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Weak Pinsker property and Markov processes

by

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SUMMARY. — In this article, we show that the ergodic Markov processes in $\mathbb{Z}^2$ have the weak Pinsker property introduced by ˇ P. Thouvenot in [8].

Key-words: Weak Pinsker Property, Markov process, Extremal.

RÉSUMÉ. — Dans cet article, nous montrons que les processus de Markov ergodiques dans $\mathbb{Z}^2$, possèdent la propriété de Pinsker faible, introduite par J.-P. Thouvenot [8].

I. INTRODUCTION

A two-parameter stochastic process is a collection of random variables:

$$(X_{i,j} : (i, j) \in \mathbb{Z}^2).$$

It is stationary if the distribution of $(X_{i_1,j_1}, X_{i_2,j_2}, \ldots, X_{i_n,j_n})$ is the same as that of $(X_{i_1+k,j_1+l}, X_{i_2+k,j_2+l}, \ldots, X_{i_n+k,j_n+l})$ for any family $(i_1,j_1), (i_2,j_2) \ldots (i_n,j_n)$ and any (k, l) in $\mathbb{Z}^2$.

Recall that for an ordinary Markov process $(X_n)_{n\in\mathbb{Z}}$, given the present X_0 the past and the future are independent. For a two-parameter stationary

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stochastic process, we say it is Markov if given ($X_{i,j}$: $(i, j) \in$ boundary of a square), the distribution in the interior is independent of that of the exterior.

$\mathbb{Z}^2$-action gives rise to two-parameter stationary stochastic processes. The $\mathbb{Z}^2$-action is called Markov if it has a generator which gives rise to a Markov process.

Pinsker's conjecture was that every ergodic dynamical system can be written as the direct product of a K-system and a system of 0-entropy. This was proved to be false in [5] by O. Ornstein. Then J.-P. Thouvenot introduced in [8] a weaker notion called weak Pinsker property: A system has this property if it can be written as the direct product of a Bernoulli and a system of arbitrary small entropy.

It is the purpose of this work to show that all the ergodic $\mathbb{Z}^2$-Markov processes have this weak Pinsker property.

Remark. — All the known measure preserving actions of $\mathbb{Z}^n$ on a Lebesgue space have this weak Pinsker property.

II. PRELIMINARIES

Let $(X, \mathscr{B}, \mu)$ be a Lebesgue space. A measure preserving action of $\mathbb{Z}^2$ on X is defined once we know two commuting automorphisms S and T of X, that generate this action.

To formally define a Markov process we will recall some definitions:

DEFINITION 1. — Let $P = (p_0, p_1, \ldots, p_t)$ a finite partition of X. For every finite $A \subset \mathbb{Z}^2$, one defines $(P)_A = \bigvee_{(k,l)\in A} S^k T^l P$ as the partition of the space whose elements are: $\bigcap_{(k,l)\in A} S^{-k}T^{-l} p_{i_{k,l}}$ with $0 \leq i_{k,l} \leqq t$.

DEFINITION 2. — $(P)_{S,T}$ is the smallest σ-algebra invariant for the $\mathbb{Z}^2$-action and for which P is measurable. We will say that P is a generating partition if $(P)_{S,T} = \mathscr{B}$.

DEFINITION 3. — Two partitions P and Q are said to be independent and we will denote it by $P \perp Q$ if:

For every $p_i \in P$ and $q_j \in Q$: $\mu(p_i \cap q_j) = \mu(p_i)\mu(q_j)$. More generally, two σ-algebras $\mathscr{B}$ and $\mathscr{C}$ are said to be independent if for every $b \in \mathscr{B}$, $c \in \mathscr{C}$; $\mu(b \cap c) = \mu(b)\mu(c)$. We will also denote it by $\mathscr{B} \perp \mathscr{C}$.

DEFINITION 4. — Let $E \in \mathscr{B}$ be a set such that $\mu(E) > 0$ and P be a partition of X. P/E will be the partition of E in $p_i \cap E$ ($p_i \in P$). The measure μ_E on E is the measure induced by μ on E and normalized so that

$$\mu_E(p_i \cap E) = \frac{\mu(p_i \cap E)}{\mu(E)}.$$

DEFINITION 5. — Let C be a square in $\mathbb{Z}^2$. $b(C)$ is the set of points in $\mathbb{Z}^2$ at the boundary of the square and $\mathring{C}$, the set of points in $\mathbb{Z}^2$ inside the square. By « square » C we will always mean: $\mathring{C} \cup b(C)$ (see figure):

0	0	0	0	0	→ $b(C)$: points with a 0.
0	X	X	X	0	
0	X	X	X	0	⟶ $\mathring{C}$: points with a X.
0	X	X	X	0	
0	0	0	0	0	$C = \mathring{C} \cup b(C)$.

With all these definitions we can define more precisely what is a Markov process.

DEFINITION 6. — A Markov process on $\mathbb{Z}^2$ is defined by a measure preserving action of $\mathbb{Z}^2$ on $(X, \mathscr{B}, \mu)$ with generators S and T and a partition P satisfying the following:

— P is a generating partition

— For any square C in $\mathbb{Z}^2$, any subset C_1 of $\mathbb{Z}^2$ whose intersection with C is empty then:

$$\bigvee_{(k,l)\in \mathring{C}} S^k T^l P/E \perp \bigvee_{(k,l)\in C_1} S^k T^l P/E$$

where E is any atom of $\bigvee_{(k,l)\in (C)} S^k T^l P$.

The independence is to be understood of course with the measure μ_E.

A more intuitive way of saying this is: The distribution of P-names inside the square is known when we know the P-name on the boundary.

DEFINITION 7 (see [8]). — One says that the dynamical system $(X, \mathscr{B}, \mu, S, T)$ satisfies the weak Pinsker property if it is ergodic with finite entropy and if there exists two sequences of partitions of X: $(H_n)_{n \geqq 1}$ and $(B_n)_{n \geqq 1}$ such that:

(1) $$(H_{n+1})_{S,T} \subset (H_n)_{S,T}$$

(2) $$E(H_n, S, T) \downarrow 0$$

(3) $$(H_n)_{S,T} \perp (B_n)_{S,T}$$

(4) $$(H_n \vee B_n)_{S,T} = \mathscr{B}$$

(5) The partitions $S^k T^l B_n$, $(k, l) \in \mathbb{Z}^2$ are independent.

Here $E(H_n, S, T)$ is the entropy for the $\mathbb{Z}^2$-action of the partition H_n. The properties of this entropy for $\mathbb{Z}^2$-action are similar to the properties for $\mathbb{Z}$-action see for instance J.-P. Conze [*1*]. This definition was introduced by J.-P. Thouvenot in [*8*], in the $\mathbb{Z}$-case but as he showed in [*9*], all his theorems extend without changes to $\mathbb{Z}^n$.

Those Markov processes in $\mathbb{Z}^2$ are relatively unknown. Most of the known examples come from the Ising model or the theory of Gibbs measure. An interesting example of a zero entropy 2-mixing but not 3-mixing, $\mathbb{Z}^2$-Markov was found by Ledrappier [*4*]. Unlike the case of $\mathbb{Z}$ [*2*], there may exist Markov processes in $\mathbb{Z}^2$ which are K and not Bernoulli. In this work, we will show that all the ergodic $\mathbb{Z}^2$-Markov processes have the weak-Pinsker property.

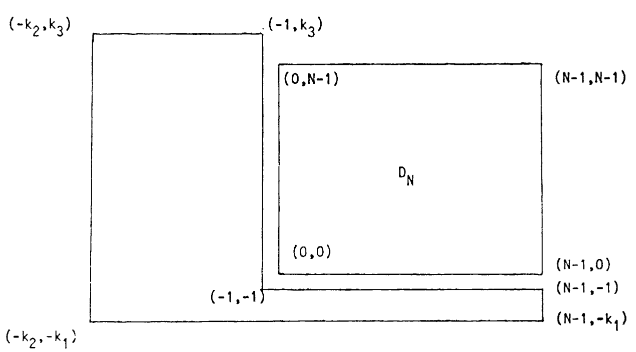
To see this we will reduce our problem to a one that implies this weak Pinsker property.

To describe this reduction we have to introduce further definitions: for N in $\mathbb{N}$, let $D_N = \{(k, l) \in \mathbb{Z}^2,\ 0 \leqq k \leqq N - 1,\ 0 \leqq l \leqq N - 1\}$.

Let N be fixed. By an element of the D_N-past for P and the $\mathbb{Z}^2$-action generated by S and T we will mean a partition $(P)_C$ (see definition 1), where C is in $\mathbb{Z}^2$ and is defined (see the picture) by:

$$C = C(k_1, k_2, k_3) = \{(k, l) \in \mathbb{Z}^2; \quad (0 \leqq k \leqq N-1 \quad \text{and} \quad -k_1 \leqq l < 0)$$

$$\text{or} (-k_2 \leqq k < 0 \quad \text{and} \quad -k_1 \leqq l \leqq k_3)\}, \text{ for any } k_1, k_2, k_3 \text{ in } \mathbb{N}^*.$$



(C is in the D_N past for any choice of k_1, k_2, k_3 in $\mathbb{N}^*$).

We recall from Conze [1], that the ordinary past in $\mathbb{Z}^2$ is obtained for $N = 1$. Let also $C_N = \{ (k, l) \in \mathbb{Z}^2 ; |k| \leqq N - 1 \text{ and } |l| \leqq N - 1 \}$.

DEFINITION 8. (see [10]). — Let $(X, \mathscr{B}, \mu, S, T)$ be an ergodic $\mathbb{Z}^2$-action, P and H two finite partitions of X. One says that P is $(H)_{S,T}$ ε-relatively very weakly Bernoulli if there exists $N \in \mathbb{N}$, such that for every partition $(P)_C$ in the D_N-past for P, there exists $m (m = m(C))$ such that for every $m' > m$, for a family $h \cap q$ of atoms with $h \in (H)_{C_{m'}}$ and $q \in (P)_C$ of measure bigger than $1 - \varepsilon$ one has:

$$(6) \qquad \bar{d}\left[\left(\bigvee_{(k,l)\in D_N} S^k T^l P/h\right), \quad \left(\bigvee_{(k,l)\in D_N} S^k T^l P/h \cap q\right)\right] < \varepsilon .$$

One says that P is $(H)_{S,T}$ relatively very weakly Bernoulli if the above property is true for every ε, with an N depending on ε.

The organization of our work is the following:

— In part III we will show that if $(X, \mathscr{B}, S, T)$ is an ergodic Markov process then: For any $\varepsilon > 0$, there exists a partition H_ε with $E(H_\varepsilon, S, T) < \varepsilon$ and P is $(H_\varepsilon)_{S,T}$ ε-relatively very weakly Bernoulli.

— In part IV we will show that any ergodic $\mathbb{Z}^2$-action satisfying the above condition has the weak Pinsker property. This part IV is more standard and in the case of $\mathbb{Z}$-action is essentially contained in Thouvenot's work ([8] [9] [10]) although it is not explicitly stated there.

III. ε-RELATIVE VERY WEAK BERNOULLICITY OF P

We now suppose given a $\mathbb{Z}^2$-Markov process $(X, \mathscr{B}, \mu, S, T)$. For the rest of the proof we assume E (P, S, T) to be nonzero otherwise the weak Pinsker property is trivially satisfied.

Let ε be fixed, we want to show the existence of a partition H such that $E(H, S, T) < \varepsilon$ and P is $\varepsilon-(H)_{S,T}$ relatively very weakly Bernoulli.

For this purpose, we choose an integer n and suppose it is fixed for the rest of this part.

Together with n, we consider two partitions: $Q = (P)_{D_n}$ and $R = (P)_{b(D_n)}$ (see definition 5 for $b(D_n)$). We recall

$$D_n = \{ (k, l) \in \mathbb{Z}^2 ; 0 \leqq k \leqq n - 1, 0 \leqq l \leqq n - 1 \}$$

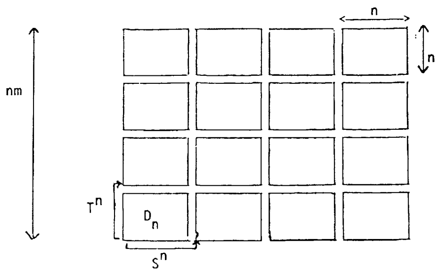
and

$$C_n = \{ (k, l) \in \mathbb{Z}^2 ; 0 \leqq |k| \leqq n - 1, 0 \leqq |l| \leqq n - 1 \} .$$

Because $D_n \supset b(D_n)$, it is clear that $Q \supset R$.

In the sequel, we will repeatedly use the following property of a $\mathbb{Z}^2$-Markov process:

Given $m > 0$, we can consider (see figure) a paving of D_{nm} by disjoint translates of D_n. This gives us a « frame ». Now the distribution of P-names inside any D_n-translate depends only on the P-name on its boundary (this is exactly the Markov property) and thus knowing the P-names on the frame, the distribution of P-names inside the D_n-translates are all independent of each other and also of any « information » on the P-names outside D_{nm}.



From that we deduce that for every m, every $m' \geqq m$, and every $(Q)_C$ in the D_m-past of Q for the $\mathbb{Z}^2$-action generated by S^n and T^n we have

$$d\left[\bigvee_{(k,l)\in D_m} S^{kn}T^{ln}Q/r\right] = d\left(\bigvee_{(k,l)\in D_m} S^{kn}T^{ln}Q/q \cap r\right) \tag{7}$$

where r is any atom of $\bigvee_{(k,l)\in C_m} S^{kn}T^{ln}R$ and q any atom of $(Q)_C$.

The equality of the two distributions implies that the $\bar{d}$ distance between them is zero.

If R is considered relative to the full $\mathbb{Z}^2$-action generated by S and T, it is clear that $(R)_{S,T} = (P)_{S,T}$ so that the entropy of R relative to the $\mathbb{Z}^2$-action generated by S and T is the same as that of P.

Our goal in the following is to obtain a partition H with small entropy « looking like » R and such that the equality in (7) becomes a small $\bar{d}$ distance when H is substituted for R.

In the sequel we will suppose that the $\mathbb{Z}^2$-action generated by S^n, T^n is ergodic. This will simplify our calculation and we will indicate at the end of this part how these calculations are modified in the case of non-ergodicity.

Let $R = (r_1, r_2, \ldots, r_s)$, we recall the following:

$$E(Q'/R') = E(Q' \vee R') - E(R')$$

and:

$$E(Q', S, T/(R')_{S,T}) = E(Q' \vee R', S, T) - E(R', S, T).$$

We will use in the sequel, properties of this conditional entropy, well known in the $\mathbb{Z}$-case, that extend without changes to $\mathbb{Z}^2$ (see again Conze [1]). Using as above the Markov property we can prove

LEMMA 1.

$$E(Q, S^n, T^n/(R)_{S^n,T^n}) = E(Q/R).$$

Proof. — Let

$$J_M = \frac{1}{M^2} E\Big(\bigvee_{(k,l)\in D_M} S^{nk}T^{nl}Q \Big/ \bigvee_{(k,l)\in C_M} S^{nk}T^{nl}R\Big)$$

$$= \sum_{r\in(R)^n_{C_M}} \mu(r) \times \frac{1}{M^2} E\Big(\bigvee_{(k,l)\in D_M} S^{nk}T^{nl}Q/r\Big).$$

the notation $(R)^n_{C_M}$ refers to the partition $\Big(\bigvee_{(k,l)\in C_M} S^{nk}T^{nl}R\Big)$.

Recall $|R| = s$, $R = (r_1, r_2, \ldots, r_s)$. For $1 \leqq i \leqq s$ let $k_i(r')$ be the number of times one « sees » r_i in $r' \in (R)^n_{D_M}$ (we recall, see definition 1 that

$$r' = \bigcap_{(k,l)\in D_M} S^{-kn}T^{-ln}r_{i_{k,l}} \quad \text{where} \quad 0 \leqq i_{k,l} \leqq s$$

and then $k_i(r')$ is the number of (k, l) in D_M such that $i_{k,l} = i$).

Because of the Markov property $J_M = \displaystyle\sum_{r'\in(R)^n_{D_M}} \mu(r') \times \frac{1}{M^2} \sum_{i=1}^{s} k_i(r')E(Q/r_i)$

(this comes again from the fact that on D_{nM} together with the « frame » of disjoint translates of D_n the distribution of P-names inside those D_n-translates is independent from everything outside once we know the P-name on its-boundary).

Because the action of S^n, T^n was assumed to be ergodic we get, using the mean ergodic theorem for the functions 1_{r_i} $(1 \leqq i \leqq s)$:

For any $\alpha > 0$, if M is large enough, for $1-\alpha$ of the $r \in (R)^n_{D_M}$ and any i, $1 \leqq i \leqq s$:

$$\left|\frac{k_i(r')}{M^2} - \mu(r_i)\right| \leqq \alpha. \tag{8}$$

Using (8) and the identity

$$E(Q/R) = \sum_{i=1}^{s} \mu(r_i)E(Q/r_i) = \sum_{r' \in (R)^n_{D_M}} \mu(r') \sum_{i=1}^{s} \mu(r_i)E(Q/r_i),$$

for any α, if M is large enough:

$$|J_M - E(Q/R)| \leqq \sum_{r' \in (R)^n_{D_M}} \mu(r') 2\alpha \sum_{i=1}^{s} |E(Q/r_i)|.$$

Because $\lim_{M \to +\infty} J_M = E(Q, S^n, T^n/(R)_{S^n,T^n})$. we conclude:

$$E(Q/R) = E((Q, S^n, T^n/(R)_{S^n,T^n}).$$

This finishes the proof.

We will now construct the partition H we are looking for. In fact H will depend on a small $\alpha > 0$ and on an integer K. We will make them precise along the proof, and specially before the proof of theorem 1 (see below).

Let $\alpha > 0$ and then K be chosen so that:

$$(9) \qquad \frac{1}{K^2} E\Big(\bigvee_{(k,l) \in D_K} S^{nk}T^{nl}R\Big) \leqq E(R, S^n, T^n) + \frac{\alpha}{2}.$$

For $1 - \alpha$ of the atoms r in $\bigvee_{(k,l) \in D_K} S^{nk}T^{nl}R$, for $1 \leqq i \leqq s$

$$(10) \qquad \left| \frac{k_i(r)}{K^2} - \mu(r_i) \right| \leqq \alpha$$

where $k_i(r)$ is as before the number of times one « sees » r_i in r (see definition above).

α and K being fixed, according to the strong Rohlin's lemma (see [3]) for the S, T action, one can find a set F such that:

a) S^kT^lF, $(k, l) \in D_{nK}$ are disjoint

b) $\mu\Big(\bigcup_{(k,l) \in D_{nK}} S^kT^lF\Big) \geqq 1 - \alpha^2/2$

c) $d\Big(\bigvee_{(k,l) \in D_K} S^{-nk}T^{-nl}Q/F\Big) = d\Big(\bigvee_{(k,l) \in D_K} S^{-nk}T^{-nl}Q\Big)$

Let then H, be the partition of the space defined as follows:

One atom of H is X − F and the other atoms of H are the atoms of $\bigvee_{(k,l)\in D_K} S^{-nk}T^{-nl}R/F$. (That is by definition we partition F, according to the « R, S^n, T^n names » of its points along the Rohlin tower) let: $H' = \bigvee_{(k,l)\in D_n} S^kT^lH$, F_0 be the partition (F, X−F), and $F' = \bigvee_{(k,l)\in D_n} S^kT^lF_0$. If K is big enough

$$E(F') \leqq \alpha . \tag{11}$$

This comes from the fact that n is fixed and $E(F') \leqq n^2E(F_0)$. We can now prove

LEMMA 2. — $E(H', S^n, T^n) \leqq E(R, S^n, T^n) + 2\alpha$.

Proof.

$$E(H', S^n, T^n) = E(H', S^n, T^n/(F')_{S^n,T^n}) + E(F', S^n, T^n)$$
$$\leqq E(H', S^n, T^n/(F')_{S^n,T^n}) + \alpha .$$

It is thus enough to prove $E(H', S^n, T^n/(F')_{S^n,T^n}) \leqq E(R, S^n, T^n) + \alpha$. If $J_M = \frac{1}{M^2} E\left(\bigvee_{(k,l)\in D_M} S^{nk}T^{nl}H' \middle/ \bigvee_{(k,l)\in C_M} S^{nk}T^{nl}F'\right)$ we have

$$J_M \leqq \frac{1}{M^2} E\left(\bigvee_{(k,l)\in D_M} S^{nk}T^{nl}H' \middle/ \bigvee_{(k,l)\in D_M} S^{nk}T^{nl}F'\right)$$
$$= \sum_{f \in \bigvee_{(k,l)\in D_M} S^{nk}T^{nl}F'} \mu(f) \times \frac{1}{M^2} E\left[\bigvee_{(k,l)\in D_M} S^{nk}T^{nl}H'/f\right].$$

Now using (9), and the inequality $E[P' \vee Q'] \leqq E(P') + (EQ')$ for any partitions P′, Q′ we obtain easily: $J_M \leqq E(R, S^n, T^n) + \alpha + 0\left(\frac{1}{M}\right)$.

The fact that $\lim_{M\to+\infty} J_M = E(H', S^n, T^n/(F')_{S^n,T^n})$ finishes the proof of the lemma.

COROLLARY 1. — *a*) $E(H, S, T) \leqq \varepsilon_n$ and $\lim_{n\to+\infty} \varepsilon_n = 0$

b) $$E(Q/R) - 2\alpha \leqq E(Q, S^n, T^n/(H')_{S^n,T^n}) .$$

(Recall that n is fixed but R, Q and H′ depend on n).

Proof. — *a*) We have $E(H, S, T) = \frac{1}{n^2} E(H', S^n, T^n) \leqq \frac{1}{n^2} E(R, S^n, T^n) + 2\alpha$.

Let $| P |$, be the number of atoms of our initial partition P. In R, there are at most $| P |^{4n}$ atoms so that:

$$E(H, S, T) \leqq \frac{1}{n^2} [E(R, S^n, T^n) + 2\alpha] \leqq \frac{1}{n^2} [4n \log p + 2\alpha]$$

and this clearly proves *a*).

b) It is enough to note that Q is a generating partition for the action of S^n and T^n so that:

$$\begin{aligned} E[Q, S^n, T^n/(H')_{S^n,T^n}] &= E(Q, S^n, T^n) - E(H', S^n, T^n) \\ &\geqq E(Q, S^n, T^n) - E(R, S^n, T^n) - 2\alpha \qquad \text{(because of lemma 2)} \\ &= E[Q, S^n, T^n/(R)_{S^n,T^n}] - 2\alpha = E(Q/R) - 2\alpha \end{aligned}$$

because of lemma 1 and this proves *b*).

By the corollary if n is big enough $E(H, S, T) \leqq \varepsilon$ (ε was fixed at the beginning of part III). We now want to prove that P is $(H)_{S,T}$, ε-relatively very weakly Bernoulli and to do that we will make use of a further notion found by J.-P. Thouvenot and exposed for instance in the work of D. Rudolph [7], that of extremality: in fact we will only use the following lemma:

LEMMA 3. — Let $(X, \mathscr{B}, \mu, S_1)$ be a system with an ergodic $\mathbb{Z}$ action, and P_1 a partition of X. If (P_1, S_1) is finitely determined, then for every positive θ, there exist an integer n_0 and $\delta_0 > 0$ such that if G is a partition of X satisfying:

For $(1 - \delta_0)$ of the atoms g of G, for $n \geqq n_0$ we have:

$$\frac{1}{n} E\left[\bigvee_{k=0}^{n-1} S_1^k P_1/g\right] \geqq E(P_1, S_1) - \delta_0 \tag{12}$$

then for a set G_1 of atoms of G with $\mu(G_1) > 1 - \theta$ we have for every $g \in G_1$:

$$\bar{d}\left[\bigvee_{k=0}^{n-1} S_1^k P_1, \bigvee_{k=0}^{n-1} S_1^k P_1/g\right] \leqq \theta. \tag{13}$$

Proof. — It is enough to note that this comes from the lemma 1 of [7] (in its proof Rudolph only uses inequality (12) and the fact that the « good » g are in a large set).

Let us see how we will use lemma 3 for our purpose: let $N > 0$, and r be an atom of $\bigvee_{(k,l)\in D_N} S^{-nk}T^{-nl}R$.

Given r, we can look at the distribution of $\bigvee_{k,l\in D_N} S^{-nk}T^{-nl}Q$. To r corresponds a paving of D_{Nn} by disjoint translates of D_n and on the boundary of each of these translates we will « see » an atom of R $\left(\text{if } r = \bigvee_{k,l\in D_N} S^{-nk}T^{-nl} r_{i_{k,l}}\right.$ on $D_n + n(k, l)$ we see the atom $\left. r_{i_{k,l}}\right)$. For any i, $1 \leqq i \leqq s$, where we recall $s = |R|$, we can look in this paving to the part $D_N^{i,r} \subset D_N$ of the translates of D_n, where we see a given boundary r_i.

Because of the Markov property, in every translate of D_n with boundary r_i, we see independently the distribution of Q/r_i, thus in $D_N^{i,r}$ we have something similar to a Bernoulli process in $\mathbb{Z}$ with independent generator a partition with distribution that of Q/r_i. This process is finitely determined.

Applying lemma 3 successively to the s processes so defined (s as well as r are fixed) we easily obtain the following: For any positive θ, there exists an integer n_0 and $\delta_0 > 0$ such that if for any i, $1 \leqq i \leqq s$, and any partition G of X we have:

(14) $\quad |D_N^{i,r}| \geqq n_0$ and for $(1 - \delta_0)$ of the atoms of G

$$(15) \qquad \frac{1}{|D_N^{i,r}|} E\left[\bigvee_{(k,l)\in D_N^{i,r}} S^{-nk}T^{-nl}Q/g \cap r\right] \geqq E[Q/r_i] - \delta_0 .$$

Then for $(1 - \theta)$ of the atoms g of G we have:

$$\bar{d}\left[\bigvee_{(k,l)\in D_N^{i,r}} S^{-nk}T^{-nl}Q/g \cap r \Big/ \bigvee_{(k,l)\in D_N^{i,r}} S^{-nk}T^{-nl}Q/r\right] \leqq \theta$$

The $\bar{d}$ distance here is to be understood of course with respect to $|D_N^{i,r}|$, that is the number of times we see in r, the atom r_i. Keeping our notations we want now to obtain similar inequalities for the global distribution given r and this is the object of the following crucial lemma.

LEMMA 4. — Let $\gamma > 0$ be given. There exists δ_1 and an integer n_1 such that:

— If r is any atom of $\bigvee_{(k,l)\in D_N} S^{-nk}T^{-nl}R$, $N \geqq n_1$ where we see $|D_N^{i,r}|$ times the atom r_i and for each $1 \leqq i \leqq s$: $|D_N^{i,r}| \geqq \gamma N^2$ then:

— For every partition G of X that satisfies for $(1 - \delta_1)$ of the atoms g of G:

$$(17)\qquad \frac{1}{N^2} E\left(\bigvee_{(k,l)\in D_N} S^{-nk}T^{-nl}Q/g \cap r\right) \geqq \sum_{i=1}^{s} \frac{|D_N^{i,r}|}{N^2} E(Q/r_i) - \delta_1$$

then for $\left(1 - \frac{\varepsilon}{2}\right)$ of the atoms g of G we have:

$$(18)\qquad \bar{d}\left[\bigvee_{(k,l)\in D_N} S^{-nk}T^{-nl}Q/g \cap r, \quad \bigvee_{(k,l)\in D_N} S^{-nk}T^{-nl}Q/r\right] \leqq \frac{\varepsilon}{2}.$$

Proof. — We can write:

$$E\left[\bigvee_{(k,l)\in D_N} S^{-nk}T^{-nl}Q/g \cap r\right]$$
$$= \sum_{i=1}^{i=s} E\left[\bigvee_{(k,l)\in D_N\cap D_N^{i,r}} S^{-nk}T^{-nl}Q/g \cap r \bigvee_{\substack{1\leqq j\leqq i-1\\ (k,l)\in D\cap D_N^{j,r}}} S^{-nk}T^{-nl}Q\right].$$

Let for $1 \leqq i \leqq s$:

$$\bigvee_{(k,l)\in D_N\cap D_1^{i,r}} S^{-nk}T^{-nl}Q = A_i$$

and

$$B_i = \bigvee_{\substack{(k,l)\in D_N\cap D_N^{j,r}\\ 1\leqq j\leqq i-1}} S^{-nk}T^{-nl}Q \qquad \text{(for } i=1,\ B_i \text{ is the trivial partition).}$$

Because of the Markov property, (18) is true if for $\left(1 - \frac{\varepsilon}{2}\right)$ of the atom g in G, for every $1 \leqq i \leqq s$, for $\left(1 - \frac{\varepsilon}{2}\right)$ of the atoms $b_i \in B_i$ we have:

$$(19)\qquad \bar{d}[A_i/r, \quad A_i/r \cap g \cap b_i] \leqq \frac{\varepsilon}{2}.$$

To obtain (19) we take $\theta = \left(\frac{\varepsilon}{2s}\right)^2$ then choose n_0 and δ_0 so that (14) and (15) imply (16). If for every $1 \leqq i \leqq s$ the following two conditions hold

(14′) $|D_N^{i,r}| \geqq n_0$ and for $(1 - \delta_0)$ of the atoms $g \cap b_i$ of GB_i

$$(15')\qquad \frac{1}{|D_N^{i,r}|} E\left[\bigvee_{(k,l)\in D_N^{i,r}} S^{-nk}T^{-nl}Q/g \cap b_i \cap r\right] \geqq E(Q/r_i) - \delta_0,$$

then for $1 - \left(\frac{\varepsilon}{2s}\right)^2$ of the atoms $g \cap b_i$ of GB_i we have:

$$\bar{d}[A_i/r, \quad A_i/r \cap g \cap b_i] \leqq \left(\frac{\varepsilon}{2s}\right)^2 .$$

Then by an easy calculation using a Fubini's like equality, we clearly have (19). So now n_0 and δ_0 are given we are left to prove (14′) and (15′):

(14′) is obtained if n_1 is big enough because $|D_N^{i,r}| \geqq \gamma N^2$.

Applying the Shannon-Mac Millan theorem to the Bernoulli process (in $\mathbb{Z}$), with an independent generator having distribution dist (Q/r_i), for any δ' if n_1 is big enough, so $|D_N^{i,r}| \geqq \gamma n_1^2$, we have a number smaller than $e^{|D_N^{i,r}|(E(Q/r_i)+\delta')}$ atoms of $\bigvee_{(k,l)\in D_N^{i,r}} S^{-nk}T^{-nl}Q/r$, that recover a set C_i of measure bigger than $\left(1 - \frac{\delta'^3}{s}\right)$ of $\bigvee_{(k,l)\in D_N^{i,r}} S^{-nk}T^{-nl}Q/r$. We thus obtain

$$(20) \quad \frac{1}{|D_N^{i,r}|} E\left[\bigvee_{(k,l)\in D_N^{i,r}} S^{-nk}T^{-nl}Q/r \cap g \cap b_i\right] \leqq E(Q/r_i) + \delta' + \delta'' .$$

for $1 - \frac{\delta'^2}{s}$ of the atoms $g \cap b_i$ of GB_i (where $\mu(g \cap b_i \cap C_i) > 1 - \delta'$, δ'' is the small correction for the part of $g \cap b_i$ not in C_i).

Thus for a set of atoms g in G of measure bigger than $1 - \delta'$ we have for each i, (20) is true for $1 - \delta'$ of the b_i in B_i. If (15′) was not true for some i_0 and (17) was true we then would get:

$$\frac{1}{N^2} E\left[\bigvee_{(k,l)\in D_N} S^{-nk}T^{-nl}Q/g \cap r\right] \leqq \sum_{\substack{i=1 \\ i\neq i_0}}^{s} \frac{|D_N^{i,r}|}{N^2} E(Q/r_i)$$
$$+ \frac{|D_N^{i,r}|}{N^2}(\delta' + 2\delta'') + \frac{|D_N^{i,r}|}{N^2}(E(Q/r_{i_0}) - \delta_0) .$$

(The term $2\delta''$ comes from the correction for the small portion δ' of the b_i that do not satisfy (20)). Thus we get comparing with (17):

$$-\delta_1 \leqq \frac{1}{N^2} \sum_{\substack{i=1 \\ i\neq i_0}}^{s} |D_N^{i,r}|(\delta' + 2\delta'') - \frac{|D_N^{i_0,r}|}{N^2}\delta_0 \text{ or using } |D_N^{i_0,r}| \geqq \gamma N^2 :$$

$$\delta_0 \leqq \frac{1}{\gamma}[\delta' + 2\delta'' + \delta_1] .$$

This if δ_1, δ', δ'' were chosen small enough (that is if n_1 is big enough as well as δ_1 small enough) we obtain a contradiction and this proves the lemma.

Before describing more precisely the choice of the parameters K and α in the Rohlin tower and then ending the proof, we will prove two general lemmas concerning the $\mathbb{Z}^2$-entropy:

LEMMA 5. — Let P a given finite partition. Then, for any integer m and real $\delta > 0$, for any set A such that $\mu(A) \leqq \delta$:

$$\frac{1}{m^2} E\left[\bigvee_{(k,l)\in D_m} S^k T^l P \cap A \right] \leqq f(\delta), \qquad \text{with} \qquad \lim_{\delta\to 0} f(\delta) = 0$$

f depending only on $|P|$ and δ.

(For a partition P′ and a set A by E(P′ ∩ A) we mean:

$$- \sum_{p_i \in P'} \mu(p_i \cap A) \log (p_i \cap A))) .$$

Proof. — In $(P)_{D_m}$ there are at most $|P|^{m^2}$ atoms and the entropy we want to compute is maximum when all these atoms have the same measure μ_A. We then obtain:

$$- \frac{1}{m^2} E[(P)_{D_m} \cap A] \leqq - \frac{1}{m^2} \mu(A) \log \frac{\mu(A)}{|P|^{m^2}}$$
$$= - \frac{1}{m^2} \mu(A) \log \mu(A) + \mu(A) \log |P| \leqq \delta \log |P| - \frac{\delta \log \delta}{m^2},$$

for δ small enough and this proves the lemma.

LEMMA 6. — Let $m > 0$, P and H two partitions of a space $(X, \mathscr{B}, \mu)$ together with a $\mathbb{Z}^2$-action with generators (S, T) on X. Then:

For any $B \subset \mathbb{Z}^2$, and every $C \subset \mathbb{Z}^2$, such that $(P)_C$ is in the D_m-past for P:

$$\frac{1}{m^2} E\left[\bigvee_{(k,l)\in D_m} S^k T^l P/(P)_C \vee (H)_B \right] \geqq E[P, S, T/(H)_{S,T}] .$$

Proof. — Let us introduce $P' = \bigvee_{(k,l)\in D_m} S^k T^l P$ and $H' = \bigvee_{(k,l)\in D_m} S^k T^l H$.

If C′ is in the D_1-past for P′ and the $\mathbb{Z}^2$-action generated by S^m, T^m and $B' \subset \mathbb{Z}^2$ we have:

$$m^2 E[P, S, T/(H)_{S,T}] = E[P', S^m, T^m/(H')_{S^m,T^m}] \leqq E[P'/(P')^m_{C'} \vee (H')^m_{B'}] .$$

Here $(P')^m_{C'}$ and $(H')^m_{B'}$ are to be understood with the action of S^m, T^m. The last inequality is easy to see and comes from the definition:

$$E[P', S^m, T^m/(H')_{S^m,T^m}] = E[P'/(H')_{S^m,T^m} \vee \text{(entire past of } P' \text{ for the action } (S^m, T^m))].$$

It is now easy to choose C′ and B′ so that

$$(P')^m_{C'} \supset (P)_C \quad \text{and} \quad (H')^m_{B'} \supset (H)_B$$

and this implies what we wanted to prove:

$$\frac{1}{m^2} E\left[\bigvee_{(k,l)\in D_m} S^k T^l P/(P)_C \vee (H)_B\right] = \frac{1}{m^2} E[P'/(P)_C \vee (H)_B]$$
$$\geqq \frac{1}{m^2} E[P'/(P')^m_{C'} \vee (H')^m_{B'}] \geqq E[P, S, T/(H)_{S,T}].$$

Precisions for the construction of the Rohlin tower (K as a function of α):

Let us now describe how to choose K, where the Rohlin tower has size D_{nK} as a function of α. (The value of α is made precise at the end of the proof of theorem 1, this value then, fixes the value of K and we can then construct our Rohlin tower satisfying the above properties *a*), *b*) and *c*) of the Rohlin tower). Let ε and n be fixed. This fixes s, the number of atoms of R. From lemma 4, we know n_1 and δ_1 that enable us to apply this lemma. Let finally

$$\gamma = \inf_{1 \leqq i \leqq s} \frac{\mu(r_i)}{2} \tag{21}$$

and K_1, be an intermediate integer with

$$\frac{4n}{K_1} \leqq \frac{\delta_1^2}{100}. \tag{22}$$

We suppose K_1 is big enough so that:

There is a set $B_R^{K_1}$ of atoms of $\bigvee_{(k,l)\in D_{K_1}} S^{-nk}T^{-nl}R$ so that $\mu(B_R^{K_1}) > 1-\alpha^2$ and if $r \in B_R^{K_1}$:

i) For each i, $1 \leqq i \leqq s$, we see the atom r_i in r, at least $\gamma K_1^2 \geqq n_1$ times. Let $|D_{K_1}^{i,r}|$ be this number; and $D_{K_1}^{i,r} \subset D_{K_1}$ the corresponding places.

ii) Let us consider the partition $\bigvee_{(k,l)\in D_{K_1}} S^{-nk}T^{nl}Q/r$ then $1 - \alpha^2$ of r is covered by at most $e^{\sum_{i=1}^{s} |D_N^{i,r}|[E(Q/r_i)+\alpha]}$ atoms of this partition.

i) comes easily, if we use the mean ergodic theorem for the $\mathbb{Z}^2$-action generated by (S^n, T^n) for the functions $1_{r_i} (1 \leqq i \leqq s)$.

ii) Follows from the Shannon-Mac-Millan theorem applied for each i to the Bernoulli process with independent generator Q_i, such that

$$\text{dist } Q_i = \text{dist } (Q/r_i):$$

If K_1 is big enough, $\left(1 - \frac{\alpha^2}{s}\right)$ of r is covered by at most $e^{|D_N^{i,r}|(E(Q/r_i)+\alpha)}$ atoms of $\bigvee_{(k,l)\in D_{K_1}^{i,r}} S^{-nk}T^{-nl}Q/r$, and so (*ii*) follows.

Let finally K be such that:

$$\frac{K_1}{K} \leqq \frac{\alpha}{2}. \tag{23}$$

If $$g(x) = \frac{1}{K^2} \sum_{(k,l)\in D_K} 1_{B_R}^{K_1}(S^{nk}T^{nl}x), \quad \int_X g d\mu \geqq 1 - \alpha^2.$$

Thus there exists a set C_R^K of atoms of $\bigvee_{(k,l)\in D_K} S^{-nk}T^{-nl}R$ of measure bigger than $(1 - \alpha)$ so that for any r' in C_R^K and $x \in r'$, $g(x) > 1 - \alpha$.

Restricting the summation to the (k, l) so that: $n(k, l) + D_{nK_1} \subset D_{nK}$ in the definition of $g(x)$ we obtain $g_1(x)$ and by (23) we have: if $r' \in C_R^K$, $x \in r'$, $g_1(x) > 1 - 2\alpha$. With a given value of α we find K_1 and K so that we can construct the Rohlin tower and the partition H with those values of K and α.

We are now ready to prove:

THEOREM 1. — P is ε-very weakly Bernoulli relative to $(H)_{S,T}$.

Proof. — We want to show the following: For any $B \subset \mathbb{Z}^2$ such that $D_{nK} \subset B$, any $C \subset \mathbb{Z}^2$ with $(P)_C$ in the D_{nK_1} past for P and the action (S, T):

For a family $h \cap p$, $h \in (H)_B$, $p \in (P)$ of atoms whose union has measure bigger than $1 - \varepsilon$ one has:

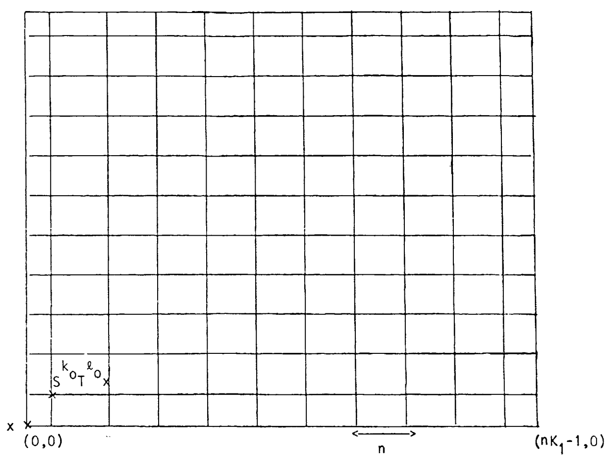
$$\bar{d}\left[\left(\bigvee_{(k,l)\in D_{nK_1}} S^{-k}T^{-l}P/h\right), \left(\bigvee_{(k,l)\in D_{nK_1}} S^{-k}T^{-l}P/h \cap p\right)\right] < \varepsilon.$$

To prove this it is clearly enough to prove that for any $B' \subset \mathbb{Z}^2$, $B' \supset D_K$, any C' in $\mathbb{Z}^2$ with $\bigvee_{(k,l)\in C'} S^{-nk}T^{-nl}Q$ in the D_{K_1}-past for Q and the action generated by (S^n, T^n), for a family $h' \cap q$, $h' \in \bigvee_{(k,l)\in B'} S^{-nk}T^{-nl}H'$, $q \in \bigvee_{(k,l)\in C'} S^{-nk}T^{-nl}Q$ of atoms whose union has measure bigger than $1 - \varepsilon$:

$$(24) \quad \bar{d}\left[\left(\bigvee_{(k,l)\in D_{K_1}} S^{-nk}T^{-nl}Q/h'\right), \left(\bigvee_{(k,l)\in D_{K_1}} S^{-nk}T^{-nl}Q/h' \cap q\right)\right] \leqq \varepsilon .$$

Because $(1 - \alpha)$ of the space is covered by the Rohlin tower and $\frac{K_1}{K} \leqq \frac{\alpha}{2}$, restricting us to $1 - 2\alpha$ of the space we can suppose that: Given h', there exists $k_{h'}$, $l_{h'}$ such that $h' \subset S^{k_{h'}}T^{l_{h'}}F$ and furthermore $(k_{h'}, l_{h'}) + D_{nK_1} \subset D_{nK}$.

For such a fixed h', we can also define (k_0, l_0) and (k_1, l_1) with $0 \leqq k_0 \leqq n-1$, $0 \leqq l_0 \leqq n-1$, $(k_1, l_1) \in D_K$ such that for any x in h', $S^{k_0}T^{l_0}x \in S^{nk_1}T^{nl_1}F$



Restricting further to $(1 - 4\alpha)$ of the space we can suppose that for any x in h', $S^{k_0}T^{l_0}x$ is in some $r \in B_R^{K_1}$.
This comes from the definition of $B_R^{K_1}$ in « precisions for the construction » and from the fact (c) in the properties of the Rohlin tower:

$$d\left(\bigvee_{(k,l)\in D_K} S^{-nk}T^{-nl}Q/F\right) = d\left(\bigvee_{(k,l)\in D_K} S^{-nk}T^{-nl}Q\right).$$

We will try to obtain the above inequality (24) for those h', and write for h'

fixed: $h' = S^{k_0}T^{l_0}r \cap h'' = \tilde{r} \cap h''$ where h'' enables us naturally to define h' when knowing $S^{k_0}T^{l_0}r$. From lemma 6 and corollary 1 we have:

$$(25) \qquad E\left[\bigvee_{(k,l)\in D_{K_1}} S^{nk}T^{nl}Q/(H')^n_{B'} \vee (Q)^n_{C'}\right] \geqq E(Q/R) - 2\alpha$$

(here the index n in $(H')^n_{B'}$ or $(Q)^n_{C'}$ recall that we are considering the action generated by S^n and T^n). To obtain (24) we will use lemma 4 it is then clear that it is enough to have: if $h' = \tilde{r} \cap h''$, for $1 - \frac{\delta_1\varepsilon}{4}$ of the $\tilde{r} \cap h'' \cap q$ and $1 - \frac{\delta_1\varepsilon}{4}$ of the $\tilde{r} \cap h''$:

$$(26) \quad \frac{1}{K_1^2} E\left[\bigvee_{(k,l)\in D_{K_1}} S^{-nk}T^{-nl}Q/\tilde{r} \cap h'' \cap q\right] \geqq \sum_{i=1}^{s} \frac{|D^{i,\tilde{r}}|}{K_1^2} E(Q/r_i) - \frac{\delta_1}{2}$$

$$\text{and} \quad \frac{1}{K_1^2} E\left[\bigvee_{(k,l)\in D_{K_1}} S^{-nk}T^{-nl}Q/\tilde{r} \cap h''\right] \geqq \sum_{i=1}^{s} \frac{|D^{i,\tilde{r}}_{K_1}|}{K_1^2} E(Q/r_i) - \frac{\delta_1}{2}.$$

Because then we have

$$\bar{d}\left[\left(\bigvee_{(k,l)\in D_{K_1}} S^{-nk}T^{-nl}Q/\tilde{r}\right), \left(\bigvee_{(k,l)\in D_{K_1}} S^{-nk}T^{-nl}Q/\tilde{r} \cap h''\right)\right] \leqq \frac{\varepsilon}{2}$$

and

$$\bar{d}\left[\left(\bigvee_{(k,l)\in D_{K_1}} S^{-nk}T^{-nl}Q/\tilde{r}\right), \left(\bigvee_{(k,l)\in D_{K_1}} S^{-nk}T^{-nl}Q/\tilde{r} \cap h'' \cap q\right)\right] \leqq \frac{\varepsilon}{2}.$$

For $1 - \frac{\varepsilon}{2}$ of the atoms $\tilde{r} \cap h''$ and also $1 - \frac{\varepsilon}{2}$ of the atoms $\tilde{r} \cap h'' \cap q$ and this implies what we want.

(Where as usual $D^{i,\tilde{r}}_{K_1}$ is by definition the places in D_{K_1}, where in the atom $\tilde{r}$ we see the atom r_i).

But for any h' as above we have by definition of $B^R_{K_1}$:

$$(27) \quad \frac{1}{K_1^2} E\left[\bigvee_{(k,l)\in D_{K_1}} S^{-nk}T^{-nl}Q/\tilde{r} \cap h'' \cap q\right] \leqq \sum_{i=1}^{s}\left[\frac{|D^{i,\tilde{r}}_{K_1}|}{K_1^2}(E(Q/r_i) + \alpha)\right] + f_1(\alpha).$$

For $1-\alpha$ of the atoms $h'' \cap q$ ($\tilde{r}$ is fixed), where $f_1(\alpha) < 4n\frac{\text{Log}\,|P|}{K_1}$, take account the fact that we used $S^{k_0 l_0} x$ instead of x, so that we have to count the atoms on the boundary.

Thus if B is the set of the atoms $\tilde{r} \cap h'' \cap q$ where (26) is not true we obtain:

$$(28) \quad E(Q/R) - 2\alpha \leqq (1 - m(B))[E(Q/R) + 2\alpha + f_1(\alpha)] + m(B)[E(Q/R) - \delta_1] + f(6\alpha)$$

because of the ergodic theorem, we have in most of the atoms:

$$(29) \qquad \sum_{i=1}^{s} \frac{|D_{K_1}^{i,\tilde{r}}|}{K_1^2} E(Q/r_i) - \frac{\delta_1}{2} \geqq E(Q/R) - \delta_1$$

and the terms $f(6\alpha)$ comes (see lemma 5) from the differents parts not accounted for, 4α of the space to restrict to the « good » h' and also 2α of the space for the atoms where (27) is not true.

We then have: $m(B) \leqq \dfrac{2\alpha + f_1(\alpha) + f(6\alpha)}{\delta_1 + 2\alpha + f_1(\alpha)} \leqq \dfrac{2\alpha + f_1(\alpha) + f(6\alpha)}{\delta_1}$.

This last expression can be made smaller than $\delta_1 \varepsilon$ if α is small enough and thus we finished the proof.

Case (S^n, T^n) *not ergodic.*

In that case, there exists a set A whose measure is $\dfrac{1}{n^2}$ such that all the $S^k T^l A$ $(k, l) \in D_n$ are disjoint and $X = \bigcup_{(k,l) \in D_n} S^k T^l A$. This follows from the ergodicity of the $\mathbb{Z}^2$-action generated by (S, T).

From the mean ergodic theorem we then have:

$$\frac{1}{m^2} \sum_{(k,l) \in D_m} 1_{r_i}(S^k T^l x) \xrightarrow{L^2(X,\mu)} \sum_{(k,l) \in D_n} \frac{\mu(r_i \cap S^k T^l A)}{\mu(A)} 1_{S^k T^l A}(x).$$

In lemma 1 we can replace

(8) by (8′) $$\left| \frac{k_i(r)}{M^2} - \frac{\mu(r_i \cap S^k T^l A)}{\mu(A)} \right| \leqq \alpha$$

for (k, l) such that $\mu(r \cap S^kT^lA) > 0$.

$$J_M = \sum_r \mu(r) \sum_i \frac{k_i(r)}{M^2} E(Q/r_i) = \sum_{\substack{r \\ (k,l)\in D_n}} \mu(r \cap S^kT^lA) \sum_i \frac{k_i(r)}{M^2} E(Q/r_i)$$

$$\simeq \sum_{\substack{r \\ (k,l)\in D_n}} \mu(r \cap S^kT^lA) \sum_i \frac{\mu(r_i \cap S^kT^lA)}{\mu(A)} E(Q/r_i)$$

$$= \sum_{\substack{i \\ (k,l)\in D_n}} \frac{\mu(r_i \cap S^kT^lA)}{\mu(A)} E(Q/r_i) \sum_r \mu(r \cap S^kT^lA) = \sum_{\substack{i \\ (k,l)\in D_n}} \mu(r_i \cap S^kT^lA) E(Q/r_i)$$

$$= \sum_i \mu(r_i) E(Q/r_i) = E(Q/R).$$

Thus conclusion of lemma 1 remains the same. Instead of (10) in the construction of the Rohlin tower we will have:

$$(10') \qquad \left| \frac{k_i(r)}{K^2} - \frac{\mu(r_i \cap S^kT^lA)}{\mu(A)} \right| \leqq \alpha \qquad \text{for} \qquad (k, l)$$

depending on r.

Now up to and including lemma 6 everything remains the same.

In the « precisions », we choose instead of (21):

$$(21') \qquad \gamma = \inf_{\substack{1 \leqq i \leqq s \\ (k,l);\, r_i \cap S^kT^lA \neq \phi}} \frac{\mu(r_i \cap S^kT^lA)}{2\mu(A)}.$$

Then theorem 1 has a similar proof using now the inequality (10′) and the above calculation for E(Q/R) to replace E(Q/R) in inequality (28).

This ends our proof.

It remains to see the justification of our reduction.

IV. JUSTIFICATION

Using the proof of Proposition 1 of [*8*] one can deduce from Thouvenot's work the following: to show that a process satisfies the weak Pinsker property it is enough to have: (see also lemma 7 of [*8*]), (H_n), $(B_n)_{n\geqq 1}$, finite partitions of X, as well as a sequence $(\varepsilon_n)_{n\geqq 1}$ of positive numbers tending to zero such that:

i) $$(P)_{S,T} = X$$

ii) $(H_n)_{S,T} \perp (B_n)_{S,T}$ for $n \geqq 1$

iii) $S^kT^lB_n$, $(k, l) \in \mathbb{Z}^2$ are independent

iv) $P \overset{\varepsilon_n}{\subset} (H_n \vee B_n)_{S,T}$ $n \geqq 1$

v) $E(H_n, S, T) < \varepsilon_n$.

To obtain the above property (*iv*) (ε_n-decomposition), we will use part III.

DEFINITION 9. — Let (S, T) be generators of a $\mathbb{Z}^2$-action on X. H and P two finite partitions of X. P is H_ε ε-relatively finitely determined if there exists $\delta > 0$ and $n \in \mathbb{N}$ such that for every pair of generators (S′, T′) of a $\mathbb{Z}^2$-action on a Lebesgue space Y the following conditions:

There exists two partitions P′ and H′ of Y such that:

i) For every m, $d\left(\bigvee_{(k,l)\in D_m} S'^kT'^lH'\right) = d\left(\bigvee_{(k,l)\in D_m} S^kT^lH\right)$.

ii) $d\left(\bigvee_{(k,l)\in D_n} S'^kT'^l(P' \vee H'),\ \bigvee_{(k,l)\in D_n} S^kT^l(P \vee H)\right) < \delta$

iii) $| E(P \vee H, S, T) - E(P' \vee H', S', T') | < \delta$

implies there exists a Lebesgue space Z and for every integer $p > 0$, sequences of partitions of $Z: H_{k,l}, P_{k,l}, P'_{k,l}, (k, l) \in D_p$ such that:

$$- \quad d\bigvee_{(k,l)\in D_p} S^kT^l(P \vee H) = d\left(\bigvee_{(k,l)\in D_p} (P_{k,l} \vee H_{k,l})\right)$$

$$- \quad d\left(\bigvee_{(k,l)\in D_p} S'^kT'^l(P' \vee H')\right) = d\left(\bigvee_{(k,l)\in D_p} (P'_{k,l} \vee H_{k,l})\right)$$

$$- \quad | P_{k,l} - P'_{k,l} | < \varepsilon \quad \text{for every} \quad (k, l) \in D_p.$$

We say that P is H relatively finitely determined if P is H ε-relatively finitely determined for every ε.

LEMMA 7. — If P is $\frac{\varepsilon^2}{10}$ very weakly Bernoulli relatively to $(H)_{S,T}$ then P is H ε-relatively finitely determined.

Proof. — This lemma is explicitely contained in the proof of the fact: H-relatively very weakly Bernoulli implies H-relatively finitely determined (see lemma 6 of [*10*] for the case of $\mathbb{Z}$, the case of $\mathbb{Z}^2$ being similar).

LEMMA 8. — If P is H ε-relatively finitely determined, there exists two finite partitions $\hat{B}$ and $\tilde{H}$ such that:

(30) $$(\tilde{B})_{S,T} \perp (\tilde{H})_{S,T}$$

(30) the $$S^k T^l \tilde{B}, \quad (k, l) \in \mathbb{Z}^2 \text{ are independent}$$

(32) $$P \overset{3\varepsilon}{\subset} (\tilde{H} \vee \tilde{B})_{S,T}$$

(33) $$| E(\tilde{H}, S, T) - E(H, S, T) | \leq \varepsilon .$$

Remark. — In the case in which $H \subset (P)_{S,T}$, J.-P. Thouvenot has showed us that we can take $\tilde{H} = H$.

Proof. — Let I be an abstract partition such that

$$E(I) = E(P, S, T) - E(H, S, T) .$$

Let Y_0 be the space $(0, 1, \ldots, i - 1)^{\mathbb{Z}^2}$ if $I = (h_0, \ldots, h_{i-1})$. On Y_0 we consider the Bernoulli $\mathbb{Z}^2$-process naturally associated with the product measure μ_0 defined by

$$\mu_0 [y_k = \alpha_k, \ldots, y_l = \alpha_l] = \prod_{j=k}^{j=l} \mu(h_{\alpha j}) .$$

Let $Y = Y_0 \times (H)_{S,T}$. On Y we consider the $\mathbb{Z}^2$-action product of the $\mathbb{Z}^2$-action on Y_0 and $(H)_{S,T}$, its generator will be denoted by S′ and T′.

Using the proof of lemma 4 in [*9*], we conclude that for n and δ corresponding to ε in the definition 9 of P, H ε-relatively finitely determined, there exists $\tilde{P}$, a partition of Y such that

$$d\Big(\bigvee_{(k,l)\in D_n} S'^k T'^l(\tilde{P} \vee H)\Big), \quad \bigvee_{(k,l)\in D_n} S^k T^l (P \vee H) < \delta$$

and $$| E(P \vee H, S, T) - E(\tilde{P} \vee H, S', T') | < \delta .$$

We conclude that there exists a space Z with a $\mathbb{Z}^2$-action whose generators are S_1, T_1 and partitions H_1, P_1, $\tilde{P}_1$ such that

i) $$(P \vee H, S, T) \sim (P_1 \vee H_1, S_1, T_1)$$

ii) $$(\tilde{P} \vee H, S', T') \sim (\tilde{P}_1 \vee H_1, S_1, T_1)$$

iii) $$| P_1 - \tilde{P}_1 | \leqq \varepsilon .$$

(*iii*) is obtained as in the $\mathbb{Z}$-case from the equivalence of the different definition of the $\bar{d}$ distance (see appendix C of [*6*], to do everything relative to $(H)_{S,T}$ does not change the conclusion).

Then according to proposition 5 of [*9*], there exists a partition B′ in $\mathbb{Z}$ such that:

iv) $$(\tilde{P}_1 \vee H_1)_{S_1,T_1} = (H_1 \vee B')_{S_1,T_1}$$

v) $(H)_{S_1,T_1} \perp (B')_{S_1,T_1}$

vi) the $S_1^k T_1^l B'$ for $(k, l) \in \mathbb{Z}^2$ are independent.

From *iii)* and *iv)* we then conclude that $P_1 \overset{\varepsilon}{\subset} (H_1 \vee B')_{S_1,T_1}$.

Lemma 4 of [*8*] allows us to conclude that there exists two partitions $(P_1)_{S_1,T_1}$-measurable $\overline{H}_1$ and $\overline{B}_1$ satisfying (30) to (33), with P replaced by P_1 and then using the isomorphism given by *i*), we obtain the desired conclusion.

Lemma 7, 8 and the remark at the beginning of this part proved that if for any ε, there exists H_ε with $E(H_\varepsilon, S, T) < \varepsilon$ and P is H ε-relatively very weakly Bernoulli then (X, B, μ, S, T) has the weak Pinsker property. This ends our proof.

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