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Characterization of probability distributions by Poincaré-type inequalities

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ABSTRACT. — We define a functional related to a Poincaré-type inequality and use it to characterize the infinitely divisible distribution on the Euclidean space. Some related results including a limit theorem are proved. This leads us to the problem of uniquely determining the distribution of a random vector X in terms of the constant c and the class of functions g for which the inequality $E[g(X)]^2 \leqslant cE|\nabla g(X)|^2$ with $Eg(X) = 0$ becomes an equality. We give a solution to this problem in some special cases and consider a discrete analog of it. Our results generalize substantially known results in the literature.

List of key-words: Poincaré-type inequality, infinitely divisible distribution, weak limit theorem, characterization of probability distributions.

RÉSUMÉ. — Nous définissons une fonctionnelle reliée à une inégalité de type Poincaré et nous l'utilisons pour caractériser les lois infiniment divisibles sur l'espace euclidien. Nous démontrons des résultats reliés comprenant un théorème limite. Cela nous conduit au problème de déterminer uniquement la loi d'un vecteur aléatoire X en donnant la constante c et la classe des fonctions g pour lesquelles l'inégalité $E[g(X)]^2 \leqslant cE|\nabla g(X)|^2$

avec $Eg(X) = 0$ devient une égalité. Nous donnons une solution à ce problème dans quelques cas spéciaux et nous considérons un analogue discret. Nos résultats généralisent substantiellement les résultats connus dans la littérature.

1. INTRODUCTION

In [4] Chernoff proved that if X has a normal distribution with variance $\sigma^2 > 0$ and g is an absolutely continuous function such that $E[g(X)]^2 < \infty$, then $\text{Var}[g(X)] \leqslant \sigma^2 E[g'(X)]^2$ and equality holds if and only if $g(x) = ax + b$ for some real numbers a and b. Borovkov and Utev [1] defined the functional $U_X = \sup \frac{\text{Var}[g(X)]}{\sigma^2 E[g'(X)]^2}$ for any random variable X with variance $\sigma^2 > 0$, where the supremum is taken over a suitable class of absolutely continuous functions g, and proved that if $U_X = 1$ then X has a normal distribution. Combining the result of Chernoff and of Borovkov and Utev, one obtains an interesting characterization of the normal distribution: X is normally distributed if and only if $U_X = 1$.

The Chernoff inequality has been generalized to higher dimensions and then to the multivariate infinitely divisible distribution (see Chen [2], [3]). It is therefore natural to ask whether the above characterization of the normal distribution can be extended to the multivariate infinitely divisible distribution. In Section 2 we give a positive answer to this question using a much simpler method than that of Borovkov and Utev.

In connection with the characterization theorem, Borovkov and Utev obtained a list of properties of the functional R_X (which is related to U_X by $U_X = R_X/\text{Var}(X)$), and proved a limit theorem involving U_X. We generalize these results in Section 3.

The proof of the characterization theorem leads us to the following question which is analogous to the well known « Can one hear the shape of a drum? » (see Kac [5] and Remark (1) in Section 4): If c is given and the inequality $E[g(X)]^2 \leqslant cE|\nabla g(X)|^2$ where $Eg(X) = 0$ becomes an equality for g belonging to a given class of functions, does this uniquely determine the distribution of the random vector X? In Section 4, we show that the answer is positive in some special cases.

In the last section, we consider a discrete analog of this problem. The inequality in this case becomes $\operatorname{Var}[g(X)] \leqslant c \sum_{i=1}^{m} E[\Delta_i g(x)]^2$ where $\Delta_i g(x) = g(x_1, \ldots, x_i + 1, \ldots, x_m) - g(x_1, \ldots, x_i, \ldots, x_m)$.

Throughout the paper all functions are real-valued unless otherwise stated. We denote by $C_B^k(\Omega)$ or $C_0^k(\Omega)$ respectively the class of C^k functions on an open set Ω which are bounded or have compact support. The transpose of a matrix C is denoted by C^*. All vectors are taken to be column vectors and we write $x = (x_1, \ldots, x_m)^*$ for a column vector. The usual inner product of two vectors x and y is denoted by (x, y) and the norm of x by $|x|$.

2. CHARACTERIZATION OF THE INFINITELY DIVISIBLE DISTRIBUTION

In their proof of the characterization theorem [*1*, Theorem 3], Borovkov and Utev used the properties of the functional R_X and an ingenious construction to characterize the normal distribution by the method of moments. In this section, we show that the proof of Borovkov and Utev can be substantially simplified and the theorem generalized to the multivariate infinitely divisible distribution. This is done by reducing the problem to a system of first order partial differential equations whose unique solution is the characteristic function of an infinitely divisible distribution. In our proof, the use of R_X is avoided.

Let $X = (X_1, \ldots, X_m)^*$ be a random vector with distribution γ and mean vector b such that $\operatorname{Var}(X_i) < \infty$ for $i = 1, \ldots, m$. Denote by $L^2(X)$ the class of Borel measurable functions g defined on $\mathbb{R}^m$ such that $E[g(X)]^2 < \infty$ and Q(X) the class of such functions such that $\operatorname{Var}[g(X)] > 0$. Define $\mathscr{H}_X = C^1(\mathbb{R}^m) \cap L^2(X) \cap Q(X)$.

Let $\Sigma = (\tau_1, \ldots, \tau_m) = (\tau_{ij})$ be an $m \times m$ positive semidefinite matrix and μ a measure on $\mathscr{B}(\mathbb{R}^m)$ such that it has no atom at 0 and $\int_{\mathbb{R}^m} |x|^2 \mu(dx) < \infty$. Define

$$U(X, \Sigma, \mu) = \sup_{g \in \mathscr{H}_X} \frac{\operatorname{Var}[g(X)]}{E[\nabla^* g(X) \Sigma \nabla g(X)] + \int_{\mathbb{R}^m} E[\Delta^\xi g(X)]^2 \mu(d\xi)} \tag{2.1}$$

where $\Delta^\xi g(x) = g(x + \xi) - g(x)$.

THEOREM 2.1. — *Suppose* $\mathrm{Var}(X_j)=\tau_{jj}+\int_{\mathbb{R}^m} x_j^2\mu(dx)>0$ *for* $j=1, \ldots, m$. *Then* $U(X, \Sigma, \mu) \geqslant 1$ *and* $U(X, \Sigma, \mu) = 1$ *if and only if the distribution* γ *is infinitely divisible with characteristic function given by*

$$\hat{\gamma}(t) = \exp\left\{ ib^*t - \frac{1}{2}t^*\Sigma t + \int_{\mathbb{R}^m} (e^{it^*x} - 1 - it^*x)\mu(dx) \right\} \tag{2.2}$$

for $t \in \mathbb{R}^m$.

Proof. — The fact $U(X, \Sigma, \mu) \geqslant 1$ follows trivially from the definition (2.1) and the substitution $g(x) = x_j$ for any $j = 1, \ldots, m$. The necessity of $U(X, \Sigma, \mu) = 1$ is implied by Theorem 4.1 of Chen [*3*]. For the sufficiency, let $g(x)=x_j+\lambda f(x)$ where $f\in C_B^1(\mathbb{R}^m)$ and $j=1, \ldots, m$. Then $U(X, \Sigma, \mu)=1$ implies that for $\lambda \in \mathbb{R}$,

$$\mathrm{Var}\,[X_j + \lambda f(X)] \leqslant E\,[\nabla^*(X_j + \lambda f(X))\Sigma\nabla(X_j + \lambda f(X))] + \int_{\mathbb{R}^m} E\,[\Delta^\xi(X_j + \lambda f(X))]^2\mu(d\xi)\,.$$

This yields

$$\lambda^2 \,\mathrm{Var}\,[f(X)] + 2\lambda\,\mathrm{Cov}\,(X_j, f(X)) \leqslant \lambda^2 \left\{ E\,[\nabla^* f(X)\Sigma f(X)] + \int_{\mathbb{R}^m} E\,[\Delta^\xi f(X)]^2\mu(d\xi) \right\} + 2\lambda \left\{ E\,[\tau_j^*\nabla f(X)] + \int_{\mathbb{R}^m} E\,[\xi_j\Delta^\xi f(X)]\mu(d\xi) \right\}.$$

Since this inequality holds for all $\lambda \in \mathbb{R}$, it follows that

$$\mathrm{Cov}\,(X_j, f(X)) = E\,[\tau_j^*\nabla f(X)] + \int_{\mathbb{R}^m} E\,[\xi_j\Delta^\xi f(X)]\mu(d\xi)\,. \tag{2.3}$$

Substituting $f(x) = \mathrm{Re}\, e^{it^*x}$ and then $\mathrm{Im}\, e^{it^*x}$, $t \in \mathbb{R}^m$, in (2.3), we obtain

$$\frac{\partial\,\hat{\gamma}(t)}{\partial t_j} = \left\{ ib_j - \tau_j^*t + i\int_{\mathbb{R}^m} \xi_j(e^{it^*\xi} - 1)\mu(d\xi) \right\} \hat{\gamma}(t)$$

for $t \in \mathbb{R}^m$ and $j = 1, \ldots, m$, where $\hat{\gamma}$ is the characteristic function of γ. Since $\hat{\gamma}(0) = 1$ and the characteristic function of the infinitely divisible distribution never vanishes, (2.4) has a unique solution which is given by (2.2). This proves the sufficiency of $U(X, \Sigma, \mu) = 1$ and hence the theorem.

Two special cases of Theorem 2.1 are of interest. We state them without proof.

COROLLARY 2.1. — *Let* $X = (X_1, \ldots, X_m)^*$ *be a random vector such that* $\operatorname{Var}(X_j) = \sigma_j^2 > 0$ *for* $j = 1, \ldots, m$. *Define*

$$U(X, \Sigma) = \sup_{g \in \mathscr{H}_X} \frac{\operatorname{Var}[g(X)]}{E[\nabla^* g(X) \Sigma \nabla g(x)]}$$

where Σ *is an* $m \times m$ *positive semidefinite matrix with* $\sigma_1^2, \ldots, \sigma_m^2$ *as its diagonal elements. Then* $U(X, \Sigma) \geqslant 1$ *and* $U(X, \Sigma) = 1$ *if and only if* X *has a multivariate normal distribution with covariance matrix* Σ.

For the next corollary we defined $\mathscr{H}_X$ to be the class of functions $g : \mathbb{Z}^m \to \mathbb{R}$ such that $E[g(X)]^2 < \infty$ and $\operatorname{Var}[g(X)] > 0$ where X is a random vector taking values in $\mathbb{Z}^m$.

COROLLARY 2.2. — *Let* $X = (X_1, \ldots, X_m)^*$ *be a random vector taking values in* $\mathbb{Z}^m$ *such that* $\operatorname{Var}(X_j) = \lambda_j > 0$ *for* $1, \ldots, m$. *Define*

$$U_X = \sup_{g \in \mathscr{H}_X} \frac{\operatorname{Var}[g(X)]}{\sum_{j=1}^{m} \lambda_j E[\Delta_j g(X)]^2}$$

where $\Delta_j g(x) = g(x_1, \ldots, x_j + 1, \ldots, x_m) - g(x_1, \ldots, x_j, \ldots, x_m)$. *Then* $U_X = 1$ *if and only if each* X_j *has a translated Poisson distribution and* $X_1, \ldots, X_m$ *are independent.*

3. SOME RELATED RESULTS

Although we have avoided the use of R_X in the proof of Theorem 2.1, its properties proved and used by Borovkov and Utev [*1*] are nevertheless of interest in themselves. Borovkov and Utev also proved a limit theorem involving U_X. The objective of this section is to generalize these results.

As in section 2, let $\mathscr{H}_X = C^1(\mathbb{R}^m) \cap L^2(X) \cap Q(X)$. Let $\Sigma = (\tau_{ij})$ be an $m \times m$ positive semidefinite matrix and μ a measure on $\mathscr{B}(\mathbb{R}^m)$ such that μ has no atom at 0 and $\int_{\mathbb{R}^m} |x|^2 \mu(dx) < \infty$. Assume that $\tau_{jj} + \int x_j^2 \mu(dx) > 0$

for $j = 1, \ldots, m$. For each random vector $X = (X_1, \ldots, X_m)^*$, define $U(X, \Sigma, \mu)$ by (2.1). Also define

$$U_0(X, \Sigma, \mu) = \sup_{g \in C_0^\infty(\mathbb{R}^m) \cap Q(X)} \frac{\mathrm{Var}\,[g(X)]}{E\,[\nabla^* g(X) \Sigma \nabla g(X)] + \int_{\mathbb{R}^m} E\,[\Delta^\xi g(X)]^2 \mu(d\xi)}.$$

Note that $U(X, \Sigma, \mu)$ and $U_0(X, \Sigma, \mu)$ correspond to R_X and r_X in [*1*] respectively. Although $C^1(\mathbb{R})$ is smaller than the class of absolutely continuous functions, there is no loss of generality in our definition of $U(X, \Sigma, \mu)$, when specialized to one dimension, in view of *i*) of Theorem 2 in [*1*].

We abbreviate $U(X, \Sigma, \mu)$ and $U_0(X, \Sigma, \mu)$ to $U(X, \Sigma)$ and $U_0(X, \Sigma)$ respectively if μ is the zero measure. Also write

$$M_{\Sigma,\mu} g(x) = \nabla^* g(x) \Sigma \nabla g(x) + \int_{\mathbb{R}^m} [\Delta^\xi g(x)]^2 \mu(d\xi).$$

The following lemma is a generalization of Lemma 1 in [*1*]. We omit its proof here as the proof of the latter carries over easily to this general case

LEMMA 3.1. — *Let* $g \in C^1(\mathbb{R}^m)$ *and* $g_n \in C^1(\mathbb{R}^m) \cap L^2(X)$. *Suppose*

a) $g_n \to g$ *pointwise as* $n \to \infty$;

b) $\mathrm{Var}\,[g_n(X)] \leqslant c E M_{\Sigma,\mu} g_n(X)$ *for* $n \geqslant 1$;

c) $\limsup_{n \to \infty} E M_{\Sigma,\mu} g_n(X) \leqslant E M_{\Sigma,\mu} g(X) < \infty$.

Then $g \in L^2(X)$ *and* $\mathrm{Var}\,[g(X)] \leqslant c E M_{\Sigma,\mu} g(X)$.

THEOREM 3.1. — i) $U(X, \Sigma, \mu) = U_0(X, \Sigma, \mu)$.

ii) *Let* $g \in C^1(\mathbb{R}^m)$. *If* $U(X, \Sigma, \mu) < \infty$ *and* $E M_{\Sigma,\mu} g(X) < \infty$, *then* $g \in L^2(X)$ *and* $\mathrm{Var}\,[g(X)] \leqslant U(X, \Sigma, \mu) E M_{\Sigma,\mu} g(X)$.

iii) *If* $U(X, \Sigma) < \infty$, *then the distribution of* X *has a continuous (but not necessarily absolutely continuous) component.*

iv) *If* $U(X, \Sigma, \mu) < \infty$ *and there exists* a $(0 < a \leqslant 1)$ *such that* $\int_{\mathbb{R}^m} (e^{a|\xi|} - 1) |\xi| \mu(d\xi) < \infty$, *then* $E\,[\exp(t_1 X_1 + \ldots + t_m X_m)] < \infty$ *for* $|t_i| < (a^2 / b_i c_i m^2)^{1/2}$, $i = 1, \ldots, m$, *where*

$$b_i = \max\left(1, \theta_i^{-1}\left(\tau_{ii} + a^{-1} \int_{\mathbb{R}^m} \xi_i \sinh(a\xi_i) \mu(d\xi)\right)\right),$$

$$c_i = \max(1, 2\theta_i U(X, \Sigma, \mu))$$

and

$$\theta_i = \tau_{ii} + \int_{\mathbb{R}^m} x_i^2 \, \mu(dx) .$$

v) *Let* C *be an* $m \times m$ *nonsingular matrix and* $b \in \mathbb{R}^m$. *Then*

$$U(CX + b, \Sigma) = U(X, C\Sigma C^*) .$$

vi) $U(X, \Sigma, \mu) \geqslant \dfrac{v^* \operatorname{Cov}(X) v}{v^*\left(\Sigma + \int_{\mathbb{R}^m} xx^* \mu(dx)\right)v}$ *for* $v \in \mathbb{R}^m$, $v \neq 0$, *where* Cov (X) *is the covariance matrix of* X.

vii) *If* X *and* Y *are independent, then*

$$U(X + Y, \Sigma, \mu) \leqslant U(X, \Sigma, \mu) + U(Y, \Sigma, \mu) .$$

viii) *If* X_n *converges in distribution to* X, *then*

$$\liminf_{n \to \infty} U(X_n, \Sigma, \mu) \geqslant U(X, \Sigma, \mu) .$$

Proof. — We note that Lemma 4.2 of Chen [*3*] is true for any probability distribution γ if $D = \phi$. By an application of this lemma, *i*) follows. For *ii*), let $g_n = ng/(n^2 + g^2)^{1/2}$. Then $g_n \to g$ pointwise, $|g_n| \leqslant n$, $\nabla^* g_n \Sigma \nabla g_n \leqslant \nabla^* g \Sigma \nabla g$ and $|\Delta^\xi g_n(x)| \leqslant |\Delta^\xi g(x)|$. So g_n satisfies the conditions in Lemma 3.1 and *ii*) follows. By letting g depend on one variable, part *iii*) of Theorem (2) in [*1*] implies that each X_i has an absolutely continuous component and this proves *iii*). To see that the distribution of X may be singularly continuous, let $m = 2$, X_1 have the standard normal distribution and $X_2 = X_1$. Then by the Chernoff inequality,

$$\operatorname{Var}[g(X)] \leqslant E\left[\frac{\partial}{\partial x_1} g(X_1, X_2) + \frac{\partial}{\partial x_2} g(X_1, X_2)\right]^2 \leqslant 2E\,|\nabla g(X)|^2$$

for $g \in C^1(\mathbb{R}^2)$ such that $E[g(X)]^2 < \infty$. The proofs of *v*)-*viii*) are easy extensions of those of the corresponding parts of Theorem 2 in [*1*], and are therefore omitted here. But we remark that for *vii*) and *viii*) Lemma 4.2 of [*3*] is used.

It remains to prove *iv*). Since

$$E[\exp(t_1 X_1 + \ldots + t_m X_m)] \leqslant m^{-1} \sum_{i=1}^{m} [\exp(mt_i X_i)] ,$$

it suffices to consider each X_i separately. Abbreviate $U(X, \Sigma, \mu)$ to U. By taking g to depend on one variable, we have

$$\operatorname{Var}[g(X_i)] \leqslant U\left\{ \tau_{ii} E[g'(X_i)]^2 + E\int_{\mathbb{R}^m} [\Delta^{\xi_i} g(X_i)]^2 \mu(d\xi) \right\}$$

for $g \in C^1(\mathbb{R}) \cap L^2(X_i)$ and $i = 1, \ldots, m$. Let X_i' be an independent copy of X_i and let $W_i = X_i - X_i'$. Then for $g \in C_0^1(\mathbb{R})$, we have

$$\begin{aligned} \operatorname{Var}[g(W_i)] &= E \operatorname{Var}^{X_i}[g(W_i)] + \operatorname{Var}[E^{X_i} g(W_i)] \\ &\leqslant UE\left\{ \tau_{ii} E^{X_i}[g'(W_i)]^2 + E^{X_i}\int_{\mathbb{R}^m} [g(W_i - \xi_i) - g(W_i)]^2 \mu(d\xi) \right\} \\ &\quad + U\left\{ \tau_{ii} E[E^{X_i} g'(W_i)]^2 + E\int_{\mathbb{R}^m} [E^{X_i} g(W_i + \xi_i) - E^{X_i} g(W_i)]^2 \mu(d\xi) \right\}. \end{aligned}$$

This yields

$$\begin{aligned} (3.1)\quad \operatorname{Var}[g(W_i)] \leqslant U\Big\{ 2\tau_{ii} E[g'(W_i)]^2 &+ E\int_{\mathbb{R}^m} [g(W_i - \xi_i) - g(W_i)]^2 \mu(d\xi) \\ &+ E\int_{\mathbb{R}^m} [g(W_i + \xi_i) - g(W_i)]^2 \mu(d\xi) \Big\}. \end{aligned}$$

By Lemma 4.2 of [*3*], (3.1) holds for $g \in C^1(\mathbb{R}) \cap L^2(X_i)$. By Jensen's inequality and Fubini's theorem as in the proof of Theorem 4.3 in [*3*], and then by a change of variable, (3.1) yields

$$\begin{aligned} (3.2)\quad & \operatorname{Var}[g(W_i)] \\ & \leqslant U\left\{ 2\tau_{ii} E[g'(W_i)]^2 + E\int_{-\infty}^{\infty} [g'(W_i + t)]^2 [\phi_i(t) + \phi_i(-t)] dt \right\} \end{aligned}$$

where

$$\phi_i(t) = \begin{cases} \displaystyle\int_{\xi_i \geqslant t} \xi_i \mu(d\xi), & t \geqslant 0, \\ \displaystyle -\int_{\xi_i < t} \xi_i \mu(d\xi), & t < 0. \end{cases}$$

Let $\theta_i = \tau_{ii} + \int_{\mathbb{R}^m} x_i^2 \mu(d\xi) = \tau_{ii} + \frac{1}{2}\int_{-\infty}^{\infty} [\phi_i(t) + \phi_i(-t)] dt$ and define Y_i to be a random variable independent of W_i and distributed as $\tilde{\nu}_i$ where

$$\tilde{\nu}_i(dt) = \theta_i^{-1}\left\{ \tau_{ii} \varepsilon_0(dt) + \frac{1}{2}[\phi_i(t) + \phi_i(-t)] dt \right\}.$$

Then (3.2) may be written as

$$\text{(3.3)} \qquad \operatorname{Var}[g(\mathrm{W}_i)] \leqslant c_i \mathrm{E}[g'(\mathrm{W}_i + \mathrm{Y}_i)]^2 .$$

By *ii*) and induction, $\mathrm{U} < \infty$ and the condition $\int_{\mathbb{R}^m} (e^{a|\xi|} - 1)|\xi|\mu(d\xi) < \infty$ imply that all moments of X_i exist for $i = 1, \dots, m$, which in turn implies the same for W_i for $i = 1, \dots, m$. We now prove by induction that for $n \geqslant 1$,

$$\text{(3.4)} \qquad \mathrm{EW}_i^{2n} \leqslant (2n\,!)(b_i c_i / a^2)^n .$$

By a simple calculation, $\mathrm{E}e^{a\mathrm{Y}_i} = \theta_i^{-1}\left[\tau_{ii} + a^{-1}\int_{\mathbb{R}^m} \xi_i \sinh(a\xi_i)\mu(d\xi)\right]$ which is finite by virtue of the condition $\int_{\mathbb{R}^m} (e^{a|\xi|} - 1)|\xi|\mu(d\xi) < \infty$, and so $b_i = \max(1, \mathrm{E}e^{a\mathrm{Y}_i})$.

For the rest of the proof, we drop the subscript i for brevity, but will pick it up whenever it is necessary to avoid ambiguity. Since $\mathrm{EW}^2 = 2\mathrm{U}\theta \leqslant c$, (3.4) holds for $n = 1$. For $n \geqslant 2$, (3.3) yields

$$\begin{aligned} \mathrm{EW}^{2n} &\leqslant (\mathrm{EW}^n)^2 + cn^2\mathrm{E}(\mathrm{W} + \mathrm{Y})^{2(n-1)} \\ &= (\mathrm{EW}^n)^2 + cn^2 \sum_{k=0}^{n-1} \binom{2(n-1)}{2k} \mathrm{EW}^{2k}\mathrm{EY}^{2(n-k-1)} \end{aligned}$$

where we have used the fact that all odd moments of W vanish. By the induction hypothesis

$$\begin{aligned} \text{(3.5)} \quad \mathrm{EW}^{2n} &\leqslant (\mathrm{EW}^n)^2 + cn^2 \sum_{k=0}^{n-1} \binom{2(n-1)}{2k} (2k\,!)\left(\frac{bc}{a^2}\right)^k \mathrm{EY}^{2(n-k-1)} \\ &\leqslant (\mathrm{EW}^n)^2 + \frac{cn^2(2(n-1)\,!)(bc)^{n-1}}{a^{2(n-1)}} \sum_{k=0}^{n-1} \frac{\mathrm{E}(a\mathrm{Y})^{2(n-k-1)}}{(2(n-k-1)\,!)} \\ &\leqslant (\mathrm{EW}^n)^2 + \frac{n}{2(2n-1)} \cdot (2n\,!)(bc/a^2)^n \\ &\leqslant (\mathrm{EW}^n)^2 + 3^{-1}(2n\,!)(bc/a^2)^n. \end{aligned}$$

If n is odd, then $EW^n = 0$. On the other hand, if n is even, then $(EW^n)^2 \leqslant [n!(bc/a^2)^{n/2}]^2 = (n!)^2(bc/a^2)^n \leqslant \frac{4n^{1/2}(2n!)}{2^{2n}}\left(\frac{bc}{a^2}\right)^n \leqslant 2^{-1}(2n!)(bc/a^2)^n$.
In either case the right hand side of the last inequality in (3.5) is less than $(2n!)(bc/a^2)^n$ and (3.4) is proved. By Jensen's inequality,

$$E\,|\,W\,|^{2n-1} \leqslant (EW^{2n})^{(2n-1)/2n} \leqslant (2n!)(bc/a^2)^{(2n-1)/2}.$$

So for $|\,t_i\,| < (a^2/b_ic_im^2)^{1/2}$,

$$E\exp(mt_i\,|\,W_i\,|) \leqslant \sum_{k=0}^{\infty}\frac{(m\,|\,t_i\,|)^k}{k!}(k+1)!(b_ic_i/a^2)^{k/2}$$

$$= \sum_{k=0}^{\infty}(k+1)[(b_ic_im^2/a^2)^{1/2}\,|\,t_i\,|\,]^k < \infty.$$

This implies that $E\exp(mt_iW_i) \leqslant E\exp(mt_i\,|\,W_i\,|) < \infty$. As

$$E\exp(mt_iW_i) = [E\exp(mt_iX_i)]\,[E\exp(-mt_iX_i)],$$

iv) is proved. Hence the theorem.

For the next theorem, let $X_n = (X_{n1}, \ldots, X_{nm})^*$ be a random vector with mean vector b_n and $0 < \mathrm{Var}(X_{ni}) < \infty$ for $i = 1, \ldots, m$. Let Σ_n be an $m \times m$ positive semidefinite matrix and μ_n a measure on $\mathscr{B}(\mathbb{R}^m)$ such that it has no atom at 0 and $\int_{\mathbb{R}^m} |\,x\,|^2\mu_n(dx) < \infty$.

THEOREM 3.2. — *Assume the diagonal elements of* $\Sigma_n + \int_{\mathbb{R}^m} xx^*\mu_n(dx)$ *are respectively* $\mathrm{Var}(X_{ni})$, $i = 1, \ldots, m$. *Suppose as* $n \to \infty$, $b_n \to b$, $\Sigma_n \to \Sigma$ *and* $|\,x\,|^2\mu_n(dx)$ *converges weakly to* $|\,x\,|^2\mu(dx)$ *where without loss of generality* μ *may be assumed to have no atom at* 0. *If* $U(X_n, \Sigma_n, \mu_n) \to 1$ *as* $n \to \infty$, *then* $\mathscr{L}(X_n)$ *converges weakly to the infinitely divisible distribution* γ *whose characteristic function is given by (2.2) as* $n \to \infty$.

Proof. — Since $b_n \to b$ and $\Sigma_n \to \Sigma$, the distributions of X_n, $n \geqslant 1$ are tight. Hence it suffices to prove that every weakly convergent subsequences of $\{ \mathscr{L}(X_n) \}$ converges to γ. Suppose $X_{n'} \xrightarrow{\mathscr{L}} X$. Then for $g \in C_0^1(\mathbb{R}^m)$

$$(3.6) \qquad \begin{aligned} \operatorname{Var}[g(X)] &= \lim_{n' \to \infty} \operatorname{Var}[g(X_{n'})] \\ &\leqslant \lim_{n' \to \infty} U(X_{n'}, \Sigma_{n'}, \mu_{n'}) EM_{\Sigma_{n'}, \mu_{n'}} g(X_{n'}) \\ &= EM_{\Sigma, \mu} g(X). \end{aligned}$$

By Theorem 3.1 *i*), $U(X, \Sigma, \mu) \leqslant 1$. By Theorem 3.1 *iv*), $\{ X_{ni}^2 \}$ is uniformly integrable for $i = 1, \ldots, m$ and so the diagonal elements of $\Sigma + \int_{\mathbb{R}^m} xx^* \mu(dx)$ are $\operatorname{Var}(X_i)$, $i = 1, \ldots, m$, where $X = (X_1, \ldots, X_m)^*$. It follows from Theorem 2.1 that $U(X, \Sigma, \mu) = 1$ and so the distribution of X is γ. This proves the theorem.

4. MORE ON CHARACTERIZATION. THE CONTINUOUS CASE

In Theorem 2.1, although the infinitely divisible distribution is characterized by using the functional $U(X, \Sigma, \mu)$, the proof depends crucially on the knowledge of the functions for which the inequality becomes an equality. One may therefore restate a special case of Theorem 2.1 as follows: If the inequality $E[g(X)]^2 \leqslant cE|\nabla g(X)|^2$ where $Eg(X) = 0$ becomes an equality for $g(x) = x_i$, $i = 1, \ldots, m$, then X has the multivariate normal distribution with zero mean vector and covariance matrix cI. This leads us to the following question: If c is given and the inequality $E[g(X)]^2 \leqslant cE|\nabla g(X)|^2$ where $Eg(X) = 0$ becomes an equality for g belonging to a given class of functions (not necessarily linear), does this uniquely determine the distribution of X? The objective of this section is to give an answer to this question. We show that the answer is positive in some special cases but negative in general.

As in Section 2, we reduce the problem to a system of first order partial differential equations. But unlike that in Section 2, the method here is more akin to a method of normal approximation due to Stein [*6*].

LEMMA 4.1. — *Let* $I = (a, b)$ *be an open (possibly infinite) interval. Let* X *be a random variable taking values in* I, *and* γ *the distribution of* X. *Suppose for all absolutely continuous functions* g *on* I *such that* $E[g(X)]^2 < \infty$ *and* $Eg(X) = 0$, *there is a constant* c *such that* $E[g(X)]^2 \leqslant cE[g'(X)]^2$ *and* *equality holds if* $g(x) = \psi(x)$, *where* $\psi \in C^1(I)$. *If* $\gamma(\psi' = 0) = 0$, *then* X *is absolutely continuous.*

Proof. — Decompose γ into $\gamma = \alpha\mu + (1 - \alpha)\nu$, where μ and ν are probability measures on I such that μ is absolutely continuous with respect to the Lebesgue measure and ν is singular, and $0 \leqslant \alpha \leqslant 1$. Let A be the set of points of increase of ν. Then

$$\begin{aligned}
\operatorname{Var} \psi(X) &= c\mathrm{E}[\psi'(X)]^2 \\
&= c\left\{ \alpha \int_I \psi'^2(x)\mu(dx) + (1 - \alpha)\int_I \psi'^2(x)\nu(dx) \right\} \\
&\geqslant c\alpha \int_I \psi'^2(x)\mu(dx) \\
&= c\mathrm{E}[\psi'(X)\mathrm{I}_{A^c}(X)]^2 \geqslant \operatorname{Var} \psi(X)
\end{aligned}$$

where the last inequality follows from the observation that $\psi' \mathrm{I}_{A^c}$ is a Borel measurable derivative of ψ. Hence we have $(1 - \alpha)\int_I \psi'^2(x)\nu(dx) = 0$. This implies that either $\alpha = 1$ or $\int_I \psi'(x)\nu(dx) = 0$. If $\int_I \psi'^2(x)\nu(dx) = 0$, then $\psi' = 0$ ν-a. e., that is $\nu(\psi' = 0) = 1$. The assumption $\gamma(\psi' = 0) = 0$ then forces α to be equal to 1. In this case, $\gamma = \mu$ and is therefore absolutely continuous with respect to the Lebesgue measure.

LEMMA 4.2. — *Under the condition of Lemma 4.1, suppose further that* ψ' *never vanishes on* I. *Then the density function of* X *is given by* $f_X = \tilde{\phi} \Big/ \int_I \tilde{\phi}$ *a. e., where* $\tilde{\phi} = \frac{1}{\psi'} \exp\left(-\frac{1}{c}\int \frac{\psi}{\psi'}\right)$ *and* $\int \frac{\psi}{\psi'}$ *is an indefinite integral of* $\frac{\psi}{\psi'}$.

Proof. — Let h be an absolutely continuous function on I such that $\mathrm{E}[h(X)]^2 < \infty$. Put $g = \lambda(h - \mathrm{E}h(X)) + \psi$, $\lambda \in \mathbb{R}$ in the inequality. By the same argument as in the proof of Theorem 2.1, we obtain

$$\mathrm{E}[\psi(X)h(X)] = c\mathrm{E}[\psi'(X)h'(X)],$$

or equivalently, $\int_I \psi(x)h(x)f_X(x)dx = c\int_I \psi'(x)h'(x)f_X(x)dx$. In particular, when $h(x) = (x - x_0)\mathrm{I}_{(x \geqslant x_0)}$, where $x_0 \in \mathrm{I}$, the above identity can be rewritten as

$$\int_{x_0}^{b} \psi(x)(x - x_0) f_X(x)dx = c \int_{x_0}^{b} \psi'(x) f_X(x)dx .$$

Taking both integrals as functions of x_0 and then taking Borel measurable derivatives with respect to x_0 on both sides, we have the following identity

$$\int_{x_0}^{b} \psi(x) f_X(x)dx = c\psi'(x_0) f_X(x_0)$$

for almost all x_0 w. r. t. the Lebesgue measure. Let $\mathrm{H}(x) = \int_x^b \psi(y) f_X(y)dy$. We observe that H is absolutely continuous and is a version of $c\psi' f_X$ on I. Now

$$\mathrm{H}(x) = \int_x^b \frac{\psi(y)}{\psi'(y)} \psi'(y) f_X(y)dy = \frac{1}{c} \int_x^b \frac{\psi(y)}{\psi'(y)} \mathrm{H}(y)dy .$$

This implies H is differentiable everywhere on I and

$$\mathrm{H}'(x) = -\frac{1}{c}\frac{\psi(x)}{\psi'(x)} \mathrm{H}(x) .$$

Solving for H, we obtain a solution given by $\exp\left(-\frac{1}{c}\int \frac{\psi}{\psi'}\right)$ up to a multiplicative constant. It is now clear that $f_X = \tilde{\phi} \Big/ \int_{\mathrm{I}} \tilde{\phi}$ a. e., where $\int_{\mathrm{I}} \tilde{\phi}$ is the normalizing constant to make f_X a density function.

For the following theorem, let $\Omega = \mathrm{I}_1 \times \ldots \times \mathrm{I}_m$ where each $\mathrm{I}_i = (a_i, b_i)$ is an open (possibly infinite) interval. Let $X = (X_1, \ldots, X_m)^*$ be a random vector taking values in Ω. For $\psi_i \in C^1(\mathrm{I}_i)$ and $c > 0$, define

$$\tilde{\phi}_i = \frac{1}{\psi_i'} \exp\left(-\frac{1}{c}\int \frac{\psi_i}{\psi_i'}\right) \tag{4.1}$$

where $\int \frac{\psi_i}{\psi_i'}$ is an indefinite integral of $\frac{\psi_i}{\psi_i'}$.

THEOREM 4.1. — *Suppose that there is a constant c such that for all $g \in C^1(\Omega)$ with $E[g(X)]^2 < \infty$ and $Eg(X) = 0$,*

$$E[g(X)]^2 \leqslant cE|\nabla g(X)|^2 \tag{4.2}$$

and that equality holds if $g(x) = \psi_i(x_i)$ for $i = 1, \ldots, m$, where $\psi_i \in C^1(\mathrm{I}_i)$,

$i = 1, \ldots, m$. *Assume that* ψ_i' *does not vanish on* I_i *for* $i = 1, \ldots, m$. *Then* X *is absolutely continuous with density function given by* $f_X(x) = \prod_{i=1}^{m} \phi_i(x_i)$ *where* $\phi_i(x_i) = \tilde{\phi}(x_i) \Big/ \int_{I_i} \tilde{\phi}_i$.

Proof. — By letting g depend on one variable, (4.2) and [*1*, Theorem 2 *i*)] imply $E[g(X_i)]^2 \leqslant cE[g'(X_i)]^2$ for all absolutely continuous g defined on I_i such that $E[g(X_i)]^2 < \infty$ and $Eg(X_i) = 0$, $i = 1, \ldots, m$. By Lemmas 4.1 and 4.2, X_i is absolutely continuous with density function ϕ_i. It remains to prove that $X_1, \ldots, X_m$ are independent. Let $f \in C_B^1(\Omega)$ and substitute $g = \lambda(f - Ef(X)) + \psi_i$ in (4.2) where $\lambda \in \mathbb{R}$. By the same argument as in the proof of Theorem 2.1, we obtain

$$E\left\{ c\psi_i'(X_i)\frac{\partial}{\partial x_i} f(X) - \psi_i(X_i)f(X) \right\} = 0 \tag{4.3}$$

for $i = 1, \ldots, m$. For each i, let $v_i \in C_0(\Omega)$ such that $\int_{I_i} v_i\tilde{\phi}_i = 0$. The differential equation

$$c\psi_i'(x_i)\frac{\partial}{\partial x_i} f(x) - \psi_i(x_i)f(x) = v_i(x)$$

where f is defined on Ω has a solution given by

$$f(x) = \frac{\int_{a_i}^{x_i} v_i(x_1, \ldots, y, \ldots, x_m)\tilde{\phi}_i(y)dy}{\psi_i'(x_i)\tilde{\phi}_i(x_i)} = -\frac{\int_{x_i}^{b_i} v_i(x_1, \ldots, y, \ldots, x_m)\tilde{\phi}_i(y)dy}{\psi_i'(x_i)\tilde{\phi}_i(x_i)} \tag{4.4}$$

where the second equality follows from the condition $\int_{I_i} v_i\tilde{\phi}_i = 0$. Since $v_i \in C_0(\Omega)$, we have $f \in C_0(\Omega)$ and hence $f \in C_0^1(\Omega)$ by virtue of the equation

$$\frac{\partial}{\partial x_i} f(x) = (\psi_i(x_i)f(x) + v_i(x))/c\psi_i'(x_i).$$

Substituting (4.4) in (4.3), we obtain $Ev_i(X) = 0$. Now let u, $h_i \in C_0(\Omega)$, and let $K_1 \subset K_2 \subset \Omega$ be two compact sets such that $h_i = 1$ on K_1, $h_i = 0$ on K_2^c, and $0 \leqslant h_i \leqslant 1$ on $K_2 - K_1$ and $\int_{I_i} h_i \phi_i \neq 0$. By letting $v_i = u - \frac{h_i \int_{I_i} u\phi_i}{\int_{I_i} h_i \phi_i}$, we obtain

$$(4.5) \qquad \mathrm{E}u(\mathrm{X}) = \mathrm{E}\, \frac{h_i(\mathrm{X}) \int_{I_i} u(\mathrm{X}_1, \ldots, y, \ldots, \mathrm{X}_m)\phi_i(y)dy}{\left\{ \int_{I_i} h_i(\mathrm{X}_1, \ldots, y, \ldots, \mathrm{X}_m)\phi_i(y)dy \right\}} .$$

Now let K_1 increase to Ω. Then (4.5) yields

$$(4.6) \qquad \mathrm{E}u(\mathrm{X}) = \mathrm{E} \int_{I_i} u(\mathrm{X}_1, \ldots, y, \ldots, \mathrm{X}_m)\phi_i(y)dy .$$

By approximating functions of $C_B(\Omega)$ by those of $C_0(\Omega)$, (4.6) also holds for $u \in C_B(\Omega)$. Now take u to be a function of $(x_1, \ldots, x_i)$ for $i = 1, \ldots, m$. Then it follows that $\mathrm{X}_1, \ldots, \mathrm{X}_m$ are independent with joint density function given by $f_{\mathrm{X}}(x) = \prod_{i=1}^{m} \phi_i(x_i)$. This proves the theorem.

By applying Theorem 4.1 to a number of inequalities, we obtain the following corollaries.

COROLLARY 4.1. — *Let* γ *be a probability measure on* $\mathscr{B}(\mathbb{R}^m)$. *Then* $\gamma(dx) = (2\pi\sigma^2)^{-m/2} \exp\left(- \frac{|x|^2}{2\sigma^2} \right) dx$ *if and only if the following inequality holds: For* $g \in C^1(\mathbb{R}^m)$ *such that* $\int_{\mathbb{R}^m} g^2 d\gamma < \infty$ *and* $\int_{\mathbb{R}^m} g d\gamma = 0$, $\int_{\mathbb{R}^m} g^2 d\gamma \leqslant \sigma^2 \int_{\mathbb{R}^m} |\nabla g|^2 d\gamma$ *and equality holds if and only if* $g(x) = a^* x$ *for* $a \in \mathbb{R}^m$.

Proof. — Combine Theorem 4.1 with [3, Corollary 5.1].

COROLLARY 4.2. — *Let* γ *be a probability measure on* $\mathscr{B}(\mathbb{R}^m_+)$ *such that* $\gamma(\partial \mathbb{R}^m_+) = 0$. *Then* $\gamma(dx) = \left[\prod_{i=1}^{m} (x_i/2)^{2\alpha_i - 1} / \theta^{\alpha_i} \Gamma(\alpha_i) \right] \exp\left(- \frac{|x|^2}{4\theta} \right) dx$, *where* $\theta, \alpha_1, \ldots, \alpha_m > 0$, *if and only if the following inequality holds: For* $g \in C^1(\mathbb{R}^m_+)$ *such that* $\int_{\mathbb{R}^m_+} g^2 d\gamma < \infty$ *and* $\int_{\mathbb{R}^m_+} g d\gamma = 0$, $\int_{\mathbb{R}^m_+} g^2 d\gamma \leqslant \theta \int_{\mathbb{R}^m_+} |\nabla g|^2 d\gamma$ *and* *equality holds if and only if* $g(x) = \sum_{i=1}^{m} a_i(x^2 - 4\alpha_i\theta)$ *for* $a_1, \ldots, a_m \in \mathbb{R}$.

Proof. — Combine Theorem 4.1 with [*3*, Corollary 5.5].

COROLLARY 4.3. — *Let* γ *be a probability measure on* $\mathscr{B}([0, \pi]^m)$ *such that* $\gamma(\partial [0, \pi]^m) = 0$. *Then*

$$\gamma(dx) = \left[\prod_{i=1}^{m} \frac{\Gamma(\lambda)}{\Gamma(\alpha_i)\Gamma(\beta_i)} \cos^{2\alpha_i - 1}\frac{x_i}{2} \sin^{2\beta_i - 1}\frac{x_i}{2}\right] dx, \tag{4.7}$$

where $\alpha_i > 0$, $\beta_i > 0$ *and* $\lambda = \alpha_i + \beta_i$ *for* $i = 1, \ldots, m$; *if and only if the following inequality holds: For* $g \in C^1([0, \pi]^m)$ *such that* $\int_{[0,\pi]^m} g^2 d\gamma < \infty$ *and* $\int_{[0,\pi]^m} g d\gamma = 0$,

$$\int_{[0,\pi]^m} g^2 d\gamma \leqslant \frac{1}{\lambda} \int_{[0,\pi]^m} |\nabla g|^2 d\gamma \tag{4.8}$$

and equality holds if and only if

$$g(x) = \sum_{i=1}^{m} a_i\left(\cos^2\frac{x_i}{2} - \frac{\alpha_i}{\lambda}\right) \quad \textit{for} \quad a_1, \ldots, a_m \in \mathbb{R}.$$

Proof. — We first give a sketch of the proof of the inequality (4.8) under the assumption (4.7). Let X_1 and X_2 be independent random variables having the gamma distributions $\Gamma(\alpha, \theta)$ and $\Gamma(\beta, \theta)$ respectively. Write $X = (X_1, X_2)^*$. By [*3*, Corollary 5.3], we have for $g \in C^1(\mathbb{R}_+^2)$ such that $E[g(X)]^2 < \infty$ and $Eg(X) = 0$,

$$E[g(X)]^2 \leqslant \frac{1}{\theta} \sum_{i=1}^{2} EX_i\left[\frac{\partial}{\partial x_i} g(X)\right]^2. \tag{4.9}$$

Equality holds if and only if $g(x) = a_1\left(x_1 - \frac{\alpha}{\theta}\right) + a_2\left(x_2 - \frac{\beta}{\theta}\right)$ for $a_1, a_2 \in \mathbb{R}$. Let $Y_1 = X_1/(X_1 + X_2)$ and $Y_2 = X_1 + X_2$. Then Y_1 and Y_2 are independent, Y_1 has the beta distribution $B(\alpha, \beta)$ and $Y_1 + Y_2$ the gamma distribution $\Gamma(\lambda, \theta)$ where $\lambda = \alpha + \beta$. Let $f \in C^1([0, 1])$ such that $Ef(Y_1) = 0$.

Substituting $g(X) = Y_2 f(Y_1)$ in (4.9), we obtain, after some calculations,

$$E[f(Y_1)]^2 \leqslant \frac{1}{\lambda} EY_1(1 - Y_1)[f'(Y_1)]^2 \tag{4.10}$$

which reduces to an equality if and only if $f(y) = a\left(y - \frac{\alpha}{\lambda}\right)$ for $a \in \mathbb{R}$. By suitable transformations in the spirit of [3, Corollary 5.5], (4.10) yields the one dimensional special case of (4.8) under the assumption that $\gamma(dx) = \frac{\Gamma(\lambda)}{\Gamma(\alpha)\Gamma(\beta)} \cos^{2\alpha-1}\frac{x}{2} \sin^{2\beta-1}\frac{x}{2} dx$. A martingale argument as in Chen [2] then extends it to the higher dimensions. Combining this with Theorem 4.1, we prove the corollary.

We now give an example to show that Theorem 4.1 is false if the dimension of the class of functions g for which (4.2) becomes an equality is less than the dimension of the domain Ω even though other conditions of the theorem are satisfied.

Example. — Let X_1 and X_2 be independent random variables with X_1 distributed as $N(0, 1)$ and X_2 as $N(0, \sigma^2)$ where $0 < \sigma^2 < 1$. Then by [3, Corollary 5.1], for $g \in C^1(\mathbb{R}^2)$ such that $E[g(X)]^2 < \infty$ and $Eg(X) = 0$, $E[g(X)]^2 \leqslant E|\nabla g(X)|^2$ and equality holds if and only if $g(x) = ax_1$ for $a \in \mathbb{R}$. Clearly the distribution of $X = (X_1, X^2)^*$ is not uniquely determined.

Remarks. — 1) In characterizing the distribution of X by the inequality $E[g(X)]^2 \leqslant cE|\nabla g(X)|^2$, suppose, in addition, the distribution of X is given to be uniform on a bounded domain Ω. Then the inequality becomes $\int_\Omega g^2 \leqslant c \int_\Omega |\nabla g|^2$ where $\frac{1}{c}$ is the first eigenvalue of the Laplacian Δ on Ω in the Neumann problem, and the class of functions for which the inequality becomes an equality is the first eigenspace. Furthermore, finding the distribution of X is equivalent to finding the shape of Ω. Hence the problem of characterizing the distribution of X becomes that of determining the shape of Ω given the first eigenvalue and the « form » of the functions in the first eigenspace of Δ on Ω. This is analogous to the problem « Can one hear the shape of a drum? » (see Kac [5]) where the shape of Ω is to be determined given all the eigenvalues of Δ on Ω. The answer to this question has been shown to be negative for dimension of Ω greater than or equal to 4 in both the Dirichlet and the Neumann problem (see Urakawa [7]).

2) In the case Ω is an arbitrary open set and the class of functions for

which the inequality becomes an equality is also arbitrary, the following question remains unanswered: Under what conditions is the distribution of X uniquely determined?

5. MORE ON CHARACTERIZATION. THE DISCRETE CASE

In this section we consider a discrete analog of the problem in Section 4. Let $X = (X_1, \ldots, X_m)^*$ be a random vector taking values in $\mathbb{Z}^m$ and let $p_i(z) = P(X_i = z)$. Define $e_i = (0, \ldots, 1, \ldots, 0)$ where the number 1 occupies the i-th position. Suppose for $g : \mathbb{Z}^m \to \mathbb{R}$ such that $E[g(X)]^2 < \infty$ and $Eg(X) = 0$, we have

$$E[g(X)]^2 \leqslant cE\sum_{i=1}^{m}[\Delta_i g(X)]^2 \tag{5.1}$$

and equality holds if and only if $g(x) = \psi_i(x_i)$ for $i = 1, \ldots, m$, where each $\psi_i : \mathbb{Z} \to \mathbb{R}$. We show that if $\Delta\psi_i(x_i) = \psi_i(x_i + 1) - \psi_i(x_i) \neq 0$ for $x_i \in \mathbb{Z}$, then the distribution of X is uniquely determined and $X_1, \ldots, X_m$ are independent.

By the same argument as before, we have the identities

$$E\left\{\Delta\psi_i(X_i)f(X + e_i) - \left[\Delta\psi_i(X_i) + \frac{1}{c}\psi_i(X_i)\right]f(X)\right\} = 0 \tag{5.2}$$

for $i = 1, \ldots, m$, where $f : \mathbb{Z}^m \to \mathbb{R}$ and is bounded. By choosing f to be the indicator function of $\{(x_1, \ldots, x_m) \in \mathbb{Z}^m : x_i = z\}$, (5.2) yields

$$\Delta\psi_i(z-1)p_i(z-1) - \left[\Delta\psi_i(z) + \frac{1}{c}\psi_i(z)\right]p_i(z) = 0\,. \tag{5.3}$$

Let a_i be the supremum of the set of zeros of $\Delta\psi_i + \frac{1}{c}\psi_i$. It is easy to see from (5.3) that $-\infty \leqslant a_i < \infty$. If $a_i > -\infty$, then $p_i(a_i) > 0$ and

$$p_i(z) = \frac{p_i(a_i)\Delta\psi_i(a_i)}{\Delta\psi_i(z)}\prod_{k=a_i+1}^{z}\left[1 + \frac{\psi_i(k)}{c\Delta\psi_i(k)}\right]^{-1}, \quad z > a_i\,. \tag{5.4}$$

If $a_i = -\infty$, then $p_i(0) > 0$ and

$$(5.5) \quad p_i(z) = \begin{cases} \dfrac{p_i(0)\Delta\psi_i(0)}{\Delta\psi_i(z)} \displaystyle\prod_{k=1}^{z} \left[1 + \frac{\psi_i(k)}{c\Delta\psi_i(k)}\right]^{-1}, & z \geqslant 1, \\ \dfrac{p_i(0)\Delta\psi_i(0)}{\Delta\psi_i(z)} \displaystyle\prod_{k=z+1}^{0} \left[1 + \frac{\psi_i(k)}{c\Delta\psi_i(k)}\right], & z \leqslant -1. \end{cases}$$

Note that (5.3) implies that $\Delta\psi_i(x_i - 1)$ and $\Delta\psi_i(x_i) + \frac{1}{c}\psi_i(x_i)$ have the same sign for $x_i > a_i$. Without loss of generality we assume that $a_i > -\infty$ for $i = 1, \ldots, m'$ and that $a_i = -\infty$ for $i = m' + 1, \ldots, m$.

Now let f be the indicator function of $\{ (x'_1, \ldots, x'_m) \in \mathbb{Z}^m : (x'_1, \ldots, x'_i) = (x_1, \ldots, x_i) \}$. Substitution of this in (5.2) yields a difference equation whose solution implies that X_i is independent of $(X_1, \ldots, X_{i-1})$. By induction on i, $X_1, \ldots, X_m$ are independent. Hence we have the following theorem.

THEOREM 5.1. — *Suppose the inequality (5.1) holds and equality is achieved when* $g(x) = \psi_i(x_i)$ *for* $i = 1, \ldots, m$, *where each* $\psi_i : \mathbb{Z} \to \mathbb{R}$. *If* $\Delta\psi_i$ *never vanishes for* $i = 1, \ldots, m$, *then* $X_1, \ldots, X_m$ *are independent and their individual density functions are given by (5.4) and (5.5).*

Combining this theorem with [3, Corollary 5.2], we have the following corollary.

COROLLARY 5.1. — *The random variables* $X_1, \ldots, X_m$ *are independent each having the Poisson distribution with mean* λ *if and only if the following inequality holds: For* $g : \mathbb{Z}^m \to \mathbb{R}$ *such that* $E[g(X)]^2 < \infty$ *and* $Eg(X) = 0$,

$$E[g(X)]^2 \leqslant \lambda E \sum_{i=1}^{m} [\Delta_i g(X)]^2$$ *and equality holds if and only if*

$$g(x) = a_1(x_1 - \lambda) + \ldots + a_m(x_m - \lambda)$$

for $a_1, \ldots, a_m \in \mathbb{R}$.

Added in Proof. — A. A. Borovkov and S. A. Utev pointed out an error in the proof of Theorem 3.2 and gave a correction of it as follows: The uniform integrability of $\{ X_{ni}^2 \}$ be proved by substituting $g(x) = x_i - kx_i/(k^2 + x_i^2)^{1/2}$ in the Poincaré-type inequality for X_n instead of using Theorem 3.1 *iv*).

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