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Construction of semimartingales from pieces by the method of excursion point processes

by

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RÉSUMÉ. — Nous étudions la construction d'une semimartingale à partir de ses pièces qui, regardée à la direction opposée, donne une décomposition de la semimartingale dans ses pièces comme excursions. La collection de ces pièces est définie comme un processus ponctuel aux valeurs dans un espace fonctionnel et la construction est basée sur le calcul stochastique des processus ponctuels.

ABSTRACT. — We study the construction of a semimartingale from its pieces which, regarded in the opposite direction, gives a decomposition of the semimartingale into pieces like excursions. The collection of these pieces is formulated to be a point process with values in a function space and the construction is based on the stochastic calculus of point processes.

Key words : Brownian motion, martingale, point process, stochastic integral.

1. INTRODUCTION

The notion of Poisson point processes has been introduced by P. Lévy in his study of additive processes (processes with independent increments)

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and it plays an important role in the Lévy-Itô theorem describing the structure of their sample paths ([8], [9], [4], [5]). This is a case of Poisson point processes taking values in Euclidean spaces but, of course, the notion of Poisson point processes is meaningful on arbitrary state spaces. Actually, Itô [6] noticed that the collection of excursions from a point to it of a strong Markov process forms a Poisson point process with values in a function space (path space). We know that excursions of Brownian motions have been studied first by Lévy [9], then by Itô-McKean [7] and the study has been expanded and generalized in several directions by many people. Now the collections of these excursions can be best described by using the notion of Poisson and more general point processes on path spaces, *cf.* [3], [10], [14].

In studying fine structure of stochastic processes, it is often useful to decompose a sample path into pieces like excursions and the theory of point processes is a useful tool for this purpose. An approach is given in [13] where we try to formulate the collection of these pieces as a point process on a path space and then construct a whole stochastic process by putting these pieces together. Purpose of the present paper is to give a detailed proof of results in [13]. In particular, the results of this paper supplement the materials of Chapter III, 4.3 of [3]; construction of a Brownian motion from its pieces will be discussed in more general cases than the excursions from a point to it and also, an obscure statement concerning the continuity of the process constructed will be made precise.

2. CONSTRUCTION OF A BROWNIAN MOTION FROM ITS PIECES

For simplicity, we discuss the case of one-dimensional Brownian motion but the generalization to the multi-dimensional case is straightforward. First we formulate the notion of "pieces" of a Brownian motion and then, give a general procedure to construct a whole Brownian motion from these pieces.

Let $(\mathrm{W}, \mathscr{B}_{\mathrm{W}}, n)$ be a σ-finite but infinite measure space and $(\mathscr{B}_t)$ be a right-continuous increasing family of sub σ-fields of $\mathscr{B}_{\mathrm{W}}$. Suppose we are given a system of real valued functions $\mathrm{F}(t, w)$ defined on $[0, \infty) \times \mathrm{W}$ and

$\sigma(w)$ defined on W such that:

for each $t \geqq 0$,

$$\{w;\ \sigma(w)>t\} \in \mathscr{B}_t \text{ and } w \to F(t, w) \text{ is } \mathscr{B}_t\text{-measurable}, \tag{2.1}$$

for almost all $w(n)$, $0<\sigma(w)<\infty$, $F(0, w)=0$,

$$t \to F(t, w) \text{ is continuous and}$$
$$F(t, w)=F(t \wedge \sigma(w), w) \text{ for all } t \geqq 0, \tag{2.2}$$

$$\int_W [\sigma(w) \wedge 1]\, n(dw) < \infty. \tag{2.3}$$

We suppose furthermore that

$$F(t, w) \in L^2(W, n) \quad \text{for every } t>0, \tag{2.4}$$

and, for each $0<t_1<t_2$ and $H \in L^2(W, n) \cap L^\infty(W, n)$ which is $\mathscr{B}_{t_1}$-measurable,

$$\int_W [F(t_2, w)-F(t_1, w)]\, H(w)\, n(dw)=0 \tag{2.5}$$

$$\int_W [F(t_2, w)-F(t_1, w)]^2\, H(w)\, n(dw)$$
$$=\int^W [t_2 \wedge \sigma(w)-t_1 \wedge \sigma(w)]\, H(w)\, n(dw). \tag{2.6}$$

Remark 2.1. — (i) (2.3) implies that

$$\int_W [\sigma(w) \wedge t]\, n(dw) < \infty \quad \text{for every } t>0 \tag{2.3$'$}$$

because, as is easily seen,

$$\int_W [\sigma(w) \wedge t]\, n(dw) \leqq \int_{\{\sigma(w) \leqq 1\}} \sigma(w)\, n(dw) + t n\{\sigma(w)>1\} < \infty.$$

(ii) (2.6) implies that

$$\int_W F(t, w)^2\, n(dw) = \int_W [t \wedge \sigma(w)]\, n(dw), \qquad t>0. \tag{2.6$'$}$$

LEMMA 2.1:
(1) $n\{w;\ \sigma(\mathrm{w})>1\}<\infty$.
(2) $\displaystyle\int_{\mathrm{W}} |\mathrm{F}(1, w)|\,\mathrm{I}_{\{\sigma(w)>1\}}\,n(dw)<\infty$.

(3) $\displaystyle\int_{\mathrm{W}} \mathrm{F}(\sigma(w), w)^2\,\mathrm{I}_{\{\sigma(w)\leqq 1\}}\,n(dw)<\infty$.

Proof. — (1) is an immediate consequence of (2.3). (2) is also immediate from $\mathrm{F}(1, w)\in\mathrm{L}^2(\mathrm{W}, n)$ and $\mathrm{I}_{\{\sigma(w)>1\}}\in\mathrm{L}^2(\mathrm{W}, n)$. (3) follows from

$$\int_{\mathrm{W}} \mathrm{F}(\sigma, w)^2\,\mathrm{I}_{\{\sigma\leqq 1\}}\,n(dw)$$

$$=\int_{\mathrm{W}} \mathrm{F}(\sigma\wedge 1, w)^2\,\mathrm{I}_{\{\sigma\leqq 1\}}\,n(dw)=\int_{\mathrm{W}} \mathrm{F}(1, w)^2\,\mathrm{I}_{\{\sigma\leqq 1\}}\,n(dw)<\infty. \qquad \square$$

For some examples of such $(\mathrm{W}, \mathscr{B}_{\mathrm{W}}, n)$ and F, σ, *see* [13], Examples 1 and 2. We can always construct, on a suitable probability space $(\Omega, \mathscr{F}, \mathrm{P})$ with a filtration $(\mathscr{F}_t)$, an $(\mathscr{F}_t)$-stationary Poisson point process $p(t)$ taking values in W with the characteristic measure n, *cf.* [3]. Now, $\{[\mathrm{F}(s, p(t))]_{0\leqq s\leqq \sigma[p(t)]};\ t\in\mathrm{D}_p\}$ is what we would formulate to be a collection of "pieces" of a Brownian motion. [Here, we follow the notations of [3]: D_p is, for each $\omega\in\Omega$, a countable subset of $(0, \infty)$ and $p(t)\in\mathrm{W}$ is defined for each $t\in\mathrm{D}_p$.] A Brownian motion can be constructed from these pieces by the following procedure: First of all, we introduce the following:

CONVENTION. — Δ *is an extra point attached to* W. *We set* $p(t)=\Delta$ *if* $t\notin\mathrm{D}_p$ *and* $\mathrm{F}(t, \Delta)=0$ *for all* $t\geqq 0$.
Also, we set $\sigma(\Delta)=0$.
As in [3], a random measure $\mathrm{N}_p(ds, dw)$ is defined by

$$\left.\begin{array}{c}\mathrm{N}_p((0, t]\times\mathrm{E})=\#\{s\in\mathrm{D}_p;\ 0<s\leqq t,\ p(s)\in\mathrm{E}\},\\ t>0, \qquad \mathrm{E}\in\mathscr{B}_{\mathrm{W}}\end{array}\right\} \tag{2.7}$$

and its compensator $\hat{\mathrm{N}}_p(ds, dw)$ is given by

$$\hat{\mathrm{N}}_p((0, t]\times\mathrm{E})=t\,.\,n(\mathrm{E}). \tag{2.8}$$

The compensated random measure (martingale random measure) $\tilde{N}_p(ds, dw)$ is defined by

$$\tilde{N}_p(ds, dw) = N_p(ds, dw) - ds\, n(dw) \qquad (2.9)$$

and stochastic integrals with respect to these random measures N_p and $\tilde{N}_p$ are defined as in [3], *see* section 3 below for classes $\mathbf{F}_p$, $\mathbf{F}_p^1$, $\mathbf{F}_p^{1, \mathrm{loc}}$, $\mathbf{F}_p^2$ and $\mathbf{F}_p^{2, \mathrm{loc}}$ of predictable integrands.

We define two processes $\xi(t)$ and $A(t)$ by

$$\xi(t) = \int_0^{t+} \int_W F(\sigma(w), w) . I_{\{\sigma(w) > 1\}} N_p(ds, dw) + \int_0^{t+} \int_W F(\sigma(w), w) . I_{\{\sigma(w) \leqq 1\}} \tilde{N}_p(ds, dw) - t \int_W F(1, w) . I_{\{\sigma(w) > 1\}} n(dw) \qquad (2.10)$$

and

$$A(t) = \int_0^{t+} \int_W \sigma(w) N_p(ds, dw). \qquad (2.11)$$

$\xi(t)$ and $A(t)$, being well-defined by Lemma 2.1, are $(\mathscr{F}_t)$-adapted time homogeneous Lévy processes [a right-continuous process with left-hand limits having stationary independent increments with respect to $(\mathscr{F}_t)$] and (2.10) and (2.11) give their Lévy-Itô representation. Furthermore, $A(t)$ is strictly increasing since $n(w; \sigma(\mathrm{w}) > 0) = \mathrm{n}(\mathrm{W}) = \infty$. The corresponding Lévy-Kchinchine characteristics are given by

$$E(\exp\{\sqrt{-1}\,\eta . \xi(t)\}) = \exp\{t . \psi_1(\eta)\}, \qquad \eta \in \mathbf{R},$$

where

$$\psi_1(\eta) = -\sqrt{-1} . \eta \int_W F(1, w) I_{\{\sigma > 1\}} n(dw) + \int_W (\exp\{\sqrt{-1}\,\eta . F(\sigma, w)\} - 1 - \sqrt{-1} . I_{\{\sigma \leqq 1\}} \eta . F(\sigma, w)) n(dw)$$

and

$$E(\exp\{-\lambda A(t)\}) = \exp\{-t\psi_2(\lambda)\}, \qquad \lambda > 0,$$

where

$$\psi_2(\lambda) = \int_W (1 - \exp\{-\lambda\sigma(w)\})\, n(dw).$$

Let $\Omega_0 = \{\omega \in \Omega;\ A(0+, \omega) = 0,\ t \to A(t, \omega)$ is strictly increasing, right-continuous with left-hand limits and $\lim_{t\uparrow\infty} A(t, \omega) = \infty\}$. Then $P(\Omega_0) = 1$. In the following discussions, we always assume $\omega \in \Omega_0$ to simplify arguments. Then, for every $t \in [0, \infty)$, there exists unique $s \in [0, \infty)$ such that

$$A(s-) \leqq t \leqq A(s). \tag{2.12}$$

This s is denoted by $s = \varphi(t)$. Clearly, $t \to \varphi(t)$ is continuous and $\lim_{t\uparrow\infty} \varphi(t) = \infty$. We set, noting the above convention,

$$\left.\begin{array}{c} B(t) = \xi(\varphi(t)-) + F(t - A(\varphi(t)-),\ p\{\varphi(t)\}), \\ 0 \leqq t < \infty. \end{array}\right\} \tag{2.13}$$

THEOREM 2.1. — *With probability one, $t \to B(t)$ is continuous and $B(t)$ is a one-dimensional Brownian motion such that* $B(0) = 0$.

Proof. — 1° First we prove the a. s. continuity of $t \to B(t)$. For this, we need the following

LEMMA 2.2. — *Let*

$$M(w) = \max\{|F(t, w)|;\ 0 \leqq t \leqq \sigma(w)\}, \qquad w \in W.$$

Then, for every $\varepsilon > 0$,

$$n(w;\ M(w) > \varepsilon) < \infty. \tag{2.15}$$

Proof. — Since

$$\lim_{\eta\downarrow 0} n(w;\ \sigma(w) > \eta) = n(w;\ \sigma(w) > 0) = \infty,$$

$\eta_0 > 0$ exists such that $n(w;\ \sigma(w) > \eta) > 0$ for all $0 < \eta < \eta_0$. For such η, set

$$P^\eta(B) = \frac{n(B \cap \{\sigma > \eta\})}{n(\sigma > \eta)}, \qquad B \in \mathscr{B}_W.$$

Then P^η is a probability measure on $(W, \mathscr{B}_W)$ and it follows from (2.5) that $X^\eta(t, w) = F(t+\eta, w)$ is a martingale with respect to $(\mathscr{B}_t^\eta)$ where the filtration $(\mathscr{B}_t^\eta)$ is defined by $\mathscr{B}_t^\eta = \mathscr{B}_{t+\eta}$. Hence, by Doob's inequality and (2.6)′, we have for every $T > \eta$,

$$\int_{\{\sigma>\eta\}} \max_{\eta \leqq t \leqq \sigma\wedge T} |F(t, w)|^2 n(dw)$$

$$\leqq 4 \int_{\{\sigma>\eta\}} |F(\sigma\wedge T, w)|^2 n(dw)$$

$$\leqq 4 \int_W F(\sigma\wedge T, w)^2 n(dw)$$

$$= 4 \int_W [\sigma(w)\wedge T] n(dw) := K(T) < \infty.$$

Letting $\eta \downarrow 0$,

$$\int_W \max_{0<t\leqq\sigma\wedge T} |F(t, w)|^2 n(dw) \leqq K(T)$$

and hence,

$$n\{M(w) > \varepsilon\}$$

$$\leqq n\{M(w) > \varepsilon,\ \sigma(w) \leqq T\} + n\{\sigma(w) > T\}$$

$$\leqq \varepsilon^{-2} K(T) + n\{\sigma(w) > T\} < \infty. \quad \square$$

Now it is easy to see the a.s. continuity of $t \to B(t)$. If $t \in (A(s-), A(s))$ for some $s \in D_p$, then

$$B(t) = \xi(s-) + F(t - A(s-), p(s))$$

is clearly continuous at t. Consider the case that

$$t \in (0, \infty) \setminus \bigcup_{s\in D_p} (A(s-), A(s)).$$

By Lemma 2.2, $N_p((0, T] \times \{w;\ M(w) > \varepsilon\}) < \infty$ a.s. for every $T > 0$ and $\varepsilon > 0$ and hence, if we set

$$\Omega^1 = \{\omega \in \Omega^0;\ N_p((0, T] \times \{w\colon M(w) > \varepsilon\}) < \infty \text{ for all } T > 0 \text{ and } \varepsilon > 0\}$$

then $P(\Omega^1)=1$. Let

$$t \in (0,\infty) \setminus \bigcup_{s \in D_p} (A(s-), A(s)).$$

Then there exists $s_n \in D_p$ such that, either $t < A(s_n-) < A(s_n)$ and $A(s_n) \downarrow t$ as $n \to \infty$ or $A(s_n-) < A(s_n) < t$ and $A(s_n) \uparrow t$ as $n \to \infty$. In both cases $M\{p(s_n)\} \to 0$ as $n \to \infty$ if $\omega \in \Omega^1$ and the continuity of B at t is easily concluded.

2° Here, we prepare two lemmas; Lemmas 2.3 and 2.4.

LEMMA 2.3. — *For every* $t > \varepsilon > 0$,

$$E[\varphi(t) - \varphi(t-\varepsilon)]\, n(w;\ \sigma(w) > \varepsilon) \leqq 2. \tag{2.16}$$

Proof. — We have

$$\begin{aligned} \varepsilon . \#\{s \in D_p;\ \varphi(t-\varepsilon) < s \leqq \varphi(t) :\ \sigma\{p(s)\} > \varepsilon\} \\ \leqq \sum_{s \in D_p,\ \varphi(t-\varepsilon) < s \leqq \varphi(t)} \varepsilon \wedge \sigma\{p(s)\} \\ \leqq A(\varphi(t)-) - A(\varphi(t-\varepsilon)) + \varepsilon \leqq t - (t-\varepsilon) + \varepsilon = 2\varepsilon. \end{aligned}$$

Noting that $\varphi(t)$ and $\varphi(t-\varepsilon)$ are $(\mathscr{F}_t)$-stopping times, we have by taking expectations of the above,

$$\begin{aligned} \varepsilon E[N((\varphi(t-\varepsilon), \varphi(t)) \times \{w;\ \sigma(w) > \varepsilon\})] \\ = \varepsilon E[\varphi(t) - \varphi(t-\varepsilon)]\, n\{w;\ \sigma(w) > \varepsilon\} \leqq 2\varepsilon \end{aligned}$$

Hence (2.16) is obtained. □

Let $T > 0$ be fixed and consider $(\mathscr{F}_s)$-predictable process $f_T(s, w, \omega)$ defined by

$$f_T(s, w, \omega) = \begin{cases} F(T - A(s-), w) & \text{if} \quad s \leqq \varphi(T) \\ 0 & \text{if} \quad s > \varphi(T). \end{cases}$$

We have $f_T \in \mathbf{F}_p^2$ since, for every $t > 0$,

$$\begin{aligned} E\left[\int_0^t \int_W f_T(s, w, .)^2\, n(dw)\, ds\right] \\ = E\left[\int_0^{t \wedge \varphi(T)} ds \int_W [(T - A(s)) \wedge \sigma(w)]\, n(dw)\right] \\ = E\Big[\sum_{\substack{s \in D_p \\ s \leqq t \wedge \varphi(T)}} (T - A(s-)) \wedge \sigma(w)\Big] \end{aligned}$$

$$\leqq E\Big[\sum_{\substack{s \in D_p \\ s \leqq \varphi(T)}} (T - A(s-)) \wedge \sigma(w)\Big] = T.$$

Hence, a right-continuous process $M_T(t)$ with left-hand limits is defined by the stochastic integral

$$M_T(t) = \int_0^{t+} \int_W f_T(s, w, .)\, \tilde{N}_p(ds, dw)$$

and $M_T(t)$ is an $(\mathscr{F}_t)$-square integrable martingale.

LEMMA 2.4. — *Let* $T > 0$ *be arbitrary but fixed. Then, with probability one,*

$$B(T) = M_T(\varphi(T)) = \int_0^{\varphi(T)+} \int_W F(T - A(s-), w)\, \tilde{N}_p(ds, dw). \quad (2.17)$$

Proof. — We have

$$\begin{aligned} B(T) = F(T - A(\varphi(T)-), p\{\varphi(T)\}) + \xi(\varphi(T)-) \\ = F(T - A(\varphi(T)-), p\{\varphi(T)\}) \\ + \int_0^{\varphi(T)-} \int_W F(\sigma, w)\, I_{\{\sigma > 1\}} N_p(ds, dw) \\ - \varphi(T) \int_W F(1, w)\, I_{\{\sigma > 1\}} n(dw) \\ + \lim_{\varepsilon \downarrow 0} \int_0^{\varphi(T)-} \int_W F(\sigma, w)\, I_{\{\varepsilon < \sigma \leqq 1\}} \tilde{N}_p(ds, dw) \\ := F(T - A(\varphi(T)-), p\{\varphi(T)\}) \\ + J_1(T) - J_2(T) + \lim_{\varepsilon \downarrow 0} J_3(T). \end{aligned}$$

On the other hand,

$$\begin{aligned} M(\varphi(T)) = \int_0^{\varphi(T)+} \int_W F(T - A(s-), w)\, \tilde{N}_p(ds, dw) \\ = \int_0^{\varphi(T)+} \int_W F(T - A(s-), w)\, I_{\{\sigma > 1\}} \tilde{N}_p(ds, dw) \\ + \lim_{\varepsilon \downarrow 0} \int_0^{\varphi(T)+} \int_W F(T - A(s-), w)\, I_{\{\varepsilon < \sigma \leqq 1\}} \tilde{N}_p(ds, dw) \end{aligned}$$

$$
\begin{aligned}
&= F(T-A(\varphi(T)-), p\{\varphi(T)\}) \\
&+ \int_0^{\varphi(T)-}\int_W F(T-A(s-), w)\, I_{\{\sigma>1\}} N_p(ds, dw) \\
&- \int_0^{\varphi(T)} ds \int_W F(T-A(s), w)\, I_{\{\sigma>1\}} n(dw) \\
&+ \lim_{\varepsilon\downarrow 0} \int_0^{\varphi(T)-}\int_W F(T-A(s-), w)\, I_{\{\varepsilon<\sigma\leqq 1\}} \tilde{N}_p(ds, dw) \\
&:= F(T-A(\varphi(T)-), p\{\varphi(T)\}) \\
&\qquad + I_1(T) - I_2(T) + \lim_{\varepsilon\downarrow 0} I_3(T)
\end{aligned}
$$

Hence, it is enough to prove that $I_1(T) = J_1(T)$ and

$$
-I_2(T) + \lim_{\varepsilon\downarrow 0} I_3(T) = -J_2(T) + \lim_{\varepsilon\downarrow 0} J_3(T).
$$

The first equality is obvious if we notice that

$$
F(T-A(s-), p(s)) = F(\sigma\{p(s)\}, p(s))
$$

provided $s<\varphi(T)$ and $s\in D_p$. By the same reason, we have

$$
\begin{aligned}
&\int_0^{\varphi(T)-}\int_W F(T-A(s-), w)\, I_{\{\varepsilon<\sigma\leqq 1\}} N_p(ds, dw) \\
&\qquad = \int_0^{\varphi(T)-}\int_W F(\sigma, w)\, I_{\{\varepsilon<\sigma\leqq 1\}} N_p(ds, dw).
\end{aligned}
$$

Hence, in order to show the second equality, it is sufficient to to prove that

$$
\begin{aligned}
&\lim_{\varepsilon\downarrow 0} \int_0^{\varphi(T)} ds \int_W \{(F(T-A(s), w) - F(1, w))\, I_{\{\sigma>1\}} \\
&\qquad + (F(T-A(s), w) - F(\sigma, w))\, I_{\{\varepsilon<\sigma\leqq 1\}}\} n(dw) \\
&\qquad = \lim_{\varepsilon\downarrow 0} \int_0^{\varphi(T)} ds \int_W \{F(T-A(s), w) - F(1, w)\}\, I_{\{\varepsilon<\sigma\}} n(dw) = 0.
\end{aligned}
$$

But, if $0<\varepsilon<1$,

$$
\int_0^{\varphi(T)} ds \int_W \{F(T-A(s), w) - F(1, w)\}\, I_{\{\sigma>\varepsilon\}} n(dw)
$$

$$= \int_0^T d\varphi(s) \int_W \{F(T-s, w) - F(1, w)\} I_{\{\sigma > \varepsilon\}} n(dw)$$

$$= \left(\int_0^{T-\varepsilon} d\varphi(t) + \int_{T-\varepsilon}^T d\varphi(t) \right)$$

$$\times \int_W \{F(T-s, w) - F(1, w)\} I_{\{\sigma > \varepsilon\}} n(dw) := K_1 + K_2$$

and $K_1 = 0$ because, by (2.5),

$$\int_W \{F(T-s, w) - F(1, w)\} I_{\{\sigma > \varepsilon\}} n(dw) = 0$$

provided $s \leqq T - \varepsilon$. As for K_2, we have by (2.5)

$$K_2 = \int_{T-\varepsilon}^T d\varphi(s) \int_W \{F(T-s, w) - F(\varepsilon, w)\} I_{\{\sigma > \varepsilon\}} n(dw)$$

and hence,

$$|K_2| \leqq \int_{T-\varepsilon}^T d\varphi(s)$$

$$\times \left[\int_W \{F(T-s, w) - F(\varepsilon, w)\}^2 n(dw) \right]^{1/2} n(\sigma > \varepsilon)^{1/2}$$

$$\leqq E[\varphi(T) - \varphi(T-\varepsilon)]$$

$$\times \left[\int_W \{\sigma(w) \wedge \varepsilon\} n(dw) \right]^{1/2} n(\sigma > \varepsilon)^{1/2}$$

by (2.6). Since $n(\sigma > \varepsilon) \to \infty$ as $\varepsilon \downarrow 0$, we may assume that $n(\sigma > \varepsilon)^{1/2} < n(\sigma > \varepsilon)$ for all small $\varepsilon > 0$. Hence by (2.16),

$$E(K_2) \leqq E[\varphi(T) - \varphi(T-\varepsilon)]$$

$$\times \left[\int_W \{\sigma(w) \wedge \varepsilon\} n(dw) \right]^{1/2} n(\sigma > \varepsilon)$$

$$\leqq 2 \left[\int_W \{\sigma(w) \wedge \varepsilon\} n(dw) \right]^{1/2} \to 0 \quad (2.18)$$

as $\varepsilon \downarrow 0$ by the dominated convergence theorem. □

3° We now construct a filtration $(\mathscr{H}_t)$ of $(\Omega, \mathscr{F}, \mathrm{P})$ such that $\mathrm{B}(t)$ is $(\mathscr{H}_t)$-adapted and satisfies, for $t_2 > t_1 > 0$ and $\mathrm{G}(\omega)$ which is bounded and $\mathscr{H}_{t_1}$-measurable,

$$\mathrm{E}(\{\mathrm{B}(t_2) - \mathrm{B}(t_1)\}\mathrm{G}(\omega)) = 0 \tag{2.19}$$

and

$$\mathrm{E}(\{\mathrm{B}(t_2)^2 - \mathrm{B}(t_1)^2 - (t_2 - t_1)\}\mathrm{G}(\omega)) = 0. \tag{2.20}$$

This will imply, together with the continuity of $t \to \mathrm{B}(t)$ established in 1°, that $\mathrm{B}(t)$ is a Brownian motion.

We define $\mathscr{H}_t$, for each $t > 0$, by

$$\mathscr{H}_t = \mathscr{F}_{\varphi(t)-} \vee \sigma_\Omega\{\mathrm{F}(s \wedge (t - \mathrm{A}(\varphi(t)-), p\{\varphi(t)\}); 0 \leqq s < \infty\}. \tag{2.21}$$

Note that $\varphi(t)$ is an $(\mathscr{F}_s)$-stopping time and, for any $(\mathscr{F}_s)$-stopping time T, $\mathscr{F}_{\mathrm{T}-}$ is defined, as usual, to be the σ-field generated by sets of the form $\mathrm{A} \cap \{s < \mathrm{T}\}$, $\mathrm{A} \in \mathscr{F}_s$, $s \geqq 0$, *Cf.* Dellacherie [2], Dellacherie-Meyer [3]. Also, we denote by $\sigma_\Omega(\eta_s; 0 \leqq s < \infty)$ the σ-field on Ω generated by the family of random variables η_s, $0 \leqq s < \infty$. It is easy to see that $\mathscr{H}_t \subset \mathscr{H}_{t'}$ if $t < t'$ and $\mathscr{H}_t \subset \mathscr{F}_{\varphi(t)}$. Also, it is obvious that $\mathrm{B}(t)$ is $\mathscr{H}_t$-measurable for each $t \geqq 0$.

Now we prove (2.19). For this, it is sufficient to show that,

$$\mathrm{E}[\mathrm{B}(t_2)\mathrm{H}_1(\omega)\mathrm{H}_2(\omega)] = \mathrm{E}[\mathrm{B}(t_1)\mathrm{H}_1(\omega)\mathrm{H}_2(\omega)] \tag{2.22}$$

for $0 < t_1 < t_2$. Here, $\mathrm{H}_1(\omega)$ is $\mathscr{F}_{\varphi(t_1)-}$-measurable and bounded and $\mathrm{H}_2(\omega)$ is of the form

$$\mathrm{H}_2(\omega) = \Phi(f^{t_1}(\varphi(t_1), p\{\varphi(t_1)\}, \omega))$$

where Φ is a bounded Borel function on $\mathbf{C}([0, \infty) \to \mathbf{R})$ and $f^{t_1}(s, w, \omega) \in \mathbf{C}([0, \infty) \to \mathbf{R})$ is defined by

$$f^{t_1}(s, w, \omega)[u] = \begin{cases} \mathrm{F}(u \wedge (t_1 - \mathrm{A}(s-)), w) & \text{if } t_1 \geqq \mathrm{A}(s-) \\ 0 & \text{if } t_1 < \mathrm{A}(s-) \end{cases}.$$

By Theorem 2.1 of [2] (*cf.* [3] also), there exists and $(\mathscr{F}_t)$-predictable bounded process $\mathrm{G}_t(\omega)$ such that $\mathrm{H}_1(\omega) = \mathrm{G}_{\varphi(t_1)}(\omega)$. Now, by Lemma 2.4, we have

$$\mathrm{I} := \mathrm{E}[\mathrm{B}(t_2)\mathrm{H}_1\mathrm{H}_2] = \mathrm{E}\left[\int_0^{\varphi(t_2)+}\int_{\mathrm{W}} \mathrm{F}(t_2 - \mathrm{A}(s-), w)\tilde{\mathrm{N}}_p(ds, dw)\mathrm{H}_1\mathrm{H}_2\right]$$

and, by the martingale property of the stochastic integral, this is equal to

$$\mathrm{E}\left[\int_0^{\varphi(t_1)+}\int_{\mathrm{W}} \mathrm{F}(t_2-\mathrm{A}(s-),\ w)\,\tilde{\mathrm{N}}_p(ds,\ dw)\ \mathrm{H}_1\,\mathrm{H}_2\right]$$

since $\mathrm{H}_1\,\mathrm{H}_2$ is $\mathscr{F}_{\varphi(t_1)}$-measurable. Hence,

$$\mathrm{I}=\lim_{\varepsilon\downarrow 0}\mathrm{E}\left[\int_0^{\varphi(t_1)+}\int_{\mathrm{W}} \mathrm{F}(t_2-\mathrm{A}(s-),\ w)\,\mathrm{I}_{\{\sigma(w)>\varepsilon\}}\tilde{\mathrm{N}}_p(ds,\ dw)\,\mathrm{H}_1\,\mathrm{H}_2\right]$$

$$=\lim_{\varepsilon\downarrow 0}\left\{\mathrm{E}\left[\int_0^{\varphi(t_1)+}\int_{\mathrm{W}} \mathrm{F}(t_2-\mathrm{A}(s-),\ w)\,\mathrm{I}_{\{\sigma(w)>\varepsilon\}}\mathrm{N}_p(ds,\ dw).\,\mathrm{H}_1\,\mathrm{H}_2\right]\right.$$

$$\left.-\mathrm{E}\left[\int_0^{\varphi(t_1)} ds\int_{\mathrm{W}} \mathrm{F}(t_2-\mathrm{A}(s),\ w)\,\mathrm{I}_{\{\sigma(w)>\varepsilon\}}\,n(dw).\,\mathrm{H}_1\,\mathrm{H}_2\right]\right\}$$

$$:=\lim_{\varepsilon\downarrow 0}\{\mathrm{I}_1^\varepsilon-\mathrm{I}_2^\varepsilon\}.$$

Now

$$\mathrm{I}_2^\varepsilon=\mathrm{E}\left[\int_0^{t_1} d\varphi(s)\int_{\mathrm{W}} \mathrm{F}(t_2-s,\ w)_{\{\sigma(w)>\varepsilon\}}\,n(dw).\,\mathrm{H}_1\,\mathrm{H}_2\right]$$

$$=\mathrm{E}\left[\int_{t_1-\varepsilon}^{t_1} d\varphi(s)\int_{\mathrm{W}}\right]+\mathrm{E}\left[\int_0^{t_1-\varepsilon} d\varphi(s)\int_{\mathrm{W}}\right]=\mathrm{I}_{2,1}^\varepsilon+\mathrm{I}_{2,2}^\varepsilon$$

and noting that

$$\mathrm{I}_{2,1}^\varepsilon=\mathrm{E}\left[\int_{t_1-\varepsilon}^{t_1} d\varphi(s)\right.$$

$$\left.\times\left\{\int_{\mathrm{W}} \mathrm{F}(\varepsilon,\ w)\,\mathrm{I}_{\{\sigma(w)>\varepsilon\}}\,n(dw)\right\}.\,\mathrm{H}_1\,\mathrm{H}_2\right],\qquad \varepsilon<t_2-t_1,$$

we have $\mathrm{I}_{2,1}^\varepsilon=o(1)$ as $\varepsilon\downarrow 0$ by the same proof as for (2.18). If $s\leqq t_1-\varepsilon$, then we have

$$\int_{\mathrm{W}} \mathrm{F}(t_2-s,\ w)\,\mathrm{I}_{\{\sigma(w)>\varepsilon\}}\,n(dw)$$

$$=\int_{\mathrm{W}} \mathrm{F}(t_1-s,\ w)\,\mathrm{I}_{\{\sigma(w)>\varepsilon\}}\,n(dw)$$

$$=\int_{\mathrm{W}} \mathrm{F}(\varepsilon,\ w)\,\mathrm{I}_{\{\sigma(w)>\varepsilon\}}\,n(dw)$$

by (2.5) and hence we can conclude that

$$\mathrm{I}_2^\varepsilon=\mathrm{E}\left[\int_0^{\varphi(t_1)} ds \int_{\mathrm{W}} \mathrm{F}(t_1-\mathrm{A}(s), w)\right.$$

$$\left.\times \mathrm{I}_{\{\sigma(w)>\varepsilon\}} n(dw) . \mathrm{H}_1 \mathrm{H}_2\right]+o(1) \quad \text{as } \varepsilon \downarrow 0. \quad (2.23)$$

Next,

$$\mathrm{I}_1^\varepsilon=\mathrm{E}\Big(\sum_{\substack{s \in \mathrm{D}_p \\ s \leqq \varphi(t_1)}} \mathrm{F}(t_2-\mathrm{A}(s-), p(s)) \mathrm{I}_{\{\sigma\{p(s)\}>\varepsilon\}} . \mathrm{H}_1 \mathrm{H}_2\Big)$$

$$=\mathrm{E}\Big(\sum_{\substack{s \in \mathrm{D}_p \\ s<\varphi(t_1)}} \mathrm{F}(t_2-\mathrm{A}(s-), p(s)) \mathrm{I}_{\{\sigma\{p(s)\}>\varepsilon\}} . \mathrm{H}_1 \mathrm{H}_2\Big)$$

$$+\mathrm{E}(\mathrm{F}(t_2-\mathrm{A}(\varphi(t_1)-), p\{\varphi(t_1)\})$$

$$\mathrm{I}_{\{\sigma\{p(\varphi(t_1))\}>\varepsilon\}} \mathrm{H}_1 \mathrm{H}_2):=\mathrm{I}_{1,1}^\varepsilon+\mathrm{I}_{1,2}^\varepsilon.$$

Then

$$\mathrm{I}_{1,1}^\varepsilon=\mathrm{E}\Big(\sum_{\substack{s \in \mathrm{D}_p \\ s<\varphi(t_1)}} \mathrm{F}(t_1-\mathrm{A}(s-), p(s)) \mathrm{I}_{\{\sigma\{p(s)\}>\varepsilon\}} \mathrm{H}_1 \mathrm{H}_2\Big) \quad (2.24)$$

because $s<\varphi(t_1)$ implies that

$$t_2-\mathrm{A}(s-)>t_1-\mathrm{A}(s-)>\mathrm{A}(s)-\mathrm{A}(s-)=\sigma\{p(s)\}$$

and hence

$$\mathrm{F}(t_2-\mathrm{A}(s-), p(s))=\mathrm{F}(t_1-\mathrm{A}(s-), p(s))=\mathrm{F}(\sigma\{p(s)\}, (s)).$$

To handle $\mathrm{I}_{1,2}^\varepsilon$, we use a trick originally due to Mainsonneuve [10]: Namely,

$$\mathrm{I}_{1,2}^\varepsilon=\mathrm{E}\Big[\sum_{\substack{s \in \mathrm{D}_p \\ s \leqq \varphi(t_1)}} \mathrm{F}(t_2-\mathrm{A}(s-), p(s))$$

$$\times \mathrm{I}_{\{\sigma\{p(s)\}>\varepsilon,\ t_1-\mathrm{A}(s-) \leqq \sigma\{p(s)\}\}} \times \mathrm{G}_s . \Phi(f^{t_1}(s, p(s), \omega))\Big]$$

because

$$s \leqq \varphi(t_1) \qquad \text{and} \qquad t_1-\mathrm{A}(s-) \leqq \sigma\{p(s)\}=\mathrm{A}(s)-\mathrm{A}(s-)$$

if and only if $s=\varphi(t_1)$. Hence,

$$I_{1,2}^{\varepsilon}=E\left[\int_0^{\varphi(t_1)} G_s\,ds\int_W F(t_2-A(s),w)\right.$$

$$\left.\times I_{\{\sigma(w)>\varepsilon,\ t_1-A(s)\leqq\sigma(w)\}}\Phi(f^{t_1}(s,w,\omega)).n(dw)\right]$$

$$=E\left[\int_0^{t_1} G_{\varphi(s)}\,d\varphi(s)\int_W F(t_2-s,w)\right.$$

$$\left.\times I_{\{\sigma(w)>\varepsilon,\ t_1-s\leqq\sigma(w)\}}\Phi(f^{t_1}(\varphi(s),w,\omega))\,n(dw)\right]$$

$$=E\left[\int_{t_1-\varepsilon}^{t_1} G_{\varphi(s)}\,d\varphi(s)\int_W\right]+E\left[\int_0^{t_1-\varepsilon} G_{\varphi(s)}\,d\varphi(s)\int_W\right],\qquad 0<\varepsilon<t_1.$$

The first term is $o(1)$ as $\varepsilon\downarrow 0$ by the same argument as for $I_{2,1}^{\varepsilon}$ or (2.18) if we notice that $\Phi(f^{t_1}(\varphi(s),w,\omega))$ is $\mathscr{B}_{t_1-s}$-measurable in w and hence

$$\int_W F(t_2-s,w)\,I_{\{\sigma(w)>\varepsilon,\ t_1-s\leqq\sigma(w)\}}$$

$$\times\Phi(f^{t_1}(\varphi(s),w,\omega))\,n(dw)=\int_W F(\varepsilon,w)\,I_{\{\sigma(w)>\varepsilon,\ t_1-s\leqq\sigma(w)]}$$

$$\Phi(f^{t_1}(\varphi(s),w,\omega))\,n(dw)$$

provided $\varepsilon<t_2-t_1$ and $t_1-\varepsilon\leqq s\leqq t_1$. Also, the second term is equal to

$$E\left[\int_0^{t_1-\varepsilon} G_{\varphi(s)}\,d\varphi(s)\int_W F(t_1-s,w)\right.$$

$$\left.\times I_{\{\sigma(w)>\varepsilon,\ t_1-s\leqq\sigma(w)\}}\Phi(f^{t_1}(\varphi(s),w,\omega))\,n(dw)\right]$$

and this can be estimated as

$$\mathrm{E}\left[\int_0^{t_1} \mathrm{G}_{\varphi(s)}\, d\varphi(s) \int_{\mathrm{W}} \mathrm{F}(t_1-s, w)\right.$$

$$\left.\times \mathrm{I}_{\{\sigma(w)>\varepsilon,\ t_1-s \leqq \sigma(w)\}} \Phi(f^{t_1}(\varphi(s), w, \omega))\, n(dw)\right] + o(1) \quad \text{as } \varepsilon \downarrow 0$$

by the same argument as for the first term. Thus

$$\mathrm{I}^{\varepsilon}_{1,2} = \mathrm{E}\left[\int_0^{\varphi(t_1)} \mathrm{G}_s\, ds \int_{\mathrm{W}} \mathrm{F}(t_1-\mathrm{A}(s), w)\right.$$

$$\left.\times \mathrm{I}_{\{\sigma(w)>\varepsilon,\ t_1-\mathrm{A}(s) \leqq \sigma(w)\}} \Phi(f^{t_1}(s, w, \omega))\, n(dw)\right] + o(1)$$

$$= \mathrm{E}[\mathrm{F}(t_1-\mathrm{A}(\varphi(t_1)-), p\{\varphi(t_1)\})$$

$$\times \mathrm{I}_{\{\sigma\{p(\varphi(t_1))\}>\varepsilon\}} \mathrm{H}_1 \mathrm{H}_2] + o(1) \quad \text{as } \varepsilon \downarrow 0 \qquad (2.25)$$

by reversing the above argument. (2.23), (2.24) and (2.25) together imply that

$$\mathrm{I} = \mathrm{E}\left[\int_0^{\varphi(t_1)+} \int_{\mathrm{W}} \mathrm{F}(t_1-\mathrm{A}(s-), w)\right.$$

$$\left.\times \mathrm{I}_{\{\sigma(w)>\varepsilon\}} \tilde{\mathrm{N}}_p(ds, dw)\, \mathrm{H}_1 \mathrm{H}_2\right] + o(1) = \mathrm{E}[\mathrm{B}(t_1) \mathrm{H}_1 \mathrm{H}_2] + o(1) \quad \text{as } \varepsilon \downarrow 0$$

and the proof of (2.22) is complete.

To prove (2.20), it is sufficient to prove that, for $0 < t_1 < t_2$,

$$\mathrm{E}[\{\mathrm{B}(t_2)^2 - t_2\} \mathrm{H}_1(\omega) \mathrm{H}_2(\omega)] = \mathrm{E}[\{\mathrm{B}(t_1)^2 - t_1\} \mathrm{H}_1(\omega) \mathrm{H}_2(\omega)]. \qquad (2.26)$$

By Itô's formula (*cf.* [3]) applied to the stochastic integral $M_t(s)$,

$$B(t)^2 = 2\int_0^{\varphi(t)+}\int_W M_t(s-)F(t-A(s-), w)\tilde{N}_p(ds, dw)$$
$$+\int_0^{\varphi(t)+}\int_W F(t-A(s-), w)^2 N_p(ds, sw)$$
$$= 2\int_0^{\varphi(t)+}\int_W M_t(s-)F(t-A(s-), w)\tilde{N}_p(ds, dw)$$
$$+\int_0^{\varphi(t)+}\int_W \{F(t-A(s-), w)^2-(t-A(s-))\wedge\sigma(w)\}\tilde{N}_p(ds, dw)$$
$$+\int_0^{\varphi(t)+}\int_W [(t-A(s-))\wedge\sigma(w)]N_p(ds, dw)$$
$$= 2\int_0^{\varphi(t)+}\int_W M_t(s-)F(t-A(s-), w)\tilde{N}_p(ds, dw)$$
$$+\int_0^{\varphi(t)+}\int_W \{F(t-A(s-), w)^2 - (t-A(s-))\wedge\sigma(w)\}\tilde{N}_p(ds, dw)+t.$$

Hence,

$$B(t)^2-t = 2\int_0^{\varphi(t)+}\int_W M_t(s-)F(t-A(s-), w)\tilde{N}_p(ds, dw)$$
$$+\int_0^{\varphi(t)+}\int_W \{F(t-A(s-), w)^2 - (t-A(s-))\wedge\sigma(w)\}\tilde{N}_p(ds, dw) := J_1(t)+J_2(t).$$

The proof of

$$E[J_2(t_2).H_1H_2] = E\Big[J_2(t_1).H_1H_2]$$

is exactly the same as above; in this case, however, the integrand is in $\mathbf{F}_p^1$ and hence the integral is absolutely convergent. So we need not introduce $I_{\{\sigma>\varepsilon\}}$ and take limit as $\varepsilon\downarrow 0$. The proof of

$$E[J_1(t_2).H_1H_2] = E[J_1(t_1)H_1H_2]$$

is also similar to the above: We need, for this, the estimate

$$E\left[\int_{\varphi(t_1-\varepsilon)}^{\varphi(t_1)} |M_{t_2}(s)|\, ds \int_W |F(\varepsilon, w)|\, I_{\{\sigma(w)>\varepsilon\}}\, n(dw)\right] = o(1)$$

as $\varepsilon \downarrow 0$ and this can be obtained, as above, from the estimate

$$E\left[\int_{\varphi(t_1-\varepsilon)}^{\varphi(t_1)} |M_{t_2}(s)|\, ds\right] n(\sigma>\varepsilon) \leqq 4\, t_2^{1/2}.$$

The last estimate can be proved as follows:

$$E\left[\int_{\varphi(t_1-\varepsilon)}^{\varphi(t_1)} |M_{t_2}(s)\, ds|\right] n(\sigma>\varepsilon)$$

$$= E\Big[\sum_{\substack{s \in D_p \\ \varphi(t_1-\varepsilon) \leqq s \leqq \varphi(t_1)}} |M_{t_2}(s-)|\,.\, I_{\{\sigma\{p(s)\}>\varepsilon\}}\Big]$$

$$\leqq 2\, E\Big(\max_{\varphi(t_1-\varepsilon) \leqq s \leqq \varphi(t_1)} |M_{t_2}(s)|\Big)$$

as in the proof of Lemma 2.3 and by Doob's inequality,

$$E\Big(\max_{0 \leqq s \leqq \varphi(t_1)} |M_{t_2}(s)|\Big) \leqq E\Big(\max_{0 \leqq s \leqq \varphi(t_1)} |M_{t_2}(s)|^2\Big)^{1/2}$$

$$\leqq 2\left[\int_0^{\varphi(t_1)} ds \int_W F(t_2 - A(s), w)^2\, n(dw)\right]^{1/2}$$

$$\leqq 2\, E\left[\int_0^{\varphi(t_2)} ds \int_W F(t_2 - A(s), w)^2\, n(dw)\right]^{1/2} = 2\, t_2^{1/2}.$$

This completes the proof of (2.20) and hence the proof of Theorem 2.1 is now finished.

3. GENERALIZATION

By generalizing the discussions of the previous section, we construct a continuous martingale from its pieces: This will provides us with basic tools to synthesize a semimartingale from its pieces like excursions or, looking this in an opposite direction, to decompose a semimartingale into pieces.

Let $(W, \mathscr{B}, n)$ be, again, a σ-finite but infinite, complete measure space. For a metric space E with metric d, we denote by $L_0(W \to E)$ the metric space consisting of all n-measurable functions $f: W \to E$ endowed with the metric ρ defined by

$$\rho(f, g) = \sum_{n=1}^{\infty} 2^{-n} \left[\int_{U_n} 1 \wedge d(f(w), g(w))\, n(dw)/n(U_n) \right]$$

where U_n is a disjoint family of sets in $\mathscr{B}_W$ such that $0 < n(U_n) < \infty$ and $\bigcup_n U_n = W$. Let $(\Omega, \mathscr{F}, P)$ be a probability space with a filtration $(\mathscr{F}_t)$. The predictable σ-field on $[0, \infty) \times \Omega$ is denoted by $\mathscr{P}$ (*cf.* [2], [3]). Suppose that we are given the following $\mathscr{P}$-measurable mappings:

$$\sigma^{t,\omega}(w):\quad (t, \omega) \in [0, \infty) \times \Omega \to [w \to \sigma^{t,\omega}(w)] \in L_0(W \to [0, \infty))$$

$$F^{t,\omega}(u, w):\quad (t, \omega) \in [0, \infty) \times \Omega \to [w \to [u \to F^{t,\omega}(u, w)]] \in L_0(W \to \mathbf{C}([0, \infty) \to \mathbf{R}^d))$$

$$\langle F^{t,\omega}, F^{t,\omega} \rangle (u, w):\quad (t, \omega) \in [0, \infty) \times \Omega \to [w \to [u \to \langle F^{t,\omega}, F^{t,\omega} \rangle (u, w)]] \in L_0(W \to \mathbf{C}([0, \infty) \to \mathfrak{S}^d_+))$$

where $\mathfrak{S}^d_+$ is the space of all symmetric and nonnegative definite matrices. Suppose also that we are given, for each $(t, \omega) \in [0, \infty) \times \Omega$, an increasing family $(\mathscr{B}^{t,\omega}_u)_{u>0}$ of sub σ-fields of $\mathscr{B}_W$ *depending predictably* on (t, ω) in the following sense. These exists a Polish space S and

$$\eta^{t,\omega}(u, w):\quad (t, \omega) \in [0, \infty) \times \Omega \to [w \to [u \to \eta^{t,\omega}(u, w)]] \in L_0(W \to \mathbf{C}([0, \infty) \to S))$$

which is $\mathscr{P}$-measurable such that

$$\mathscr{B}^{t,\omega}_u = \bigcap_{\varepsilon>0} \sigma_W\{\eta^{t,\omega}(v);\ 0 \leqq v \leqq u+\varepsilon\} \vee N$$

N being the totality of n-null sets on W. For each (t, ω), we assume that the following are satisfied:

$$\{w;\ \sigma^{t,\omega}(w) > u\} \in \mathscr{B}^{t,\omega}_u, \qquad u > 0 \tag{3.1}$$

$$F^{t,\omega}(u, w) \quad \text{and} \quad \langle F^{t,\omega}, F^{t,\omega} \rangle (u, w) \quad \text{are } (\mathscr{B}^{t,\omega}_u)\text{-adapted,} \tag{3.2}$$

for almost all $w(n)$,

$$F^{t,\omega}(0, w)=0\in\mathbf{R}^d,$$

$$\langle F^{t,\omega}, F^{t,\omega}\rangle(0, w)=0\in\mathfrak{S}^d_+, \qquad u\to\langle F^{t,\omega}, F^{t,\omega}\rangle(u, w) \tag{3.3}$$

is non-decreasing and

$$F^{t,\omega}(u, w)=F^{t,\omega}(u\wedge\sigma^{t,\omega}(w), w)$$

$$\langle F^{t,\omega}, F^{t,\omega}\rangle(u, w)=\langle F^{t,\omega}, F^{t,\omega}\rangle(u\wedge\sigma^{t,\omega}(w), w)$$

for every $u>0$.
Furthermore, we assume that the following are satisfied:

for every $T>0$,

$$\int_0^T dt\int_W [1\wedge\sigma^{t,\omega}(w)]\,n(dw)<\infty, \quad \text{a. s.} \tag{3.4}$$

for every $T>0$ and $s>0$,

$$\begin{gathered}\int_0^T dt\int_W |F^{t,\omega}(s\wedge\sigma^{t,\omega}(w), w)|^2\,n(dw)<\infty, \quad \text{a. s.}\\ \int_0^T dt\int_W \operatorname{trace}\langle F^{t,\omega}, F^{t,\omega}\rangle(s\wedge\sigma^{t,\omega}(w), w)\,n(dw)<\infty, \quad \text{a. s.}\end{gathered} \tag{3.5}$$

for every $0<s_1<s_2$, $T>0$ and every $\mathscr{P}$-measurable mapping

$$H^{t,\omega}:\quad (t, \omega)\in[0, \infty)\times\Omega\to H^{t,\omega}(w)\in L^2(W)\cap L^\infty(W)$$

such that $w\to H^{t,\omega}(w)$ is $\mathscr{B}^{t,\omega}_{s_1}$-measurable for all t, ω,

$$\int_0^T dt\int_W |H^{t,\omega}|^2(w)\,n(dw)<\infty \quad \text{a. s.} \tag{3.6}$$

and

$$\operatorname*{ess\,sup}_{[0,T]\ \ W}|H^{t,\omega}(w)|<\infty \quad \text{a. s.},$$

$$\int_0^T dt\int_W \{F^{t,\omega}(s_2, w)-F^{t,\omega}(s_1, w)\}\,H^{t,\omega}(w)\,n(dw)=0 \quad \text{a. s.} \tag*{(3.6)$_1$}$$

and

$$\int_0^T dt \int_W \{F^{t,\omega} \otimes F^{t,\omega}(s_2, w) - F^{t,\omega} \otimes F^{t,\omega}(s_1, w)\} H^{t,\omega}(w)\, n(dw) \tag{3.6$_2$}$$
$$= \int_0^T dt \int_W \{\langle F^{t,\omega}, F^{t,\omega}\rangle (s_2, w) - \langle F^{t,\omega}, F^{t,\omega}\rangle (s_1, w)\} H^{t,\omega}(w)\, n(dw) \quad \text{a. s.}$$

Here, we recall the following definition ([3], p. 61-62)

$$\mathbf{F}_p = \left\{ f(t, w, \omega);\ f \text{ is predictable and for every } t>0, \int_0^{t+} \int_W |f(s, w, \omega)|\, N_p(ds, dw) < \infty \quad \text{a. s.} \right\}$$

$$\mathbf{F}_p^1 = \left\{ f(t, w, \omega);\ f \text{ is predictable and for every } t>0, E\left[\int_0^t ds \int_W |f(s, w, \omega)|\, n(dw))\right] < \infty \right\}$$

$$\mathbf{F}_p^2 = \left\{ f(t, w, \omega);\ f \text{ is predictable and for every } t>0, E\left[\int_0^t ds \int_W |f(s, w, \omega)|^2\, n(dw)\right] < \infty \right\}$$

$$\mathbf{F}_p^{1,\,\mathrm{loc}} = \left\{ f(t, w, \omega);\ f \text{ is predictable and for every } t>0, \int_0^t ds \int_W |f(s, w, \omega)|\, n(dw) < \infty \quad \text{a. s.} \right\}$$

$$\mathbf{F}_p^{2,\,\mathrm{loc}} = \left\{ f(t, w, \omega);\ f \text{ is predictable and for every } t>0, \int_0^t ds \int_W |f(s, w, \omega)|^2\, n(dw) < \infty \quad \text{a. s.} \right\}$$

We can easily see that

$$\mathbf{F}_p^1 \subset \mathbf{F}_p^{1,\,\mathrm{loc}} \subset \mathbf{F}_p \qquad \text{and} \qquad \mathbf{F}_p^2 \subset \mathbf{F}_p^{2,\,\mathrm{loc}}.$$

LEMMA 3.1. — *For every* $\delta>0$,

(i) $I_{\{\sigma^{t,\omega}(w)>\delta\}}\in \mathbf{F}_p^{1,\,\mathrm{loc}}\subset \mathbf{F}_p$;

(ii) $\sigma^{t,\omega}(w)\in \mathbf{F}_p$;

(iii) trace $\langle F^{t,\omega}, F^{t,\omega}\rangle(\sigma^{t,\omega}(w), w)\in \mathbf{F}_p$;

(iv) $|F^{t,\omega}(\sigma^{t,\omega}, w)|\, I_{\{\sigma^{t,\omega}\leq 1\}}\in \mathbf{F}_p^{2,\,\mathrm{loc}}$;

(v) $|F^{t,\omega}(\sigma^{t,\omega}, w)|\, I_{\{\sigma^{t,\omega}>1\}}\in \mathbf{F}_p$.

The proof is easy and omitted.

Let $p(t)$ be an $(\mathscr{F}_t)$-stationary Poisson point process taking values in W with the characteristic measure n. Based on this lemma, the following stochastic integrals are well-defined and hence an $\mathbf{R}^d$-valued semimartingale $\xi(t)$ is well defined:

$$\xi(t)=\int_0^{t+}\int_W F^{s,\omega}(\sigma^{s,\omega}(w), w)\, I_{\{\sigma^{s,\omega}(w)>1\}}\, N_p(ds, dw)$$
$$+\int_0^{t+}\int_W F^{s,\omega}(\sigma^{s,\omega}(w), w)\, I_{\{\sigma^{s,\omega}(w)\leq 1\}}\, \tilde{N}_p(ds, dw)$$
$$-\int_0^t\left[\int_W F^{s,\omega}(1, w)\, I_{\{\sigma^{s,\omega}(w)>1\}}\, n(dw)\right]ds. \quad (3.7)$$

Let $\rho(s,\omega)$ be an $(\mathscr{F}_t)$-optional, nonnegative process such that, for every $t>0$, $\int_0^t \rho(s)\,ds<\infty$ a.s. Let $A(0)$ be a nonnegative $(\mathscr{F}_0)$-measurable random variable and define an $(\mathscr{F}_t)$-adapted increasing process by

$$A(t)=A(0)+\int_0^{t+}\int_W \sigma^{s,\omega}(w)\, N_p(ds, dw)+\int_0^t \rho(s)\,ds. \quad (3.8)$$

Assume that

$$t\to A(t) \text{ is strictly increasing and } \lim_{t\uparrow\infty} A(t)=\infty \quad \text{a.s.} \quad (3.9)$$

Then a continuous process $\varphi(t)$ is defined by the property that $\varphi(t)=s$ if and only if $A(s-)\leq t\leq A(s)$ $(A(0-)=0)$. We introduce the following convention: We attach to W an extra-point Δ and set $p(t)=\Delta$ if $t\notin D_p$. Also, set $F^{t,\omega}(u,\Delta)=0\in\mathbf{R}^d$ and $\langle F^{t,\omega}, F^{t,\omega}\rangle(u,\Delta)=0\in\mathfrak{S}_+^d$ for all $u\geq 0$. Let a be an extra-point attached to S [S being the Polish space appearing above in connection with $(\mathscr{B}_u^{t,\omega})$] and set $\eta^{t,\omega}(u,\Delta)=a$ for all $u\geq 0$. Also, if $u<0$, we set always $\eta^{t,\,\omega}$ to take the value a and $F^{t,\,\omega}$, $\langle F^{t,\,\omega}, F^{t,\,\omega}\rangle$ to take the values $0\in\mathbf{R}^d$ and $0\in\mathfrak{S}^d_+$, respectively.

We define a filtration $(\mathscr{H}_t)$ of sub σ-fields of $\mathscr{F}$ by

$$\mathscr{H}_t=\mathscr{F}_{\varphi(t)-}\vee\sigma_\Omega\{\eta^{\varphi(t),\,\omega}((t-A(\varphi(t)-))\wedge u, p\{\varphi(t)\}); 0\leqq u<\infty\} \quad (3.10)$$

Define an $\mathbf{R}^d$-valued process $M(t)$ by

$$M(t)=\xi(\varphi(t)-)+F^{\varphi(t),\,\omega}(t-A(\varphi(t)-), p\{\varphi(t)\}). \quad (3.11)$$

THEOREM 3.1. — *$M(t)$ is a d-dimensional continuous* (H_t)*-local martingale with*

$$\langle M, M\rangle(t)=\kappa(\varphi(t)-)$$
$$+\langle F^{\varphi(t),\,\omega}, F^{\varphi(t),\,\omega}\rangle(t-A(\varphi(t)-), p\{\varphi(t)\}) \quad (3.13)$$

where

$$\kappa(t)=\int_0^{t+}\int_W \langle F^{s,\,\omega}, F^{s,\,\omega}\rangle(\sigma^{s,\,\omega}, w)\,N_p(ds, dw). \quad (3.13)$$

The proof of this theorem is essentially the same as that of Theorem 2.1. A typical example of applications is for the processes constructed in [12]. This theorem, combined with Theorem 3 of [13], assures us that the processes constructed are actually solutions of stochastic differential equations corresponding to the given analytical data, *cf.* also [3] and [11].

Remark 3.1. — In the above, we excluded the possibility that $\sigma(w)=\infty$ or $\sigma^{t,\,\omega}(w)=\infty$. This possibility can be allowed if the following modifications are made. In the case of section 2, we assume $n\{\sigma(w)=\infty\}>0$. (2.3) implies that $n\{\sigma(w)=\infty\}<\infty$ and hence e: $=\inf\{s\in D_p;\ \sigma\{p(s)\}=\infty\}$ is exponentially distributed with mean $n\{\sigma(w)=\infty\}^{-1}$. Hence $A(e-)<\infty$ and $A(e)=\infty$. $B(t)$ is defined for $t<A(e-)$ and we set $B(t)=F(t-A(e-), p(e))$ for $t\geqq A(e-)$. Since $\sigma\{p(e)\}=\infty$, $B(t)$ is well-defined for all $t\geqq 0$. Similar modification can be given to Theorem 3.1.

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