqq 0$;

and (2) *the following are equivalent:*

(*a*) $(R_B\,m)_i(P_B^q\,u) < \infty$ *for some* $q > 0$;

(*b*) B *is u-m-cotransient;*

(*c*) $(R_B\,m)_i(P_B\,u) = 0$;

(*d*) $(R_B\,m)_i(P_B^q\,u) = 0$ *for all* $q \geqq 0$.

(7.11) *Remark.* — If $m \in \mathrm{Dis}$ and $B \in \mathcal{E}$, then it follows from (7.10) that if for some $q > 0$, $R_B^q\,m(P_B\,u) = R_B\,m(P_B^q\,u)$ is finite then B is *u*-*m*-transient and *u*-*m*-cotransient and (of course) $C(B) = \hat{C}(B)$. The assumption that $m \in \mathrm{Dis}$ is actually only used in showing that B is transient and cotransient.

In fact, (7.4), (7.6) and (7.9) imply for any $m \in \mathrm{Exc}$ with $m(u=\infty)=0$ that if $\mathrm{R}^q_{\mathrm{B}} m(\mathrm{P}_{\mathrm{B}} u)=\mathrm{R}_{\mathrm{B}} m(\mathrm{P}^q_{\mathrm{B}} u)$ is finite, then $\mathrm{C}(\mathrm{B})=\Gamma(\mathrm{B})=\hat{\mathrm{C}}(\mathrm{B})$, possibly infinite. *See* the second example in (8.3).

By means of the preceding results we are now able to prove that—at least for dissipative m—Γ coincides with the outer capacity extension of C and $\hat{\mathrm{C}}$ to $\mathscr{E}$ as mentioned earlier. Recall some definitions from (4.16) of [16]. $\mathscr{P}$ denotes the set of all $\mathrm{B} \in \mathscr{E}$ that are both transient and cotransient with $\Gamma(\mathrm{B})=\mathrm{C}(\mathrm{B})=\hat{\mathrm{C}}(\mathrm{B})$ finite. Let $\mathscr{P}_\sigma$ denote the set of countable unions of sets in $\mathscr{P}$, and for $\mathrm{A} \in \mathscr{P}_\sigma$

$$\mathrm{I}^*(\mathrm{A}) \equiv \sup\{\Gamma(\mathrm{B}) : \mathrm{B} \subset \mathrm{A},\ \mathrm{B} \in \mathscr{P}\}.$$

Then, if one sets for any $\mathrm{F} \subset \mathrm{E}$

$$\mathrm{I}^*(\mathrm{F}) \equiv \inf\{\mathrm{I}^*(\mathrm{A}) : \mathrm{A} \supset \mathrm{F},\ \mathrm{A} \in \mathscr{P}_\sigma\},$$

I^* defines an outer capacity on E which agrees with Γ on $\mathscr{P}$ (according to III-32 of [3]).

(7.12) THEOREM. — *If* $m \in \mathrm{Dis}$ *then* $\mathrm{I}^*=\Gamma$ *on* $\mathscr{E}$.

Proof. — It is clear that I^* agrees with Γ on $\mathscr{P}_\sigma$, since both are continuous on increasing sequences. We claim that $\mathrm{E} \in \mathscr{P}_\sigma$ in case $m \in \mathrm{Dis}$. If so, then there exists a sequence (E_n) in $\mathscr{P}$ increasing to E; thus for any $\mathrm{F} \in \mathscr{E}$ the sequence (F_n) where $\mathrm{F}_n \equiv \mathrm{F} \cap \mathrm{E}_n$ is in $\mathscr{P}$ and increases to F, and so

$$\mathrm{I}^*(\mathrm{F}) = \uparrow \lim_n \mathrm{I}^*(\mathrm{F}_n) = \uparrow \lim_n \Gamma(\mathrm{F}_n) = \Gamma(\mathrm{F}),$$

i.e. $\mathrm{I}^*=\Gamma$ on $\mathscr{E}$. To verify the claim fix $q>0$, and choose $f \in \mathscr{E}$ with $0<f \leqq 1$ and $m(f)<\infty$. Then $\mathrm{U}_q f \leqq \mathrm{U}_q 1 \leqq \frac{1}{q} < \infty$. Let

$$\mathrm{B}_n \equiv \left\{\mathrm{U}_q f > \frac{1}{n}\right\}$$

which increase to E as $n \to \infty$. Furthermore, since $1 \leqq n\mathrm{U}_q f$ on the fine closure of B_n, $\mathrm{P}^q_{\mathrm{B}_n} 1 \leqq n \mathrm{P}^q_{\mathrm{B}_n} \mathrm{U}_q f \leqq n \mathrm{U}_q f$. Therefore

$$\Gamma^q(\mathrm{B}_n) = \mathrm{L}^q(m, \mathrm{P}^q_{\mathrm{B}_n} 1) \leqq n\, \mathrm{L}^q(m, \mathrm{U}_q f) \leqq n \,.\, m(f) < \infty.$$

It follows by (7.4) and (7.11), since $m \in \mathrm{Dis}$, that each B_n is transient and cotransient and $\Gamma(\mathrm{B}_n) < \infty$. Consequently $\mathrm{E} \in \mathscr{P}_\sigma$. □

Of course, the result (7.12) does not apply for conservative m, because then Γ — as it is defined — is zero always whereas I^* is infinite on nonpolar sets (as the infimum over the empty set).

8. BEHAVIOR OF Γ^q AS A FUNCTION OF q

In this section we shall show that for $B \in \mathscr{E}$ the function $q \to \Gamma^q(B) \equiv \Gamma^q_{m,\, u}(B)$ as defined in (7.1) has smoothness properties on $]0, \infty[$ and moreover under some finiteness assumption behaves properly as well at $q=0$. As in section 7, $m \in \mathrm{Exc}$ and $u \in \mathrm{E}$ are regarded as fixed with $m(u=\infty)=0$.

(8.1) THEOREM. — *Let* $B \in \mathscr{E}$. *Then* $q \to \Gamma^q(B)$ *is increasing and continuous on* $]0, \infty[$, *and if it is finite for some* $q>0$, *then it is finite for all* $q \geqq 0$. *If* $R_B m(P^q_B u) < \infty$ *for some* $q>0$, *then* $\lim_{q \downarrow 0} \Gamma^q(B) = \Gamma(B)$.

(8.2) *Remarks.* — By Theorem 7.4 with $0=r<q$, if $\Gamma^q(B)<\infty$, then $R_B m(P^q_B u)<\infty$. Therefore $q \to \Gamma^q(B)$ is continuous and finite on $[0, \infty[$ whenever $\Gamma^q(B)<\infty$ for some $q>0$. Furthermore it follows from (7.6) and (7.9) — *see* also (7.11) — that Theorem 8.1 remains true if one replaces Γ by either C or $\hat{C}$ in its statement.

(8.3) *Examples.* — Our first example shows that $q \to \Gamma^q(B)$ may be discontinuous at zero. Let X be translation to the right on $\mathbb{R}$ at unit speed, m be Lebesgue measure, and $u=1$. Let $B=]-\infty, 0]$. Then according to (9.1) of [11], $C^q(B)=\hat{C}^q(B)=\Gamma^q(B)=\infty$ for $q>0$. Moreover (*see* (8.2) of [16]) : $C(B)=1$ and $\hat{C}(B)=0$. Since $\varepsilon_{-n} U \uparrow m$, it follows that $\Gamma(B)=1$. In this example $m \in \mathrm{Dis}$. For general X if $m \in \mathrm{Con}$ and $m(P_B 1)=\infty$, then from (3.27), $\Gamma_q(B)=\infty$ for $q>0$, while $\Gamma(B)=0$. Our second example shows that it is possible to have $R_B m(P^q_B u)<\infty$ for all $q>0$ and $\Gamma(B)=\infty$. *See* also (7.11). Let $E=]0,1[$ and X be translation to the right on E at unit speed. Let $a_k = 1-2^{-k}$, $k \geqq 1$ and $\mu = \sum \varepsilon_{a_k}$. Then $m=\mu U$ is σ-finite and hence excessive. Let $u=1$ and $B=\{a_k : k \geqq 1\}$. Then one easily checks that $\Gamma(B)=\mu(P_B 1)=\infty$ and that for $q>0$, $R_B m(P^q_B 1)=\mu(P_B U P^q_B 1)<\infty$.

Proof. – Using (4.8) and (5.4) for the q-subprocess and remembering that taking u-transforms and q-subprocesses commute, it suffices to prove (8.1) in the case $u=1$. *See* the proofs of (5.6) and (5.10) for a similar reduction. Suppose $q>0$. Then

(8.4) $$R_B m(P_B^q 1)=Q_m[e^{-q T_B \circ \gamma_0}; \tau_B<0]=Q_m[e^{-q T_B \circ \gamma_0}; \tau_B<0<\lambda_B],$$

since $T_B \circ \gamma_0=\infty$ if $\lambda_B \leqq 0$. We shall need to use Theorem 6.8 of [8]. Let J be the closure in $]\alpha, \beta[$ of $\{t : Y_t \in B\}$. Let G be the set of left end points contained in $]\alpha, \beta[$ of the contiguous intervals to J; that is, the maximal open intervals contained in $]\alpha, \beta[\setminus J$. Then according to (6.8) of [8], there exist a σ-finite measure ν on E and a kernel of σ-finite measures, $^*P^x$ from $(E, \mathscr{E}^*)$ to $(\Omega, \mathscr{F}^*)$ such that if $F=F(t, x, \omega)\geqq 0$ is universally measurable over $\mathscr{B}(\mathbb{R})\times\mathscr{E}\times\mathscr{F}^0$, then

(8.5) $$Q_m \sum_{r \in G} F(r, Y_r, \gamma_r)=\iint_{\mathbb{R}\times E} dt\, \nu(dx)^*P^x[F(t, x, .)].$$

We use this as follows. Let $G_s \equiv \sup\{t \leqq s : t \in J\}$. If $G_0<0<T_B \circ \gamma_0$ and $\tau_B<0<\lambda_B$, then $\alpha \leqq \tau_B<0$ and hence $G_0>\alpha$. Moreover $T_B \circ \gamma_0<\beta$ because $0<\lambda_B$. Hence $]G_0, T_B \circ \gamma_0[$ is the contiguous interval containing zero and $G_0 \in G$. Therefore $G_0<0<T_B \circ \gamma_0$ and $\tau_B<0<\lambda_B$ if and only if $r=G_0 \in G$, $r<0$ and $0<r+T_B \circ \gamma_r<\infty$, and in this situation $r+T_B \circ \gamma_r=T_B \circ \gamma_0$. Define

$$F(t, .)=e^{-q(t+T_B)}\, 1_{]-\infty,\, 0[}(t)\, 1_{\{-t<T_B<\infty\}}.$$

Then by (8.5) and the above discussion

(8.6) $$\begin{aligned} H(q) &\equiv Q_m[e^{-q T_B \circ \gamma_0}; G_0<0<T_B \circ \gamma_0, \tau_B<0<\lambda_B] \\ &= Q_m \sum_{r \in G} F(r, \gamma_r)={}^*P^\nu \int_{-\infty}^{0} e^{-q(t+T_B)}\, 1_{\{-t<T_B<\infty\}}\, dt \\ &= {}^*P^\nu\left[\int_0^{T_B} e^{-q(T_B-t)}\, dt;\ T_B<\infty\right]=q^{-1}\, {}^*P^\nu[1-e^{-q T_B}; T_B<\infty]. \end{aligned}$$

We also claim that

(8.7) $$Q_m[G_0<0=T_B \circ \gamma_0, \tau_B<0<\lambda_B]=0;$$

(8.8) $$Q_m[G_0=0<T_B \circ \gamma_0, \tau_B<0<\lambda_B]=0.$$

Before proving (8.7) and (8.8) let us use them to establish (8.1). Combining (7.4) and (8.4) together with (8.6), (8.7), and (8.8) we see that

$$(8.9) \qquad \Gamma^q(B) = \Gamma(B) + {}^*P^\vee[1 - e^{-qT_B};\ T_B < \infty] + q\,M,$$

where $M = Q_m(G_0 = 0 = T_B \circ \gamma_0,\ \tau_B < 0 < \lambda_B)$. Let $h(q)$ denote the second term on the right side of (8.9). Clearly h is increasing and by the dominated convergence theorem h is finite and continuous on $[0, r]$ with $\lim_{q \downarrow 0} h(q) = 0$ provided $r > 0$ and $h(r) < \infty$. We next claim that if $r > 0$ and $h(r) < \infty$ then $h(q) < \infty$ for all q. We need only check this for $q > r$. But $h(r) < \infty$ implies that ${}^*P^\vee \lvert t < T_B < \infty] < \infty$ for each $t > 0$ and since $(1 - e^{-qt})(1 - e^{-rt})^{-1} \to qr^{-1}$ as $t \to 0$, it follows that $h(q) < \infty$. This establishes the assertions in the second sentence of (8.1). If $R_B m(P_B^q 1) < \infty$ for some $q > 0$, then from (8.4), (8.6), (8.7), and (8.8), $h(q) < \infty$ and $M < \infty$. It now follows from (8.9) and the properties of h that $\Gamma^q(B)$ approaches $\Gamma(B)$ as q tends to zero.

Thus to complete the proof of (8.1) it suffices to establish (8.7) and (8.8). Define

$$\psi(s) \equiv Q_m[G_s < s,\ 0 = T_B \circ \gamma_s,\ \tau_B < s < \lambda_B].$$

Since $G_0 \circ \theta_s = G_s - s$ the invariance of Q_m implies that $\psi(s) = \psi(0)$ for each $s \in \mathbb{R}$. Now

$$\int_{-\infty}^{\infty} \psi(s)\,ds = Q_m \int_{\tau_B}^{\lambda_B} 1_{\{G_s < s,\ 0 = T_B \cdot \gamma_s\}}\,ds,$$

and if $G_s < s$, $T_B \circ \gamma_s = 0$, and $\tau_B < s < \lambda_B$, then s is the right endpoint of one of the contiguous intervals. But there are only a countable number of such intervals for each w, and so this last integral is zero. Consequently $\psi(0) = 0$ which proves (8.7). The argument for (8.8) is similar except that $G_s = s$, $T_B \circ \gamma_s > 0$, and $\tau_B < s < \lambda_B$ imply that s is the left end point of a contiguous interval. □

Results similar to (8.1) but under much stronger hypotheses go back to Hunt. *See* page 191 of section 19 in [17], (VI-4.16) in [2], and (2.14) in [11]. The proof of (8.1) is in the same spirit as the proof of (2.14) in [11]. In fact, if $m = \mu U$ with μ finite, one may use the argument in [11] and avoid the use of exit systems. The use of exit systems enables one to extend the proof in [11] to general excessive m.

Theorem 8.1 states that either $q \to \Gamma^q(B)$ is identically infinite or everywhere finite on $]0, \infty[$. In the latter case the conclusion of (8.1) may be strengthened.

(8.10) PROPOSITION. – *Let* $g(q) \equiv \Gamma^q(B)$ *and suppose that* $g(q) < \infty$ *for some* $q > 0$. *Then* g *has a finite continuous derivative on* $]0, \infty[$ *given by* $R_B^q m(P_B^q u)$.

Proof. – Let $\psi(r, q) \equiv R_B^q m(P_B^r u) = R_B^r m(P_B^q u)$. Then ψ is decreasing in each variable. By hypothesis and (8.1), g is finite on $[0, \infty[$, and so by (7.4) if $0 < s < r < q$, then $\psi(r, q) \leq \psi(s/2, s) \leq (s/2)^{-1} g(s/2) < \infty$. Since $P_B^q u$ is a decreasing function of q it now follows by the dominated convergence theorem that ψ is continuous in each variable separately and that $\psi(q, q)$ is continuous and bounded on $]s, \infty[$ for each $s > 0$. But $g(q) - g(r) = (q - r)\psi(r, q)$ by (7.4), and this establishes (8.10) in view of the above properties of ψ. □

9. ADDITIONAL REMARKS

There is an alternate approach to some of the results in the preceding sections that makes more use of the Kuznetsov measures. Since this approach leads to a useful formula for R_B^q, we shall briefly sketch the method in this section. As before $m \in \text{Exc}$ and $u \in E$ are fixed with $m(u = \infty) = 0$. Let ${}^q Q_m^u$ denote the Kuznetsov measure corresponding to m, u, and the semigroup $(P_t^{(q)})$. As in the case $q = 0$, this is the same as the Kuznetsov measure corresponding to um, 1, and the semigroup $(e^{-qt} P_t^{(u)})$. Let b_r and k_s be the birthing and killing operators defined in (5.9). Then by checking finite dimensional distributions one sees that for $F \in p\mathcal{G}^0$ with $F([\Delta]) = 0$

$$(9.1) \qquad {}^q Q_m^u(F) = Q_m^u \iint q^2 e^{-q(s-r)} 1_{\{r < s\}} F \circ k_s \circ b_r \, dr \, ds,$$

and this extends to $F \in p\mathcal{G}^*$. *See* also [12].

Fos simplicity let $u = 1$. The general case can be reduced to this special case as in the preceding sections. If $f \in p\mathcal{E}$ it follows from (9.1) that

$$(9.2) \quad R_B^q m(f) = {}^q Q_m[f \circ Y_0;\ \tau_B < 0]$$

$$= Q_m\left[f \circ Y_0 \iint_{r < 0 < s} q^2 e^{-q(s-r)} 1_{\{\tau_B < 0\}} \circ k_s \circ b_r \, dr \, ds \right].$$

But for $r<s$, $\tau_B \circ k_s \circ b_r$ equals τ_B if $r<\tau_B<s$, equals $r+T_B \circ \gamma_r$ if $\tau_B \leqq r$ and $r+T_B \circ \gamma_r<s$, and equals infinity in all other cases. Breaking the integral in (9.2) into integrals over $\{r<\tau_B<s\}$ and $\{\tau_B \leqq r,\ r+T_B \circ \gamma_r<s\}$, one obtains after some manipulations

$$(9.3)\quad R_B^q m(f)=Q_m[f\circ Y_0\, e^{qg_0};\ \tau_B<0]$$
$$=Q_m[f\circ Y_0\, e^{q\tau_B};\ \tau_B<0]+q\, R_B\, m(P_B^q U_q f),$$

where $g_0 \equiv \sup\{t<0 : Y_t \in B\}$. Note that it is immediate from the first equality in (9.3) that $R_B^q m \uparrow R_B m$ as $q \downarrow 0$ since $-\infty<g_0 \leqq 0$ if $\tau_B<0$. If one computes $\hat{C}_m^q(B)={}^qQ_m(0<\tau_B<1)$ in a similar manner and uses the second equality in (9.3) one obtains another proof of the identity $\hat{C}^q(B)=\hat{C}(B)+q\, R_B\, m(P_B^q 1)$ in (7.9). A similar calculation beginning with $C^q(B)={}^qQ_m(0<\lambda_B<1)$ and using the facts that for $r<s$, $\lambda_B \circ k_s \circ b_r$ equals λ_B if $r<\lambda_B<s$, equals $g_s=\sup\{t<s : Y_t \in B\}$ if $\alpha<s \leqq \lambda_B$ and $g_s>r$, and equals $-\infty$ in all other cases leads to another proof of the identity $C^q(B)=C(B)+q\, R_B^q m(P_B 1)$ in (7.9).

REFERENCES

[1] R. M. BLUMENTHAL, A Decomposition of Excessive Measures, *Sem. Stoch. Proc.*, 1985, pp. 1-8, Birkhäuser, Boston, 1986.

[2] R. M. BLUMENTHAL and R. K. GETOOR, *Markov Processes and Potential Theory*, Academic Press, New York, 1968.

[3] C. DELLACHERIE and P. A. MEYER, *Probabilities and Potential*, Vol. **I**. North Holland, Amsterdam-New York-Oxford, 1978.

[4] C. DELLACHERIE and P. A. MEYER, *Probabilités et Potentiel*, Vol. **IV**, Hermann Paris (to appear).

[5] E. B. DYNKIN, Regular Markov Processes, *Russian Math. Surv.*, Vol. **28**, 1973, pp. 33-64. Reprinted in *London Math. Soc. Lecture Notes Series*, No. **54**, Cambridge Univ. Press, Cambridge, 1982.

[6] E. B. DYNKIN and R. K. GETOOR, Additive Functionals and Entrance Laws, *J. Funct. Anal.*, Vol. **62**, 1985, pp. 221-265.

[7] P. J. FITZSIMMONS, *Homogeneous Random Measures and a Weak Order for the Excessive Measures of a Markov Process. Trans. Amer. Math. Soc.*, (to appear).

[8] P. J. FITZSIMMONS and B. MAISONNEUVE, Excessive Measures and Markov Processes with Random Birth and Death, *Probab. Th. Rel. Fields*, Vol. **72**, 1986, pp. 319-336.

[9] R. K. GETOOR, Markov Processes: Ray Processes and Right Processes, *Lecture Notes in Math.*, No. **440**, Springer, Berlin-Heidelberg-New York, 1975.

[10] R. K. GETOOR, Transience and Recurrence of Markov Processes, *Sém. de Prob. XIV*, *Lecture Notes in Math.*, No. **784**, 1980, pp. 397-409, Springer. Berlin-Heidelberg-New York.

[11] R. K. GETOOR, Capacity Theory and Weak Duality, *Sem. Stoch. Proc.* 1983, pp. 97-130, Birkhäuser, Boston, 1984.
[12] R. K. GETOOR, Killing a Markov Process Under a Stationary Measure Involves Creation, *Ann. Prob.* (to appear).
[13] R. K. GETOOR and J. GLOVER, Riesz Decompositions in Markov Process Theory, *Trans. Amer. Math. Soc.*, Vol. **285**, 1984, pp. 107-132.
[14] R. K. GETOOR and J. GLOVER, Constructing Markov Processes with Random Times of Birth and Death, *Sem. Stoch. Proc.* 1986, pp. 35-69, Birkhäuser, Boston, 1987.
[15] R. K. GETOOR and M. J. SHARPE, Naturality, Standardness, and Weak Duality for Markov Processes, *Z. Wahrscheinlichkeitstheorie verw. Geb.*, Vol. **67**, 1984, pp. 1-62.
[16] R. K. GETOOR and J. STEFFENS, Capacity Theory without Duality, *Probab. Th. Rel. Fields*, Vol. **73**, 1986, pp. 415-445.
[17] G. A. HUNT, Markov Processes and Potentials, I. II and III, *Ill. Journ. Math.*, Vol. **1**, 1957, pp. 44-93. 316-369 and Vol. **2**, 1958, pp. 151-213.
[18] S. E. KUZNETSOV, Construction of Markov Processes with Random Times of Birth and Death, *Theory Prob. and Appl.*, Vol. **18**, 1974, pp. 571-575.
[19] S. E. KUZNETSOV, Nonhomogeneous Markov Processes, *J. Soviet Math.*, Vol, **25**, 1984, pp. 1380-1498.
[20] P. A. MEYER, Note sur l'interprétation des mesures d'équilibre, *Sém. de Prob. VII*, *Lecture Notes in Math.* No. **321**, 1973, pp. 210-216, Springer. Berlin-Heidelberg-New York.
[21] J. B. MITRO, Dual Markov Processes: Construction of a Useful Auxiliary Process, *Z. Wahrscheinlichkeitstheorie verw. Geb.*, Vol. **47**, 1979, pp. 139-156.
[22] M. J. SHARPE, *The General Theory of Markov Processes*, Forthcoming book.
[23] J. STEFFENS, Some Remarks on Capacities, *Sem. Stoch. Proc.* 1986, pp. 195-213, Birkhäuser, Boston, 1987.
[24] J. B. WALSH, The Cofine Topology Revisited, *Proc. Sympos. Pure Math.*, J. L. DOOB Ed., Vol. **31**, 1976, pp. 131-152, Amer. Math. Soc., Providence.
[25] J. B. WALSH, Notes on the Duality of Markov Processes, *Paris lecture notes*, unpublished.

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