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On the distribution of random walk local time

by

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ABSTRACT. — In the paper we investigate the large deviations and the asymptotic expansions as $n \to \infty$ for the distribution of the number of times a recurrent random walk with integer values hits the point r up to time n.

RÉSUMÉ. — Dans le présent article sont étudiées les grandes déviations et les expansions asymptotiques en supposant que $n \to \infty$ pour la distribution du nombre de visites de la marche aléatoire récurrente à valeurs entières au point r en n pas.

Mots-clés : Random walk, local time, distribution, large deviations, asymptotic expansions.

1. INTRODUCTION AND RESULTS

In the paper we consider the asymptotic behaviour as $n \to \infty$ of the distribution of the number of times a recurrent random walk hits the point r up to time n.

Let $\{\xi_l\}_{l=1}^{\infty}$ be i. i. d. random variables, $\mathbf{E}\xi_1 = 0$, $\mathbf{E}\xi_1^2 = \mathrm{D}$, $\varphi(t) = \mathbf{E}\exp(it\xi_1)$. We shall assume that ξ_1 have an integer values and $|\varphi(t)| = 1$ if and only if t is a multiple of 2π. Let $\nu_0 = 0$, $\nu_k = \sum_{l=1}^{k} \xi_l$, $\varphi(n, r) = \sum_{k=1}^{n} \mathbb{1}_{\{r\}}(\nu_k)$. Here and in what follows $\mathbb{1}_A(\cdot)$ is the indicator function of set A. The function $\varphi(n, r)$ is the so called local time of the random walk v_k at point r for n steps.

The local time of Brownian motion $w(s)$ ($\mathbf{E}w^2(s) = \mathrm{D}s$, $\mathrm{D} > 0$) is a jointly continuous process $\mathfrak{t}(t, x)$ that satisfies

$$\int_A \mathfrak{t}(t, x)dx = \int_0^t \mathbb{1}_A(w(s))ds,$$

for all $t \geqslant 0$ and any measurable set A a. s.

Denote by $[a]$ the integer part of a. It is known [2] that the two-parameter process $\mathfrak{t}_n(t, x) = n^{-1/2}\varphi([nt], [x\sqrt{n}])$ converges weakly as $n \to \infty$ to Brownian local time $\mathfrak{t}(t, x)$. In particular

$$\mathbf{P}(\mathfrak{t}_n(1, x) < y) \to \mathbf{P}(\mathfrak{t}(1, x) < y). \tag{1.1}$$

It seems to be of interest to give a more detailed description of the asymptotic behaviour of $\mathbf{P}(\mathfrak{t}_n(1, x) < y)$. In the paper we consider the large deviations and the asymptotic expansions for the distribution of $\mathfrak{t}_n(1, x)$. To solve these problems we apply the saddle-point method. There are a greate variety of works using this method for solving different asymptotic problems of probability theory. We do not dwell on the history of this question since one can find it in [4] and in monographs [5] [10]. For $x = 0$ an optimal rate of convergence in (1.1) was reported in [1]. For arbitrary x as a consequence of « Strong invariance principles for local times » [3] we have an estimate $Cn^{-1/4} \ln n$.

Since the problem of asymptotic behaviour of the distribution of $\mathfrak{t}_n(1, x)$ is symmetric with respect to the replacement $x \leftrightarrow -x$, $\xi_l \leftrightarrow -\xi_l$, $l = 1, 2, \ldots$, we consider only non-negative x. In view of the lattice character of the objects involved in the consideration, x, y later will be such that $x = [x\sqrt{n}]/\sqrt{n}$, $y = [y\sqrt{n}]/\sqrt{n}$.

THEOREM 1.1. — *Let ξ_1 satisfy Cramer's condition*

$$\mathbf{E}\exp(a \mid \xi_1 \mid) < \infty, \quad a > 0. \tag{C}$$

Then there exists $\varepsilon > 0$ and a power series

$$\mu(x, y) = \sum_{\substack{k+l \geqslant 3 \\ k,l \geqslant 0}} \mu_{kl} x^k y^l,$$

convergent for $|x| \leqslant \varepsilon$, $|y| \leqslant \varepsilon$, *such that for* $0 \leqslant x \leqslant \varepsilon n^{1/2}$, $n^{-1/2} \leqslant y \leqslant \varepsilon n^{1/2}$

$$\mathbf{P}\{t_n(1, x) = y\} = \sqrt{\frac{2D}{\pi n}}\, e^{-\frac{(x+Dy)^2}{2D}} e^{n\mu(x/\sqrt{n}, y/\sqrt{n})}(1 + n^{-1/2}O(1 + x + y)), \quad (1.2)$$

REMARK 1.1. — The coefficients μ_{kl} for k, $l \leqslant m + 2$ are determined by the first $m + 2$ moments of ξ_1 and by the numbers

$$b_p = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\frac{1}{(1-\varphi(s))^{p+1}} - \left(\frac{2}{Ds^2}\right)^{p+1} - \sum_{l=1}^{2p+1} s^{l-2(p+1)} \sum_{j=1}^{l} \frac{(p+j)!}{p!\,j!}\beta_{jl}\left(\frac{2}{D}\right)^{j+p+1}\right\} ds$$

with $0 \leqslant p \leqslant (m-1)/2$, where β_{jl} are defined from the equality

$$(\varphi(s) - 1 + Ds^2/2)^j = s^{2j}\left(\sum_{l=j}^{m} \beta_{jl} s^l + o(|s|^{m+1})\right). \quad (1.3)$$

REMARK 1.2. — Let $\mu = \mathbf{E}\xi_1^3$. Then

$$\mu(x, y) = (x + Dy)^2\left\{\frac{\mu}{6D^3}x + \left(b_0 - \frac{2}{\pi^2 D} - \frac{1}{2}\right)y + \dots\right\},$$

and

$$\mu(x, 0) = \frac{x^3}{D^{3/2}}\lambda\left(\frac{x}{D^{1/2}}\right),$$

where $\lambda(z)$ is well-known Cramer's power series (see, for example, [*9*], Ch. 7, § 2).

THEOREM 1.2. — *Let condition* (C) *be satisfies. Then there exists* $\varepsilon > 0$ *such that for* $0 \leqslant x \leqslant \varepsilon n^{1/2}$, $0 \leqslant y \leqslant \varepsilon n^{1/2}$

$$\mathbf{P}\{t_n(1, x) \geqslant y\} = \sqrt{\frac{2D}{\pi}} \int_y^{\infty} e^{-\frac{(x+Dv)^2}{2D}} dv\, e^{n\mu(x/\sqrt{n}, y/\sqrt{n})}(1 + n^{-1/2}O(1 + x + y)) \quad (1.4)$$

where $\mu(x, y)$ *is the same power series as in theorem 1.1.*

THEOREM 1.3. — *Let for some* $0 < \delta < 1$ *and integer* $m \geqslant 0$

$$\mathbf{E}|\xi_1|^{m+2+\delta} < \infty. \quad (1.5)$$

Then there exists $\varepsilon > 0$ *such that for* $n^{-1/2} \leqslant x \leqslant \varepsilon n^{1/2}$, $n^{-1/2} \leqslant y \leqslant \varepsilon n^{1/2}$,

$$\mathbf{P}\{ t_n(1, x) = y \}$$
$$= \sqrt{\frac{2D}{\pi n}} e^{-\frac{(x+Dy)^2}{2D}} e^{n\mu_m(x/\sqrt{n}, y/\sqrt{n})} \left\{ \sum_{k=0}^{m} n^{-k/2} P_k^1(x, y) \right.$$
$$\left. + n^{-(m+\delta)/2} O(1 + x^{m+\delta} + y^{m+\delta}) \right\} + e^{-Dy^2/2} e^{n\mu_m(0, y/\sqrt{n})}$$
$$\times \left\{ \sum_{k=1}^{m} n^{-k/2} Q_k^1(y)(x\sqrt{n})^{k-m} \theta_{mk}^1(x\sqrt{n}) + n^{-(m+\delta)/2} O(1 + y^{m+\delta}) \right\}, \quad (1.6)$$

where $|\theta_{mk}^1(v)| \leqslant C_{mk}^1 |v|^{-\delta}$, *and* $\mu_m(x, y)$ *is some function, which can be represented for* $|x| \leqslant \varepsilon$, $|y| \leqslant \varepsilon$ *in the form*

$$\mu_m(x, y) = \sum_{\substack{3 \leqslant k+l \leqslant m+2 \\ k,l \geqslant 0}} \mu_{kl} x^k y^l + O(|x|^{m+2+\delta} + |y|^{m+2+\delta}),$$

and for $0 \leqslant y \leqslant \varepsilon n^{1/2}$

$$\mathbf{P}\{ t_n(1, 0) = y \}$$
$$= \sqrt{\frac{2D}{\pi n}} e^{-\frac{Dy^2}{2}} e^{n\mu_m(0, y/\sqrt{n})} \left\{ \sum_{k=0}^{m} n^{-k/2} R_k^1(y) + n^{-(m+\delta)/2} O(1 + y^{m+\delta}) \right\}. \quad (1.7)$$

Here and later P_k^j, Q_k^j, R_k^j, $j = 1, 2, \ldots$, are some polynomials of degree at most k, $P_0^j = Q_0^j = R_0^j = 1$, and the functions $O(\cdot)$ depend on δ, m, and the distribution of ξ_1.

THEOREM 1.4. — *Let condition (1.5) be satisfied. Then there exists* $\varepsilon > 0$ *such that for* $n^{-1/2} \leqslant x \leqslant \varepsilon n^{1/2}$, $n^{-1/2} \leqslant y \leqslant \varepsilon n^{1/2}$

$$\mathbf{P}\{ t_n(1, x) \geqslant y \} = e^{n\mu_m(x/\sqrt{n}, y/\sqrt{n})} \left\{ \sqrt{\frac{2}{\pi D}} \int_{x+Dy}^{x} e^{-\frac{v^2}{2D}} dv \sum_{k=0}^{m} n^{-k/2} P_{3k}^2(x, y) \right.$$
$$\left. + e^{-\frac{(x+Dy)^2}{2D}} \left(\sum_{k=1}^{m} n^{-k/2} Q_{3k}^2(x, y) + n^{-(m+\delta)/2} O(1 + x^{3m+\delta} + y^{3m+\delta}) \right) \right\}$$
$$+ e^{-\frac{Dy^2}{2}} e^{n\mu_m(0, y/\sqrt{n})} \left\{ \sum_{k=0}^{m-1} n^{-(k+1)/2} R_k^2(y)(x\sqrt{n})^{k+1-m} \theta_{mk}^2(x\sqrt{n}) \right.$$
$$\left. + n^{-(m+\delta)/2} O(1 + y^{m+\delta}) \right\}, \quad (1.8)$$

for $0 \leqslant y \leqslant \varepsilon n^{1/2}$

$$\mathbf{P}\{t_n(1,0) \geqslant y\} = e^{n\mu_m(0,y/\sqrt{n})}\left\{\sqrt{\frac{2D}{\pi}}\int_y^{\infty} e^{-Dv^2/2}dv \sum_{k=0}^{m} n^{-k/2}P_{3k}^3(y) + e^{-Dy^2/2}\left(\sum_{k=1}^{m} n^{-k/2}Q_{3k}^3(y) + n^{-(m+\delta)/2}O(1+y^{3m+\delta})\right)\right\}, \quad (1.9)$$

and for $n^{-1/2} \leqslant x \leqslant \varepsilon n^{1/2}$

$$\mathbf{P}\{t_n(1,x)=0\} = \sqrt{\frac{2}{\pi D}}\left[\int_0^x e^{-\frac{v^2}{2D}}dv + \int_x^{\infty} e^{-\frac{v^2}{2D}}dv \sum_{k=1}^{m} n^{-k/2}P_{3k}^4(x) + e^{-\frac{x^2}{2D}}\left(\sum_{k=1}^{m} n^{-k/2}Q_{3k}^4(x) + n^{-(m+\delta)/2}O(1+x^{3m+\delta})\right)\right]$$
$$+ \sum_{k=0}^{m-1} n^{-(k+1)/2}(x\sqrt{n})^{k+1-m}\theta_{mk}^4(x\sqrt{n}) + n^{-(m+\delta)/2}O(1), \quad (1.10)$$

where $|\theta_{mk}^j(v)| \leqslant C_{mk}^j |v|^{-\delta}$, $j = 2, 4$, *and* $\mu_m(x, y)$ *is the same function as in theorem 1.3.*

REMARK 1.3. — The coefficients at $x^k y^l$ in the decomposition of $\mu_m(x, y)$ coincide with those for $\mu(x, y)$ and the polynomials P_k^1, Q_k^1, P_{3k}^j, Q_{3k}^j, $j = 2, 3, 4$, and R_k^j, $j = 1, 2$, are determined by the first $k + 2$ moments of ξ_1 and by the numbers b_p with $0 \leqslant p \leqslant (k-1)/2$. The way of construction of $\mu_m(x, y)$ and of the polynomials is described during the proof.

2. PRELIMINARY RESULTS

Denote $\chi_{n,r}(v) = \mathbf{E}\exp(iv\varphi(n,r))$. Then

$$\chi_{n,r}(v) = 1 + \sum_{k=1}^{n} \mathbf{E}\left[(e^{iv\mathbb{1}_{\{\nu_k=r\}}} - 1)e^{iv\sum_{l=k+1}^{n}\mathbb{1}_{\{\nu_l=r\}}}\right]$$
$$= 1 + (e^{iv} - 1)\sum_{k=1}^{n} \mathbf{E}\left[\mathbb{1}_{\{\nu_k=r\}}e^{iv\sum_{l=k+1}^{n}\mathbb{1}_{\{\nu_l=r\}}}\right]$$
$$= 1 + (e^{iv} - 1)\sum_{k=1}^{n} \mathbf{P}(\nu_k = r)\chi_{n-k,0}(v). \quad (2.1)$$

For any complex $|z| < 1$ define

$$g_r(z) = \sum_{k=0}^{\infty} z^k \mathbf{P}(v_k = r), \qquad l_{v,r}(z) = \sum_{n=0}^{\infty} z^n \chi_{n,r}(v).$$

From (2.1) it follows that

$$l_{v,0}(z) = (1-z)^{-1}(e^{iv} + (1-e^{iv})g_0(z))^{-1}, \tag{2.2}$$

$$l_{v,r}(z) = (1-z)^{-1}(1 + (e^{iv}-1)g_r(z)(e^{iv} + (1-e^{iv})g_0(z))^{-1}), \quad r \neq 0. \tag{2.3}$$

Since $l_{v,r}(z)$ is analytic in $|z| < 1$ then for $0 < \varepsilon < 1$

$$\chi_{n,r}(v) = \frac{1}{2\pi i}\int_{|z|=\varepsilon} z^{-(n+1)} l_{v,r}(z)dz.$$

Applying the inversion formula one obtains

$$\mathbf{P}(\varphi(n,0) = q) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ivq}\chi_{n,0}(v)dv$$

$$= -\frac{1}{(2\pi)^2}\int_{|z|=\varepsilon}\frac{dz}{(1-z)z^{n+1}}\int_{|\omega|=1}\frac{\omega^q d\omega}{1+(\omega-1)g_0(z)}.$$

For sufficiently small z the point $\omega_0 = 1 - 1/g_0(z)$ will lie inside the circle $|\omega| \leqslant 1$. By the residue theorem

$$\int_{|\omega|=1}\frac{\omega^q d\omega}{1+(\omega-1)g_0(z)} = 2\pi i\left(1 - \frac{1}{g_0(z)}\right)^q \frac{1}{g_0(z)}, \tag{2.4}$$

and, consequently, for sufficiently small ε

$$\mathbf{P}(\varphi(n,0) = q) = \frac{1}{2\pi i}\int_{|z|=\varepsilon}\left(1 - \frac{1}{g_0(z)}\right)^q \frac{dz}{g_0(z)z^{n+1}(1-z)}. \tag{2.5}$$

In view of the fact that

$$\frac{1}{2\pi i(1-z)}\int_{|\omega|=1}\frac{\omega^q d\omega}{1+(\omega-1)g_0(z)} = \sum_{n=0}^{\infty} z^n \mathbf{P}(\varphi(n,0) = q)$$

the left-hand side of (2.4) is analytic in $|z| < 1$. The function $g_0(z)$ is analytic in $|z| < 1$ also and thus it has for any $\delta > 0$ in $|z| \leqslant 1 - \delta$ at most the finite number of zeros. Then by the uniqueness of analytical continuation the equation (2.4) holds for all $|z| < 1$ and $g_0(z) \neq 0$ if $|z| < 1$. Moreover for any $\theta \in [-\pi, \pi)$

$$\lim_{\substack{z \to e^{i\theta} \\ |z|<1}} g_0(z) \neq 0 \tag{2.6}$$

because if it is not the case then passage to the limit in (2.4) as $z \to e^{i\theta}$ leads to a contradiction.

Using the assumptions $\varphi(s) = 1 - \frac{D}{2} s^2 + 0(s^2)$, $s \to 0$,

$$|\varphi(s)| < 1 \qquad \text{if} \quad s \in [-\pi, \pi]\backslash\{0\} \tag{2.7}$$

we can choose $\Delta > 0$ so small that the open domain $\mathfrak{A}$ restricted by the contour consisting of the line segment $(1, 1 + \Delta e^{i\pi/4})$, the large circular arc (center 0) from $1+\Delta e^{i\pi/4}$ to $1+\Delta e^{-i\pi/4}$ and the line segment $(1+\Delta e^{-i\pi/4}, 1)$, has no common points with the curve $1/\varphi(s)$, $s \in [-\pi, \pi]$. By the inversion formula we have

$$g_r(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-irs}}{1 - z\varphi(z)} ds .$$

Hence the function $g_r(z)$ is analytic in the open domain $\mathfrak{A}$.

Now we should distinguish the conditions (C) and (1.5). If ξ_1 satisfies Cramer's condition (C) we set $\psi(t) = \varphi(it)$. Then $\psi(t)$ has an analytical continuation to the strip $|\operatorname{Re} z| < a$. If ξ_1 satisfies only (1.5) we set $\psi(t) = \varphi_m(it)$, where for some positive A

$$\varphi_m(s) = \mathbf{E}[e^{is\xi_1}\mathbb{1}_{\{|\xi_1| \leqslant A\}}] + \sum_{k=0}^{m+2} \frac{i^k s^k}{k!} \mathbf{E}[\xi_1^k \mathbb{1}_{\{|\xi_1| > A\}}] .$$

In this case the function $\psi(t)$ is analytic in the complex plane. The constant A will be chosen according to the property: for any $\rho_1 > 0$ we can choose A such that

$$\sup_{s \in [-\pi,\pi]} |\varphi(s) - \varphi_m(s)| < \rho_1 . \tag{2.8}$$

Moreover one can prove that for any $0 \leqslant p \leqslant m+2$ and $s \in [-\pi, \pi]$

$$\frac{d^p}{ds^p} \varphi(s) - \frac{d^p}{ds^p} \varphi_m(s) = s^{m+2-p+\delta}\theta_{mp}(s), \quad |\theta_{mp}(s)| \leqslant C_{mp} . \tag{2.9}$$

Consider the equation

$$z = 1/\psi(t) . \tag{2.10}$$

Since $(1/\psi(t))'|_{t=0} = 0$ and $(1/\psi(t))''|_{t=0} = 0$ then for any sufficiently small $\Delta > 0$ there exists such $\rho > 0$ that for any z, $|z - 1| < \Delta$, equation (2.10) has in $|t| < \rho$ exactly two solutions [6, Ch. 4, § 7]. They can be constructed as follows. Let

$$\pi(t) = t\sqrt{(1 - 1/\psi(t))/t^2}, \quad \pi'(0) = \sqrt{\frac{D}{2}}, \quad \pi''(0) = -\frac{\mu}{3\sqrt{D}},$$

and $\tau(u)$ be the inverse function to $\pi(t)$, $\tau'(0) = \sqrt{2/D}$, $\tau''(0) = 2\mu/3D^2$. Under condition (1.5), $m = 0$, we have $\mu = \mathbf{E}[\xi_1^3 \mathbb{1}_{\{|\xi_1| \leqslant A\}}]$. In other cases $\mu = \mathbf{E}\xi_1^3$. Then the solutions of (2.10) are given by

$$t_+(z) = \tau(\sqrt{1 - z}), \quad t_-(z) = \tau(-\sqrt{1 - z}).$$

Here $\sqrt{\omega} = \sqrt{|\omega|} \exp\left(\frac{i}{2} \arg \omega\right)$, $-\pi < \arg \omega \leqslant \pi$. Define the contour consisting of the line segment $(1, 1 + \Delta e^{i\pi/4})$, the large circular arc (center 1) from $1 + \Delta e^{i\pi/4}$ to $1 + \Delta e^{-i\pi/4}$ and the line segment $(1 + \Delta e^{-i\pi/4}, 1)$. Let $\mathfrak{N}$ be the open domain restricted by this contour. It is easy to prove that we can choose Δ so small that for any $z \in \mathfrak{N}$, $z \neq 1$,

$$\operatorname{Re} t_-(z) < 0, \quad \operatorname{Re} t_+(z) > 0. \tag{2.11}$$

Note that above Δ was chosen so that the set $\mathfrak{N}$ has no common points with the curve $1/\varphi(s)$, $s \in [-\pi, \pi]$.

LEMMA 2.1. — *There exists such $\gamma > 0$ independent of* A *that for any real v and t satisfying* $0 \leqslant 3|t| \leqslant |v - t| \leqslant \gamma$ *holds*

$$|\varphi_m(v) - \varphi_m(t)| \geqslant \frac{D}{8} |v - t|^2.$$

This lemma is easily proved with the help of application of Taylor's formula, and we omit the proof.

Let $\gamma_\alpha(s) = \alpha + |s| e^{i\frac{11}{16}\pi \operatorname{sign} s}$. Since $\frac{11}{16}\pi > \frac{5}{8}\pi$ then one can choose $\rho_3 > 0$ so small that $\gamma_\alpha(s) \in t_-(\mathfrak{N})$ for all $\alpha \in [-\rho_3, 0]$ and $|s| \leqslant \rho_3$. For $\sigma \in (0, \Delta)$ let $\varepsilon^+(\alpha) = \min\{ s : |\Gamma_\alpha(s)| = 1 + \sigma, s > 0 \}$,

$$\varepsilon^-(\alpha) = \min\{ |s| : |\Gamma_\alpha(s)| = 1 + \sigma, s < 0 \}, \Gamma_\alpha(s) = 1/\Psi(\gamma_\alpha(s)).$$

Now one can choose σ and α_1 so that for all $\alpha \in [\alpha_1, 0]$ and $s \in [-\varepsilon^-(\alpha), \varepsilon^+(\alpha)]$ holds $\Gamma_\alpha(s) \in \mathfrak{N}$. In addition by the choice of σ and α_1 one can obtain that for all $\alpha \in [\alpha_1, 0]$ the inequality $c_1\sqrt{\sigma} \leqslant \varepsilon^\pm(\alpha) \leqslant c_2\sqrt{\sigma}$ where c_1 and c_2 are some positive constants, be valid. This can be done in view of the relation

$$\Gamma_\alpha(s) = \frac{1}{\psi(\alpha)} - (D\alpha + O(|\alpha|)) s e^{i\frac{11}{16}\pi \operatorname{sign} s} + \frac{1}{2}(D + O(|\alpha| + |s|)) s^2 e^{i\frac{3}{8}\pi \operatorname{sign} s}.$$

Where there is no ambiguity, we shall denote $\varepsilon^+(\alpha)$ and $\varepsilon^-(\alpha)$ by ε. Denote the large circular arc (center 0) from $\Gamma_\alpha(\varepsilon^+(\alpha))$ to $\Gamma_\alpha(-\varepsilon^-(\alpha))$ by $C_\sigma(\alpha)$.

Let $\mathfrak{S}(\alpha)$ be the open domain restricted by the contour consisting of $\Gamma_\alpha(s)$, $s \in [-\varepsilon, \varepsilon]$ and $C_\sigma(\alpha)$. It is clear that $\mathfrak{S}(\alpha) \subset \mathfrak{S}(0)$.

It will be proved that $g_0(z) \to \infty$ as $z \to 1$, $z \in \mathfrak{S}(0)$. Then in view of (2.6) and the fact that $g_0(z)$ is analytic in $\mathfrak{A}$ we can choose σ so small that $g_0(z)$ has no roots in $\overline{\mathfrak{S}(0)}$. Thus the integrand in (2.5) is analytic in $\mathfrak{S}(0)$ and we may therefore deform the contour in (2.5) into $\partial\mathfrak{S}(\alpha) = \Gamma_\alpha + C_\sigma(\alpha)$. Consequently

$$\mathbf{P}(\varphi(n, 0) = q) = \frac{1}{2\pi i} \int_{\partial\mathfrak{S}(\alpha)} \left(1 - \frac{1}{g_0(z)}\right)^q \frac{dz}{g_0(z) z^{n+1}(1 - z)}. \quad (2.12)$$

Analogously to (2.12) from (2.3) for $r \neq 0$ we obtain

$$\mathbf{P}(\varphi(n, r) = q) = \frac{1}{2\pi i} \int_{\partial\mathfrak{S}(\alpha)} \left(1 - \frac{1}{g_0(z)}\right)^{q-1} \frac{g_r(z) dz}{g_0^2(z) z^{n+1}(1 - z)}, \quad q \geqslant 1, \quad (2.13)$$

$$\mathbf{P}(\varphi(n, r) = 0) = \frac{1}{2\pi i} \int_{\partial\mathfrak{S}(\alpha)} \left(1 - \frac{g_r(z)}{g_0(z)}\right) \frac{dz}{z^{n+1}(1 - z)}. \quad (2.14)$$

From (2.12)-(2.14) using the equation $\mathbf{P}(\varphi(n, r) \geqslant q) = 1 - \sum_{k=0}^{q-1} \mathbf{P}(\varphi(n, r) = k)$ one can find

$$\mathbf{P}(\varphi(n, 0) \geqslant q) = \frac{1}{2\pi i} \int_{\partial\mathfrak{S}(\alpha)} \left(1 - \frac{1}{g_0(z)}\right)^q \frac{dz}{z^{n+1}(1 - z)} \quad (2.15)$$

and for $r \neq 0$, $q \geqslant 1$

$$\mathbf{P}(\varphi(n, r) \geqslant q) = \frac{1}{2\pi i} \int_{\partial\mathfrak{S}(\alpha)} \frac{g_r(z)}{g_0(z)} \left(1 - \frac{1}{g_0(z)}\right)^{q-1} \frac{dz}{z^{n+1}(1 - z)}. \quad (2.16)$$

Now we shall prove that if $q \leqslant \varepsilon n$ and ε is sufficiently small then the integral along $C_\sigma(\alpha)$ in (2.13) is $O(e^{-n\nu})$ for some $\nu > 0$. It is majorized by

$$(2\pi\sigma)^{-1}(1 + \sigma)^{-n-1} \int_{C_\sigma(0)} \left(1 + \frac{1}{|g_0(z)|}\right)^{q-1} \frac{|g_r(z)|}{|g_0(z)|^2} |dz|. \quad (2.17)$$

Since $g_0(z) \neq 0$ for $z \in C_\sigma(0)$ then $\sup_{z \in C_\sigma(0)} |g_0(z)|^{-1} \leqslant C_\sigma$. Using the expression for $g_r(z)$ we obtain

$$\sup_{z \in C_\sigma(0)} |g_r(z)| \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \sup_{z \in C_\sigma(0)} |1 - z\varphi(s)|^{-1} ds \leqslant C.$$

Here and in what follows C with or without indexes stands for different constants. Denote $\tau = q/n$. Choose τ_0 so small that $(1+\sigma)/(1+C_\sigma)^\tau \geqslant 1 + \sigma^2$ for $\tau \leqslant \tau_0$. Then for $\tau \leqslant \tau_0$ we have the following estimate for (2.17)

$$CC_\sigma^2(2\pi(1 + C_\sigma)\sigma)^{-1}(1 + \sigma^2)^{-n} \leqslant Ce^{-n\nu}.$$

The same estimate is true for the integrals along $C_\sigma(\alpha)$ in (2.12) and (2.14)-(2.16).

Denote $\gamma_\alpha = \{ \gamma_\alpha(s) : s \in [-\varepsilon^-(\alpha), \varepsilon^+(\alpha)] \}$,

$$L_r(t) = g_r(1/\psi(t)), \quad U(t) = 1 - 1/g_0(1/\psi(t)).$$

It is clear that $L_r(t)$ is defined for $t \in t_-(\mathfrak{N})$. In the integral (2.13) taken over Γ_α we make the change $t = t_-(z)$ of the variables. For the integral (2.13) taken over $C_\sigma(\alpha)$ we use the estimate stated above. Then for $r \neq 0$

$$\mathbf{P}(\varphi(n, r) = q) = -\frac{1}{2\pi i}\int_{\gamma_\alpha} \frac{(U^\tau(t)\psi(t))^n L_r(t)\psi'(t)}{L_0(t)(L_0(t)-1)(\psi(t)-1)}\,dt + O(e^{-n\nu}). \quad (2.18)$$

Analogously

$$\mathbf{P}(\varphi(n, 0) = q) = -\frac{1}{2\pi i}\int_{\gamma_\alpha} \frac{(U^\tau(t)\psi(t))^n \psi'(t)}{L_0(t)(\psi(t) - 1)}\,dt + O(e^{-n\nu}), \quad (2.19)$$

$$\mathbf{P}(\varphi(n, 0) \geqslant q) = -\frac{1}{2\pi i}\int_{\gamma_\alpha} \frac{(U^\tau(t)\psi(t))^n \psi'(t)}{\psi(t) - 1}\,dt + O(e^{-n\nu}), \quad (2.20)$$

and for $r \neq 0$, $q \geqslant 1$

$$\mathbf{P}(\varphi(n, r) \geqslant q) = -\frac{1}{2\pi i}\int_{\gamma_\alpha} \frac{(U^\tau(t)\psi(t))^n L_r(t)\psi'(t)}{(L_0(t)-1)(\psi(t)-1)}\,dt + O(e^{-n\nu}), \quad (2.21)$$

$$\mathbf{P}(\varphi(n, r) = 0) = -\frac{1}{2\pi i}\int_{\gamma_\alpha} \left(1 - \frac{L_r(t)}{L_0(t)}\right) \frac{\psi^n(t)\psi'(t)}{(\psi(t)-1)}\,dt + O(e^{-n\nu}). \quad (2.22)$$

3. PROOFS OF THEOREMS 1.1, 1.2

Let us construct the analytical continuation of function $L_r(t)$ from $t_-(\mathfrak{N})$ to the ring $0 < |t| < \varepsilon_1$ for small ε_1. The point $t = 0$ is the pole of the first order.

If $t \in t_-(\mathfrak{N})$ then by (2.11) the equation

$$\psi(v) = \psi(t) \quad (3.1)$$

has in $(|v| < \rho) \cap (\mathrm{Re}\, v \leqslant 0)$, where ρ is defined earlier, the unique solution $v = t$. For some $0 < \gamma < a$, where a is taken from condition (C) consider the rectangular contour $L = L_1 + L_2 + L_3 + L_4$, $L_1 = [-i\pi, i\pi]$, $L_2 = (i\pi, i\pi - \gamma)$, $L_3 = [i\pi - \gamma, -i\pi - \gamma]$, $L_4 = (-i\pi - \gamma, -i\pi)$. In view of (2.7) we can choose ε_1 and γ, $0 < \varepsilon_1 < \gamma$ so small that for $|t| < \varepsilon_1$

the equation (3.1) inside the contour L has the unique solution $v = t$. By the residue theorem

$$\int_L \frac{e^{rv}}{\psi(t) - \psi(v)} dv = -2\pi i \frac{e^{rt}}{\psi'(t)}.$$

Since the function $e^{rv}(\psi(t) - \psi(v))^{-1}$ is periodic with period $2\pi i$ then the integral along $L_2 + L_4$ is zero and

$$\begin{aligned} L_r(t) &\equiv \frac{\psi(t)}{2\pi i} \int_{-i\pi}^{i\pi} \frac{e^{rv}}{\psi(t) - \psi(v)} dv \\ &= -\frac{e^{rt}\psi(t)}{\psi'(t)} + \frac{\psi(t)}{2\pi i} \int_{-i\pi-\gamma}^{i\pi-\gamma} \frac{e^{rv}}{\psi(t) - \psi(v)} dv. \end{aligned} \tag{3.2}$$

For $0 \leqslant 3|t| \leqslant |v-t| < \gamma$, if γ is sufficiently small, $|\psi(v)-\psi(t)| \geqslant C|t-v|^2$. Then taking into account (2.7) we may choose $0 < \varepsilon_1 < \gamma$ so that for $v \in [-i\pi - \gamma, i\pi - \gamma]$ and $|t| < \varepsilon_1$

$$|\psi(v) - \psi(t)| \geqslant d \tag{3.3}$$

for some $d > 0$. Thus the integral in the right-hand side of (3.2) is analytic function in $|t| < \varepsilon_1$.

Consider the question how we can calculate the coefficients in a power series expansion about $t = 0$ for function $L_0(t)$. We should find the expression for

$$J_p = \int_{-i\pi-\gamma}^{i\pi-\gamma} \frac{dv}{(1 - \psi(v))^{p+1}}, \quad p = 0, 1, 2, \ldots$$

Let $L' = L_2 + L_3 + L_4$. Then using Cauchy's theorem obtain

$$J_p = -\int_{L'} \frac{dv}{(1-\psi(v))^{p+1}} = \int_{-i\pi}^{i\pi} \left(\frac{1}{(1-\psi(v))^{p+1}} - f_p(v) \right) dv - \int_{L'} f_p(v) dv, \tag{3.4}$$

where $f_p(v)$ is chosen so that the integrand in (...) is analytic function inside the contour L. Moreover $f_p(v)$ must be chosen such that the last integral in (3.4) can be easily calculated. Let us prove an auxiliary proposition.

PROPOSITION 3.1. — For any B, G, $L = G - B$ and integer $N \geqslant 0$, $p \geqslant 0$

$$\frac{1}{B^{p+1}} = \frac{1}{G^{p+1}} \left[\sum_{j=0}^{N} \frac{(p+j)!}{p!j!} \left(\frac{L}{G}\right)^j + \left(\frac{L}{B}\right)^{N+1} \sum_{j=0}^{p} \frac{(N+j)!}{N!j!} \left(\frac{B}{G}\right)^{j+N-p} \right]. \tag{3.5}$$

Proof. — Equation (3.5) for $N = 0$ is almost obvious. We suppose that (3.5) holds for some N and every $p \geqslant 0$ and show that it then holds for $N + 1$. Using the formula, which is easily proved by induction on s

$$\sum_{j=0}^{s} \frac{(p+j)!}{p!j!} = \frac{(p+1+s)!}{(p+1)!s!}, \quad s \geqslant 0, \quad p \geqslant 0, \tag{3.6}$$

and (3.5) for $N = 0$ we have

$$\frac{1}{B^{p+1}} = \frac{1}{G^{p+1}}\left\{\sum_{j=0}^{N} \frac{(p+j)!}{p!j!}\left(\frac{L}{G}\right)^{j} + L^{N+1}\sum_{j=0}^{p} \frac{(N+j)!}{N!j!}\frac{1}{G^{j+N-p}}\left[\frac{1}{G^{p-j+1}}\right.\right.$$
$$\left.\left. + \frac{L}{B}\sum_{k=0}^{p-j}\frac{1}{G^{k+1}B^{p-j-k}}\right]\right\} = \frac{1}{G^{p+1}}\left\{\sum_{j=0}^{N+1}\frac{(p+j)!}{p!j!}\left(\frac{L}{G}\right)^{j}\right.$$
$$\left. + \left(\frac{L}{B}\right)^{N+2}\sum_{j=0}^{p}\frac{(N+j)!}{N!j!}\sum_{s=j}^{p}\left(\frac{B}{G}\right)^{s+N+1-p}\right\}.$$

Changing the order of summation in the last sum and again applying (3.6) we obtain (3.5) for $N + 1$.

Setting in (3.5) $B = 1 - \varphi(s)$, $G = Ds^2/2$, $N = 2p + 1$ and using the decomposition (1.3), $m = 2p + 1$ one can verify that the function

$$\frac{1}{(1-\varphi(s))^{p+1}} - \left(\frac{1}{Ds^2}\right)^{p+1} - \sum_{l=1}^{2p+1} s^{l-2(p+1)} \sum_{j=1}^{l} \frac{(p+j)!}{p!j!}\beta_{jl}\left(\frac{2}{D}\right)^{j+p+1}$$

is integrable. Finally substituting $v = -is$ in (3.4) we find

$$J_p = 2\pi i b_p - i\left(\frac{2}{D}\right)^{p+1} 2(2p+1)^{-1}\pi^{-2p-1}$$
$$- 2i\sum_{k=1}^{p}(2(p-k)+1)^{-1}\pi^{2(k-p)-1}\sum_{j=1}^{2k}\frac{(p+j)!}{p!j!}\beta_{jl}\left(\frac{2}{D}\right)^{j+p+1}$$
$$- \pi\sum_{j=1}^{2p+1}\frac{(p+j)!}{p!j!}\beta_{j(2p+1)}\left(\frac{2}{D}\right)^{j+p+1}. \tag{3.7}$$

So we have

$$L_0(t) = -\frac{1}{Dt} - \frac{2}{\pi^2 D} - \frac{\mu}{6D^2} + b_0 + \dots \tag{3.8}$$

Hence

$$U(t) = 1 + Dt + D^2\left(b_0 - \frac{2}{\pi^2 D} - \frac{\mu}{6D^2}\right)t^2 + \dots \tag{3.9}$$

Define $H(t) = \tau \ln U(t) + \ln \psi(t) + \beta t$, where $\beta = r/n$. Here and later $\ln \omega = \ln |\omega| + i \arg \omega$, $-\pi < \arg \omega \leqslant \pi$.

Consider the saddle-point equation

$$\frac{dH}{dt} = 0. \tag{3.10}$$

Since $\frac{dH}{dt}$ is analytic function in the variables (β, τ, t) in a small neighbourhood of the point $(0, 0, 0)$, $\left.\frac{dH}{dt}\right|_{\beta=\tau=t=0} = 0$, $\left.\frac{d^2H}{dt^2}\right|_{\beta=\tau=t=0} = D$, then the equation (3.10) has (see, for example, [*8*], Ch. 1, § B, Th. 4) in some neighbourhood of the point (0, 0) the analytical solution

$$t_0 = t_0(\beta, \tau) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \gamma_{kl}\beta^k\tau^l, \quad \gamma_{00} = 0. \tag{3.11}$$

The function $H(t)$ for $t \in \mathbf{R}^1$ is real, therefore t_0 will lie on the real axis. Taking into account (3.9) one can find

$$t_0 = (\beta + \tau D)\left\{-\frac{1}{D} + \frac{\mu}{2D^3}\beta + \left(2b_0 - 1 - \frac{4}{\pi^2 D} + \frac{\mu}{6D^2}\right)\tau + \dots\right\}. \tag{3.12}$$

The fact that t_0 is a multiple of $(\beta + \tau D)$ one can prove as follows. It is sufficient to set $\beta = -\tau D$ in (3.10) and to prove that in this case the equation (3.10) will have the solution equal to zero. Using (3.9) we obtain

$$H(t_0) = -\frac{(\beta + D\tau)^2}{2D} + (\beta + D\tau)^2\left\{\frac{\mu}{6D^3}\beta + \left(b_0 - \frac{2}{\pi^2 D} - \frac{1}{2}\right)\tau + \dots\right\}, \tag{3.13}$$

$$H''(t_0) = D + \frac{\mu}{D}\beta + \left(\mu + D^2\left(2b_0 - \frac{\mu}{3D^2} - \frac{4}{\pi^2 D} - 1\right)\right)\tau + \dots \tag{3.14}$$

We now proceed to the proof of (1.2). Denote $H_2 = H''(t_0)/2$,

$$b_r(t) = -\frac{\psi'(t)L_r(t)e^{-rt}}{L_0(t)(L_0(t) - 1)(\psi(t) - 1)}, \quad b_r(0) = 2D.$$

Making use of (3.2) we receive

$$b_r(t) = 2D + t\theta_r, \qquad |\theta_r| \leqslant Ce^{-\gamma r/2}.$$

From (2.18) it follows that for some $\lambda > 0$

$$\mathbf{P}(\varphi(n,r)=q)=\frac{e^{n\mathrm{H}(t_0)}}{2\pi i}\int_{\gamma_{t_0}} e^{n(\mathrm{H}(t)-\mathrm{H}(t_0))}b_r(t)dt+\mathrm{O}(e^{-n\nu})$$
$$=\frac{e^{n\mathrm{H}(t_0)}}{2\pi i}\left\{\int_{t_0-i\varepsilon}^{t_0+i\varepsilon} e^{n(\mathrm{H}(t)-\mathrm{H}(t_0))}b_r(t)dt+\mathrm{O}(e^{-n\lambda})\right\}+\mathrm{O}(e^{-n\nu}). \quad (3.15)$$

Here we apply Cauchy's theorem and simple estimates for the integrals along $l_1 = (\gamma_{t_0}(\varepsilon), t_0 + i\varepsilon)$, $l_2 = (t_0 - i\varepsilon, \gamma_{t_0}(-\varepsilon))$:

$$\left|\int_{l_i} e^{n(\mathrm{H}(t)-\mathrm{H}(t_0))}b_r(t)dt\right| \leqslant \mathrm{C}e^{-n\lambda}, \quad i = 1, 2. \quad (3.16)$$

This estimate is due to the fact that in the definition of $\gamma_\alpha(s)$ the angle of the slope was chosen such that it belongs to $(\pi/2, 3\pi/4)$ or to $(-3\pi/4, -\pi/2)$.

Using Taylor's theorem we have

$$\int_{t_0-i\varepsilon}^{t_0+i\varepsilon} e^{n(\mathrm{H}(t)-\mathrm{H}(t_0))}b_r(t)dt = i\int_{-\varepsilon}^{\varepsilon} e^{-n(\mathrm{H}_2s^2+\mathrm{O}(|s|^3))}b_r(is + t_0)ds$$
$$= 2\mathrm{D}i\int_{-\varepsilon}^{\varepsilon} e^{-n\mathrm{H}_2s^2}(1 + \mathrm{O}(|s| + |t_0|))ds + \int_{-\varepsilon}^{\varepsilon} e^{-n\mathrm{H}_2s^2/2}\mathrm{O}(n|s|^3)ds$$
$$= 2\mathrm{D}i\left(\sqrt{\frac{\pi}{n\mathrm{H}_2}} + \mathrm{O}\left(\frac{1}{n} + \frac{|t_0|}{\sqrt{n}}\right)\right) = 2i\sqrt{\frac{2\pi\mathrm{D}}{n}}\left(1 + \mathrm{O}\left(\frac{1}{\sqrt{n}} + \beta + \tau\right)\right).$$

Combining this with (3.15), (3.14) and letting $r = x\sqrt{n}$, $q = y\sqrt{n}$, we obtain (1.2) for $x \neq 0$. If $x = 0$ we should use the formula (2.19) instead of (2.18). Here, in the saddle-point equation (3.10), one should take $\beta = 0$ and set

$$b_0(t) = -\psi'(t)(\mathrm{L}_0(t)(\psi(t) - 1))^{-1}.$$

To prove theorem 1.2 for $x \neq 0$, $y \geqslant 1/\sqrt{n}$ we start from (2.21). Similarly to the previous case we have

$$\mathbf{P}(\varphi(n,r)\geqslant q)$$
$$=\frac{e^{n\mathrm{H}(t_0)}}{2\pi i}\left\{\int_{t_0-i\varepsilon}^{t_0+i\varepsilon} e^{n(\mathrm{H}(t)-\mathrm{H}(t_0))}(-2/t+\mathrm{O}(1))dt+\mathrm{O}(\sigma^{-1/2}e^{-n\lambda})\right\}+\mathrm{O}(e^{-n\nu})$$
$$=\frac{e^{n\mathrm{H}(t_0)}}{2\pi-}\left\{-2\int_{-\varepsilon}^{\varepsilon} e^{-n\mathrm{H}_2s^2}\frac{ds}{is+t_0}+\int_{-\varepsilon}^{\varepsilon} e^{-n\mathrm{H}_2s^2/2}\mathrm{O}\left(\frac{n|s|^3}{|is+t_0|}\right)ds+\mathrm{O}\left(\frac{1}{\sqrt{n}}\right)\right\}$$
$$=-\frac{e^{n\mathrm{H}(t_0)}}{\pi}\left\{\int_{-\infty}^{\infty} e^{-n\mathrm{H}_2s^2}\frac{ds}{is+t_0}+\mathrm{O}\left(\frac{1}{\sqrt{n}}\right)\right\}, \quad (3.17)$$

Further

$$-\int_{-\infty}^{\infty} e^{-nH_2 s^2}\frac{ds}{is+t_0} = -2t_0\int_0^{\infty} e^{-nH_2 s^2}\frac{ds}{t_0^2+s^2} = \sqrt{2\pi}e^{nH_2 t_0^2}\int_{\sqrt{2nH_2}|t_0|} e^{-v^2/2}dv.$$

Let

$$R(x) = e^{x^2/2}\int_x^{\infty} e^{-v^2/2}dv. \tag{3.18}$$

The following formula is well-known

$$R(x) = \sum_{j=0}^{k-1}(-1)^j\frac{(2j-1)!!}{x^{2j+1}} + R_k(x), \quad k = 1, 2, \ldots, \tag{3.19}$$

where $| R_k(x) | \leqslant \dfrac{(2k-1)!!}{|x|^{2k+1}}$. From (3.19), $k = 1$, we obtain

$$| R(z) - R(y) | = \left|\int_y^z R'(x)dx\right| = \left|\int_y^z (xR(x) - 1)dx\right| \leqslant \frac{|y-z|}{|y|\,|z|}.$$

Then applying (3.12), (3.14) and the estimate $1/R(y) \leqslant C(1 + y)$ we have

$$R(\sqrt{2nH_2}\,|t_0|) = R\left(\sqrt{\frac{n}{D}}(\beta + D\tau)\right)\left(1 + O\left(\frac{1}{\sqrt{n}} + \beta + \tau\right)\right).$$

Now from (3.17) it follows that

$$\mathbf{P}(\varphi(n,r) \geqslant q) = \sqrt{\frac{2}{\pi}}e^{nH(t_0)}e^{\frac{n(\beta+D\tau)^2}{2D}}\int_{\sqrt{\frac{n}{D}}(\beta+D\tau)} e^{-v^2/2}dv\left(1+O\left(\frac{1}{\sqrt{n}}+\beta+\tau\right)\right)$$

$$= \sqrt{\frac{2D}{\pi}}\int_{\sqrt{n}\tau} e^{-\frac{(\beta\sqrt{n}+Dv)^2}{2D}}dv\,e^{n\mu(\beta,\tau)}(1+n^{-1/2}O(1+\sqrt{n}(\beta+\tau))).$$

This is the required relation (1.4) if one let $r = x\sqrt{n}$, $q = y\sqrt{n}$.

Let us verify (1.4) for $x \neq 0$, $y = 0$. From (2.22) we derive

$$\mathbf{P}(\varphi(n,r) > 0) = -\frac{1}{2\pi i}\int_{\gamma_\alpha}\frac{\psi^n(t)L_r(t)\psi'(t)}{L_0(t)(\psi(t)-1)}dt + O(e^{-n\nu}). \tag{3.20}$$

This relation is similar to (2.21) when $\tau = 0$ and so it can be treated analogously. Note only that in this case the saddle-point equation

$$\frac{d}{dt}g(t) = 0, \tag{3.21}$$

where $g(t) = \ln \psi(t) + \beta t$, is coincide with that for limit theorems for large deviations for sums of i. i. d. random variables (see [*9*], Ch. 7, § 2). Thus if t_2 is a solution of this equation then

$$g(t_2) = -\frac{\beta^2}{2D} + \frac{\beta^3}{D^{3/2}}\lambda\left(\frac{\beta}{D^{1/2}}\right),$$

where $\lambda(v)$ is the so called Cramer's series.

4. PROOFS OF THEOREMS 1.3, 1.4

Consider the asymptotic behaviour as $t \to 0$, $t \in t_-(\mathfrak{N})$ of the function $L_r(t)$. Let

$$D_r(t) = \frac{L_r(t)}{\psi(t)} + \frac{e^{rt}}{\psi'(t)}.$$

LEMME 4.1. — *Let condition (1.5) be satisfied. If $t \in t_-(\mathfrak{N})$ and $|t| < \varepsilon$ for sufficiently small ε, then*

a) *for all $p \geqslant 0$*

$$\sup_r \left|\frac{d^p}{dt^p} D_r(t)\right| \leqslant C(1 + |t|^{m-1-p+\delta}), \tag{4.1}$$

b) *for $0 \leqslant p \leqslant m-1$ there exists the limit*

$$\lim_{t\to 0} \frac{d^p}{dt^p} D_r(t) \equiv D_r^{(p)}(-0), \tag{4.2}$$

which for $r \neq 0$ can be represented in the form

$$D_r^{(p)}(-0) = r^{p+1-m}\theta_{mp}(r), \quad |\theta_{mp}(r)| \leqslant C_{mp} r^{-\delta}. \tag{4.3}$$

REMARK 4.1. — The coefficients $D_0^{(p)}(-0)$ are determined by the first p moments of ξ_1 and by J_l, $l \leqslant p$ where J_l are defined in (3.4), (3.7).

Proof. — Let

$$V_r(t) = \frac{1}{2\pi i}\int_{-i\pi}^{i\pi} \frac{e^{rv}}{\psi(t) - \psi(v)}\, dv,$$

$$D_r^2(t) = \frac{1}{2\pi i}\int_{-i\pi}^{i\pi} e^{rv}\left(\frac{1}{\psi(t) - \varphi(iv)} - \frac{1}{\psi(t) - \psi(v)}\right) dv.$$

It is clear that $L_r(t) = \psi(t)(V_r(t) + D_r^2(t))$.

Let γ be the number defined in Lemma 2.1. In view of (2.7) and (2.8) one can choose A so that for all $s \in [-\pi, -\gamma] \cup [\gamma, \pi]$ and some $\eta > 0$ holds $|\varphi_m(s)| < 1 - 2\eta$. By (2.11) there exists such $\rho_A > 0$ that for $t \in t_-(\mathfrak{N}) \cup \{0\}$ the equation (3.1) has in $(|v| < \rho_A) \cap (\operatorname{Re} v \leqslant 0)$ the unique solution $v = t$. Let $D\rho_A^2/16 < \eta$. Choose ε_A, $0 < \varepsilon_A < \rho_A$, so that for $|t| < \varepsilon_A$ holds $|1 - \psi(t)| \leqslant D\rho_A^2/16$. Then applying Lemma 2.1 we obtain that for $v \in [-i\gamma, -i\rho_A] \cup [i\rho_A, i\gamma]$ and $|t| < \varepsilon_A$

$$|\psi(t) - \psi(v)| \geqslant |1 - \psi(v)| - |1 - \psi(t)| \geqslant \frac{D}{8}\rho_A^2 - \frac{D}{16}\rho_A^2 = \frac{D}{16}\rho_A^2 .$$

In addition one can choose ρ_A so small that for $0 \leqslant 3|t| \leqslant |v - t| \leqslant \rho_A$ holds $|\psi(t) - \psi(v)| \geqslant D|t - v|^2/8$. Here t and v is not necessarily real as it is in Lemma 2.1. Let $\gamma_A > 0$ be such that the rectangular

$$[-i\gamma_A, i\gamma_A] + (i\gamma_A, (i-1)\gamma_A) + [(i-1)\gamma_A, -(i+1)\gamma_A] + (-(i+1)\gamma_A, -i\gamma_A)$$

belongs to the circle $|v| < \rho_A$. Let

$$L' = (i\pi, i\gamma_A] + (i\gamma_A, (i - 1)\gamma_A) + [(i - 1)\gamma_A, -(i + 1)\gamma_A] \\ + (-(i + 1)\gamma_A, -i\gamma_A) + [-i\gamma_A, -i\pi] .$$

By the inequalities obtained above we have that for $|t| < \varepsilon_A$, $t \in t_-(\mathfrak{N})$ and $v \in L'$ the function ψ satisfies (3.3), where the constant d may depend on A. By residue theorem

$$V_r(t) = -\frac{e^{rt}}{\psi'(t)} - \frac{1}{2\pi i}\int_{L'} \frac{e^{rv}}{\psi(t) - \psi(v)}\, dv . \tag{4.4}$$

From the formula of differentiation of the composition of functions [7, p. 33] it follows that

$$\frac{d^p}{dt^p}(\psi(t) - \psi(v))^{-1} = \sum_{k=0}^{p} (\psi'(t))^k \sum_{k \leqslant l \leqslant (k+p)/2} C_{lkp}(t)(\psi(t) - \psi(v))^{-l-1}, \tag{4.5}$$

where $C_{lkp}(t)$ is a product of derivatives of function $\psi(t)$. Denote the second term in the right-hand side of (4.4) by $D_r^1(t)$. The relation (4.5) together with (3.3) imply that the function $D_r^1(t)$ satisfies conditions (4.1), (4.2) for all p. To prove (4.1) for the function $D_r^2(t)$ we should again use the formula of differentiation of composite functions and then to estimate the integrals

$$F_k=|t|^k\int_{-i\pi}^{i\pi}\frac{|\psi(v)-\varphi(iv)|\,dv}{|\psi(t)-\psi(v)|^{l_1+1}|\psi(t)-\varphi(iv)|^{l_2+1}},\quad 0\leqslant k\leqslant p,\quad k\leqslant l_1+l_2\leqslant\frac{p+k}{2}.$$

Here the factor $|t|^k$ is due to the fact that $\psi'(t)\sim Dt$.

Next we shall need the following estimates. If Δ_A is sufficiently small then for all $t\in t_-(\mathfrak{N})$, $v\in[-i\Delta_A,i\Delta_A]$,

$$|\psi(t)-\varphi(iv)|\geqslant\frac{D}{4}|t^2-v^2|,\ |\psi(t)-\psi(v)|\geqslant\frac{D}{4}|t^2-v^2|.\tag{4.6}$$

Indeed, taking into account (2.9) we can write

$$\varphi(iv)-\psi(t)=(v-t)\left[\frac{D}{2}(v+t)+O(|t|^{1+\delta}+|v+t|^{1+\delta})\right]+O(|v|^{2+\delta}).\tag{4.7}$$

Since $t\in t_-(\mathfrak{N})$ then there exists such $\eta>0$ that

$$\operatorname{Re}t\leqslant-\eta\,|\operatorname{Im}t|\tag{4.8}$$

and, consequently, $|v\pm t|\geqslant|\operatorname{Re}t|\geqslant|t|\,\eta/\sqrt{1+\eta^2}$. This and (4.7) yield the left-hand side inequality in (4.6). The right-hand side one is proved analogously. Applying (2.7), (2.8), (2.9) $p=0$, and (4.6) for all sufficiently small t we have

$$F_k\leqslant C|t|^k\int_{-\pi}^{\pi}\frac{|s|^{m+2+\delta}ds}{|t^2+s^2|^{(p+k)/2+2}}\leqslant C|t|^{m-1-p+\delta}\int_{-\pi/|t|}^{\pi/|t|}\frac{|u|^{m+2+\delta}du}{|t^2/|t|^2+u^2|^{(p+k)/2+2}}.$$

This proves (4.1) for the function $D_r^2(t)$. Relations (4.2) is due to these estimates also.

To obtain (4.3) we note that in view of (4.5) the coefficients $D_r^{(p)}(-0)$ depend only on the first p moments of ξ_1 and of the expressions

$$K_{rl}=\int_{-i\pi}^{i\pi}e^{rv}\eta_l(v)dv-\int_{L'}e^{rv}(1-\psi(v))^{-l-1}dv,\quad l\leqslant p/2,\tag{4.9}$$

where for brevity we let $\eta_l(v)=(1-\varphi(iv))^{-l-1}-(1-\psi(v))^{-l-1}$. Since $\varphi(\pi)=\varphi(-\pi)$ then integrating in (4.9) by parts N times for $r\neq0$ we obtain

$$K_{rl}=\frac{(-1)^N}{r^N}\left\{\int_{-i\pi}^{i\pi}e^{rv}\frac{d^N}{dv^N}\eta_l(v)dv-\int_{L'}e^{rv}\frac{d^N}{dv^N}(1-\psi(v))^{-l-1}dv\right\}.$$

Dividing the interval $[-i\pi, i\pi]$ into two parts $[-i/r, i/r]$, $[-i\pi, i\pi]\backslash[-i/r, i/r]$ and again applying the formula of integration by parts we have

$$K_{rl} = \frac{(-1)^N}{r^N}\left\{\int_{-i/r}^{i/r} e^{rv}\frac{d^N}{dv^N}\eta_l(v)dv + \frac{1}{r}\left[e^{rv}\eta_l(v)\,\Big|_{i/r}^{-i/r}\right.\right.$$

$$\left.\left. -\left(\int_{i/r}^{i\pi} + \int_{-i\pi}^{-i/r}\right)e^{rv}\frac{d^{N+1}}{dv^{N+1}}\eta_l(v)dv + \int_{L'} e^{rv}\frac{d^{N+1}}{dv^{N+1}}(1-\psi(v))^{-l-1}dv\right]\right\}. \quad (4.10)$$

The last integral is bounded in view of (3.3). To estimate other terms we need the estimate

$$\left|\frac{d^N}{dv^N}\eta_l(v)\right| \leqslant C\,|\,v\,|^{m-2-N-2l+\delta}. \quad (4.11)$$

To prove it one should apply the formula (3.5) with $N = 0$, $B = 1 - \varphi(iv)$, $G = 1 - \psi(v)$ and then verify that for $1 \leqslant l_1 + l_2 \leqslant l$

$$\left|\frac{d^N}{dv^N}\frac{(\varphi(iv) - \psi(v))}{(1 - \varphi(iv))^{l_1+1}(1 - \psi(v))^{l_2+1}}\right| \leqslant Cv^{m-2-N-2(l_1+l_2)+\delta}.$$

This, in turn, can be done with the help of the formula of differentiation of composite functions, (2.9) and the estimates (4.6), $t = 0$. The worst estimate in (4.11) will be for $l = p/2$. Then letting in (4.10) $N = m - 1 - p$ it is easy to conclude that $|\,K_{rl}\,| \leqslant Cr^{p-m+1-\delta}$.

To prove Remark 4.1 one should substitute in (4.9), with $r = 0$, the expression for the second integral defined by (3.4), (3.7).

As a consequence of Lemma 4.1 we have the function $U(t)$ has $m + 1$ bounded derivatives for $t \in t_-(\mathfrak{R})$, $|\,t\,| < \varepsilon$. Moreover there exists the limit $\lim_{t\to 0} U^{(p)}(t) = U^{(p)}(-0)$, $0 \leqslant p \leqslant m + 1$ and

$$|\,U^{(m+2)}(t)\,| \leqslant Ct^{\delta-1}, \quad |\,U^{(m+3)}(t)\,| \leqslant Ct^{\delta-2}. \quad (4.12)$$

Here and later simbol (p) means the derivative of the order p. Consider the saddle-point equation (3.10) provided that condition (1.5) is satisfied. We may consider (3.10) as a real-valued equation because the functions $U(t)$, $\psi(t)$ have the real values for real t. We shall solve (3.10) in two steps. First consider the equation $\beta = -\psi'(t)/\psi(t)$. Since $\psi(t)$ is analytic function and $\psi''(0) = D$, this equation for all sufficiently small β has the solution

$$t(\beta) = \sum_{k=1}^{\infty}\sigma_k\beta^k, \qquad \sigma_1 = -1/D. \quad (4.13)$$

Let

$$\mathrm{L}(t) = -\frac{\mathrm{U}(t)}{\mathrm{U}'(t)}\left[\frac{\psi'(t)}{\psi(t)} + \beta\right], \quad \mathrm{K}(t) = -\frac{\mathrm{U}(t)\psi'(t)}{\mathrm{U}'(t)\psi(t)}, \quad \mathrm{S}(t) = -\frac{\mathrm{U}(t)}{\mathrm{U}'(t)}.$$

Then $\mathrm{L}(t(\beta)) = 0$. Now we consider the equation

$$\tau = \mathrm{L}(t) \tag{4.14}$$

in a small neighbourhood of the point $t(\beta)$. For $t \in t_-(\mathfrak{N})$, $|t| < \varepsilon$, in view of (4.12), $\mathrm{K}(t)$ has $m+1$ bounded derivatives, $\mathrm{S}(t)$ has m bounded derivatives and

$$|\mathrm{K}^{(m+2)}(t)| \leqslant \mathrm{C}t^{\delta-1}, \quad |\mathrm{S}^{(m+j)}(t)| \leqslant \mathrm{C}t^{\delta-j}, \quad j = 1, 2. \tag{4.15}$$

Since $|\mathrm{L}'(t(\beta))| \geqslant 1/2$ for all sufficiently small β, the equation (4.14) has a solution $l(\tau)$ and

$$l^{(p)}(\tau) = \sum \frac{(-1)^s(p-1+s)!}{(\mathrm{L}'(l(\tau)))^{p+s}} \prod_{j=1}^{p-1} \frac{1}{k_j!}\left(\frac{\mathrm{L}^{(j+1)}(l(\tau))}{(j+1)!}\right)^{k_j}, \tag{4.16}$$

where the summation is carried out over all non-negative integer solutions of the equation $k_1+2k_2+\ldots+(p-1)k_{p-1}=p-1$ and $s=k_1+k_2+\ldots+k_{p-1}$. By Taylor's formula

$$l(\tau) = \sum_{k=0}^{m+1} \frac{l^{(k)}(0)\tau^k}{k!} + \int_0^\tau \frac{l^{(m+2)}(v)(\tau - v)^{m+1}}{(m+1)!}\,dv. \tag{4.17}$$

In view of (4.15), (4.13) and (4.16) the worst component of the remainder of this expansion is estimated by

$$\int_0^\tau \frac{|\mathrm{L}^{(m+2)}(l(v))|\,|\tau - v|^{m+1}}{(m+1)!}\,dv \leqslant \mathrm{C}\tau^{m+1}\int_0^\tau (|l(v)|^{\delta-1} + \beta|l(v)|^{\delta-2})dv$$
$$\leqslant \mathrm{C}\tau^{m+1}\int_0^\tau ((\beta+v)^{\delta-1} + \beta(\beta+v)^{\delta-2})dv \leqslant \mathrm{C}\tau^{m+1}(\beta^\delta + \tau^\delta).$$

Again applying Taylor's formula we find that for $1 \leqslant p \leqslant m+1$

$$\mathrm{L}^{(p)}(t(\beta)) = \sum_{l=0}^{m-p} \frac{\mathrm{L}^{(l+p)}(-0)}{l!}(t(\beta))^l + \frac{\mathrm{K}^{(m+1)}(-0)}{(m+1-p)!}(t(\beta))^{m+1-p} + \mathrm{O}(\beta^{m+1-p+\delta}).$$

The last two equations together with (4.17), (4.13) and (4.16) imply

$$t_0 \equiv l(\tau) = \sum_{1 \leqslant k+l \leqslant m+1} \gamma_{kl}\beta^k\tau^l + O(\beta^{m+1+\delta} + \tau^{m+1+\delta}). \qquad (4.18)$$

So t_0 is the solution of (3.10) and the coefficients γ_{kl} coincide with those in (3.11).

Let $h(t) = \tau \ln U(t) + \ln \psi(t)$ and

$$t_1 = t_0 |_{\beta=0} = \sum_{l=1}^{m+1} \gamma_{0l}\tau^l + O(\tau^{m+1+\delta}) \qquad (4.19)$$

be the solution of the saddle point equation

$$\frac{dh}{dt} = 0. \qquad (4.20)$$

Letting $a(t) = \psi(t)(L_0(t)(L_0(t) - 1)(\psi(t) - 1))^{-1}$, $a(0) = 2D$, we can represent (2.18) in the form

$$\mathbf{P}(\varphi(n, r) = q)$$
$$= \frac{1}{2\pi i}\int_{\gamma_{t_1}} e^{nH(t)}a(t)dt - \frac{1}{2\pi i}\int_{\gamma_{t_1}} e^{nh(t)}D_r(t)a(t)\psi'(t)dt + O(e^{-n\nu}). \qquad (4.21)$$

Consider the first integral in the right-hand side of this equation. Denote it by I_0. Applying Cauchy's theorem and the estimates analogous to (3.16) along $l_1 = (\gamma_{t_1}(\varepsilon), \gamma_{t_0}(\varepsilon))$, $l_2 = (\gamma_{t_0}(-\varepsilon), \gamma_{t_1}(-\varepsilon))$, we obtain

$$I_0 = \frac{e^{nH(t_0)}}{2\pi i}\left\{\int_{\gamma_{t_0}} e^{n(H(t) - H(t_0))}a(t)dt + O(e^{-n\lambda})\right\}. \qquad (4.22)$$

For brevity write $H_k = H^{(k)}(t_0)/k!$, $a_k = a^{(k)}(t_0)/k!$, $e_m = \dfrac{(\ln \psi(t_0))^{(m+2)}}{(m+2)!}$. By virtue of (4.12) and Lemma 4.1

$$H(t) - H(t_0) = \sum_{k=2}^{m+1} H_k(t - t_0)^k + e_m(t - t_0)^{m+2}$$
$$+ O(|t - t_0|^{m+1}(\tau(|t|^\delta + |t_0|^\delta) + |t - t_0|^2)), \qquad (4.23)$$

$$a(t) = \sum_{k=0}^{m} a_k(t - t_0)^k + O(|t - t_0|^m(|t|^\delta + |t_0|^\delta)). \qquad (4.24)$$

Let f_{pj} be the coefficient before v^p in the decomposition

$$B_{mj}(v) \equiv \frac{1}{j!}\left(\sum_{k=0}^{m} a_k v^k\right)\left(\sum_{k=3}^{m+1} H_k v^{k-2} + e_m v^m\right)^j .$$

It is clear that $f_{pj} = 0$ for $j > p$. Expanding the exponential function by Taylor's formula, we obtain

$$\int_{\gamma_{t_0}} e^{n(H(t)-H(t_0))}a(t)dt = \int_{\gamma_{t_0}} e^{nH_2(t-t_0)^2} \sum_{j=0}^{m} n^j (t-t_0)^{2j} B_{mj}(t-t_0)dt$$
$$+ \int_{-\varepsilon}^{\varepsilon} e^{-nH_2 s^2(1-\eta^2)/2} O(|s|^m(|s|^\delta + |t_0|^\delta)$$
$$+ n|s|^{m+1}(\tau(|s|^\delta + |t_0|^\delta) + s^2) + [ns^2]^{m+1}|s|^{m+1})ds, \quad (4.25)$$

where η is taken from (4.8). Applying to the first integral in the right-hand side of this equation Cauchy's theorem and the estimates analogous to (3.16) along $l_1 = (\gamma_{t_0}(\varepsilon), t_0 + i\varepsilon)$, $l_2 = (t_0 - i\varepsilon, \gamma_{t_0}(-\varepsilon))$ we find

$$I_0 = \frac{e^{nH_0}}{2\pi i}\left\{ \int_{-i\varepsilon}^{i\varepsilon} e^{nH_2 v^2} \sum_{j=0}^{m} n^j v^{2j} B_{mj}(v)dv + n^{-m} O(n^{-(1+\delta)/2} + (\beta+\tau)^{1+\delta}) \right\}$$
$$= \frac{e^{nH_0}}{2\pi}\left\{ \int_{-\infty}^{\infty} e^{-nH_2 s^2} \sum_{0 \leqslant l \leqslant m/2} \sum_{j=0}^{2l} f_{2lj}(-1)^{l+j} s^{2l} (ns^2)^j ds \right.$$
$$\left. + n^{-(m+1+\delta)/2} O(1 + (\sqrt{n}\beta)^{1+\delta} + (\sqrt{n}\tau)^{1+\delta}) \right\}$$
$$= \frac{e^{nH_0}}{2\pi}\left\{ \sum_{0 \leqslant l \leqslant m/2} \sum_{j=0}^{2l} f_{2lj}(-1)^{l+j} \Gamma(l+j+1/2) \frac{n^j}{(nH_2)^{l+j+1/2}} \right.$$
$$\left. + n^{-(m+1+\delta)/2} O(1 + (\sqrt{n}\beta)^{1+\delta} + (\sqrt{n}\tau)^{1+\delta}) \right\}. \quad (4.26)$$

Next taking into account (4.12) and Lemma 4.1 we have that for $0 \leqslant p \leqslant m + 1$

$$H_p = \frac{1}{p!} \sum_{k=0}^{m+1-p} \frac{H^{(k+p)}(-0)}{k!} t_0^k + \frac{(\ln \psi(0))^{(m+2)}}{(m+2-p)!} t_0^{m+2-p}$$
$$+ O(|t_0|^{m+1-p}(\tau |t_0|^\delta + t_0^2)), \quad (4.27)$$

$$a_p = \frac{1}{p!} \sum_{k=0}^{m-p} \frac{a^{(k+p)}(-0)}{k!} t_0^k + O(|t_0|^{m-p+\delta}). \quad (4.28)$$

Hence one can derive that

$$f_{2lj}=\sum_{k=0}^{m-2l} C_{lj}t_0^k+O(|t_0|^{m-1-2l}(\tau|t_0|^\delta+|t_0|^{1+\delta})),\quad 0\leqslant j\leqslant 2l. \quad (4.29)$$

Setting $\beta = x/\sqrt{n}$, $\tau = y/\sqrt{n}$ from (4.26)-(4.29) and (4.18) we deduce

$$I_0 = \sqrt{\frac{2D}{\pi n}} e^{-\frac{(x+Dy)^2}{2D}} e^{n\mu_m(x/\sqrt{n},y/\sqrt{n})} \left\{ \sum_{k=0}^{m} n^{-k/2} P_k^1(x, y) + n^{-(m+\delta)/2} O(1 + x^{m+\delta} + y^{m+\delta}) \right\}. \quad (4.30)$$

Consider the second integral in the right-hand side of (4.21). Denote it by I_1. Let $a(r, t) = -D_r(t)a(t)\psi'(t)$, $h_k = h^{(k)}(t_1)/k!$, $a_k(r) = a^{(k)}(r, t_1)/k!$. Then using the analogy of (4.24) for $a(r, t)$ with t_1 instead of t_0 it is easy to understand that I_1 will have the expansion (4.26) with h_0 instead of H_0 and g_{2lj} instead of f_{2lj}, where g_{pj} is the coefficient at v^p in the decomposition

$$\frac{1}{j!}\left(\sum_{k=0}^{m} a_k(r)v^k\right)\left(\sum_{k=3}^{m+1} h_k v^{k-2} + e_m v^{m+1}\right)^j.$$

By virtue of Lemma 4.1

$$a_p(r) = \frac{1}{p!}\sum_{k=0}^{m-p}\frac{1}{k!}a^{(k+p)}(r, -0)t_1^k + O(|t_1|^{m-p+\delta}).$$

Note that $a^{(0)}(r, -0) = 0$. Since h_p has for $p \geqslant 2$ the same decomposition as H_p

$$g_{2lj} = \sum_{k=0}^{m-2l} t_1^k r^{2l+k-m}\theta_{klj}(r) + O(|t_1|^{m-1-2l}(\tau|t_1|^\delta + |t_1|^{1+\delta})),$$

$0\leqslant j\leqslant 2l$, where $|\theta_{klj}(r)|\leqslant C_{klj}r^{-\delta}$, $\theta_{000}(r)=0$. Setting $\beta=x/\sqrt{n}$, $\tau=y/\sqrt{n}$, we obtain

$$I_1 = \frac{e^{-Dy^2/2}}{2\pi\sqrt{n}} e^{n\mu_m(0,y\sqrt{n})} \left\{ \sum_{k=1}^{m} n^{-k/2} Q_k^1(y)(\sqrt{n}x)^{k-m}\theta_{mk}(\sqrt{n}x) + n^{-(m+\delta)/2} O(1 + y^{m+\delta}) \right\},$$

where $|\theta_{mk}(v)| \leqslant C_{mk}|v|^{-\delta}$. Now (4.21) jointly with the expressions for I_0 and I_1 imply (1.6). Equation (1.7) can be derived in a similar manner from (2.19).

Now we pass to the prove of Theorem 1.4. To prove (1.8) set

$$C(t) = t\psi(t)((L_0(t) - 1)(\psi(t) - 1))^{-1}, \quad C(0) = -2.$$

From (2.21) it follows that

$$P(\varphi(n, r) \geqslant q) = \frac{1}{2\pi i}\int_{\gamma_{t_1}} e^{nH(t)}t^{-1}C(t)dt - \frac{1}{2\pi i}\int_{\gamma_{t_1}} e^{nh(t)}D_r(t)C(t)t^{-1}\psi'(t)dt + O(e^{-n\nu}). \quad (4.31)$$

Letting $C(t) = C(0) + (C(t) - C(0))$ we obtain that it is sufficient only to consider the integral

$$I = -\frac{1}{\pi i}\int_{\gamma_{t_1}} e^{nH(t)}t^{-1}dt.$$

Indeed for $m \geqslant 1$ the second integral in (4.31) is treated analogously to I_1 because the limit $D_r(t)C(t)\psi'(t)t^{-1} \to -2DD_r(-0)$ exists as $t \to 0$, $t \in t_-(\mathfrak{N})$. The distinction of this case consists in the fact that the analog of $\theta_{000}(r)$ is not equal to zero. The integral

$$\tilde{I}_0 = \frac{1}{2\pi i}\int_{\gamma_{t_1}} e^{nH(t)}(C(t) - C(0))t^{-1}dt$$

is similar to I_0. The case $m = 0$ is special. For it we obtain

$$|\tilde{I}_0| = \frac{e^{nH(t_0)}}{2\pi}\left|\int_{\gamma_{t_0}} e^{n(H(t)-H(t_0))}(C(t) - C(0))t^{-1}dt\right| + O(\sigma^{-1/2}e^{-n\lambda})$$
$$\leqslant \frac{e^{nH(t_0)}}{2\pi n^{\delta/2}}\int_{-\infty}^{\infty} e^{-\frac{Dv^2}{6}}|\sqrt{n}t_0 + |v|e^{i\frac{11}{16}\pi \operatorname{sign} v}|^{\delta-1}dv + O(\sigma^{-1/2}e^{-n\lambda})$$
$$\leqslant \frac{Ce^{nH(t_0)}}{n^{\delta/2}}(1 + x^\delta + y^\delta).$$

As to the integral I, analogously to (4.25) we have

$$I = -\frac{e^{nH(t_0)}}{\pi i}\left[\int_{\gamma_{t_0}} t^{-1}e^{nH_2(t-t_0)^2}\sum_{j=0}^{m}\frac{n^j}{j!}(t-t_0)^{2j}\left(\sum_{k=3}^{m+1}H_k(t-t_0)^{k-2}\right.\right.$$
$$\left. + e_m(t-t_0)^m\right)^j dt + \int_{-\varepsilon}^{\varepsilon} e^{-nH_2s^2(1-\eta^2)/2}O(n|s|^{m+1}(\tau(|s|^\delta + |t_0|^\delta) + s^2)$$
$$\left. + [ns^2]^{m+1}|s|^{m+1})|t_0 + |s|e^{i\frac{11}{16}\pi \operatorname{sign} s}|^{-1}ds\right].$$

The last integral has an estimate

$$n^{-(m+\delta)/2}\mathrm{O}(1 + (\sqrt{n\beta})^{1+\delta} + (\sqrt{n\tau})^{1+\delta}).$$

The previous one we break into two integrals. The first of them does not contain the term t^{-1} and can be calculated similarly to (4.26). The distinction is only in that the decomposition of type (4.30) will have a polynomial of degree at most $3k$ at $n^{-k/2}$. The second one is

$$\sum_{j=0}^{m} \frac{n^j}{j!}(-t_0)^{2j}\left(\sum_{k=3}^{m+1} \mathrm{H}_k(-t_0)^{k-2} + e_m(-t_0)^m\right)^j \int_{\gamma_{t_0}} t^{-1} e^{n\mathrm{H}_2(t-t_0)^2} dt.$$

Using Cauchy's theorem with the contour of integration

$$\gamma_{t_0} + l_1 + (t_0 + i\varepsilon^+, t_0 - i\varepsilon^-) + l_2,$$

where $l_1 = (\gamma_{t_0}(\varepsilon^+), t_0 + i\varepsilon^+)$, $l_2 = (t_0 - i\varepsilon^-, \gamma_{t_0}(-\varepsilon^-))$, and the estimates analogous to (3.16), we obtain

$$\int_{\gamma_{t_0}} t^{-1} e^{n\mathrm{H}_2(t-t_0)^2} dt = i\int_{-\varepsilon^-}^{\varepsilon^+} \frac{e^{-n\mathrm{H}_2 v^2}}{iv+t_0} dv + \mathrm{O}(\sigma^{-1/2}e^{-n\lambda})$$

$$= i\int_{-\infty}^{\infty} \frac{e^{-n\mathrm{H}_2 v^2}}{iv+t_0} dv + \mathrm{O}(\sigma^{-1/2}e^{-n\lambda}) = -i\sqrt{2\pi}\mathrm{R}(\sqrt{2n\mathrm{H}_2}\,|t_0|) + \mathrm{O}(\sigma^{-1/2}e^{-n\lambda}),$$

where $\mathrm{R}(x)$ is the function defined by (3.18).

Since $\mathrm{R}^{(p)}(x) \leqslant \mathrm{C}_p$ then by Taylor's formula

$$\mathrm{R}(z) = \mathrm{R}(z_0) + \sum_{k=1}^{m} (\mathrm{R}(z_0)\mathrm{P}_k^1(z_0) - \mathrm{Q}_{k-1}^1(z_0)) + \mathrm{O}(|z - z_0|^{m+1}).$$

Here and later P_k^j, Q_k^j, $j = 1, 2, 3$, are some polynomials of degree at most k. Applying (4.18), (4.27), one can find

$$\mathrm{R}(\sqrt{2n\mathrm{H}_2}\,|t_0|)$$
$$= \mathrm{R}\left(\sqrt{\frac{n}{\mathrm{D}}}(\beta+\mathrm{D}\tau)\right) + \sum_{k=1}^{m} n^{-k/2}\left[\mathrm{R}\left(\sqrt{\frac{n}{\mathrm{D}}}(\beta+\mathrm{D}\tau)\right)\mathrm{P}_{2k}^2(\sqrt{n}\beta, \sqrt{n\tau})\right.$$
$$\left. + \mathrm{Q}_{2k}^2(\sqrt{n\beta}, \sqrt{n\tau})\right] + n^{-(m+\delta)/2}\mathrm{O}(1 + \sqrt{n\beta})^{2m+\delta} + (\sqrt{n\tau})^{2m+\delta}).$$

Taking into account the fact that

$$\sum_{j=0}^{m} \frac{n^j}{j!}(-t_0)^{2j}\left(\sum_{k=3}^{m+1} H_k(-t_0)^{k-2} + e_m(-t_0)^m\right)^j$$

$$= \sum_{k=0}^{m} n^{-k/2} P_{3k}^3(\sqrt{n}\beta, \sqrt{n}\tau) + n^{-(m+\delta)/2} O(1 + (\sqrt{n}\beta)^{3m+\delta} + (\sqrt{n}\tau)^{3m+\delta}),$$

and substituting $\beta = x/\sqrt{n}$, $\tau = y/\sqrt{n}$, from the relations obtained above one can deduce (1.8).

The proof of (1.9) proceeds along similar lines. The initial equation here is (2.10).

To prove (1.10) rewrite (3.20) in the form

$$\mathbf{P}(\varphi(n, r) > 0) = \frac{1}{2\pi i}\int_{\gamma_{t_2}} e^{ng(t)} C(t) t^{-1} dt - \frac{1}{2\pi i}\int_{\gamma_{t_2}} e^{n \ln \psi(t)} D_r(t) C(t) t^{-1} \psi'(t) dt + O(e^{-n\nu}),$$

where

$$C(t) = \psi(t)(L_0(t)(\psi(t) - 1))^{-1}, \quad C(0) = -2,$$

and t_2 is the solution of the equation (3.21). This relation is very similar to (4.31), with $\tau = 0$, and can be treated analogously.

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