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# Semistable convolution semigroups on measurable and topological groups

by

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ABSTRACT. — We consider semistable convolution semigroups which constitute a natural generalization of stable convolution semigroups. Several examples are discussed, a purity law is proved, and the supports of the corresponding measures are investigated. Moreover we study the relation between semistability and holomorphy of semigroups. Special attention is given to the classical semistable semigroups on Euclidean spaces.

RÉSUMÉ. — On considère des semi-groupes de convolution semi-stables qui sont des généralisations naturelles des semi-groupes de convolution stables. De nombreux exemples seront discutés, une loi de pureté sera démontrée, et les supports des mesures correspondantes seront explorés. De plus, on étudie la relation entre la semi-stabilité et l'holomorphie des semi-groupes. Les semi-groupes semi-stables classiques, sur les espaces Euclidiens apparaissent comme des exemples d'un intérêt spécial.

## INTRODUCTION

Let $(X_n)_{n \geq 1}$ be a sequence of independent identically distributed random vectors with values in some Euclidean space $\mathbb{R}^d$. Moreover let $(k_n)_{n \geq 1}$ be an increasing sequence of positive integers such that the sequence

$(k_n/k_{n+1})_{n\geq 1}$ converges to some positive real number $c$; let $(A_n)_{n\geq 1}$ be a sequence of non-singular linear operators on $\mathbb{R}^d$; and let $(a_n)_{n\geq 1}$ be a sequence in $\mathbb{R}^d$. If the normed sequence $(A_n(X_1 + \ldots + X_n) + a_n))_{n\geq 1}$ of random vectors converges in distribution to some probability measure $\mu$ on $\mathbb{R}^d$ then $\mu$ is said to be operator-semistable with coefficient $c$.

On the real line semistability was first considered by P. Lévy. The definition above is due to R. Jajte [*8*]. He proved the following characterization theorem: A full probability measure $\mu$ on $\mathbb{R}^d$ is operator-semistable if and only if it is infinitely divisible and if there exist a number $c \in ]0, 1[$, a vector $b \in \mathbb{R}^d$, and a non-singular linear operator B on $\mathbb{R}^d$ with spectral radius less or equal $\sqrt{c}$ such that $\mu^c = B(\mu) * \varepsilon_b$. Obviously this implies $\mu^{ct} = B(\mu^t) * \varepsilon_{bt}$ for all $t > 0$. Moreover if $c$ does not belong to the spectrum of B then there exists some $a \in \mathbb{R}^d$ such that $(B - cI)a = b$; and this yields $B((\mu * \varepsilon_a)^t) = (\mu * \varepsilon_a)^{ct}$ for all $t > 0$.

Operator-semistability is a generalization of operator-stability (cf. [*19*]). Since the definition of stability demands a one-parameter group of automorphisms the concept of semistability is the more elementary one for it involves only a single automorphism. Hence it was tempting to study semistable measures on a topological or measurable group where often one has only poor information about the structure of the associated automorphism group. In fact there already exist some results for this general situation [*23*] [*24*]. Moreover on topological vector spaces semistable measures have been investigated to some extent [*2*] [*11*] [*13*] [*14*] [*23*]. On the other hand on locally compact groups stable measures are by now a well established quantity as may be seen from the basic paper [*3*] of W. Hazod and from its references.

Following a suggestion of A. Tortrat we place ourselves into a rather general framework. Instead of topological groups we consider measurable groups as far as possible. Instead of topological automorphisms we choose measurable homomorphisms to define semistability. Hence we are often forced to distinguish between the cases $c < 1$ and $c > 1$ for the coefficient $c$ of semistability.

There are at least two reasons why to consider semistability on measurable groups. Firstly there exist vector spaces that are measurable groups in some natural way but that are not topological groups (e. g. D [*0, 1*]; cf. [*1*], pp. 181). Secondly it is the measurable (and not the topological) structure that is responsible for many of our results.

The paper is organized as follows: In Section 1 the definition of semistability is given and several examples are discussed. In Section 2 a general purity law for semistable measures is proved (Theorem 1) which in particular yields that on a locally compact group a semistable continuous convolution semigroup is either absolutely continuous or continuous singular or degenerate. Moreover a general result related to zero-one laws is proved for semistable convolution semigroups (Theorem 2).

In Section 3 the concept of a quasi-analytic convolution semigroup is introduced; it is shown that on topological groups they admit a common support semigroup (Theorem 3). Moreover it is proved that semistable convolution semigroups are quasi-analytic provided they are not orthogonal (Theorem 4). Finally under some additional assumptions such a semigroup is even holomorphic (Theorem 5). In Section 4 we consider semistability on Euclidean spaces as an illustration. It is proved that a full semistable convolution semigroup on $\mathbb{R}^d$ is holomorphic (Theorem 6). Moreover the support of a full operator-semistable measure is studied more precisely.

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## PRELIMINARIES

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}_+^*$, $\mathbb{C}$ denote the sets of positive integers, integers, real numbers, positive real numbers, and complex numbers respectively. The cardinality of a set A we denote by $|A|$. If B is a subset of A then $1_B$ denotes the indicator function of B i. e. $1_B(x) = 1$ if $x \in B$ and 0 if $x \in A \backslash B$.

Let G be a group with identity $e$ and let $\mathscr{B}$ be a $\sigma$-field on G. If the mapping $(x, y) \to xy^{-1}$ of $(G \times G, \mathscr{B} \otimes \mathscr{B})$ into $(G, \mathscr{B})$ is measurable then $(G, \mathscr{B})$ is called a measurable group. $\mathscr{M}^b(G, \mathscr{B})$ denotes the vector space of real valued bounded signed measures defined on $\mathscr{B}$. Furnished with the norm $\|\cdot\|$ of total variation $\mathscr{M}^b(G, \mathscr{B})$ becomes a Banach space. Moreover with respect to its natural ordering $\mathscr{M}^b(G, \mathscr{B})$ becomes a Banach lattice; the absolute value of $\mu \in \mathscr{M}^b(G, \mathscr{B})$ is denoted by $|\mu|$.

If $m$ denotes the (measurable) mapping $(x, y) \to xy$ of $G \times G$ into G then the convolution $\mu * \nu$ of $\mu$, $\nu \in \mathscr{M}^b(G, \mathscr{B})$ is defined as the image of the product measure $\mu \otimes \nu$ with respect to $m$:

$$\mu * \nu(B) = \mu \otimes \nu(\{(x, y) \in G \times G : xy \in B\});$$

by Fubini's theorem we have $x^{-1}B$, $By^{-1} \in \mathscr{B}$ (all $x$, $y \in G$) and

$$\mu * \nu(B) = \int \nu(x^{-1}B)\mu(dx) = \int \mu(By^{-1})\nu(dy)$$

for all $B \in \mathscr{B}$. This multiplication turns $\mathscr{M}^b(G, \mathscr{B})$ into a Banach algebra.

$\mathcal{M}^1(G, \mathcal{B})$ denotes the subsemigroup (with respect to convolution) of probability measures in $\mathcal{M}^b(G, \mathcal{B})$. For every $x \in G$ the unit mass $\varepsilon_x$ in $x$ belongs to $\mathcal{M}^1(G, \mathcal{B})$. If $\mu \in \mathcal{M}^1(G, \mathcal{B})$ the adjoint $\tilde{\mu} \in \mathcal{M}^1(G, \mathcal{B})$ is defined by $\tilde{\mu}(B) = \mu(B^{-1})$, $B \in \mathcal{B}$; and $\mu$ is said to be symmetric if $\mu = \tilde{\mu}$. The orthogonality of $\mu$, $\nu \in \mathcal{M}^1(G, \mathcal{B})$ is denoted by $\mu \perp \nu$, their equivalence by $\mu \sim \nu$. If $\delta$ is a measurable mapping of $(G, \mathcal{B})$ into itself the image $\delta(\mu)$ of $\mu \in \mathcal{M}^1(G, \mathcal{B})$ is defined by $\delta(\mu)(B) = \mu(\delta^{-1}(B))$, $B \in \mathcal{B}$.

Let D denote an additive subsemigroup of $\mathbb{R}_+^*$ such that $t - s \in D$ if $s, t \in D$ and $s < t$. A convolution semigroup in $\mathcal{M}^1(G, \mathcal{B})$ (with index set D) is a family $(\mu_t)_{t \in D}$ in $\mathcal{M}^1(G, \mathcal{B})$ such that $\mu_s * \mu_t = \mu_{s+t}$ for all $s, t \in D$. If $D = \mathbb{R}_+^*$ we write $(\mu_t)_{t>0}$ instead of $(\mu_t)_{t \in D}$. The convolution semigroup $(\mu_t)_{t \in D}$ is said to be diffuse (discrete, or degenerate) if all the measures $\mu_t$ are diffuse (discrete, or degenerate i. e. unit masses).

If G is a topological Hausdorff group then $\mathfrak{B} = \mathfrak{B}(G)$ denotes the $\sigma$-field of Borel subsets of G and $\mathfrak{V}(G)$ the system of neighbourhoods of $e \in G$ which are in $\mathfrak{B}(G)$. Moreover Aut (G) denotes the group of topological automorphisms of G.

Obviously $(G, \mathfrak{B})$ is a measurable group. By $\mathcal{M}^b(G)$ (resp. $\mathcal{M}^1(G)$) we denote the subset of $\tau$-regular signed measures in $\mathcal{M}^b(G, \mathfrak{B})$ (resp. in $\mathcal{M}^1(G, \mathfrak{B})$). As is well known $\mathcal{M}^b(G)$ is a Banach lattice and a Banach algebra in itself; and $\mathcal{M}^1(G)$ is a sub-semigroup of $\mathcal{M}^1(G, \mathfrak{B})$. The support of $\mu \in \mathcal{M}^1(G)$ is denoted by supp $(\mu)$. A convolution semigroup $(\mu_t)_{t \in D}$ in $\mathcal{M}^1(G)$ is said to be continuous if D is dense in $\mathbb{R}_+^*$ and if $\lim_{t \downarrow 0, t \in D} \mu_t(U) = 1$ for all $U \in \mathfrak{V}(G)$. If G is a locally compact group $(\mu_t)_{t \in D}$ is said to be absolutely continuous (singular) if all the measures $\mu_t$ are absolutely continuous (singular) with respect to a left Haar measure $\lambda$ on G.

## 1. DEFINITION OF SEMISTABILITY AND EXAMPLES

Let $(G, \mathcal{B})$ be a measurable group and let $(\mu_t)_{t \in D}$ be a convolution semigroup in $\mathcal{M}^1(G, \mathcal{B})$. Moreover let $\delta$ be a measurable homomorphism of $(G, \mathcal{B})$ into itself and let $c \in ]0, 1[ \cup ]1, \infty[$.

DEFINITION. — The convolution semigroup $(\mu_t)_{t \in D}$ in $\mathcal{M}^1(G, \mathcal{B})$ is said to be *$\delta$-semistable* (in the strict sense) with *coefficient* $c$ if $cD \subset D$ and if $\delta(\mu_t) = \mu_{ct}$ for all $t \in D$.

*Remark.* — This definition has been suggested by A. Tortrat. In fact there exist more general concepts of semistability namely for probability measures that are not necessarily embedded into a convolution semigroup (cf. [*15*] [*23*] [*24*]).

EXAMPLES. 1. — Let G be a topological group. Following W. Hazod [*3*] a convolution semigroup $(\mu_t)_{t>0}$ in $\mathscr{M}^1(G)$ is said to be stable (in the strict sense) if there exists a subgroup $(\delta_t)_{t>0}$ of Aut (G) in the sense of $\delta_s \circ \delta_t = \delta_{st}$ for all $s, t > 0$ such that $\mu_t = \delta_t(\mu_1)$ for all $t > 0$. Obviously this implies $\delta_s(\mu_t) = \mu_{st}$ for all $s, t > 0$. Consequently $(\mu_t)_{t>0}$ is $\delta_s$-semistable with coefficient $s$ for every $s \in ]0, 1[ \cup ]1, \infty[$.

2. Let $\mu \in \mathscr{M}^1(\mathbb{R}^d)$ be a full operator-stable measure with exponent $A \in \mathrm{Aut}(\mathbb{R}^d)$ in the sense of M. Sharpe [*19*]. Then there exists a continuous mapping $b$ of $\mathbb{R}_+^*$ into $\mathbb{R}^d$ with $b(1)=0$ such the measures $\mu_t:=t^A(\mu)*\varepsilon_{b(t)}$, $t > 0$, constitute a convolution semigroup in $\mathscr{M}^1(\mathbb{R}^d)$.

It is well known that every $\mu_t$ is absolutely continuous ([*7*], Theorem 1). Now if 1 is not in the spectrum of A then there exists some $a \in \mathbb{R}^d$ such that $b(t) = ta - t^A a$ for all $t > 0$ ([*19*], Theorem 6). In fact if the expectation vector $E(\mu)$ of $\mu$ exists (which is the case iff the real part of every eigenvalue of A is less then 1 ; cf. [*12*]) then one has $a = E(\mu)$. Thus $(\mu_t * \varepsilon_{-ta})_{t>0}$ is a stable convolution semigroup (in the sense of example 1) which is absolutely continuous.

3. Let $(\nu_t)_{t>0}$ be a stable convolution semigroup in $\mathscr{M}^1(G)$ with respect to the automorphism group $(\delta_t)_{t>0}$ (in the sense of example 1). Let $H = G \times \mathbb{R}$, $\mu_t = \nu_t \otimes \varepsilon_t$ and $\sigma_t(x, r) = (\delta_t(x), tr)$ for all $(x, r) \in H$ and $t \in \mathbb{R}_+^*$. Then $(\mu_t)_{t>0}$ is a convolution semigroup in $\mathscr{M}^1(H)$, $(\sigma_t)_{t>0}$ is a subgroup of Aut (H), and $\sigma_t(\mu_1) = \mu_t$ for all $t > 0$. Thus $(\mu_t)_{t>0}$ is a stable convolution semigroup such that $\mathrm{supp}(\mu_t) \subset G \times \{ t \}$. Hence $\mu_s \perp \mu_t$ if $s \neq t$. Moreover if G is locally compact then the semigroup $(\mu_t)_{t>0}$ is singular ; if $(\nu_t)_{t>0}$ is absolutely continuous then $(\mu_t)_{t>0}$ is diffuse.

4. Let $(G, \mathscr{B})$ be an Abelian measurable group and let $\delta$ denote the measurable homomorphism $x \to 2x$ of $(G, \mathscr{B})$ into itself. Now let $\mu \in \mathscr{M}^1(G, \mathscr{B})$ be a symmetric Gaussian measure in the sense of T. Byczkowski [*1*] i. e. one has $\psi(\mu \otimes \mu) = (\mu * \mu) \otimes (\mu * \mu)$ where $\psi$ denotes the mapping $(x, y) \to (x + y, x - y)$ of $G \times G$ into itself. Obviously one has $\delta(\mu) = \mu^{*4}$ ; hence $(\mu^{*n})_{n \in \mathbb{N}}$ is a $\delta$-semistable convolution semigroup with coefficient 4 (cf. [*23*], p. 549).

Moreover if $\delta$ is injective and bimeasurable (i. e. $\delta(B) \in \mathscr{B}$ for all $B \in \mathscr{B}$) then there exists a unique convolution semigroup $(\mu_t)_{t \in D}$ in $\mathscr{M}^1(G, \mathscr{B})$, $D = \{ k/2^n : k, n \in \mathbb{N} \}$, such that every $\mu_t$ is a symmetric Gauss measure and such that $\mu_1 = \mu$ [*1*]. Then $(\mu_t)_{t \in D}$ is a $\delta$-semistable convolution semigroup with coefficient 4.

5. Let G be a locally convex topological vector space and let $r \in ]0, 1[$, $\alpha \in ]0, 2[$. Then by $\delta(x) = r^{1/\alpha}x$, $x \in G$, there is given a topological automorphism of G. Now let $\mu \in \mathscr{M}^1(G)$ be $r$-semistable with exponent $\alpha$ in the sense of [*2*] [*14*]. Then $\mu$ is embeddable into a continuous convolution semigroup $(\mu_t)_{t>0}$ in $\mathscr{M}^1(G)$ such that $\varepsilon_{x(t)} * \delta(\mu_t) = \mu_{rt}$ with appropriate $x(t) \in G$ for all $t > 0$. If $\alpha \neq 1$ then there exists some $x \in G$ such that $\delta(\mu_t * \varepsilon_{-tx}) = \mu_{rt} * \varepsilon_{-rtx}$ for all $t > 0$. Thus the convolution semigroup $(\mu_t * \varepsilon_{-tx})_{t>0}$ is $\delta$-semistable with coefficient $r$.

6. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, exponential mapping exp, and Hunt function $\varphi$ ([*5*], 4.2). Let $\delta \in \mathrm{Aut}(G)$ such that the associated automorphism Ad$(\delta)$ of $\mathfrak{g}$ has norm $\alpha < 1$. Choose $X \in \mathfrak{g}$ and $c \in ]\alpha, 1[$. Then put $x = \exp X$ and $\eta = \Sigma_{n \in \mathbb{Z}} c^{-n} \varepsilon_{\delta^n(x)}$. Obviously $\eta$ is a measure on $\mathfrak{B}(G)$ such that $\delta(\eta) = c\eta$.

It is not hard to prove that $\lim_{n \geq 1} \delta^n(x) = e$ and that $\varphi(\delta^k(x)) \leq \alpha^{2k}\beta$ for all $k \in \mathbb{N}$ and $x \in G$ (where $\beta \in \mathbb{R}_+^*$ is an appropriate constant). Hence one easily verifies that $\eta$ is a Lévy measure on G i. e. $\eta(\complement U) < \infty$ for all $U \in \mathfrak{B}(G)$ and $\int \varphi d\eta < \infty$ ([*5*], 4.4.14). In fact even $\int \sqrt{\varphi}\, d\eta < \infty$ holds. Hence by $A(f) := \int (f - f(e))d\eta$, $f \in \mathscr{D}(G)$, there is given the generating functional of a continuous convolution semigroup $(\mu_t)_{t>0}$ in $\mathscr{M}^1(G)$ ([*5*], 4.2.8). But $\delta(\eta) = c\eta$ implies $A(f \circ \delta) = cA(f)$, $f \in \mathscr{D}(G)$, and thus $\delta(\mu_t) = \mu_{ct}$ for all $t > 0$. Consequently $(\mu_t)_{t>0}$ is a $\delta$-semistable convolution semigroup with coefficient $c$.

7. We keep the notations of Example 6. Let $X \in \mathfrak{g}$, $X \neq 0$, and put $x_t = \exp tX$ as well as $\mu_t = \varepsilon_{x_t}$ for all $t > 0$. Moreover let $\delta \in \mathrm{Aut}(G)$ and $c \in ]0, 1] \cup ]1, \infty[$. Then the convolution semigroup $(\mu_t)_{t>0}$ is $\delta$-semistable with coefficient $c$ iff $\delta(x_t) = x_{ct}$ for all $t > 0$ and hence iff Ad$(\delta)X = cX$ i. e. iff X is an eigenvector of Ad$(\delta)$ for the eigenvalue $c$. Obviously the semigroup $(\mu_t)_{t>0}$ is degenerate.

## 2. PURITY OF SEMISTABLE CONVOLUTION SEMIGROUPS

We start with a rather general concept of purity. By way of specialization we then arrive at more familiar situations.

DEFINITION. — Let $\delta$ be a measurable homomorphism of the measurable group $(G, \mathscr{B})$ into itself and let $c \in ]0, 1[ \cup ]1, \infty[$. Moreover let $\mathbb{B}$ be a band of the Banach lattice $\mathscr{M}^b(G, \mathscr{B})$ (cf. [*18*], II.2) such that the orthogonal band $\mathbb{B}^\perp$ is in addition an ideal of the Banach algebra $\mathscr{M}^b(G, \mathscr{B})$.

Let us call $\mathbb{B}$ a $(\delta, c)$-band if « $c < 1$ and $\delta\mathbb{B}^\perp \subset \mathbb{B}^\perp$ » or « $c > 1$ and $\delta\mathbb{B} \subset \mathbb{B}$ ».

THEOREM 1. — Let $(\mu_t)_{t\in D}$ be a $\delta$-semistable convolution semigroup with coefficient $c$ in $\mathscr{M}^1(G, \mathscr{B})$ and let $\mathbb{B}$ be a $(\delta, c)$-band in $\mathscr{M}^b(G, \mathscr{B})$.

Then either $\mu_t \in \mathbb{B}$ for all $t \in D$ or $\mu_t \in \mathbb{B}^\perp$ for all $t \in D$.

*Proof.* — For every $t \in D$ let $\mu_t = \mu_t^{(1)} + \mu_t^{(2)}$ be the unique decomposition of $\mu_t$ such that $\mu_t^{(1)} \in \mathbb{B}$ and $\mu_t^{(2)} \in \mathbb{B}^\perp$. Since $\mathbb{B}^\perp$ is an ideal we have $\mu_{s+t}^{(1)} \leq \mu_s^{(1)} * \mu_t^{(1)}$ and hence

$$\| \mu_{s+t}^{(1)} \| \leq \| \mu_s^{(1)} \| \; \| \mu_t^{(1)} \| \leq \| \mu_t^{(1)} \|$$

for all $s, t \in D$. Consequently the mapping $t \to \| \mu_t^{(1)} \|$ is non-increasing.

If $c < 1$ then we have $\delta\mathbb{B}^\perp \subset \mathbb{B}^\perp$, hence $\delta(\mu_t^{(2)}) \leq \mu_{ct}^{(2)}$ and thus $\delta(\mu_t^{(1)}) \geq \mu_{ct}^{(1)}$. Consequently

$$\| \mu_{ct}^{(1)} \| \leq \| \delta(\mu_t^{(1)}) \| = \| \mu_t^{(1)} \| \leq \| \mu_{ct}^{(1)} \| .$$

If $c > 1$ then we have $\delta\mathbb{B} \subset \mathbb{B}$ and hence $\delta(\mu_t^{(1)}) \leq \mu_{ct}^{(1)}$. Consequently

$$\| \mu_{ct}^{(1)} \| \leq \| \mu_t^{(1)} \| = \| \delta(\mu_t^{(1)}) \| \leq \| \mu_{ct}^{(1)} \| .$$

Thus in any case we have $\| \mu_t^{(1)} \| = \| \mu_{ct}^{(1)} \|$ for all $t \in D$. Therefore $t \to \| \mu_t^{(1)} \|$ has to be constant $= \alpha \in [0, 1]$. But $\alpha = \| \mu_{s+t}^{(1)} \| \leq \| \mu_s^{(1)} \| \; \| \mu_t^{(1)} \| = \alpha^2$ yields $\alpha = 0$ or $\alpha = 1$. Hence the assertion. □

REMARK 1. — Bands $\mathbb{B}$ of $\mathscr{M}^b(G, \mathscr{B})$ such that $\mathbb{B}^\perp$ is an ideal and $\mathbb{B}$ is an algebra (so-called prime L-subalgebras) have been applied in the context of zero-one laws (and even purity laws) for convolution semigroups (cf. [*10*]). But there are bands $\mathbb{B}$ such that $\mathbb{B}^\perp$ is an ideal but $\mathbb{B}$ is not an algebra (cf. Lemma 2 below).

LEMMA 1. — Let H be a measurable normal subgroup of $(G, \mathscr{B})$ and let C be a cross section for the H-cosets. Moreover let $\delta$ be a measurable homomorphism of $(G, \mathscr{B})$ into itself and let $c \in ]0, 1[ \cup ]1, \infty[$. If $c < 1$ then let $(\delta^{-1}(H)H)/H$ be countable; if $c > 1$ then let $\delta^{-1}(H) \supset H$.

Then $\mathbb{B} = \{ \mu \in \mathscr{M}^b(G, \mathscr{B}) : \| \mu \| = \Sigma_{x\in C} | \mu | (xH) \}$ is a $(\delta, c)$-band and one has $\mathbb{B}^\perp = \{ \mu \in \mathscr{M}^b(G, \mathscr{B}) : | \mu | (xH) = 0 \text{ for all } x \in G \}$.

*Proof.* — Straightforward.

THEOREM 2. — Let $(G, \mathscr{B})$ be a measurable group such that $\{ x \} \in \mathscr{B}$ for all $x \in G$ and let $\delta$ be a measurable homomorphism of $(G, \mathscr{B})$ into itself. Moreover let $(\mu_t)_{t\in D}$ be a $\delta$-semistable convolution semigroup in $\mathscr{M}^1(G, \mathscr{B})$ such that some and hence each measure $\mu_t$ is discrete. If the coefficient $c$ of $(\mu_t)_{t\in D}$ is less than 1 then let $\delta$ be injective.

Then there exist a finite subgroup F of G and elements $x_t \in G$ such that $\mu_t = \varepsilon_{x_t} * \omega = \omega * \varepsilon_{x_t}$ for all $t \in D$ where $\omega = |F|^{-1}\Sigma_{x\in F}\varepsilon_x$ denotes the uniform distribution on F.

*Proof.* — For abbreviation we put $\mu(x) = \mu(\{ x \})$, $x \in G$, and $S(\mu) = \{ x \in G : \mu(x) > 0 \}$, $\mu \in \mathscr{M}^1(G, \mathscr{B})$.

1. Let $c < 1$; hence $\delta$ is injective by assumption. We put $\mu = \mu_1, \nu = \mu_{1-c}$; thus $\mu = \delta(\mu) * \nu$. Let $A = \{ a \in G : \mu(a) \geq \mu(x) \text{ for all } x \in G \}$. Obviously A is finite; put $k = |A|$. Now we fix $a \in A$. Then $\mu = \delta(\mu) * \nu$ yields :

$$\Sigma_{b\in G} \{ \mu(a) - \mu(\delta^{-1}(ab^{-1})) \} \nu(b) = 0 .$$

Hence $\delta^{-1}(ab^{-1}) \in A$ for all $b \in S(\nu)$. Consequently $|S(\nu)| \leq k$ in view of the injectivity of $\delta$.

Let $s\in D$ such that $s<1-c$ and put $t=(1-c)-s$. Then $\mu_s*\mu_t=\mu_{1-c}=\nu$ yields $S(\mu_s)S(\mu_t) = S(\nu)$; hence $|S(\mu_s)| \leq k$. Moreover $S(\mu_r)S(\mu_{r'}) = S(\mu_{r+r'})$ for all $r, r' \in D$ implies $|S(\mu_s)| < \infty$ for all $s \in D$ and the isotony of the mapping $s \to |S(\mu_s)|$. Finally

$$|S(\mu_{cs})| = |S(\delta(\mu_s))| = |\delta(S(\mu_s))| = |S(\mu_s)|, \qquad s \in D,$$

yields $|S(\mu_s)| = l \in \mathbb{N}$ for all $s \in D$; whence $l \leq k$.

On the other hand we have $A \subset S(\mu_1)$; hence $k = |A| \leq |S(\mu_1)| = l$ i. e. $k = l$ and consequently $A = S(\mu_1)$. But this yields $\mu_1 = \frac{1}{k}\Sigma_{a\in S(\mu)}\varepsilon_a$.

Starting with $\mu_s$ instead of $\mu_1$ now gives us $\mu_s = \frac{1}{k}\Sigma_{x\in S(\mu_s)}\varepsilon_x$ for all $s \in D$.

2. Let $c > 1$. Define $\mu$, A, $k$ as in 1. and put $\nu = \mu_{c-1}$. Then $\delta(\mu) = \mu * \nu$ yields for $a \in A$ :

$$\Sigma_{b\in G} \{ \mu(\delta^{-1}(\delta(a))) - \mu(\delta(a)b^{-1}) \} \nu(b) = 0 .$$

But $\mu(\delta(a)b^{-1}) \leq \mu(a) \leq \mu(\delta^{-1}(\delta(a)))$ implies $\delta(a)b^{-1} \in A$ for all $b \in S(\nu)$. Consequently $|S(\nu)| \leq k$.

As in 1. we conclude that the mapping $s \to |S(\mu_s)|$ is finite and increasing such that $|S(\mu_s)| \leq k$ if $s \leq c - 1$. Moreover

$$|S(\mu_s)| \geq |\delta(S(\mu_s))| = |S(\delta(\mu_s))| = |S(\mu_{cs})| \geq |S(\mu_s)|$$

yields $|S(\mu_s)| = l \leq k$ for all $s \in D$. Hence we arrive again at $\mu_s = \frac{1}{k}\Sigma_{x\in S(\mu_s)}\varepsilon_x$ for all $s \in D$.

3. Put $T_s = S(\mu_s)$ for all $s \in D$. Let $s, t \in D$ and $x \in T_t$. In view of 1. or 2. respectively we have $|T_s| = |T_{s+t}| = k$. Hence $T_s x \subset T_s T_t = T_{s+t}$ yields $T_s x = T_{s+t}$. Consequently

$$T_s T_s^{-1} = (T_s x)(T_s x)^{-1} = T_{s+t} T_{s+t}^{-1}.$$

Thus $F = T_s T_s^{-1}$ is independent of $s \in D$.

Moreover $T_t F T_t^{-1} = T_t T_s T_s^{-1} T_t^{-1} = T_{s+t} T_{s+t}^{-1} = F$ yields $xFy^{-1} = F$ and hence $xF = Fy$ for all $x, y \in T_t$ or $\in T_t^{-1}$, $t \in D$. Thus

$$F^2 = T_t T_t^{-1} F = T_t F T_t^{-1} = F.$$

Together with $F = F^{-1}$ this proves that F is a group.

4. Let $s \in D$ and $a, b \in T_s$. As seen in 3. we have $T_s b = T_{2s} = a T_s$. Hence

$$\begin{aligned} \mu_s * \tilde{\mu}_s(a^{-1}T_s) &= \Sigma_{b\in T_s} \mu_s(a^{-1}T_s b)\,\mu_s(b) \\ &= \Sigma_{b\in T_s} \mu_s(T_s)\,\mu_s(b) = 1. \end{aligned}$$

In view of $S(\mu_s * \tilde{\mu}_s) = F$ this yields $F \subset a^{-1}T_s$. But $|F| \geq |T_s|$ implies $aF = Fa = T_s$; hence the assertion. □

REMARKS. 2. — In the situation of Theorem 2 we have $\delta(F) = F$; hence $\delta$ operates as a permutation on F.

$$[\delta(F) = \delta(T_s T_s^{-1}) = \delta(T_s)\delta(T_s)^{-1} = T_{cs} T_{cs}^{-1} = F].$$

3. If D is a submonogeneous semigroup in $\mathbb{R}_+^*$ then the $x_t$ in Theorem 2 can be chosen in such a way that $x_s x_t = x_{s+t}$ for all $s, t \in D$ (cf. [5], 3.6.2).

4. In its generality Theorem 2 is the best result possible : Let F be a finite group (furnished with the discrete topology) and $\omega$ the uniform distribution on F. Then $G = \mathbb{R} \times F$ is a topological group. Let $\mu_t = \varepsilon_t \otimes \omega$ for all $t \in \mathbb{R}_+^*$ and $\delta_c(r, x) = (cr, x)$ for all $(r, x) \in G$ (and $c \in \mathbb{R}_+^*$). Then $(\mu_t)_{t>0}$ is a convolution semigroup in $\mathcal{M}^1(G)$ stable with respect to $(\delta_c)_{c>0}$ (cf. Example 1.1). Obviously the measures $\mu_t$ are discrete but non-degenerate.

COROLLARY 1. — Let $(G, \mathscr{B})$ be a measurable group and $\delta$ a measurable homomorphism of $(G, \mathscr{B})$ into itself. Moreover let H be a measurable normal subgroup of G such that $\delta(H) \subset H$. Finally let $(\mu_t)_{t\in D}$ be a $\delta$-semistable convolution semigroup in $\mathcal{M}^1(G, \mathcal{B})$ with coefficient $c$. If $c < 1$ we assume additionally $\delta^{-1}(H) = H$. Then one of the following two alternatives does hold :

*i)* $\mu_t(xH) = 0$ for all $x \in G$ and $t \in D$.

*ii)* There exist a measurable subgroup $H_1 \supset H$ of G such that $|H_1/H| = k$ is finite and elements $x_t \in G$ such that $\mu_t(x_t H_1) = 1$ and $x_t H_1 = H_1 x_t$ for all $t \in D$. Moreover if $H_1 = y_1 H \cup \ldots \cup y_k H$ then $\mu_t(x_t y_j H) = \frac{1}{k}$ for all $j = 1, \ldots, k$ and $t \in D$.

*Proof.* — Let $p$ denote the natural projection of G onto $\dot{G} = G/H$. Furnished with the $\sigma$-field $\mathcal{C} = \{ B' \subset \dot{G} : p^{-1}(B') \in \mathcal{B} \}$ one turns $\dot{G}$ into a measurable group such that $\{ p(x) \} \in \mathcal{C}$, $x \in G$. By $\dot{\delta} \circ p = p \circ \delta$ there is defined a measurable homomorphism $\dot{\delta}$ of $(\dot{G}, \mathcal{C})$ into itself; $\dot{\delta}$ is injective iff $\delta^{-1}(H) = H$. We put $\dot{x} = p(x)$, $x \in G$, and $\dot{\mu} = p(\mu)$, $\mu \in \mathcal{M}^1(G, \mathcal{B})$. Then $(\dot{\mu}_t)_{t \in D}$ is a $\dot{\delta}$-semistable convolution semigroup in $\mathcal{M}^1(\dot{G}, \mathcal{C})$. According to Theorem 1 and Lemma 1 either all the measures $\dot{\mu}_t$, $t \in D$, are diffuse or discrete. In the latter case we are in the situation of Theorem 2. Hence there exist a finite subgroup F of $\dot{G}$ and $x_t \in \dot{G}$ such that $\dot{\mu}_t = |F|^{-1} \Sigma_{\dot{x} \in F} \varepsilon_{\dot{x}_t \dot{x}}$, $t \in D$. Now the assertion follows with $H_1 = p^{-1}(F)$. □

REMARK 5. — Corollary 1 and Theorem 2 are closely related to a theorem of A. Tortrat ([*23*], p. 550). Under additional assumptions (concerning the group G, the homomorphism $\delta$, or the semigroup $(\mu_t)_{t \in D}$) Corollary 1 can be strengthened to a zero-one law i. e. one can arrive at $H_1 = H$ (cf. [*1*] [*10*] [*11*] [*15*] [*23*] [*24*]). In particular D. Louie, B. S. Rajput and A. Tortrat [*15*] have proved such a result in the situation of Corollary 1 (even for a weaker concept of semistability); but under additional assumptions for the homomorphism $\delta$.

COROLLARY 2. — Let G be a topological group and $\delta$ a measurable homomorphism of $(G, \mathfrak{B})$ into itself having a countable kernel. Moreover let $(\mu_t)_{t \in D}$ be a $\delta$-semistable continuous convolution semigroup in $\mathcal{M}^1(G)$ with coefficient $c$. If $c < 1$ we assume additionally $\delta$ to be injective.

Then either all the measures $\mu_t$, $t \in D$, are diffuse or degenerate.

*Proof.* — We apply Corollary 1 with $H = \{ e \}$. Hence we may restrict ourselves to the case that $\mu_t = \frac{1}{k} \Sigma_{x \in F} \varepsilon_{x_t x}$ where $x_t \in G$, $t \in D$, and F a finite subgroup of G with $k = |F|$. In view of [*20*], Lemma 2 we must have

$\lim_{t\downarrow 0} \mu_t(\{x_t\}) = 1$. But $\mu_t(\{x_t\}) = \frac{1}{k}$ for all $t \in D$ yields $k = 1$ hence $\mu_t = \varepsilon_{x_t}$. □

LEMMA 2. — Let G be a locally compact group and $\lambda$ a left Haar measure on G. Moreover let $\delta \in \mathrm{Aut}(G)$ and $c \in ]0, 1[ \cup ]1, \infty[$.

Then the space $\mathbb{B}$ of signed measures in $\mathcal{M}^b(G)$ that are singular with respect to $\lambda$ is a $(\delta, c)$-band; and $\mathbb{B}^\perp$ is the space of signed measures in $\mathcal{M}^b(G)$ that are absolutely continuous with respect to $\lambda$. Hence Theorem 1 applies to $\mathbb{B}$.

*Proof.* — As is well known $\mathcal{M}^b(G) = \mathbb{B} + \mathbb{B}^\perp$ is a band decomposition (cf. [*4*], (14.22)) and $\mathbb{B}^\perp$ is an ideal ([*4*], (19.18)). (But $\mathbb{B}$ is not an algebra in general.) Moreover there exists some $k \in \mathbb{R}_+^*$ such that $\delta(\lambda) = k\lambda$ ([*4*], (15.26)). Thus $\delta\mathbb{B} \subset \mathbb{B}$, $\delta\mathbb{B}^\perp \subset \mathbb{B}^\perp$. Hence the assertion. □

REMARK 6. — In view of Corollary 2 of Theorem 2, Lemma 2 and Theorem 1 a $\delta$-semistable continuous convolution semigroup on a locally compact group G with $\delta \in \mathrm{Aut}(G)$ is either absolutely continuous or diffuse and singular or discrete and then even degenerate. In fact there exist also examples for each of these three types (cf. Examples 1.2, 1.3, 1.7).

## 3. QUASI-ANALYTICITY AND HOLOMORPHY

Let E be a Banach space over $\mathbb{C}$, E′ its topological dual, and $(T(s))_{s>0}$ a semigroup of linear contractions on E. For every $u \in E$ and for every $\varphi \in E'$ we put $F_{u,\varphi}(s) = \varphi(T(s)u)$, $s \in \mathbb{R}_+^*$. By $\mathcal{F}$ we denote the family of those functions $F_{u,\varphi}$ that are continuous on $\mathbb{R}_+^*$ ($u \in E$, $\varphi \in E'$).

The semigroup $(T(s))_{s>0}$ is said to be $r$-quasi-analytic if the family $\mathcal{F}$ is quasi-analytic on the interval $\left]\frac{r}{2}, \infty\right[$ i. e. two functions in $\mathcal{F}$ coinciding on an open non-void subset of $\left]\frac{r}{2}, \infty\right[$ already coincide on $\left]\frac{r}{2}, \infty\right[$ ($r \in \mathbb{R}_+^*$). Moreover $(T(s))_{s>0}$ is said to be quasi-analytic if it is $r$-quasi-analytic for every $r \in \mathbb{R}_+^*$.

There is the following sufficient condition for $(T(s))_{s>0}$ to be $r$-quasi-analytic ([*17*], Theorem 2) :

(QA) There exist a sequence $(s(k))_{k\geq 1}$ in $\mathbb{R}_+^*$ converging to 0 and $h, \varepsilon \in \mathbb{R}_+^*$ such that for all $k \in \mathbb{N}$ one has

$$\sup \{ \| (T(s(k)) - I)^n \|^{1/n} : r \leq ns(k) \leq r + h \} \leq 2 - \varepsilon .$$

This criterion will be applied in the following situation: Let $(G, \mathscr{B})$ be a measurable group and let $E = \mathscr{L}(G, \mathscr{B})$ be the Banach space of complex valued bounded measurable functions on $(G, \mathscr{B})$ (furnished with the supremum norm). Moreover let $(\mu_t)_{t>0}$ be a convolution semigroup in $\mathscr{M}^1(G, \mathscr{B})$. For all $s \in \mathbb{R}_+^*$ we put $T(s)f(x) = \int f(xy)\mu_s(dy)$ $(f \in \mathscr{L}(G, \mathscr{B}), x \in G)$. Then $(T(s))_{s>0}$ is a semigroup of linear contractions on $\mathscr{L}(G, \mathscr{B})$. Now the convolution semigroup $(\mu_t)_{t>0}$ is called *quasi-analytic* if the associated operator semigroup $(T(s))_{s>0}$ is quasi-analytic. Moreover let us call $(\mu_t)_{t>0}$ *measurable* if $(T(s))_{s>0}$ is weakly measurable i. e. if $F_{u,\varphi}$ is measurable for all $u \in E$, $\varphi \in E'$.

Now let G be a topological group and let $\mathscr{C}_{lu}(G)$ denote the subspace of functions in $\mathscr{L}(G, \mathfrak{B})$ that are uniformly continuous with respect to the left uniform structure on G. If the convolution semigroup $(\mu_t)_{t>0}$ in $\mathscr{M}^1(G)$ is continuous then the associated operator semigroup $(T(s))_{s>0}$ is strongly continuous on $\mathscr{C}_{lu}(G)$ (cf. [5], 4.1.1). Hence we have $F_{u,\varphi} \in \mathscr{F}$ for every $u \in \mathscr{C}_{lu}$ and for all $\varphi \in E'$. The following result is a modified version of Theorem 1 in [*21*].

THEOREM 3. — Let G be a topological group and let $(\mu_t)_{t>0}$ be a quasi-analytic continuous convolution semigroup in $\mathscr{M}^1(G)$.

Then there exists a closed subsemigroup S of G with $e \in S$ such that $\operatorname{supp}(\mu_t) = S$ for all $t > 0$.

*Proof.* — We put $S_t = \operatorname{supp}(\mu_t)$ for all $t > 0$. Let us assume $e \notin S_t$ for some $t > 0$. Then there exists some $V \in \mathfrak{B}(G)$ such that $\mu_t(V^2) = 0$. We choose some $f \in \mathscr{C}_{lu}(G)$ such that $1_{\{e\}} \leq f \leq 1_V$. (The existence of such a function can be shown with the aid of Theorem 8.2 in [*4*].) Now $\mu_t(V^2) \geq \mu_s(V)\mu_{t-s}(V)$ implies $\int f d\mu_s = 0$ or $\int f d\mu_{t-s} = 0$ for all $s \in ]0, t[$. On the other hand the continuity of the convolution semigroup yields $\lim_{s\downarrow 0} \int f d\mu_s = f(e) = 1$. In view of $\int f d\mu_s = T(s)f(e)$ this contradicts the quasi-analyticity of the semigroup. Consequently $e \in S_r$ and thus $S_r \subset S_rS_s \subset S_{r+s}$ for all $r, s > 0$.

Now let us assume $S_r \neq S_{r+s}$ for some $r, s > 0$. Then there exist $x \in S_{r+s}$ and $V \in \mathfrak{B}(G)$ such that $Vx \cap S_r = \emptyset$. This time we choose some $f \in \mathscr{C}_{lu}(G)$ such that $1_{\{x\}} \leq f \leq 1_{Vx}$. Consequently $\int f d\mu_t = 0$ if $0 < t \leq r$ and

$\int f d\mu_{r+s} > 0$. Again this contradicts the quasi-analyticity. Hence the assertion. □

THEOREM 4. — Let $(G, \mathscr{B})$ be a measurable group and let $\delta$ be a measurable homomorphism of $(G, \mathscr{B})$ into itself. Moreover let $(\mu_t)_{t>0}$ be a $\delta$-semistable convolution semigroup with coefficient $c < 1$ in $\mathscr{M}^1(G, \mathscr{B})$. Finally let there exist $t > t' > 0$ such that $\| \mu_t - \mu_{t'} \| < 2$.

Then the convolution semigroup $(\mu_t)_{t>0}$ is quasi-analytic.

*Proof.* — We have to prove that $(\mu_t)_{t>0}$ is $r$-quasi-analytic for every $r > 0$. Thus let $r > 0$ be fixed. By assumption there exist $t > t' > 0$ and $0 < a < 2$ such that $\| \mu_t - \mu_{t'} \| = 2 - a$. Thus there are some $s > 0$ and $m \in \mathbb{N}$, $m \neq 1$, such that $\| \mu_{(m-1)s} - \mu_{ms} \| \leq 2 - a$. [Apply the same simple argument as in the proof of Theorem 4 in [*22*].]

Obviously there exist $\varepsilon \in ]0, 2[$ and $n_0 \in \mathbb{N}$, $n_0 \geq m$, such that $2^{m/n}(2^m - a)^{1/m} \leq 2 - \varepsilon$ for all $n \geq n_0$. Moreover there exists some $l \in \mathbb{N}$ such that $r \geq n_0 s c^l$. We put $s(k) = sc^{l+k}$ for all $k \in \mathbb{N}$ and $h = 1$. Then the semistability yields:

$$\begin{aligned}\|\mu_{(m-1)s(k)} - \mu_{ms(k)}\| &= \| \delta^{l+k}(\mu_{(m-1)s}) - \delta^{l+k}(\mu_{ms}) \| \\ &\leq \| \mu_{(m-1)s} - \mu_{ms} \| \leq 2 - a .\end{aligned}$$

Hence $\| (\mu_{s(k)} - \varepsilon_e)^m \| \leq 2^m - a$.

Given $k \in \mathbb{N}$ let $n \in \mathbb{N}$ such that $r \leq ns(k) \leq r + h$. Thus $n \geq n_0$. Moreover there are $p \in \mathbb{N}$, $q \in \{ 0, 1, \ldots, m - 1 \}$ such that $n = pm + q$. Consequently:

$$\begin{aligned}\| (\mu_{s(k)} - \varepsilon_e)^n \|^{1/n} &\leq 2^{q/n} \| (\mu_{s(k)} - \varepsilon_e)^m \|^{p/n} \\ &\leq 2^{m/n} \| (\mu_{s(k)} - \varepsilon_e)^m \|^{1/m} \leq 2^{m/n}(2^m - a)^{1/m} \leq 2 - \varepsilon .\end{aligned}$$

Let $(T(t))_{t>0}$ be the operator semigroup on $\mathscr{L}(G, \mathscr{B})$ associated with $(\mu_t)_{t>0}$. Obviously $\| (T(s(k)) - I)^n \| = \| (\mu_{s(k)} - \varepsilon_e)^n \|$. Hence the assertion follows from (QA). □

REMARK 1. — The proof of Theorem 4 is a modification of the proof of Theorem 3 in [*22*].

Combining Theorems 3 and 4 we now obtain the following result:

COROLLARY. — Let G be a topological group and let $\delta$ be a measurable homomorphism of $(G, \mathfrak{B})$ into itself. Moreover let $(\mu_t)_{t>0}$ be a $\delta$-semistable continuous convolution semigroup with coefficient $c < 1$ in $\mathscr{M}^1(G)$. Finally let there exist $t > t' > 0$ such that $\| \mu_t - \mu_{t'} \| < 2$.

Then there exists a closed subsemigroup S of G with $e \in S$ such that $\operatorname{supp}(\mu_t) = S$ for all $t > 0$.

REMARK 2. — There exist (semi-) stable continuous convolution semigroups $(\mu_t)_{t>0}$ with common support semigroup and such that $\|\mu_s - \mu_t\| = 2$ for all $s, t > 0$ with $s \neq t$. For example let G be the topological direct product of countably many copies of $\mathbb{R}$. Moreover let $(\nu_t)_{t>0}$ be a stable continuous convolution semigroup in $\mathcal{M}^1(\mathbb{R})$ with common support $\mathbb{R}$ (e. g. a Gaussian semigroup) and let $\mu_t$ be the product of countably many copies of $\nu_t$, $t > 0$. Then it is easy to check that $(\mu_t)_{t>0}$ is a stable continuous convolution semigroup in $\mathcal{M}^1(G)$ with $\operatorname{supp}(\mu_t) = G$ for all $t > 0$. But $\mu_s \perp \mu_t$ if $s \neq t$ by Kakutani's theorem.

Let $(G, \mathcal{B})$ be a measurable group and $(\mu_t)_{t>0}$ a convolution semigroup in $\mathcal{M}^1(G, \mathcal{B})$. Roughly speaking $(\mu_t)_{t>0}$ is said to be *holomorphic* if the mapping $t \to \mu_t$ of $\mathbb{R}_+^*$ into the Banach space $\mathcal{M}^b(G, \mathcal{B})$ is holomorphic. For the exact definition we refer to [*22*].

THEOREM 5. — Let $\delta$ be a measurable homomorphism of $(G, \mathcal{B})$ into itself and let $(\mu_t)_{t>0}$ be a $\delta$-semistable convolution semigroup with coefficient $c < 1$ in $\mathcal{M}^1(G, \mathcal{B})$. Then the following assertions are equivalent:

*i)* $t \to \mu_t$ is a continuous mapping of $\mathbb{R}_+^*$ into the Banach space $\mathcal{M}^b(G, \mathcal{B})$ such that $\|\mu_s - \mu_t\| < 2$ for all $s, t > 0$.

*ii)* $(\mu_t)_{t>0}$ is measurable; moreover there exist $a, b, d \in \mathbb{R}_+^*$ such that $c \geq 2a/b$ and such that $\|\mu_s - \mu_t\| \leq 2 - d$ for all $s, t \in [a, b]$.

*iii)* $(\mu_{t>0})$ is holomorphic.

*Proof.* — '*(i)* ⇒ *(ii)*' is evident.

'*(ii)* ⇒ *(iii)*'. Let $r \in \left[a, \frac{b}{2}\right]$ and $k \in \mathbb{N} \cup \{0\}$. Then

$$\|\mu_{c^k r} - \mu_{2c^k r}\| = \|\delta^k(\mu_r) - \delta^k(\mu_{2r})\| \\ \leq \|\mu_r - \mu_{2r}\| \leq 2 - d .$$

In view of $c \geq 2\frac{a}{b}$ we have $\cup \left\{ \left[c^k a, c^k \frac{b}{2}\right] : k \geq 0 \right\} = \left]0, \frac{b}{2}\right]$ and thus $\|\mu_s - \mu_{2s}\| \leq 2 - d$ for all $s \in \left]0, \frac{b}{2}\right]$. Consequently $\|(\mu_s - \varepsilon_e)^2\| \leq 4 - d$ for all $s \in \left]0, \frac{b}{2}\right]$. Hence the assertion in view of [*22*], Criterion 1.

'*(iii)* ⇒ *(i)*'. This implication holds for every holomorphic convolution semigroup (cf. [*22*], Section 3). □

COROLLARY. — Let G be a locally compact group and let $\delta$ be a measurable homomorphism of (G, $\mathfrak{B}$) into itself. Moreover let $(\mu_t)_{t>0}$ be a $\delta$-semistable continuous convolution semigroup with coefficient $c < 1$ in $\mathcal{M}^1(\mathrm{G})$. Finally let the semigroup $(\mu_t)_{t>0}$ be absolutely continuous.

Then the convolution semigroup $(\mu_t)_{t>0}$ is holomorphic.

*Proof.* — Since $(\mu_t)_{t>0}$ is absolutely continuous the mapping $t \to \mu_t$ of $\mathbb{R}_+^*$ into the Banach space $\mathcal{M}^b(\mathrm{G})$ is continuous. (This is well known; it may be proved by applying Theorem 20.4 in [*4*] to the densities of the measures $\mu_t$.) Moreover by the Corollary above all the measures $\mu_t$ have a common support which is a semigroup containing $e$. Hence in view of [*9*], Satz III.2.8, we have $\mu_s \sim \mu_t$ and thus $\| \mu_s - \mu_t \| < 2$ for all $s, t > 0$. Hence the assertion in view of Theorem 5. □

## 4. SEMISTABILITY ON EUCLIDEAN SPACES

A measure $\mu \in \mathcal{M}^1(\mathbb{R}^d)$ is said to be *full* if its support is not contained in an affine hyperplane of $\mathbb{R}^d$. If $\mu$ is not full then it can be easily shown that there exist a proper linear subspace V of $\mathbb{R}^d$ and a vector $a$ in the orthogonal complement of V such that $\mu$ is the direct product of its projection $\nu$ onto V and of the point measure $\varepsilon_a$ where $\nu$ is full on V. Thus only full measures will be considered in the sequel.

Now let $\mu \in \mathcal{M}^1(\mathbb{R}^d)$ be full and infinitely divisible with corresponding convolution semigroup $(\mu_t)_{t>0}$. Moreover let B be a linear operator on $\mathbb{R}^d$, $b \in \mathbb{R}^d$ and $c \in \mathbb{R}_+^* \setminus \{1\}$. $\mu$ is said to be (B, $b$, $c$)-decomposable if $\mu_c = \mathrm{B}(\mu) * \varepsilon_b$. Since with $\mu$ also $\mu_c$ is full first of all B has to be non-singular. Moreover it follows $\mu_{ct} = \mathrm{B}(\mu_t) * \varepsilon_{tb}$ for all $t > 0$ and hence $\mu_{1/c} = \mathrm{B}^{-1}(\mu) * \varepsilon_{-\mathrm{B}^{-1}b/c}$. Thus $\mu$ is also ($\mathrm{B}^{-1}$, $-\mathrm{B}^{-1}b/c$, $1/c$)-decomposable. Without loss of generality it will therefore always be assumed that $0 < c < 1$.

Every full operator-semistable probability measure on $\mathbb{R}^d$ is (B, $b$, $c$)-decomposable for an appropriate triple (B, $b$, $c$) (cf. Introduction). Conversely every full (B, $b$, $c$)-decomposable probability measure on $\mathbb{R}^d$ is operator-semistable (cf. [*8*] [*16*]).

Let $\mu \in \mathcal{M}^1(\mathbb{R}^d)$ be full and (B, $b$, $c$)-decomposable with corresponding convolution semigroup $(\mu_t)_{t>0}$. Moreover let there exist some $a \in \mathbb{R}^d$ such that $(\mathrm{B} - c\mathrm{I})a = b$. [Sufficient conditions are (*i*) $c$ is no eigenvalue of B; (*ii*) the expectation vector $\mathrm{E}(\mu)$ of $\mu$ exists (which is the case iff $|\beta| > c$ for all eigenvalues $\beta$ of B [*12*] [*16*]): then $a = -\mathrm{E}(\mu)$ fulfills the equation.] Then one has $\mathrm{B}(\mu_t * \varepsilon_{ta}) = \mu_{ct} * \varepsilon_{cta}$ for all $t > 0$; i. e. $(\mu_t * \varepsilon_{ta})_{t>0}$ is a B-semistable convolution semigroup with coefficient $c$. Hence the results on semistable semigroups also apply to most of the operator-semistable measures.

THEOREM 6. — Let $(\mu_t)_{t>0}$ be a $\delta$-semistable continuous convolution semigroup in $\mathcal{M}^1(\mathbb{R}^d)$ such that some (and hence each) measure $\mu_t$ is full. Then $(\mu_t)_{t>0}$ is a holomorphic semigroup. Moreover $\delta$ is a topological automorphism of $\mathbb{R}^d$.

*Proof.* — In view of [*4*], (22.18) the homomorphism $\delta$ is continuous and hence linear. Since the measures $\mu_t$ are full $\delta$ has to be surjective. Hence $\delta \in \mathrm{Aut}(\mathbb{R}^d)$. Thus $(\mu_t)_{t>0}$ is also $\delta^{-1}$-semistable. Hence without loss of generality the coefficient of $(\mu_t)_{t>0}$ can be chosen less than 1.

According to [*16*], Theorem 2.2 every measure $\mu_t$ is absolutely continuous. Hence the first assertion follows from the Corollary of Theorem 5. □

Let $(\mu_t)_{t>0}$ as in Theorem 6. Then by the Corollary following Theorem 4 there exists a closed subsemigroup S of $\mathbb{R}^d$ with $0 \in S$ such that $\mathrm{supp}(\mu_t) = S$ for all $t > 0$. For the rest of the section this support semigroup will be studied to some extent. At first some preparations:

LEMMA 3. — Let G be a locally compact group with a left Haar measure $\lambda$ and let $(\mu_t)_{t>0}$ be a convolution semigroup in $\mathcal{M}^1(G)$ such that every $\mu_t$ admits a bounded and continuous $\lambda$-density $f_t$. Put $S_t = \mathrm{supp}(\mu_t)$, $P_t = \{ x \in G : f_t(x) > 0 \}$ and denote by $K_t$ the interior of $S_t$ $(t > 0)$.

Then $K_t = P_t$. Moreover $S_t$ is the closure of $K_t$ and $\mu_t(K_t) = 1$ (all $t > 0$).

*Proof.* — First of all $P_t \subset S_t$ implies $P_t \subset K_t$. Assume that there exists some $x \in K_t \backslash P_t$. This yields with $s := t/2$:

$$0 = f_t(x) = \int f_s(xy) f_s(y^{-1}) \lambda(dy) .$$

Consequently $f_s(xy)=0$ and hence $xy \notin P_s$ for all $y \in P_s^{-1}$. Thus $x \notin P_s P_s \subset K_t$. But this is a contradiction. Hence $K_t = P_t$. Now $\mu_t(P_t) = 1$ yields the remaining assertions. □

An *angular semigroup* A in $\mathbb{R}^d$ is an open subset of $\mathbb{R}^d$ with $0 \in \overline{A}$ (where the bar means closure) and $A + A \subset A$. Obviously every open convex cone in $\mathbb{R}^d$ with vertex 0 is an angular semigroup. If $d = 1$ these are the ontly ones. But for $d > 1$ there exist also other examples; e. g. $\{ (x, y) \in \mathbb{R}^2 : x > 0, |y| < x^2 \}$ is an angular semigroup in $\mathbb{R}^2$. Further informations about angular semigroups can be found in [6], 8.7. The following result is probably well known:

LEMMA 4. — An angular semigroup A in $\mathbb{R}^d$ is connected.

*Proof.* — Apply induction following the dimension $d$. If $d = 1$ the assertion is true in view of the remarks above. Let $d > 1$. There exists some $b \in \mathbb{R}^d$, $b \neq 0$, such that $A + \rho b \subset A$ for all $\rho \geq 0$ ([6], 8.7.4). For every $x \in A$ put $[x] = \{ x + \rho b : \rho \in \mathbb{R} \} \cap A$ and $[x, \sigma] = \{ x + \rho b : \rho \geq \sigma \}$ $(\sigma \in \mathbb{R})$. Since $[x]$ is the union of all subsets $[x, \sigma]$ such that $\sigma \leq 0$ and $x + \sigma b \in A$ the set $[x]$ is connected. Let $p$ denote the projection of $\mathbb{R}^d$ onto the hyperplane H orthogonal to the vector $b$ and containing 0. Then $p(A)$ is an angular semigroup in H hence is connected by the induction assumption. Together with $[x] = p^{-1}(p(x)) \cap A$, $x \in A$, this yields the assertion. □

THEOREM 7. — Let $(\mu_t)_{t>0}$ be a $\delta$-semistable convolution semigroup in $\mathcal{M}^1(\mathbb{R}^d)$ such that every measure $\mu_t$ is full. Then there exists an angular semigroup A in $\mathbb{R}^d$ such that $\delta(A) = A$, $\mu_t(A) = 1$ and $\overline{A} = \operatorname{supp}(\mu_t)$ for all $t > 0$. Moreover every $\mu_t$ is on $\overline{A}$ equivalent to the Lebesgue measure $\lambda^d$ on $\mathbb{R}^d$.

*Proof.* — According to the remark following Theorem 6 there exists a closed subsemigroup S of $\mathbb{R}^d$ with $0 \in S$ such that $\operatorname{supp}(\mu_t) = S$ for all $t > 0$. Since $\delta \in \operatorname{Aut}(\mathbb{R}^d)$ (in view of Theorem 6) we must have $\delta(S) = S$.

Let A denote the interior of S. Obviously $\delta(A) = A$. Moreover every measure $\mu_t$ admits a bounded and continuous $\lambda^d$-density ([16], Theorem 2.2). Thus Lemma 3 yields $\overline{A} = S$ and $\mu_t(A) = 1$ for all $t > 0$. Furthermore $A + A \subset S$ implies $A + A \subset A$; hence A is an angular semigroup.

In view of Lemma 3 the measures $\mu_t$ and $\lambda^d$ are equivalent on A. But as is well known the boundary of an angular semigroup in $\mathbb{R}^d$ is a zero set for $\lambda^d$ (cf. [9], Lemma III.2.6). Hence the last assertion follows. □

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