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On the distribution of the norm for a gaussian measure

by

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SUMMARY. — Let E be an infinite dimensional Banach space with norm $\|\cdot\|$. Then for each $\varepsilon > 0$, there exists a norm N which is $(1 + \varepsilon)$-equivalent to $\|\cdot\|$, and a centered Gaussian measure μ on E such that the distribution of $N(\cdot)$ for μ has an unbounded density with respect to Lebesgue measure.

RÉSUMÉ. — Soit E un espace de Banach de dimension infinie avec la norme $\|\cdot\|$. Alors, pour chaque $\varepsilon > 0$, il y a une norme N qui est $(1 + \varepsilon)$-équivalente à $\|\cdot\|$, et une mesure gaussienne centrée μ sur E telle que la distribution de $N(\cdot)$ pour μ ait une densité non bornée par rapport à la mesure de Lebesgue.

1. INTRODUCTION

Consider an infinite dimensional Banach space, and μ a centered Gaussian measure on E, that is a Radon measure on E such that for each $x^* \in E^*$ the law of x^* is Normal centered. For $t \in R^+$, let $B_t = \{ x \in E ; \| x \| \leqq t \}$, and $\phi(t) = \mu(B_t)$. The function $\phi(t)$ has remarkable properties. Let $\Phi(u)$ given by $\Phi(u) = (1/\sqrt{2\pi}) \int_{-\infty}^{u} \exp(-x^2/2)dx$. A remarkable recent result of A. Ehrhard [1] asserts that $\Psi = \Phi^{-1} \circ \phi$ is concave. It follows that for each $t_0 > 0$, there is a constant C_0 such that $|\Psi(t)-\Psi(u)| \leqq C_0(t-u)$ for $t, u \geqq t_0$. It follows that $|\phi(t)-\phi(u)| \leqq C_0(t-u)$ for $t, u \geqq t_0$ since Φ is lipschutz of constant $1/\sqrt{2\pi}$. This shows that the distribution of $\|\cdot\|$ has a bounded density with respect to Lebesgue measure on each interval $[t_0, \infty[$. (M. Talagrand recently showed that this density is continuous [8].) Let us consider the problem whether this density is bounded on $[0, \infty[$, that is whether there is C such that for $0 \leqq u \leqq t$, we have

$$(*) \qquad \mu(B_t \backslash B_u) \leqq C(t-u)$$

It has been shown by J. Kuelbs and T. Kurtz [3] that condition (*) holds for each gaussian μ when $E = l_2(N)$ provided with the usual norm. These results have been considerably generalized by the author and M. Talagrand, who showed that it is enough to assume the norm of E is uniformly convex and that the modulus of uniform convexity is of power type (that is $\geqq \varepsilon^p$ for some p and $\varepsilon > 0$).

In the opposite direction, it has been shown independently by V. Paulaskas and by the author and M. Talagrand that condition (*) fails in general [5]. A further example by the author and M. Talagrand exhibits a $\mathscr{C}^\infty$ renorming of $l_2(N)$, such that all the differentials of the norm remain bounded on the unit sphere, and still condition (*) fails for this renorming [6].

Closely connected to condition (*) is the problem of the rate of convergence in the central limit theorem (C. L. T.). If X is an E-valued r. v. with zero expectation and moments of order 2, we say that X is pregaussian if there exists a gaussian measure μ on E with the same covariance as S, that is

$$E(x(X)y(X)) = \int x(t)y(t)d\mu(t) \qquad \text{for} \qquad x^*, y^* \in E^*$$

If $(X_i)_{i \leqq n}$ are i. i. d. copies of X, the rate of convergence in the C. L. T. is often estimated by

$$\Delta_n = \sup_t \left| P\left\{ \left\| n^{-1/2} \sum_{i \leqq n} X_i \right\| \leqq t \right\} - \mu(B_t) \right|.$$

J. Kuelbs and T. Kutz showed that if condition (*) holds and the norm $\|\cdot\|$ is three times differentiable with these differentials bounded on the unit sphere, then $\Delta_n = 0(n^{-1/6})$ if X has a third moment. F. Gotze [1] reduced this bound to the best estimate $0(n^{-1/2})$ under slightly stronger conditions.

For $\alpha > 1$, a linear isomorphism T from E to F is called an α-isomorphism if for $x \in E$ we have $\|x\|/\alpha \leqq \|T(x)\| \leqq \alpha \|x\|$. We say that E and F are α-isomorphic if there exists an α-isomorphism between E and F. We say that two norms $\|\cdot\|$, $N(\cdot)$ on E are α-equivalent if the identity is an α-isomorphism from $(E, \|\cdot\|)$ to $(E, N(\cdot))$.

THEOREM. — Let $(E, \|\cdot\|)$ be an infinite dimensional Banach space. Let $\varepsilon > 0$ and (ξ_n) be a sequence converging to zero. Then there exists a norm $N(\cdot)$ on E and an E valued r. v. X such that

a) $N(\cdot)$ is $(1+\varepsilon)$-equivalent to $\|\cdot\|$,
b) X is bounded and pregaussian,
c) if μ is the gaussian measure on E with the same covariance as X, μ fails condition (*) for the norm $N(\cdot)$,
d) the inequality

$$\Delta_n = \sup_t \left| P\left\{ N\left(n^{-1/2} \sum_{i \leqq n} X_i\right) \leqq t \right\} - \mu(N(x) \leqq t) \right| \geqq \xi_n$$

holds for infinitely many n.

2. SOME TOOLS

Let l_2^n be the n dimensional Hilbert space, and $(e_i)_{i \leqq n}$ be the canonical basis. Let γ_n be the gaussian measure on l_2^n such that the dual functionals e_i^* are independent and standard nomally distributed. The following observations are crucial.

OBSERVATION 1. — Since the variable $(e_i^*)^2$ are equidistributed independent of expectation 1 and variance 3, the one-dimensional C. L. T. asserts that the distribution of $\| x \|^2 = \sum_{i \leq n} (x_i(x))^2$ is close to $N(n, \sqrt{3n})$. In particular $\gamma_n\{x;\, n^{1/2} - 10 < \| x \| < n^{1/2}\} > 1/3$ for n large and $\gamma_n\{x;\, \|x\| < 2n^{1/2}\} \to 1$. Notice also that $\int \| x \|^2 d\gamma_n(x) = n$.

OBSERVATION 2. — Let Y_n be a r. v. valued in l_2^n such that for $i \in \{1, 2, 3, \ldots, n\}$ and $j \in \{-1, 1\}$, it takes the value $jn^{1/2}e_i$ with probability $1/2n$. Let (Y_n^i) be i. i. d. like Y_n. If q is much smaller than n, with probability close to 1, the r. v. $S_{n,q} = q^{-1/2} \sum_{1 \leqq i \leqq q} Y_n^i$ takes values of the type $\sum_{i \in I} a_i e_i$ where card I $= q$ and $|a_i| = n^{1/2}q^{-1/2}$, so $\|S_{n,q}\| = n^{1/2}$ in this case. So for q fixed,

$$\lim_{n\to\infty} P\{\|S_{n,q}\| = n^{1/2}\} = 1.$$

We shall also make essential use of the following Banach space result.

THEOREM 1. — Let E be an infinite dimensional Banach space, and F be a finite dimensional subspace of E, $\tau > 1$ and $n \in \mathrm{N}$. Then there is an n-dimensional subspace G of E of dimension n, that is τ-isomorphic to l_2^n and such that for $x \in \mathrm{G}$, $y \in \mathrm{F}$ we have $\|x\| \leqq \tau \|x + y\|$.

We shall need the following version of Dvoretzski's theorem: Given $\alpha > 1$, and $p \in \mathrm{N}$, there is a number $q(p, \alpha)$ such that any finite dimensional Banach space H of dimension $\geqq q(p, \alpha)$ contains a subspace α-isomorphic to l_2^p [*4*].

Let H be a complement of F. Let $\alpha = \tau^{1/4}$. We can assume $n \geqq 1 + \dim \mathrm{F}$. Let G_1 be a subspace of H that is α-isomorphic to l_2^q with $q = q(2n, \alpha)$. On G_1 consider the norm $\|\cdot\|_1$ given by $\|x\|_1 = \mathrm{Inf}\{\|x+y\|\,;\, y \in \mathrm{F}\}$. Dvoretzski's theorem gives a subspace G_2 of G_1 such that $(\mathrm{G}_2, \|\cdot\|_1)$ is α-isomorphic to l_2^{2n}.

Let T_1 (resp. T_2) be an α-isomorphism from $(\mathrm{G}_2, \|\cdot\|)$ (resp. $(\mathrm{G}_2, \|\cdot\|_1)$ to l_2^{2n} and let $\mathrm{T} = \mathrm{T}_2 \circ \mathrm{T}_1^{-1}$). The quadratic form $\mathrm{Q}(x) = \|\mathrm{T}(x)\|^2$ on l_2^{2n} can be diagonalized in an orthonormal basis $f_1, f_2, \ldots, f_{2n}$. We can assume the eigenvalues are such that $\lambda_1 \leqq \lambda_2 \leqq \ldots \leqq \lambda_{2n}$. For $i \leqq n$, there exist u_i, v_i with $u_i^2 + v_i^2 = 1$, $u_i\lambda_i + v_i(\lambda_{2n-i-1}) = \lambda_n$. Let G′ be the space generated by the vectors $u_ie_i + v_ie_{2n-i-1}$. For $x \in \mathrm{G}'$, we have $\|\mathrm{T}(x)\|^2 = \lambda_n \|x\|^2$. Let $\mathrm{G} = \mathrm{T}_1^{-1}(\mathrm{G}')$. For $x \in \mathrm{G}$, we have

$$\|x\| \leqq \alpha \|\mathrm{T}_1(x)\| = \lambda_n^{1/2}\alpha \|\mathrm{T}_2(x)\| \leqq \lambda_n^{1/2}\alpha^2 \|x\|_1$$

and similarly $\lambda_n^{1/2} \|x\|_1 \leqq \alpha^2 \|x\|$. Since $\dim \mathrm{G} > \dim \mathrm{F}$, it follows from [*4*], lemma 2.8 C that there is $x_0 \in \mathrm{G}$ with $\|x_0\| = 1$ and $\|x_0\|_1 = 1$. This shows that $\lambda_n^{1/2} \leqq \alpha^2$. Hence $\|x\| \leqq \tau \|x\|_1$ for $x \in \mathrm{G}$.

3. CONSTRUCTION

Let β_p be a sequence with $\beta_p > 1$, $\prod_{1 \leqq i \leqq \infty} \beta_i < 1 + \varepsilon$. By induction over p, we construct sets B_p of E, integers $q(p)$, real numbers a_p, δ_p and r. v. Z_p such that the following conditions are satisfied.

(1) B_p is convex balanced; B_1 is the unit ball of E; for $p \geqq 2$,

$$\mathrm{B}_{p-1} \subset \mathrm{B}_p \subset \beta_p \mathrm{B}_{p-1}.$$

(2) Z_p is valued in a finite dimensional space; for each ω, $\| Z_p(\omega) \| \leqq 2^{-p}$ and the sequence (Z_p) is independent.

(3) If η_p is the gaussian measure with the same covariance as Z_p, then

$$\int \| x \|^2 d\eta_p(x) \leqq 2^{-p}.$$

(4) If ν_p is the gaussian measure with the same covariance as $X_p = \sum_{l \leqq p} Z_l$, and N_p is the gauge of B_p, we have for $r \leqq p$,

$$\nu_p \{ x ; a_r - \delta_r < N_p(x) < a_r \} > 2\xi_{q(r)}.$$

(5) If $(X_p^i)_i$ are i. i. d. copies of X_p, for $r \leqq p$, we have

$$P \left\{ N_p\left(q(r)^{-1/2} \sum_{i \leqq q(r)} X_p^i \right) \leqq a_r \right\} < \xi_{q(r)}.$$

(6) $\delta_p = \xi_{q(p)}/p$.

We proceed to the first step of the construction. We choose $q(1)$ such that $\xi_{q(1)} < 1/6$, and $\delta_1 = \xi_{q(1)}$. It follows from observations 1 and 2 that there exists n such that $10n^{-1/2} < \delta_1$ and that

(7) $$\gamma_n \{ n^{1/2} - 10 < \| x \| < n^{1/2} \} > 1/3 > 2\xi_{q(1)}$$

(8) $$P \left\{ \left\| q(1)^{-1/2} \sum_{i \leqq q(1)} Y_n^i \right\| < n^{1/2} \right\} < \xi_{q(1)}.$$

There exists d with $n^{1/2} > d > n^{1/2}/2$ such that

(9) $$\gamma_n \{ n^{1/2} - 10 < \| x \| < d \} > 2\xi_{q(1)}$$

and automatically we have

(10) $$P \left\{ \left\| q(1)^{-1/2} \sum_{i \leqq q(1)} Y_n^i \right\| \leqq d \right\} < \xi_{q(1)}.$$

There is $1 < \alpha < 2$ such that

(11) $$\gamma_n \{ (n^{1/2} - 10)\alpha < \| x \| < d/\alpha \} > 2\xi_{q(1)}.$$

(12) $$P \left\{ \left\| q(1)^{-1/2} \sum_{i \leqq q(1)} Y_n^i \right\| < \alpha d \right\} < \xi_{q(1)}.$$

From Dvoretzski's theorem, there is a subspace G of E and an α-isomorphism T from l_2^n to G. Let $b = 1/(8d)$ and $Z_1 = bT(Y_n)$.

(2) follows from $\| Z_1(\omega) \| \leqq 2bn^{1/2} \leqq 1/2$.

We check (3). Since $\eta_1 = bT(\gamma_n)$, we have

$$\int \| x \|^2 d\eta_1(x) = b^2 \int \| T(x) \|^2 d\gamma_n(x) \leqq 4b^2 n \leqq 2^{-1},$$

so (3) holds. Let $a_1 = 1/8$. We check (4). Since $\delta_1/b \geqq 10$, we have

$$\begin{aligned} \eta_1 \{ x; a_1 - \delta_1 < \| x \| < a_1 \} &= \gamma_n \{ y; a_1 - \delta_1 < b \| T(y) \| < a_1 \} \\ &\geqq \gamma_n \{ y; n^{1/2} - 10 < \| T(y) \| < d \} \\ &\geqq \gamma_n \{ y; \alpha(n^{1/2} - 10) \leqq \| y \| \leqq d/\alpha \} \\ &> 2\xi_{q(1)} \end{aligned}$$

and hence (4) holds. To check (5), we note that

$$\left\| q(1)^{-1/2} \sum_{i \leqq q(r)} X_1^i \right\| \leqq a_1 \Rightarrow \left\| T\left(q(1)^{-1/2} \sum_{i \leqq q(r)} Y_1^i \right) \right\| \leqq d$$

so (5) follows from (12). Finally (6) holds by construction. The first step is completed.

Let us now assume that the first p steps have been completed. There exist two numbers $1 < \alpha < \beta_p$ and $b > 0$ such that for $r \leqq p$ we have

$$(13) \qquad (1-b)\nu_p \{ x; \alpha(a_r - \delta_r + 16b) < N_p(x) < a_r - 16b \} > 2\xi_{q(r)}.$$

$$(14) \qquad P \left\{ N_p \left(q(r)^{-1/2} \sum_{i \leqq q(r)} X_p^i \right) \leqq \alpha a_r + 8b \right\} < \xi_{q(r)}.$$

We can assume $b \leqq 2^{-p-4}$. Let $c = b/q(p)$. Let $q(p + 1)$ be large enough that

$$(15) \qquad 6\xi_{q(p+1)} < \nu_p \{ x; \| x \| < c(\alpha - 1)/2 \} \quad \text{and} \quad 12\xi_{q(p+1)} < c.$$

Let $\delta_{p+1} = \xi_{q(p+1)}/(p + 1)$. From observations 1 and 2, there exists n with $\delta_{p+1} \geqq 20n^{-1/2}$ and

$$(16) \qquad \gamma_n \{ x; n^{1/2} - 10 < \| x \| < n^{1/2} \} > 1/3,$$

$$(17) \qquad \gamma_n \{ x; \| x \| \geqq 2n^{1/2} \} \leqq b,$$

$$(18) \qquad \int \| x \|^2 d\gamma_n(x) = n,$$

$$(19) \qquad P \left\{ \left\| q(p + 1)^{-1/2} \sum_{i \leqq q(p+1)} Y_n^i \right\| < n^{1/2} \right\} < \xi_{q(p+1)}.$$

Let d with $n^{1/2}/2 < d < n^{1/2}$ and

$$\gamma_n\{x; n^{1/2} - 10 < \| x \| < d\} > 1/3. \tag{20}$$

Let τ with $\tau^3 = (\alpha + 1)/2$.

Let F be a finite dimensional space of E in which X_p is valued. We use Theorem 1 for (E, N_p). So there is a finite dimensional space G of E and τ-isomorphism T from l_2^n to G such that for $x \in G$, $y \in F$ we have

$$N_p(x) \leqq \tau N_p(x + y).$$

We define B_{p+1} as the closed convex hull of the set

$$B_p \cup \{x + y; x \in G, y \in F, \| T^{-1}(x) \| = \tau^2, \| y \| \leqq (\alpha - 1)/2\}.$$

For $x \in G$, $y \in F$, $\| T^{-1}(x) \| \leqq \tau^2$, $\| y \| \leqq (\alpha - 1)/2$, we have $N_p(x) \leqq \tau^3$, so $N_p(x + y) \leqq N_p(x) + N_p(y) \leqq N_p(x) + \| y \| \leqq \alpha$.

In particular $B_p \subset B_{p+1} \subset \alpha B_p$, so (1) holds since $\alpha < \beta_p$.

Moreover for $x \in E$ we have $N_p(x)/\alpha \leqq N_{p+1}(x) \leqq N_p(x)$.

We now propose the following fact.

Fact. — For $x \in G$, $\| T^{-1}(x) \| = \tau^2$, $y \in F$, $\| y \| \leqq (\alpha - 1)/2$, we have $N_{p+1}(x + y) = 1$.

We already know that $N_{p+1}(x + y) \leqq 1$. There is a linear functional ϕ_1 on G such that $\phi_1(x) = 1$ while $\phi_1(x') \leqq 1$ when $\| T^{-1}(x') \| \leqq \tau^2$, so there is a linear functional ϕ_2 on F + G such that $\phi_2(x) = 1$, $\phi_2(x') \leqq 1$ whenever $\| T^{-1}(x') \| \leqq 1$, $x' \in G$ and $\phi_2 = 0$ on F. In particular $\phi_2(x+y)=1$. If $x'+y' \in B_p$, then $\| x' \| \leqq \tau$, so $\| T^{-1}(x') \| \leqq \tau^2$, so $\phi_2(x'+y') \leqq 1$. This shows that $N_p(\phi_2) \leqq 1$. So ϕ_2 can be extended on E by a ϕ with $N_p(\phi) \leqq 1$. Since $\phi(x' + y') \leqq 1$ for $x' \in G$, $\| T^{-1}(x') \| \leqq \tau^2$ and $y' \in F$ and since $\phi \leqq 1$ on B_p, the definition of B_{p+1} shows that $N_{p+1}(\phi) \leqq 1$. As $\phi(x+y) = 1$, the fact is proved.

Remark. — This fact motivated the choice of B_{p+1}.

We set $Z_{p+1}(\omega) = (c/d)T(Y(\omega))$. Now (2) follows from

$$\begin{aligned} \| Z_{p+1}(\omega) \| &\leqq (1 + \varepsilon)N_p(Z_{p+1}(\omega)) \leqq 2(c/d) \| T \| \cdot \| Y(\omega) \| \\ &\leqq 4cn^{1/2}/d \leqq 8b/q(p) \leqq 2^{-p-1}. \end{aligned}$$

Also, (3) follows from

$$\int \| x \|^2 d\eta_{p+1}(x) \leqq 4(c/d)^2 \int \| y \|^2 d\gamma_n(y) \leqq 2^{-p-1} \quad \text{with} \quad \eta_{p+1} = 2(c/d)T^{-1}(\gamma_n).$$

We now check (4). We first show that for $r \leqq p$, we have

$$\nu_{p+1}\{ z ; a_r - \delta_r < \mathrm{N}_{p+1}(z) < a_r \} > 2\xi_{q(r)} .$$

We notice that ν_{p+1} is a measure on F+G, that identifies to $\nu_p \otimes \eta_{p+1}$. Let $\mathrm{A}=\{ x\in\mathrm{G};\ \|x\|\leqq 16c\}$. For $z\in l_2^n$, $\|z\|\leqq 2n^{1/2}$, we have $\|(c/d)\mathrm{T}(z)\|\leqq 16c$. It follows from (17) and the fact that $\eta_{p+1}=(c/d)\mathrm{T}(\gamma_n)$ that $\eta_{p+1}(\mathrm{A})\geqq 1-b$. Let

$$\mathrm{B} = \{ y\in \mathrm{F};\ \alpha(a_r-\delta_r+16b) < \mathrm{N}_p(y) < a_r-16b \} .$$

For $x\in\mathrm{A}$, since $\mathrm{N}_{p+1}(x) \leqq \| x \|$, we have $\mathrm{N}_{p+1}(x) \leqq 16c \leqq 16b$. For $y\in\mathrm{B}$, since $\mathrm{N}_{p+1}(y) \leqq \mathrm{N}_p(y) \leqq \alpha\mathrm{N}_{p+1}(y)$, we have

$$a_r - \delta_r + 16b < \mathrm{N}_{p+1}(y) < a_r - 16b .$$

So, for $x\in\mathrm{A}$, $y\in\mathrm{B}$, we have $a_r - \delta_r < \mathrm{N}_{p+1}(x + y) < a_r$.

It follows from (13) that

$$\nu_{p+1}\{ z;\ a_r - \delta_r < \mathrm{N}_{p+1}(z) < a_r \} \geqq \nu_p(\mathrm{B})\eta_{p+1}(\mathrm{A}) > 2\xi_{q(p+1)} .$$

Let $a_{p+1} = c/\tau^2$. To finish the proof that (4) holds at rank $p + 1$, it remains to show if

$$\mathrm{H} = \{ z;\ a_{p+1} - \delta_{p+1} < \mathrm{N}_{p+1}(z) < a_{p+1} \}$$

then $\nu_{p+1}(\mathrm{H}) > 2\xi_{q(p+1)}$. Let

$$\mathrm{C} = \{ x\in\mathrm{G};\ a_{p+1} - \delta_{p+1} < \mathrm{N}_{p+1}(z) < a_{p+1} \} .$$

For $x\in\mathrm{G}$, $\mathrm{N}_{p+1}(x) = \| \mathrm{T}^{-1}(x) \|/\tau^2$, so

$$\mathrm{C} \supset \{ x\in\mathrm{G};\ c-\delta^{p+1} < \| \mathrm{T}^{-1}(x) \| < c \}$$
$$\supset \{ x\in\mathrm{G};\ d - 10 < \| (d/c)\mathrm{T}^{-1}(x) \| < d \}$$

since $(d/c)\delta_{p+1} \geqq d\delta_{p+1} \geqq 10$. In particular, $\eta_{p+1}(\mathrm{C}) \geqq 1/3$ from (20).

Let $\mathrm{D} = \{ y\in\mathrm{F};\ \| y \| \geqq c(\alpha-1)/3 \}$. We have $\delta_{p+1} \leqq c/6$, and we can assume $\tau^2 < 4/3$. We then have $c(\alpha-1)/3 \leqq a_{p+1} - \delta_{p+1}$.

It follows that for $x\in\mathrm{C}$, $y\in\mathrm{D}$, we have $\mathrm{N}_{p+1}(x+y)=\mathrm{N}_{p+1}(x)$, so $x+y\in\mathrm{H}$. Hence $\nu_{p+1}(\mathrm{H})>\nu_p(\mathrm{D})\eta_{p+1}(\mathrm{C})>2\xi_{q(p+1)}$ from (15), so (4) holds.

We now check (5). We first show that for $r \leqq p$, we have

$$\mathrm{P}\left\{ \mathrm{N}_{p+1}\left(q(r)^{-1/2} \sum_{i\leqq q(r)} \mathrm{X}_{p+1}^i \right) \leqq a_r \right\} < \xi_{q(r)} .$$

We have seen that $\| Z_{p+1}(\omega) \| \leqq 8b/q(p)$, so since $X^i_{p+1} = X^i_p + Z^i_{p+1}$, we get

$$N_p\Big(q(r)^{-1/2} \sum_{i \leqq q(r)} X^i_{p+1} \leqq N_p\Big(q(r)^{-1/2} \sum_{i \leqq q(r)} X^i_p\Big) + 8b$$
$$\leqq \alpha N_{p+1}\Big(q(r)^{-1/2} \sum_{i \leqq q(r)} X^i_p\Big) + 8b$$

so the result follows from (14). To check (5), it remains to show that

$$P\Big\{ N_{p+1}\Big(q(p+1)^{-1/2} \sum_{i \leqq q(p+1)} X^i_{p+1}\Big) \leqq a_{p+1} \Big\} < \xi_{q(p+1)}.$$

We first note that for $x \in G$, $y \in F$, we have $N_{p+1}(x+y) \geqq N_{p+1}(x)$ since $N_{p+1}(x + \lambda y)$ is a convex function of λ that is equal to $N_{p+1}(x)$ for small λ. Hence

$$N_{p+1}\Big(q(p+1)^{-1/2} \sum_{i \leqq q(p+1)} X^i_{p+1}\Big) \geqq N_{p+1}\Big(q(p+1)^{-1/2} \sum_{i \leqq q(p+1)} Z^i_{p+1}\Big)$$

and the result follows from (19) and the definition of Z. The construction is complete since (6) holds by construction.

4. PROOF OF THE THEOREM

It follows from (2) that one can define a bounded r. v. X by $X(\omega) = \sum_l Z_l(\omega)$.

For each l, let U_l be a Gaussian r. v. with the same covariance as Z_l, and such that the sequence (U_l) is independent. It follows from (3) that the series (U_l) is summable in $L_2(E)$. Its sum V is Gaussian, and has the same covariance as X, so X is pregaussian. Let μ be the distribution of V.

Let $N(x) = \lim_p N_p(x)$ and $\theta_p = \prod_{i \leqq p} \beta_i$. From condition (1) we get $N(x) \leqq N_p(x) \leqq \theta_p N(x)$. It follows from (4) that for $q \leqq p$, $r \leqq p$ we get

$$\nu_p \{ x ; a_r - \delta_r < N(x) < \theta_q a_r \} > 2\xi_{q(r)}.$$

Since ν_p is the distribution of $V_p = \sum_{l \leqq p} U_l$, we get by letting $p \to \infty$

$$\mu \{ x ; a_r - \delta_r \leqq N(x) \leqq \theta_q a_r \} \geqq 2\xi_{q(r)},$$

and letting $q \to \infty$ gives

$$\mu\{x; a_r - \delta_r \leqq N(x) \leqq a_r\} \geqq 2\xi_{q(r)}.$$

It follows from (6) that condition (*) fails for μ. It follows from (5) that for $r \leqq p$,

$$P\left\{\theta_p N\left(q(r)^{-1/2} \sum_{i \leqq q(r)} X_p^i\right) \leqq a_r\right\} \leqq \xi_{q(r)}.$$

Letting $p \to \infty$ gives

$$P\left\{N\left(q(r)^{-1/2} \sum_{i \leqq q(r)} X^i\right) \leqq a_r\right\} \leqq \xi_{q(r)}.$$

In particular,

$$\left|\mu\left\{x; N(x) \leqq a_r\right\} - P\left\{N\left(q(r)^{-1/2} \sum_{i \leqq q(r)} X^i\right) \leqq a_r\right\}\right| \geqq \xi_{q(r)}$$

which completes the proof of the theorem.

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