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On Non-Euclidean Harmonic Measure

by

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ABSTRACT. — Let H_ε be the harmonic measure of the ε-sphere in a Riemannian manifold. We compute $\lim_{\varepsilon \downarrow 0} \frac{H_\varepsilon - H_{0^+}}{\varepsilon^3}$ in terms of the gradient of the Gaussian curvature.

RÉSUMÉ. — Soit H_ε la mesure harmonique de la sphère du rayon ε d'une variété riemannienne de dimension deux. On exprime $\lim_{\varepsilon \downarrow 0} (H_\varepsilon - H_{0^+})/\varepsilon^3$ en fonction de la dérivée de la courbure gaussienne.

1. INTRODUCTION

Let (M, g) be an n-dimensional Riemannian manifold with Brownian motion process $\{ X_t, t \geq 0 \}$. According to the Onsager-Machlup formula [*3*], the « most probable path » is that of a classical particle in a conservative force field whose potential energy is one-twelfth the scalar curvature. This suggests that the « most probable path » follows the negative gradient of the scalar curvature. The purpose of this note is to make this precise in terms of the harmonic measure of a small sphere. For technical reasons we restrict the discussion to surfaces. The technique follows the method used to study the mean exit time [*1*].

2. NOTATIONS AND DEFINITIONS

Let (M, g) be an n-dimensional Riemannian manifold. We use the following notations:

$\overline{M}_m$ is the tangent space at $m \in M$.
$B_m(\varepsilon)$ is the ball of radius ε in M with center at $m \in M$.
$\overline{B}_m(\varepsilon)$ is the ball of radius ε in $\overline{M}_m$ with center at $\dot{0} \in \overline{M}_m$.
$\exp_m$ is the exponential mapping (which is defined on all of $\overline{M}_m$ in case M is complete; otherwise it is a mapping) from $\overline{B}_m(\varepsilon)$ to $B_m(\varepsilon)$ for sufficiently small $\varepsilon > 0$.
Φ_ε is the mapping on functions defined by

$$(\Phi_\varepsilon f)(\exp_m \varepsilon x) = f(x) \,;$$

Φ_ε maps from $C^\infty(\overline{B}_m(1))$ to $C^\infty(B_m(\varepsilon))$ for sufficiently small $\varepsilon > 0$.
Δ is the Laplace-Beltrami operator of the Riemannian manifold:

$$\Delta f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i}\left(\sqrt{g} g^{ij} \frac{\partial f}{\partial x_j}\right) \qquad \text{where} \qquad g^{ij} = (g^{-1})^{ij}, \qquad g = \det(g_{ij}).$$

The following result, which will be used repeatedly, was proved in [*1*].

PROPOSITION 2.0. — *There exist second order differential operators* $(\Delta_{-2}, \Delta_0, \Delta_1, \dots)$ *on* $C^\infty(\overline{M}_m)$ *such that for each* $N \geq 0$ *and each* $f \in C^\infty(\overline{M}_m)$

$$(2.1) \qquad \Phi_\varepsilon^{-1}\Delta\Phi_\varepsilon f = \varepsilon^{-2}\Delta_{-2} f + \sum_{j=0}^{N} \varepsilon^j \Delta_j f + 0(\varepsilon^{N+1}) \qquad (\varepsilon \downarrow 0).$$

Δ_j *maps polynomials of degree* k *to polynomials of degree* $k + j$. *In any normal coordinate chart* $(x_1, \dots, x_n)$ *we have*

$$(2.2) \qquad \Delta_{-2} f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

$$(2.3) \qquad \Delta_0 f = (1/3) \sum_{i,a,j,b=1}^{n} R_{iajb} x_a x_b \frac{\partial^2 f}{\partial x_i \partial x_j} - (2/3) \sum_{i,a=1}^{n} \rho_{ia} x_a \frac{\partial f}{\partial x_i}$$

Here R_{iajb} is the Riemann tensor and $\rho_{ij} = \sum_{a=1}^{n} R_{iaja}$ is the Ricci tensor at $m \in M$.

Let (X_t, P_x) be the Brownian motion process with infinitesimal generator Δ. For each $m \in M$ let T_ε be the exit time from the geodesic ball $B_m(\varepsilon)$.

In case $n = 2$ we introduce geodesic polar coordinates (r, θ) by $x_1 = r\cos\theta$, $x_2 = r\sin\theta$. The coordinate formulas for the first three operators become

$$\Delta_{-2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \tag{2.4}$$

$$\Delta_0 = -\frac{K_0}{3}\left(r\frac{\partial}{\partial r} - \frac{\partial^2}{\partial\theta^2}\right) \tag{2.5}$$

$$\begin{aligned}\Delta_1 = & -(K_1\cos\theta + K_2\sin\theta)\left[\frac{r^2}{4}\frac{\partial}{\partial r} - \frac{r}{6}\frac{\partial^2}{\partial\theta^2}\right] \\ & -(K_2\cos\theta - K_1\sin\theta)\left[\frac{r}{12}\frac{\partial}{\partial\theta}\right]\end{aligned} \tag{2.6}$$

where K is the Gaussian curvature and $K_0 = K(m)$, $K_1 = \frac{\partial K}{\partial x_1}(m)$, $K_2 = \frac{\partial K}{\partial x_2}(m)$. This computation is carried out in the Appendix, Sec. 5.

3. STATEMENT OF RESULTS

To study the harmonic measure we let $f \in C^\infty(S_1)$, a smooth function on the circle. Extend f to $R^2 \setminus \{0\}$ by making f constant along rays through the origin. Equivalently f can be thought of as a function on $\overline{M}_m \setminus \{0\}$. Then $\Phi_\varepsilon f$ is a smooth function on $M \setminus \{m\}$ which is constant along geodesics eminating from m. The harmonic measure operator is

$$H_\varepsilon f(x) = E_x\{(\Phi_\varepsilon f)(X_{T_\varepsilon})\} \qquad \varepsilon > 0, \qquad x \in B_m(\varepsilon).$$

THEOREM 3.1. — *When $\varepsilon \downarrow 0$ we have*

$$H_\varepsilon f(m) = \int_{-\pi}^{\pi} f(\theta)\left[1 - \frac{\varepsilon^3}{32}\langle \nabla K, u_\theta\rangle\right]\omega(d\theta) + 0(\varepsilon^4)$$

where ω is normalized Lebesgue measure and $\langle \nabla K, u_\theta\rangle = K_1\cos\theta + K_2\sin\theta$, $-\pi \le \theta \le \pi$.

Proof. — We follow the perturbation method introduced in [*1*]. Let $u_i \in C^\infty(\overline{B}_m(1))$ $(i = 0, 2, 3)$ be defined by

$$\Delta_{-2}u_0 = 0 \qquad u_0\,|_{\partial\overline{B}_m(1)} = f \tag{3.1}$$

$$(3.2) \qquad \Delta_{-2}u_2 + \Delta_0 u_0 = 0 \qquad u_2 |_{\partial \bar{B}_m(1)} = 0$$

$$(3.3) \qquad \Delta_{-2}u_3 + \Delta_1 u_0 = 0 \qquad u_3 |_{\partial \bar{B}_m(1)} = 0$$

The proof depends on the following three lemmas:

LEMMA 1. — $u_0(0) = \int_{-\pi}^{\pi} f(\theta)\omega(d\theta)$.

LEMMA 2. — $u_2(0) = 0$.

LEMMA 3. — $u_3(0) = -\frac{1}{32}\int_{-\pi}^{\pi} (K_1 \cos\theta + K_2 \sin\theta) f(\theta)\omega(d\theta)$.

Proof of Lemma 1. — u_0 is the solution of the classical Dirichlet problem in the unit disc. thus

$$u_0 = \int_{-\pi}^{\pi} \frac{(1 - r^2)}{(1 + r^2 - 2r\cos(\theta - \phi))} f(\phi)\omega(d\phi).$$

In particular $u_0(0) = \int_{-\pi}^{\pi} f(\phi)\omega(d\phi)$. □

Proof of Lemma 2. — Equivalently we write u_0 as a Fourier series

$$u_0 = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta).$$

Thus

$$\Delta_0 u_0 = -\frac{K_0}{3}\sum_{n=1}^{\infty}(n^2 + n) r^n (A_n \cos n\theta + B_n \sin n\theta).$$

This is a sum of homogeneous polynomials $\sum_1^{\infty} h_n$. By Lemma 6.3 of [1] we have $u_2(0) = \sum_1^{\infty} \frac{1}{(n+2)^2}\int_{-\pi}^{\pi} h_n(\theta)\omega(d\theta)$. But from the orthogonality relations for the functions $(\cos n\theta, \sin n\theta)$, all of these integrals are zero, hence $u_2(0) = 0$. □

(*Note*: The above depends on the fact that $\Delta_0 u_0$ is a harmonic function which vanishes at the origin. This may also be seen from the relation $\Delta_{-2}\Delta_0 - \Delta_0\Delta_{-2} = -\frac{2K_0}{3}\Delta_{-2}$ applied to u_0.)

Proof of Lemma 3. — We have

$$\left[\frac{r^2}{4}\frac{\partial}{\partial r} - \frac{r}{6}\frac{\partial^2}{\partial\theta^2}\right]u_0 = \sum_1^\infty r^{n+1}\left(\frac{n}{4}+\frac{n^2}{6}\right)(A_n\cos n\theta + B_n \sin n\theta)$$

$$\frac{r}{12}\frac{\partial u_0}{\partial\theta} = \sum_1^\infty r^{n+1}\left(\frac{n}{12}\right)(-A_n\sin n\theta + B_n\cos n\theta).$$

Referring to the formula (2.6), we see that

$$\Delta_1 u_0 = -\sum_1^\infty \left(\frac{n}{4}+\frac{n^2}{6}\right)r^{n+1}(K_1\cos\theta + K_2\sin\theta)(A_n\cos n\theta + B_n\sin n\theta)$$

$$-\sum_1^\infty\left(\frac{n}{12}\right)r^{n+1}(K_2\cos\theta - K_1\sin\theta)(-A_n\sin n\theta + B_n\cos n\theta)$$

a sum of homogeneous polynomials $\sum_2^\infty h_k$. By Lemma 6.3 of [*1*] we have $u_3(0) = \sum_2^\infty \frac{1}{(k+2)^2}\int_{-\pi}^{\pi} h_k(\theta)\omega(d\theta)$. But from the orthogonality relations of Fourier series all of these integrals are zero except for $k = 2$, i. e. $n = 1$. This gives

$$\begin{aligned} u_3(0) &= -\left(\frac{1}{4}+\frac{1}{6}\right)\frac{1}{4^2}(K_1A_1 + K_2B_2)/2 \\ &\quad -\left(\frac{1}{12}\right)\frac{1}{4^2}(K_2B_2 + K_1A_1)/2 \\ &= -\frac{1}{32}(K_1A_1 + K_2B_2)/2\,. \end{aligned}$$

On the other hand $f(\theta) = A_0 + \sum_1^\infty (A_n\cos n\theta + B_n\sin n\theta)$ and thus

$$\int_{-\pi}^{\pi} f(\theta)(K_1\cos\theta + K_2\sin\theta)\omega(d\theta) = \frac{1}{2}(K_1A_1 + K_2B_2)\,.$$

We have proved that $u_3(0) = -\frac{1}{32}\int_{-\pi}^{\pi} f(\theta)(K_1\cos\theta + K_2\sin\theta)\omega(d\theta)$ as required. □

To complete the proof of the theorem we appeal to Proposition 2.0 with $f = u_0 + \varepsilon^2 u_2 + \varepsilon^3 u_3$; thus

$$\begin{aligned}\Phi_\varepsilon^{-1}\Delta\Phi_\varepsilon(u_0 + \varepsilon^2 u_2 + \varepsilon^3 u_3) &= \varepsilon^{-2}\Delta_{-2}(u_0 + \varepsilon^2 u_2 + \varepsilon^3 u_3) \\ &\quad + \Delta_0(u_0 + \varepsilon^2 u_2 + \varepsilon^3 u_3) \\ &\quad + \varepsilon\Delta_1(u_0 + \varepsilon^2 u_2 + \varepsilon^3 u_3) + 0(\varepsilon^2) \\ &= 0(\varepsilon^2),\end{aligned}$$

where we have used the defining relations (3.1)-(3.3). Hence $\Delta U = 0(\varepsilon^2)$ where $U = \Phi_\varepsilon(u_0 + \varepsilon^2 u_2 + \varepsilon^3 u_3)$. Noting that $U = \Phi_\varepsilon f$ on $\partial B_m(\varepsilon)$ we have that for any $x \in B_m(\varepsilon)$ by Dynkin's formula

$$\begin{aligned}H_\varepsilon f(x) &= E_x U(X_{T_\varepsilon}) \\ &= U(x) + E_x \int_0^{T_\varepsilon} \Delta U(X_s) ds \\ &= U(x) + 0(\varepsilon^4).\end{aligned}$$

Setting $x = m$ we have $U(m) = u_0(0) + \varepsilon^2 u_2(0) + \varepsilon^3 u_3(0)$ from which the result follows. □

4. COMPARISON WITH THE NON-STOCHASTIC MEAN VALUE

The mean-value operator of a Riemannian manifold is defined by

$$M_\varepsilon f(m) = \frac{\int_{S_\varepsilon} (\Phi_\varepsilon f) d\sigma_\varepsilon}{\int_{S_\varepsilon} 1 d\sigma_\varepsilon}$$

where σ_ε is the surface measure on the ε-sphere S_ε and f is a smooth function on the unit sphere. We have the following result.

THEOREM 4.1. — *When* $\varepsilon \downarrow 0$

$$M_\varepsilon f(m) = \int_{-\pi}^{\pi} f(\theta)\left[1 - \frac{\varepsilon^3}{12}\langle \nabla K, u_\theta \rangle\right]\omega(d\theta) + 0(\varepsilon^4).$$

Proof. — From the method of Gray and Willmore [2], we have

$$M_\varepsilon f = \frac{I_\varepsilon f}{I_\varepsilon 1}$$

where

$$I_\varepsilon f = \int_{-\pi}^{\pi} f(\theta) G(\varepsilon, \theta) d\theta .$$

Using the expansion $G(\varepsilon, \theta) = \varepsilon - \frac{\varepsilon^3}{6} K_0 - \frac{\varepsilon^4}{12}(K_1 \cos\theta + K_2 \sin\theta) + 0(\varepsilon^5)$ we have

$$I_\varepsilon f = \left(\varepsilon - \frac{\varepsilon^3}{6} K_0\right) \int_{-\pi}^{\pi} f(\theta) d\theta - \frac{\varepsilon^4}{12} \int_{-\pi}^{\pi} f(\theta)(K_1 \cos\theta + K_2 \sin\theta) d\theta + 0(\varepsilon^5).$$

In particular

$$I_\varepsilon 1 = 2\pi\left(\varepsilon - \frac{\varepsilon^3}{6} K_0\right) + 0(\varepsilon^5) .$$

Thus

$$\frac{I_\varepsilon f}{I_\varepsilon 1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\left[1 - \frac{\varepsilon^3}{12}(K_1 \cos\ \theta + K_2 \sin\theta)\right] d\theta + 0(\varepsilon^4)$$

which was to be proved.

COROLLARY. — *If* $H_\varepsilon f(m) = M_\varepsilon f(m)$ *for every* $m \in M$, $\varepsilon > 0$, $f \in C^\infty(S^1)$, *then* (M, g) *has constant curvature.*

Proof. — We have $\lim\limits_{\varepsilon \downarrow 0} \dfrac{M_\varepsilon f - H_\varepsilon f}{\varepsilon^3} = \dfrac{-5}{96} \displaystyle\int_{-\pi}^{\pi} \langle \nabla K, u_\theta \rangle f(\theta) \omega(d\theta)$.

If this is zero for all f, then $K_1 = K_2 = 0$, i. e. $\nabla K(m) = 0$, and hence K is constant.

5. APPENDIX. COMPUTATION OF Δ_{-2}, Δ_0, Δ_1

In geodesic polar coordinates we have

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & G^2 \end{pmatrix} \qquad G = G(r, \theta)$$

(5.1)
$$\Delta = \frac{1}{G}\frac{\partial}{\partial r}\left(G \frac{\partial}{\partial r}\right) + \frac{1}{G}\frac{\partial}{\partial \theta}\left(\frac{1}{G}\frac{\partial}{\partial \theta}\right).$$
$$= \frac{\partial^2}{\partial r^2} + \frac{G_r}{G}\frac{\partial}{\partial r} + \frac{1}{G^2}\frac{\partial^2}{\partial \theta^2} - \frac{G_\theta}{G^3}\frac{\partial}{\partial \theta}$$

The Jacobi equation connecting the metric with the curvature is

(5.2) $\qquad G_{rr} + KG = 0, \qquad G(0^+, \theta) = 0, \qquad G_r(0^+, \theta) = 1 .$

In a neighborhood of m, we can write

$$K = K_0 + r(K_1 \cos\theta + K_2 \sin\theta) + 0(r^2) \qquad (r \downarrow 0). \tag{5.3}$$

This yields

$$G = r - \frac{r^3}{6} K_0 - \frac{r^4}{12}(K_1 \cos\theta + K_2 \sin\theta) + 0(r^5) \qquad (r \downarrow 0).$$

Performing the indicated operations, we have

$$\frac{G_r}{G} = \frac{1}{r} - \frac{r}{3} K_0 - \frac{r^2}{4}(K_1 \cos\theta + K_2 \sin\theta) + 0(r^3)$$

$$\frac{1}{G^2} = \frac{1}{r^2} + \frac{K_0}{3} + \frac{r}{6}(K_1 \cos\theta + K_2 \sin\theta) + 0(r^2)$$

$$\frac{G_\theta}{G^3} = \frac{r}{12}(K_1 \sin\theta - K_2 \cos\theta) + 0(r^2).$$

Substituting in (5.1) and equating coefficients of r yields (2.4)-(2.6).

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