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Tightness results for laws of diffusion processes application to stochastic mechanics

by

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ABSTRACT. — This paper follows our joint work [*3*] with P. A. Meyer « Tightness criteria for laws of semimartingales » (preceding issue of this journal). We construct the diffusions with nice « brownian » part and very singular drifts which Nelson associates with quantum mechanical wave functions.

RÉSUMÉ. — Ce travail prolonge un article publié dans le numéro précédent [*3*] de P. A. Meyer « Tightness criteria for laws of semimartingales ». Nous construisons des diffusions ayant une partie brownienne régulière et des « drifts » très singuliers du type de ceux que Nelson associe aux fonctions d'onde de la mécanique quantique.

This paper follows our joint work [*3*] with P. A. Meyer « Tightness criteria for laws of semimartingales » (preceding issue of this journal). In that paper, we show that the set of laws of quasimartingales with uniformly bounded stochastic variation is compact in a metrizable topology weaker than Skorohod's, which we call the « pseudo-path topology ».

However, in the present paper, we deal only with sets of laws of diffusion processes, which are tight in the usual sense (uniform topology), and we use the results of [3] only to identify the limit as a diffusion.

After giving some general results, we come to the subject which provided the motivation for the whole work, namely, the construction of the diffusions with nice « brownian » part and very singular drifts which Nelson associates with quantum mechanical wave functions. While we were working on this problem, E. Carlen kindly communicated to us the remarkable results he had proved on the existence of the Nelson processes for wave functions arising from a « Rellich potential ». It seems that the scopes of the two methods are somewhat different: we demand far more regularity of the wave function (his wave functions are only weakly differentiable of order one in space and order 1/2 in time, while we need strong differentiability). On the other hand, our method is probabilistic, and shows that the process will avoid the « nodes » set of the wave function in space time. It also depends less than Carlen's on global estimates.

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1. NOTATIONS AND GENERAL RESULTATS ON TIGHTNESS

We denote by $\mathscr{C}$ the space of continuous functions defined on the interval [0, 1] and taking values in $\mathbb{R}^d$. $\mathscr{C}$ will be equipped with two different metrizable topologies. The first one, the « pseudo-path topology » of [3], is simply that of convergence in measure on [0, 1]. The second one is the usual uniform topology.

All our processes will be $\mathbb{R}^d$ valued and indexed by [0, 1], unless the contrary is specified. If (X_t^n) is a sequence of processes, each one on its own probability space, and μ^n is the law of X^n on $\mathscr{C}$ ($n \leqq \infty$), we write $X^n \overset{L}{\to} X^\infty$, $X^n \overset{CL}{\to} X^\infty$ to express that μ^n tends weakly to μ^∞ w. r. to the first (second) topology. Similarly, we say that the sequence (X^n) is « tight », « C-tight » if the sequence (μ^n) is tight in the first (second) topology.

2. A SUFFICIENT CONDITION FOR C-TIGHTNESS

We recall a result of Rebolledo ([6], p. 29).

LEMMA 1. — *Let (M_t^n) be a sequence of continuous local martingales. If the random variables M_0^n are uniformly bounded in probability, and the sequence $(\langle M^n, M^n \rangle_t)$ is C-tight, then the sequence (M_t^n) is C-tight.*

As usual, each local martingale is defined on its own probability space, and uniform boundedness in probability means that $P^n \{ | M_0^n | > a \}$ tends to 0 uniformly in n as $a \to \infty$.

For C-tightness of processes of bounded variation, we may use the following easy lemma:

LEMMA 2. — *Let (a_t^n) be measurable processes (on possibly different probability spaces $(\Omega^n, \mathscr{F}^n, P^n)$) such that, for some exponent $p > 1$, the random variables $\int_0^1 | a_s^n |^p ds$ are uniformly bounded in probability. Then the sequence of processes $X_t^n = \int_0^t a_s^n ds$ is C-tight, and under any one of its limit laws on $\mathscr{C}$, the canonical (coordinate) process (X_t) on $\mathscr{C}$ a. s. has absolutely continuous paths $X_t = \int_0^t a_s ds$ with $\int_0^1 | a_s |^p ds < \infty$.*

Proof. — We have proved similar results in [3] for the pseudo-path topology. Namely, theorem 10 under the assumption that $E^n\left[\int_0^1 | a_s^n |^p ds\right]$ is uniformly bounded (and assuming $p = 2$, but this makes no difference). The case where boundedness in probability is assumed instead of boundedness in L^1 can be reduced to it by stopping, as in theorem 7. So the only point to be proved here is that C-tightness obtains instead of just tightness. This is due to the familiar Hölder inequality:

$$| X_{t+h}^n - X_t^n | \leqq h^{1-1/p}\left(\int_0^1 | a_s^n |^p ds\right)^{1/p} .$$

Combining these two results, we get a theorem which will be quite convenient to study diffusions.

THEOREM 3. — *Let (on possibly different probability spaces $(\Omega^n, \mathscr{F}^n, P^n, (\mathscr{F}_t^n))$ (X_t^n) be continuous semimartingales with canonical decompositions*

$$X_t^n = X_0^n + M_t^n + A_t^n .$$

Assume that $U_t^n = \langle M^n, M^n \rangle_t = \int_0^t u_s^n ds$, $A_t^n = \int_0^t a_s^n ds$, and that for some $p > 1$ the random variables

$$X_0^n , \qquad \int_0^1 (u_s^n)^p ds , \qquad \int_0^1 | a_s^n |^p dt$$

are uniformly bounded in probability. Then the vector valued processes $(X_t^n, M_t^n, A_t^n, U_t^n)$ *are C-tight on* $\mathscr{C}^4$, *and under any one of their limit laws* P *on this space, the coordinate process* (X_t, M_t, A_t, U_t) *on* $\mathscr{C}^4$ *has the following properties:*

1) (A_t) *is a continuous process with finite variation,* (U_t) *a continuous increasing process,* (M_t) *a continuous local martingale;* $M_0 = A_0 = U_0 = 0$; $X_t = X_0 + M_t + A_t$ *and* $\langle M, M \rangle_t = U_t$ (P-*a. s. in* $\mathscr{C}^4$).

2) (A_t) *and* (U_t) *are absolutely continuous with densities* a_s, u_s *such that*

$$\int_0^1 | a_s |^p ds < \infty , \qquad \int_0^1 (u_s)^p ds < \infty \qquad \text{P-}a.\ s.$$

Proof. — Lemma 2 implies the C-tightness of (A_t^n), (U_t^n), then lemma 1 gives the C-tightness of (M_t^n), and the usual Prohorov criterion the tightness of (X_0^n) on $\mathbb{R}$. The C-tightness of (X_t^n) then follows since $X_t^n = X_0^n + M_t^n + A_t^n$. Statement 2) already appears in lemma 2, as well as the sample function properties of (A_t) and (U_t). So the only point left is to show that (M_t) is a continuous local martingale with $M_0 = 0$ and $\langle M, M \rangle = U$: this is theorem 12 of [*3*].

Remark. — In all our applications, we shall assume stronger conditions than boundedness in probability on (a_n^s) and (u_s^n). For instance

$$E^n\left[\int_0^1 | a_s^n |^p ds\right] \leqq K , \qquad E^n\left[\int_0^1 (u_s^n)^p ds\right] \leqq K' .$$

These conditions simplify the proof, since they are explicitly considered in [*3*], and do not require the stopping argument mentioned in lemma 2. On the other hand, the same inequalities are valid for the limit process, as mentioned in the proof of theorem 10 in [*3*].

The following result will be our main technical lemma. It will allow us to identify the weak limit of a sequence of diffusions as a diffusion.

LEMMA 4. — *Let* (X_t^n), (X_t) *be* $\mathbb{R}^d$ *valued processes such that* $X^n \overset{\text{CL}}{\to} X$. *Let* $f^n(x, t)$ *and* $f(x, t)$ *be functions on* $\mathbb{R}^d \times [0, 1]$, U *be an open set in* $\mathbb{R}^d \times [0, 1]$. *Assume that*

i) *The random variables* $\int_0^1 | f^n(X_s^n, s) |^p ds$ *are uniformly bounded in probability for some* $p > 1$.

ii) *In the open set* U, *the functions* $f^n(x, t)$ *converge locally uniformly to* $f(x, t)$, *and are continuous.*

iii) *For a. e. t,* $(X_t, t) \in U$ *a. s.*

Then the pairs $\left(X_t^n, \int_0^t f^n(X_s^n, s)ds\right)$ *converge in law to* $\left(X_t, \int_0^t f(X_s, s)ds\right)$.

Proof. — Set $A_t^n = \int_0^1 f^n(X_s^n, s)ds$. Condition *i*) implies that the processes A^n are C-tight, so by extracting a subsequence if necessary we may assume that the pairs (X^n, A^n) C-converge. This extraction will not disturb the conclusion, since we'll show that all these subsequences will converge to the same limit. Condition *i*) and lemma 2 also imply that the limit process (X_t, A_t) has an absolutely continuous second component. According to the celebrated Skorohod theorem (Billingsley [*0*], th. 3.3), we may realize all our processes (X_t^n, A_t^n, X_t, A_t) on the same probability space $(\Omega, \mathscr{F}, P)$ —we do not care about filtrations in this context—in such a way that $X_.^n(\omega) \to X_.(\omega)$, $A_.^n(\omega) \to A_.(\omega)$ uniformly.

Let t be such that $(X_t(\omega), \omega) \in U$. Since U is open and the path is continuous, for some $\varepsilon > 0$ we have $(X_s(\omega), s) \in U$ for $t \leqq s \leqq t + \varepsilon$. Let V be a compact neighbourhood of this (compact) path in U: according to the uniform convergence above, we have $(X_s^n(\omega), s) \in V$ for n large enough, over the same interval. According to the Arzelà-Ascoli theorem, the f^n are equicontinuous on V, and therefore $f^n(X_s^n(\omega), s)$ converges uniformly to $f(X_s(\omega), s)$ on $[t, t + \varepsilon]$, i. e. the derivative of $A_.^n(\omega)$ converges uniformly, while $A_.^n(\omega)$ itself converges to $A_.(\omega)$. From elementary analysis, $A_.(\omega)$ is derivable with derivative $f(X_.(\omega), .)$ on $[t; t + \varepsilon]$, and in particular at t.

Since we have assumed *iii*), an application of Fubini's theorem shows that the derivative of $A_.(\omega)$ is equal to $f(X_t(\omega), t)$ for a. e. t. On the other hand, we know in advance that $A_.(\omega)$ is absolutely continuous, so by Lebesgue's theorem it is the integral of its derivative, and we are finished.

Remark. — One could prove the same result under the weaker assumption that $X^n \xrightarrow{L} X$. In this case, Skorohod's theorem will give us versions $X_.^n(\omega) \to X_.(\omega)$ in measure for every ω, and by an additional extraction we may reduce this to a. s. convergence. Then the derivatives $A_t^n(\omega)' = f^n(X_t^n(\omega), t)$ will converge a. e. in t to $f(X_t(\omega), \omega)$ for a. e. ω (no longer uniformly), and to prove that $A_t(\omega)$ is derivable with derivative $f(X_t(\omega), t)$ a. e. we need only some uniform integrability on $[0, 1]$.

To achieve this, we strengthen assumption *i*), demanding *boundedness in* L^1 instead of simply in probability. Then (remember that we use a Skorohod representation) we have that on $(\Omega \times [0, 1], dP \times dt)$ the functions $f^n(X_t^n(\omega), t)$ converge a. s. to $f(X_t(\omega), t)$, and remain bounded in L^p,

$p > 1$. It follows that for fixed $t A_t^n$ converges weakly in L^1 to $\int_0^t f(X_s, s)ds$. On the other hand, it converges in measure to A_t. So these r. v. must be equal, and A_t has the required density [1].

3. APPLICATION TO DIFFUSIONS

We understand the word « diffusion » in the weak sense introduced by Stroock and Varadhan: given a second order differential operator $L(x, t)$ depending on time, not necessarily everywhere defined on $\mathbb{R}^d \times [0, 1]$, we say that a continuous adapted $\mathbb{R}^d$-valued process is a *diffusion governed by* L, if, for every C^∞ function f on $\mathbb{R}^d$, the process

$$M_t^f = f(X_t) - f(X_0) - \int_0^t Lf(X_s, s)ds \tag{1}$$

is a local martingale. The meaningfulness of the integral in (1) is understood as part of this definition: the set of (x, t) such that $L(x, t)$ is undefined should be a. s. visited by the process (X_s, s) along a set of times which is negligible, and the integral $\int_0^1 |Lf(X_s, s)|\, ds$ should be a. s. finite. This definition can be extended to C^∞ manifolds.

Applying (1) to the product of two C^∞ functions f, g, it isn't difficult to compute $\langle M^f, M^g \rangle$:

$$\langle M^f, M^g \rangle_t = \int_0^t \Gamma(f, g)(X_s, s)ds \tag{2}$$

where $\Gamma(f, g) = L(fg) - fLg - gLf$.

We are particularly interested in the case of the differential operator

$$L(x, t) = \frac{\nu}{2}\Delta + b(x, t)\cdot\nabla \qquad \text{in } \mathbb{R}^d \tag{3}$$

where ν is a positive constant, and $b(x, t)$ is a (possibly singular) vector field. Applying (1) to the coordinate mappings in $\mathbb{R}^d$, it is easy to see that,

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[1] Note that this proof no longer requires U to be open in (t, x), nor continuity of the functions in t.

in its natural filtration, (X_t) is a semimartingale with canonical decomposition

$$X_t = X_0 + \int_0^t b(X_s, s)ds + \sqrt{\nu}\, W_t \tag{4}$$

where W_t is a standard d-dimensional Wiener process.

To apply the above to stochastic mechanics, we summarize it as follows.

THEOREM 5. — 1) *Let* (X_t^n) *be diffusions governed by operators* $\frac{\nu}{2}\Delta + b_n(x, t)\cdot\nabla$, *such that, for some* $p > 1$

$$\sup_n E^n\left[\int_0^1 | b^n(X_s^n, s) |^p ds\right] < \infty \tag{5}$$

Assume that the sequence (X_0^n) *is tight on* $\mathbb{R}^d$. *Then the sequence* (X_t^n) *is C-tight.*

Let (X_t) *be a process such that some subsequence* $X^{n_k} \overset{CL}{\to} X$. *Then, in its natural filtration,* X *is a semimartingale with canonical decomposition*

$$X_t = X_0 + \int_0^t H_s ds + \sqrt{\nu}\, W_t, \qquad E\left[\int_0^1 | H_s |^p ds\right] < \infty \tag{6}$$

where W *is a standard Wiener process.*

2) *Let* U *be open in* $\mathbb{R}^d \times [0, 1]$, *and such that* in U $b_n(x, t)$ *converges locally uniformly to* $b(x, t)$. *Assume that* $(X_t, t) \in U$ *a. s. for a. e.* $t \in [0, 1]$. *Then* $H_s = b(X_s, s)$ *a. s.* $dP \times ds$, *and* X *is a diffusion governed by the operator* $\frac{\nu}{2}\Delta + b(x, t)\cdot\nabla$.

Proof. — The first statement follows from theorem 3: each process X_t^n has a canonical decomposition $X_t^n = X_0^n + A_t^n + M_t^n$, where M^n is a brownian motion with generator $\frac{\nu}{2}\Delta$ and $A_t^n = \int_0^t b^n(X_s^n, s)\,ds$. The only difference with theorem 3 is the fact that X is a vector process, and we must check the C-tightness for each component X^{ni} ($i = 1, \ldots, d$), i. e. the C-tightness of the sequences $(\langle M^{ni}, M^{ni}\rangle_t)$, (A_t^{ni}) (lemma 1). Since the brackets are kept fixed, and the finite variation parts are covered by condition (5), there is no difficulty.

As for the second statement, we need a slight extension of theorem 3 to the vector valued case: the process U_t^n of theorem 3 should denote now. not just a scalar bracket, but the bracket *matrix*: the extension is nearly obvious (and it is sufficient to bound the *diagonal* elements of the matrix to get the desired result). Then we know that

$$(X_t^n, A_t^n, M_t^n, U_t^n) \text{ C-converge to } (X_t, A_t, M_t, U_t)$$

Since U_t^n is simply vtI, the same is true for U_t, and M_t can be written as $\sqrt{v}\, W_t$. On the other hand, lemma 4 identifies A_t as $\int_0^t b(X_s, s)ds$. Finally, the fact that X is a diffusion governed by the operator $\frac{v}{2}\Delta + b \cdot \nabla$ follows from Ito's formula.

Remark. — We might consider as well diffusions governed by operators

$$L^n(x, t) = \Sigma_{ij} a^{n,ij}(x, t) D_i D_j + \Sigma_i b^{n,i}(x, t) D_i$$

provided we demand also that

$$\text{(7)} \qquad \sup_n E^n\left[\int_0^1 (a^{n,ii}(X_s, s))^p ds\right] < \infty \qquad (i = 1, \ldots, d)$$

In the applications to stochastic mechanics, these coefficients usually won't depend on n, and will be continuous functions of (x, t).

We shall discuss later on the case of processes taking values in a manifold.

4. THE PROBLEM OF NODES FOR A SEMIMARTINGALE

The results in this section are independent from the preceding ones, and from tightness considerations. They will lead later on to a good choice of the open set U in theorem 5, when we apply it to stochastic mechanics.

We denote by (X_t) a $\mathbb{R}^d$-valued continuous stochastic process, which is a semimartingale on $(\Omega, \mathcal{F}, P)$ w. r. to a filtration $(\mathcal{F}_t)$, indexed by $[0, 1]$. We shall assume that it belongs to the class considered by Stricker in [7], namely that in its canonical decomposition

$$X_t^i = X_0^i + A_t^i + M_t^i$$

the finite variation parts are absolutely continuous with locally L^2 densities,

$$A_t^i = \int_0^t H_s^i ds \qquad \text{with} \qquad \int_0^1 |H_s^i|^2 ds < \infty \quad \text{a. s.}$$

and the brackets are absolutely continuous with locally bounded densities

$$\langle M^i, M^i \rangle_t = \int_0^t m_s^i ds \qquad \text{with} \qquad \text{ess sup}_{s \leq 1}\, m_s^i < \infty \quad \text{a. s.}$$

LEMMA 6. — *There exist subsets* $A_n \uparrow \Omega$ *and constants* C_n *such that, for every pair of stopping times* S, T *with* $S \leqq T \leqq 1$ *we have*

$$E[\,|X_T - X_S|^2 I_{A_n}] \leqq C_n E[T - S] \tag{8}$$

Proof. — Set

$$R_n = \inf\left\{ t : \sup_i \operatorname{ess\,sup}_{s \leqq t} m_s^i > n \text{ or } \sup_i \int_0^t |H_s^i|^2 ds > n \right\} \wedge 1 ,$$

and let Y be the process $X_{t \wedge R_n}$. One checks very easily that

$$E[\,|Y_T - Y_S|^2] \leqq C(d) n E[T - S]$$

$C(d)$ depending only on the dimension. On the other hand, on $A_n = \{R_n = 1\}$ we have $Y_T - Y_S = X_T - X_S$, and $P(A_n)$ tends to 1 as $n \to \infty$.

We denote by μ a bounded measure on $\mathbb{R}^d$, and assume that for every t, the distribution of X_t is absolutely continuous with respect to μ. Then we may define the density $\rho(x, t)$ of this distribution at time t. In stochastic mechanics, $\rho(x, t)$ is always given as the squared modulus $|\psi(x, t)|^2$ of a reasonably nice « wave function », and the set where ψ vanishes is called the *nodes* set. In a preliminary version of this work we assumed ψ to be once differentiable. After reading Carlen's paper, we think it more natural to assume a *local Hölder condition*

$$|\psi(x, s) - \psi(y, t)| \leqq C_K(|y - x| + |t - s|^{1/2}) \quad \text{for} \quad \begin{array}{l} s, t \in [0, 1] \\ x, y \in K \text{ compact} \end{array} \tag{9}$$

The restriction that μ is bounded may seen disturbing, since the most usual reference measure is Lebesgue's measure λ. However, if (9) is satisfied w. r. to λ, and if μ is a bounded measure with C^∞ strictly positive density h, the new « wave function » $\psi/\sqrt{h}$ again satisfies (9).

THEOREM 9. — *Let* X *belong to the Stricker class, and have a density* $\rho(x, t) = |\psi(x, t)|^2$, *where* ψ *satisfies* (9). *Then the space-time process* $Y_t = (X_t, t)$ *never visits the nodes set* N

$$P\{\exists t \in [0, 1] : \psi(X_t, t) = 0\} = 0.$$

Proof. — We are going to prove that Y never visits N on [0, 1[. This isn't a difficulty, because we may extend the process after 1 by $X_t = X_1$, $\psi(x, t) = \psi(x, 1)$, and the result on [0, 2[will give the theorem on [0, 1].

Consider the stopping time

$$S = \inf\{t : |\psi(X_t, t)| = 0\} \wedge 1$$

We want to prove that $P\{S < 1\} = a$ is equal to 0. So we assume it to be > 0 and derive a contradiction. Since the function ψ satisfies (9), the process $\psi(Y_t) = \psi(X_t, t)$ is continuous, and therefore $\psi(Y_S) = 0$ on $\{S < 1\}$. On the other hand, the set N is such that $E\left[\int_0^1 I_N(Y_s) = 0\right]$, and therefore for sufficiently small $\varepsilon > 0$, the stopping time

$$T_\varepsilon = \inf\{t > S : |\psi(Y_t)| = \varepsilon\} \wedge 1$$

is smaller than 1 on $\{S < 1\}$. From now on, ε will be so small that $P\{T_\varepsilon < 1\} > 3a/4$. We also choose n so large in (8), and a compact K so large that the set

$$\{X_t \in K \text{ for } 0 \leqq t \leqq 1\} = B$$

satisfies $P(B \cap A_n) > 1 - a/4$. Then we have

$$\begin{aligned}\frac{a}{2}\varepsilon^2 &\leqq E[|\psi(X_{T_\varepsilon}, T_\varepsilon) - \psi(X_S, S)|^2 I_{A_n} I_B] \\ &\leqq C_K E[(|X_{T_\varepsilon} - X_S|^2 + |T_\varepsilon - S|) I_{A_n}] \\ &\leqq C_K(C_n + 1)E[S - T_\varepsilon]\end{aligned}$$

On the other hand, between T_ε and S, the process $Y_t = (X_t, t)$ remains in the set $\{|\psi(x, t)| \leqq \varepsilon\} = \{\rho(x, t) \leqq \varepsilon^2\}$. So we have

$$\begin{aligned}\frac{a}{2}\varepsilon^2 &\leqq CE\left[\int_0^1 I_{\{\rho(Y^t) \leqq \varepsilon^2\}} dt\right] = C\int\int I_{\{\rho(x,t) \leqq \varepsilon^2\}}\rho(x, t)\mu(dx)dt \\ &\leqq \varepsilon^2 C\int\int I_{\{0 < \rho(x,t) \leqq \varepsilon^2\}}\mu(dx)dt = o(\varepsilon^2)\end{aligned}$$

which is the desired contradiction (we have used at the last step the fact that μ is bounded).

5. CONSTRUCTION OF NELSON'S DIFFUSIONS. I

We are going to test the weak convergence methods on the construction on the Nelson processes in $\mathbb{R}^d$, associated with a wave function $\psi(x, t)$. Our presentation of these results has been strongly influenced by Carlen's paper [*1*], and our hypotheses are parallel to his in the sense that whenever we have a *strong* differentiability assumption (for instance, the Hölder (1, 1/2) condition and existence of the gradient in x) he has a *weak* assumption. On the other hand, our probabilistic method lends itself to easy generalizations to operators whose second order part has variable coefficients.

We shall not talk at all about wave functions. We give ourselves a function $\rho(x, t)$ on $\mathbb{R}^d \times [0, 1]$, such that $\int \rho(x, t)dx = 1$ for every t, and two vector fields $b(x, t)$, $\hat{b}(x, t)$. We make the following hypotheses.

a) *Regularity*. The function $\sqrt{\rho(x, t)}$ satisfies a local Hölder condition (9), and in particular is continuous. The fields ρb and $\rho\hat{b}$ are continuous.

The first property will be used to apply theorem 9: it is satisfied whenever $\rho = |\psi|^2$ with ψ satisfying (9), and conversely, if *a*) is satisfied we may apply theorem 9 with $\psi = \sqrt{\rho}$. The second property implies that b, $\hat{b}$ are continuous away from the nodes. It will be used to apply theorem 5 in the complement of the nodes (and is a little too strong: continuity in x alone would suffice).

b) *Weak Fokker-Planck equation*. We assume that

$$\frac{\partial \rho}{\partial t} = \frac{\nu}{2}\Delta\rho - \operatorname{div}(\rho b) = -\frac{\nu}{2}\Delta\rho - \operatorname{div}(\rho\hat{b}) \tag{10}$$

in the following weak sense (we don't want to ask for the separate existence of $\dot{\rho}$ and $\Delta\rho$): if f is a C^∞ function with compact support in $\mathbb{R}^d$, and $\langle\, ,\, \rangle$ is the usual scalar product w. r. to Lebesgue measure

$$\langle f, \rho(.,t) - \rho(.,s) \rangle = \int_s^t \left\langle \frac{\nu}{2}\Delta f + b(.,u)\cdot\nabla f, \rho(.,u) \right\rangle du \tag{11}$$

and the similar relation with $\hat{b}$.

c) *Duality*. This is the relation

$$\rho(b - \hat{b}) = \nu \operatorname{grad} \rho\,. \tag{12}$$

If we have (12), the first equality in (11) implies the second one. On the other hand, the assumed continuity of ρb and $\rho\hat{b}$ and (12) imply together that ρ has a gradient (in x) depending continuously on (x, t).

d) *Finite energy*. This is the condition

$$\iint (|b(x, s)|^2 + |\hat{b}(x, s)|^2)\rho(x, t)dxdt < \infty\,. \tag{13}$$

In this section, we'll prove the following theorem:

THEOREM 10. — *There exists a diffusion* (X_t) *governed by the operator* $\frac{\nu}{2}\Delta + b\cdot\nabla$ *which has at every time t the absolute probability density* $\rho(.,t)$.

The process $Y_t = (X_t, t)$ *in space-time never hits the nodes set* $\{ \rho(.,.) = 0 \}$. *The reversed process* $X'_t = X_{1-t}$ *is a diffusion governed by the operator* $\frac{\nu}{2}\Delta - \hat{b}(x, 1-t)\cdot\nabla$.

The main idea of the proof consists in reducing the problem to the case where the density $\rho(x, t)$ is everywhere > 0. So we have three parts: constructing the strictly positive approximants, solving the problem for them, and using weak convergence to pass to the limit. For clarity reasons, we present the second part at the end. The proof will also allow us to give weaker sufficient conditions for existence, not formally stated above.

STEP 1. — We choose a C^∞ strictly positive probability density σ on $\mathbb{R}^d$, such that

$$\int \frac{\text{grad}^2\, \sigma(x)}{\sigma(x)}\, dx < \infty \tag{14}$$

(a non degenerate gaussian density will satisfy this condition). Then we set

$$\rho_n(x, t) = \left(1 - \frac{1}{n}\right)\rho(x, t) + \frac{1}{n}\sigma(x) \tag{15}$$

and define the fields b_n, $\hat{b}_n$ by

$$\rho_n b_n = \left(1 - \frac{1}{n}\right)\rho b + \frac{1}{2n}\,\text{grad}\,\sigma, \qquad \rho_n \hat{b}_n = \left(1 - \frac{1}{n}\right)\rho\hat{b} - \frac{1}{2n}\,\text{grad}\,\sigma. \tag{15'}$$

There is no difficulty in checking conditions *a*), *b*), *c*). So we just verify the energy condition, considering only the case of b. For simplicity we set $(x, t) = y$, $\rho b = h$, $\rho_n b_n = h_n$. The energy condition can be written

$$\int |h(y)|^2 dy/\rho(y) < \infty,$$

so we set $h/\sqrt{\rho} = f$, which belongs to $L^2(\mathbb{R}^d \times [0, 1])$. Now we have from (15)

$$h_n = \left(1 - \frac{1}{n}\right)h + \frac{1}{n}k \qquad \text{with} \qquad k = \frac{1}{2}\,\text{grad}\,\sigma$$

$$|h_n| \leqq \left(1 - \frac{1}{n}\right)|h| + \frac{1}{n}|k| = \left(1 - \frac{1}{n}\right)\sqrt{\rho} f + \frac{1}{n}\sqrt{\sigma}\frac{|k|}{\sqrt{\sigma}}$$

$$\leqq \left(\left(1 - \frac{1}{n}\right)\rho + \frac{1}{n}\sigma\right)^{1/2}\left(\left(1 - \frac{1}{n}\right)|f|^2 + \frac{1}{n}\frac{|k|^2}{\sigma}\right)^{1/2}$$

$$= \sqrt{\rho_n}\left(\left(1 - \frac{1}{n}\right)|f|^2 + \frac{1}{n}\frac{|k|^2}{\sigma}\right)^{1/2}$$

and finally

$$\int \frac{|h_n|^2}{\rho_n} dy \leqq \left(1 - \frac{1}{n}\right) \int |f|^2 dy + \frac{1}{n}\left(\frac{1}{2}\right)^2 \int \frac{\text{grad}^2 \sigma}{\sigma} dy,$$

bounded in n.

Remark. — We shall also discute briefly the case (not mentioned in the statement of theorem 10) of an energy condition with exponent $p > 1$ instead of exponent 2 in (13). Then in the proof above we take

$$h = \rho^{1/q} f \qquad \left(\frac{1}{p} + \frac{1}{q} = 1\right),$$

with

$$f \in \mathrm{L}^p, |h_n| \leqq \left(1 - \frac{1}{n}\right)\rho^{1/q} |f| + \frac{1}{n} \sigma^{1/q} |k| / \sigma^{1/q},$$

and applying Hölder's inequality one gets

$$(|h_n| / \rho_n^{1/q})^p \leqq \left(1 - \frac{1}{n}\right) |f|^p + \frac{1}{n} |k|^p / \sigma^{p/q}$$

Hence the relevant condition on σ (also satisfied by a gaussian law) would be $|\text{grad}\ \sigma|^p / \sigma^{p/q} \in \mathrm{L}^1$.

STEP 2. — As we said before, we leave for the end of the proof the study of strictly positive densities. So we denote by $\mathscr{C}$ the set of all paths as in section 1, by (X_t) the coordinate process, by P_n the law on $\mathscr{C}$ corresponding to the diffusion governed by $\frac{1}{2}\Delta + b_n\nabla$ and with density ρ_n, taking for granted the existence of this diffusion. Then the energy condition (or the similar condition with exponent $p \neq 2$) and step 1 above imply that condition (5) in theorem 5 is satisfied. The first part of theorem 5 allows us to choose some limit law P, under which (X_t) is a semimartingale as in (6). By weak convergence, X_t has the law $\rho(x, t)dx$ for a. e. t, and eventually for all t by continuity.

Then we apply the second part of theorem 5, taking for U the complement of the nodes set (that is, using the continuity of ρb; we don't use the fact that the nodes set is never hit by the process, and the Hölder condition hasn't been needed yet). We find that every limit law is a diffusion governed by $\frac{1}{2}\Delta + b\nabla$, and admitting the prescribed densities. Since the same reasoning applies to the reversed process, and weak convergence is preserved under time reversal, we find that the reversed diffusion also has the correct generator.

Next we assume that $p = 2$ (true energy condition), or just that we have an energy condition (13) with exponent p instead of 2, but a *local* energy condition with exponent 2, that is

$$\int (\,|\, b(x, t)\,|^2 + |\, \hat{b}(x, t)\,|^2)\rho(x, t)a(x)dxdt < \infty \tag{16}$$

for some continuous, bounded, everywhere strictly positive function $a(x)$. This property will be preserved for the approximants in step 1 (just replace in the proof the measure dy by $a(y)dy$) and its probabilistic meaning will imply that the limit process belongs to the Stricker class. Then using for the first time the Hölder condition on $\rho(x, t)$, we apply theorem 9, and deduce from it that the process doesn't hit the nodes set.

STEP 3. — So really we are reduced to constructing the diffusion under the additional hypothesis that ρ doesn't vanish. We shall use a method which is very similar to that of our paper [4]. *We take* $v = 1$ *for simplicity.*

We denote by λ the Lebesgue measure on $\mathbb{R}^d$, by Λ the measure on $\mathscr{C}$ under which (X_t) is a brownian motion with diffusion coefficient 1 and initial measure λ. However, we slightly modify the definition of $\mathscr{C}$ by allowing explosion at a time ζ (2). We now construct a measure P on $\mathscr{C}$ by Girsanov's theorem. The process

$$\rho(X_0, 0) \exp\left(\int_0^t b(X_s, s)dX_s - \frac{1}{2}\int_0^t |\, b(X_s, s)\,|^2 ds\right) = M_t$$

is well defined, because ρb is continuous and ρ locally bounded from below [incidently, if we hadn't been using the stronger assumption of continuity, this would be a place to use (16)]. This is a positive local martingale with expectation 1 at time 0, hence a positive supermartingale of expectation ≤ 1. We define the law P as its Föllmer measure. That is, for any stopping time T with values in $[0, 1] \cup \{+\infty\}$ and any r. v. H on $\mathscr{C}$, $\mathscr{F}_T$-measurable and positive

$$E_P[HI_{\{T<\zeta\}}] = E_\Lambda[HM_T I_{\{T<\infty\}}]\,. \tag{17}$$

(2) ζ takes its values in $[0, 1] \cup \{+\infty\}$.

Remark that M_t is the Doléans exponential of the local martingale

$$L_t = \log \rho(X_0, 0) + \int_0^t b(X_s, s)dX_s$$

whose bracket is $\langle L, L \rangle_t = \int_0^t \frac{1}{2} | b(X_s, s) |^2 ds$. Let us for a moment take for granted the following fact

(18) *The subprobability law μ_t defined by $\mu_t(f) = E_P[f(X_t), t < \zeta]$ is dominated by $\rho(x, t)dx$.*

We are going to prove that $\zeta = +\infty$ P-a. s. Then it will follow from classical considerations (Girsanov's theorem...) that under P, (X_t) is a diffusion governed by $\frac{1}{2}\Delta + b\nabla$. Also, μ_t being a *probability* law dominated by $\rho_t dx$, they will be equal, and X will have the prescribed densities.

Consider the stopping times

$$T_n = \inf \{ t : \langle L, L \rangle_t \geqq n \}, \qquad \tau = \lim_n T_n$$

Since L has a bounded bracket up to T_n, we have $E_\Lambda[M_{T_n \wedge 1}] = E_\Lambda[M_0] = 1$. Applying (17) to $T_n \wedge 1$, with $H = 1$, we find that $T_n \wedge 1 < \zeta$ P-a. s. for every n. Otherwise stated, we have $\langle L, L \rangle_1 = +\infty$ P-a. s. on the set $\{ \zeta < \infty \}$. So we must only prove that $\langle L, L \rangle_1 < \infty$ a. s.

Under the energy condition (13), this is very simple: we have from (18)

$$E_P[\langle L, L \rangle_t] = \int\int \frac{1}{2} | b(x, t) |^2 \mu_t(dx)dt \leqq \frac{1}{2} \int\int | b(x, t) |^2 \rho(x, t)dxdt < \infty .$$

But this isn't the last word: let us assume that we have an energy condition (13) with exponent $p \neq 2$, and a *local* energy condition (16). This condition implies that

$$E_P\left[\int_0^1 a(X_s) | b(X_s, s) |^2 ds\right] < \infty$$

with a continuous strictly positive. Since at time $\zeta \int_0^\cdot | b(X_s, s) |^2 ds$ diverges, $a(X_s)$ cannot remain bounded from below near ζ, i. e. X_s must wander to infinity. On the other hand

$$X_t - X_0 - \int_0^t b(X_s)ds$$

is a local martingale on $[0, \zeta[$ with brownian brackets, so it remains bounded near ζ, and the wandering near infinity implies that $\int_0^{\cdot} |b(X_s, s)| ds$ diverges. This is excluded by the energy condition (13) as above. So we are finished, provided we prove the following.

STEP 4. — (18) *is satisfied.* The meaning of this can be stated as follows: the family (μ_t) is in some sense the minimal positive solution of the weak Fokker-Planck equation. So the proof shouldn't require a duality argument. In that case, our set of hypotheses could be reduced to purely « forward » assumptions. However, we shall use duality here.

We denote by Ω the set of all continuous functions from $\mathbb{R}_+$ to $\mathbb{R}^d$, by (X_t) the coordinate process, by W_x the law on Ω under which (X_t) is a Wiener process with diffusion coefficient 1 and initial law ε_x. We also set for $s < t$

$$L_{s,t} = \int_0^{t-s} b(X_u, s+u).dX_u \qquad \langle L, L \rangle_{s,t} = \int_0^{t-s} |b(X_u, s+u)|^2 du$$

$$M_{s,t} = \exp\left(L_{s,t} - \frac{1}{2}\langle L, L \rangle_{s,t}\right)$$

$$Q_{s,t}(x, f) = W_x[f(X_{t-s})M_{s,t}]$$

Then it is well known that $Q_{s,t}$ is a submarkovian transition function (non homogeneous) which satisfies identically the Chapman-Kolmogorov relation $Q_{s,t}Q_{t,u} = Q_{s,u}$ for $s < t < u$. On the other hand, the process we have constructed above by Girsanov's method admits this transition function, and in particular $\mu_t = \mu_0 Q_{0t}$.

We now perform the same construction in the reverse direction, starting from 1 and using $-\hat{b}$ instead of b. We have again a Markov process, with non homogeneous transition probability $\hat{Q}_{t,s}$ (still $s < t$!), possibly submarkovian, and given by a similar formula

$$\hat{L}_{t,s} = \int_0^{t-s} -\hat{b}(X_u, t-u)dX_u \qquad \hat{M}_{t,s} = \ldots \quad \hat{Q}_{t,s}(x, f) = W_x[f(X_{t-s})\hat{M}_{t,s}]$$

Of course, the new initial measure is $\rho(x, 1)dx$. Our claim is that for $s < t$

$$(19) \qquad \int dx\rho(x, s)f(x)Q_{s,t}(x, dy)g(y) = \int f(x)\hat{Q}_{t,s}(y, dx)g(y)\rho(y, t)dy$$

If this is proved, then we are finished. Indeed taking $s = 0$, $g \geqq 0$, $f = 1$, on the left side we have $\mu_t(g)$. On the right side, since $\hat{Q}_{t,s}$ is submarkovian, the integral is dominated by $\int g(y)\rho(y, t)dy$, and this is the result we need.

We write (19) as follows

$$(20) \qquad E_\Lambda[f(X_s)H_{st}g(X_t)] = E_\Lambda[f(X_s)\hat{H}_{ts}g(X_t)] \qquad \begin{array}{l} H_{st} = \rho(X_s, s)M_{st} \\ \hat{H}_{ts} = \hat{M}_{ts}\rho(X_t, t) \end{array}$$

and since the measure Λ is invariant by time reversal, this amounts to the fact that H_{st} and $\hat{H}_{st}$ are exchanged by time reversal. Note that this property involves computation on brownian motion, not on the perturbed diffusions.

We first perform a computation, *assuming that ρ is twice differentiable and > 0, b and $\hat{b}$ once differentiable.* We write Y_t for (X_t, t), and we compute $\log H_{st}$, that is

$$(21) \qquad \log \rho(Y_s) + \int_s^t b(Y_u)dX_u - \frac{1}{2}\int_s^t |b(Y_u)|^2 du$$

We write the Ito formula for $\log \rho$ as follows

$$(22) \quad \log \rho(Y_s) = \log \rho(Y_t) - \int_s^t \operatorname{grad} \log \rho(Y_u)dX_u - \int_s^t \left(D_t + \frac{1}{2}\Delta\right) \log \rho(Y_u)du$$

Of course $D_t \log \rho = \dot{\rho}/\rho$, $\Delta (\log \rho) = \frac{1}{\rho}\Delta\rho - \frac{1}{\rho^2}\operatorname{grad}^2 \rho$. We substitute this in (21), and substitute also $\hat{b} + \operatorname{grad} \rho/\rho$ for b, and $-\operatorname{div}(\rho\hat{b}) = -\rho \operatorname{div} b - \operatorname{grad} \rho . \hat{b}$ for $\dot{\rho} + \frac{1}{2}\Delta\rho$ (Fokker-Planck equation). After some computation it remains that

$$\log H_{st} = \log \rho(Y_t) + \int_s^t \hat{b}(Y_u)dX_u - \frac{1}{2}\int_s^t |\hat{b}(Y_u)|^2 du + \int_s^t \operatorname{div} \hat{b}(Y_u)du$$

but the second and last terms combine together to become a *backward stochastic integral*, and by time reversal one gets the $\log \hat{H}_{ts}$ for the reversed path.

We must now get rid of the additional differentiability assumptions. To this end, we remark that the above is a *pathwise computation*, depending on the strict positivity ρ in a neighbourhood of the path (so that the Ito formula can be applied to $\log \rho$), on the Fokker-Planck and duality relations, but not demanding anything about ρ being a probability density, or b, $\hat{b}$ satisfying energy conditions. So the problem becomes the following: can we approximate on a large compact set K our functions b, $\hat{b}$, ρ by very regular functions b_n, $\hat{b}_n$, ρ_n, with $\rho_n > 0$ on K, satisfying the Fokker-Planck and duality relations on K ? Note also that it is sufficient to prove the relation for $0 < s < t < 1$, and then pass to the limit.

In our present situation, the approximation is very easy: we simply set $\rho_n = \varphi_n * \rho$, $\rho_n b_n = \varphi_n * (\rho b)$, $\rho_n \hat{b}_n = \varphi_n * (\rho \hat{b})$, where (φ_n) is the usual approximation of the Dirac measure in $\mathbb{R}^d \times \mathbb{R}$ by positive, C_c^∞ functions. More precisely, we shall assume that $\varphi_n(x, t) = u_n(x)v_n(t)$, where u_n, v_n approximate the Dirac measure on $\mathbb{R}^d$, $\mathbb{R}$ separately (this will be better behaved w. r. to the weak Fokker-Planck relation). We leave the details aside.

However, we have used here the fact that a smoothing procedure exists which commutes with the operators Δ on functions and *div* on fields, and it is not clear for us (3) how we would do with variable coefficients. Though getting rid of minor differentiability assumptions isn't the essential point in the proof, we would be happy to know how to do it.

Remark. — We have used duality only in the last step of the proof (4), to imply that the approximating diffusions with $\rho > 0$ were non-explosive. If it is known from some other reason that non-explosion obtains, then the assumptions can be stated in purely « forward » terms: weak F-P equation and energy condition on b only. The main case where this will happen concerns the construction of Nelson processes in a *compact* manifold. We shall do this in the next section, to illustrate the essential simplicity of the weak convergence method.

6. CONSTRUCTION OF NELSON'S PROCESSES. II

Here we shall consider the case of a compact riemannian manifold M. We still denote by $\lambda(dx)$ or simply dx the riemannian measure on M. We aren't going to minimize differentiability hypotheses: the main point here is the way one deals with variable coefficients.

We consider a wave function $\psi(x, t)$ on M, which we assume to be n class $C^{2,1}$ (twice in x, once in t), and solution of a Schrödinger equation

$$i\dot{\psi} = H\psi$$

where the hamiltonian $H\psi = \frac{1}{2}(-\operatorname{div} + ia\cdot)(\operatorname{grad} - ia)\psi + p\psi$ (we keep

(3) The standard trick is to use the second order part (if non degenerate) to define a Riemannian structure, and use the heat equation regularizer.

(4) Of course, one wishes to know what the reversed diffusion is, but once the diffusions are known to exist, this follows rather easily from the pair of F-P equations.

the notation of Meyer [2], p. 199) involves a scalar potential p and a vector potential a. As usual, $\rho(x, t)$ will be the probability density $|\psi|^2$, and will satisfy the Fokker-Planck equations

$$\dot{\rho} = \frac{1}{2}\Delta\rho - \operatorname{div}(\rho b) = -\frac{1}{2}\Delta\rho - \operatorname{div}(\rho \hat{b})$$

where the fields b, $\hat{b}$ are defined by

$$b = u + v,\ \hat{b} = u - v,\ u + iv = \frac{1}{\psi}\operatorname{grad}\psi - ia\,.$$

At this stage, we are in a situation close to that of theorem 10, and we would need much less than class $C^{2,1}$ on ρ. However, we shall refrain again from minimizing hypotheses, since it isn't clear how the weakened statements might be expressed in terms of ψ (it is mainly to keep the derivation of the above F-P equation valid that we assumed so much on ψ).

THEOREM 11. — *The forward and backward diffusions with density* ρ, *generators* $\frac{1}{2}\Delta + b\cdot\nabla$, $\frac{1}{2}\Delta - \hat{b}\cdot\nabla$ *exist, and never hit the nodes set.*

Proof. — We shall follow the steps in the proof of theorem 10, indicating the appropriate modifications.

STEP 1. — This operation consists in defining approximate densities and fields ρ_n, b_n, $\hat{b}_n$ satisfying the Fokker-Planck equation (and if necessary the energy relation) and such that ρ_n is strictly positive. Here the construction is simpler than in (15), (15′), because we take simply σ to be the constant 1 (the density of the normalized Riemann measure λ). We give no details.

Since the last step is also much simpler than in theorem 10 because of compactness, we present it now.

STEP 3. — *Construction of the diffusion when* $\rho > 0$. We use Girsanov's theorem, the reference measure Λ being now that of Riemannian brownian motion with initial measure λ. The Girsanov density for the forward diffusion is known to be (Meyer [2], p. 201)

$$\frac{dP}{d\Lambda} = \rho(X_0, 0)\exp\left(L_1 - \langle L, L\rangle_1\right)$$

where L_t is a local martingale which is represented in [2] as the Stratonovich integral along the path of $X_{\cdot}$ of the differential form β associated to the vector field b

$$L_t = \int_{X_0^t} \beta + \frac{1}{2}\int_0^t \delta\beta_s \qquad (\delta \text{ is the codifferential: } \delta\beta = -\operatorname{div} b)$$

If b isn't sufficiently differentiable, the Stratonovich integral should be expressed in Ito's form, which will cancel the higher order term div b. Since we are on a *compact* Riemann manifold, and b here is continuous, $\langle L, L \rangle_1$ is bounded, and therefore the exponential martingale representing the density is uniformly integrable. No problem arises.

STEP 2. — *Passage to the limit using tightness.* To apply conveniently the tightness results, we imbed V into some $\mathbb{R}^{d'}$ (we don't need to know that d' can be taken to be $2d + 1$, so this is a rather easy theorem), and so we are reduced to prove tightness of the laws P_n of the approximating diffusions *as $\mathbb{R}^{d'}$-valued processes.* It is obvious that the property of taking values in the closed set V will be preserved under weak convergence. Denote by A^n, A the generators $\frac{1}{2}\Delta + b_n \cdot \nabla$, $\frac{1}{2}\Delta + b \cdot \nabla$, considered as mapping C^∞ functions on $\mathbb{R}^{d'}$ to functions on V. Then under P_n, for every C^∞ function f on $\mathbb{R}^{d'}$ the process

$$f(X_t) - f(X_0) - \int_0^t A^n f(X_s, s)ds = C_t^f$$

is a local martingale, with bracket

$$\langle C^f, C^g \rangle_t = \int_0^t (\operatorname{grad} f \,.\, \operatorname{grad} g)(X_s)ds$$

not depending on n and, due to compactness, $| \langle C^f, C^f \rangle_1 |$ is a *bounded* random variable. Taking for f the coordinate mappings on $\mathbb{R}^{d'}$, we see that the tightness property will follow if we show that

$$E^n\left[\int_0^1 | A^n f(X_s^n, s) |^2 ds\right] \qquad (f = x^1, \ldots, x^{d'})$$

and this is easily reduced to the energy condition on b (13), and in turn to a condition on the wave function

$$\iint_{V \times [0,1]} (| \operatorname{grad} \psi |^2 + | \psi a |^2) dxdt < \infty .$$

Then we forget about $\mathbb{R}^{d'}$, and apply theorem 5 in V: the reasoning is essentially the same, and there is no need to give details.

7. MARKOV PROPERTY OF NELSON'S DIFFUSIONS

We return now to the set-up of theorem 10. Since we have taken the word « diffusion » in the generalized sense of solution of a martingale problem, Markov property doesn't follow automatically.

Remember that in theorem 10 we had assumed Hölder continuity of $\sqrt{\rho}$ (implying that the diffusion X never hits the nodes) and ordinary continuity of ρb, $\rho \hat{b}$.

THEOREM 12. — *Under the same hypotheses as in theorem 10, the law* P *of the forward diffusion* (X_t) *is absolutely continuous w. r. to the measure* Λ *of brownian motion* ([5]), *and* (X_t) *is a Markov process* (its non-homogeneous transition function is made explicit in the proof).

Proof. — We set on $\mathscr{C}$ (continuous functions from [0, 1] to $\mathbb{R}^d$)

$$S_k = \inf\left\{ t : \int_0^t |\, b(X_s, s)\,|^2 ds \geqq k \right\}, \quad S = \lim_k S_k$$

S, S_k take values in $[0, 1] \cup \{ +\infty \}$. We define on [0, 1], under the measure Λ

$$M_t = \rho(X_0, 0) \exp\left\{ \int_0^t b(X_s, s) \cdot dX_s - \frac{\nu}{2} \int_0^t |\, b(X_s, s)\,|^2 ds \right\} I_{\{t < S\}}$$

a positive supermartingale, which is a local martingale on [0, S[and has expectation 1 at time 0. We know (theorem 5) that under the law P, $X_t - X_0 - \int_0^t b(X_s, s)ds$ is a brownian motion with diffusion coefficient ν. According to Girsanov's theorem, for any stopping time T such that the r. v. $\int_0^T |\, b(X_s, s)\,|^2 ds$ is *bounded*, P is absolutely continuous w. r. to Λ on $\mathscr{F}_{T_n}$, with density M_T. This is true in particular for the stopping times

$$T_n = \inf\{ t : |\, b(X_s, s)\,| > n \}.$$

On the other hand, $P\{ T_n < \infty \}$ is small for n large. Indeed, we know that ρb and ρ are continuous, and that under P the process (X_t, t) never hits

([5]) With diffusion coefficient ν as usual.

the nodes set $\{ \rho = 0 \}$. So P-a. s. ρ is bounded from below on the path, ρb from above, and finally b from above.

Let A be some Λ-negligible set. Then $A \cap \{ T_n = \infty \}$ is $\mathscr{F}_{T_n}$-measurable, and since P is absolutely continuous w. r. to Λ on $\mathscr{F}_{T_n}$, it is also P-negligible. Letting $n \to \infty$, we see that A itself is P-negligible. So P is absolutely continuous, and we see that really M_t is a true martingale of expectation 1. Note however that we may have $S < \infty$ with strictly positive Λ-measure, and the two measures P, Λ usually won't be equivalent.

The Markov property of (X_t) under P now reduces to the fact that M_t is a (non homogeneous) multiplicative functional. Given $s < t$, $A \in \mathscr{F}_s$, f a positive function on $\mathbb{R}^d$, we have (in abbreviated notation)

$$E_P [f(X_t)I_A] = E_\Lambda [I_A f(X_t)M_t]$$
$$= E_\Lambda\left[I_A M_s \exp\left\{ \int_s^t b \cdot dX_u - \frac{v}{2}\int_s^t | b |^2 du \right\} f(X_t)\right]$$

Then we apply the Markov property of brownian motion, and this becomes

$$E_\Lambda [I_A M_s Q_{st}(X_s, f)] = E_P [I_A Q_{st} f(X_s)]$$

where $Q_{st}(x, f)$ is the expectation

$$E_x\left[f(X_{t-s}) \exp\left\{ \int_s^t b(X_u, s + u) \cdot dX_u - \frac{v}{2}\int_s^t | b(X_u, s + u) |^2 du \right\}\right]$$

computed on brownian motion, with diffusion coefficient v, starting at x.

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