ANNALES DE L'I. H. P., SECTION B

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Annales de l'I. H. P., section B, tome 19, nº 1 (1983), p. 91-100

<http://www.numdam.org/item?id=AIHPB_1983__19_1_91_0>

Ann. Inst. Henri Poincaré, Section B :
Vol. XIX, n° 1, 1983, p. 91-100. *Calcul des Probabilités et Statistique.*

Exchanging the order of taking suprema and countable intersections of σ-algebras

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SUMMARY. — Let $\mathcal{F}, \mathcal{G}_n \downarrow$ be σ-algebras in a probability space $(\Omega, \mathcal{A}, P)$. We give necessary and sufficient conditions for the equality

$$\mathcal{F} \vee \bigcap_{n=1}^{\infty} \mathcal{G}_n = \bigcap_{n=1}^{\infty} \mathcal{F} \vee \mathcal{G}_n$$

to hold up to sets of measure 0. Roughly speaking they say that the tail behaviour described by $\bigcap_{n=1}^{\infty} \mathcal{G}_n$ should not depend too much on the $\mathcal{F}$-part.

RÉSUMÉ. — Soient $\mathcal{F}, \mathcal{G}_n \downarrow$ des tribus dans un espace probabilisé $(\Omega, \mathcal{A}, P)$. On donne des conditions nécessaires et suffisantes pour l'égalité

$$\mathcal{F} \vee \bigcap_{n=1}^{\infty} \mathcal{G}_n = \bigcap_{n=1}^{\infty} \mathcal{F} \vee \mathcal{G}_n \mod P.$$

L'idée essentielle de ces conditions est que l'influence de $\mathcal{F}$ sur le comportement asymptotique décrit par $\bigcap_{n=1}^{\infty} \mathcal{G}_n$ soit modéré.

1. INTRODUCTION

Let $(\Omega, \mathcal{A}, P)$ be a probability space. Let $\mathcal{F}$ and $\mathcal{G}_n$ $(n \in \mathbb{N})$ be sub σ-algebras of $\mathcal{A}$ such that $\mathcal{G}_1 \supseteq \mathcal{G}_2 \supseteq \ldots$. Then generally the σ-algebra $\mathcal{F} \vee \bigcap_{n=1}^{\infty} \mathcal{G}_n$

is strictly smaller than $\bigcap_{n=1}^{\infty} \mathcal{F} \vee \mathcal{G}_n$ even modulo P-nullsets. A number of problems in different probabilistic situations are related to this fact. Some instances are mentioned at the end of this note. Our aim is to reformulate the phenomenon in other measure theoretic terms and to contribute in this way to a better under-standing of such situations. The main idea is that the two σ-algebras are equal if and only if the tail behaviour described by $\bigcap_{n=1}^{\infty} \mathcal{G}_n$ does not depend too much on the $\mathcal{F}$-part.

2. NOTATION

For a σ-algebra $\mathcal{B}$ the symbol $\mathcal{B}^b$ denotes the set of all real bounded $\mathcal{B}$-measurable functions on the underlying space. Let $(\Omega, \mathcal{A}, \mathrm{P})$ be a probability space. If $\mathcal{G}$, $\mathcal{H}$ are sub-σ-algebras of $\mathcal{A}$ we write $\mathcal{G} = \mathcal{H} \bmod \mathrm{P}$ whenever $\mathcal{G}$ and $\mathcal{H}$ induce the same sets of P-equivalence classes of sets. Suppose $\mathcal{F}$ and $\mathcal{G}$ are sub-σ-algebras of $\mathcal{A}$ and that $(\mathrm{P}_\omega^{\mathcal{F}})_{\omega \in \Omega}$ is a conditional probability kernel on $\mathcal{G}$ given $\mathcal{F}$, i. e. $\mathrm{P}_\omega^{\mathcal{F}}(\cdot)$ is a probability measure on $\mathcal{G}$ for each ω and $\mathrm{P}_\cdot^{\mathcal{F}}(\mathrm{G})$ is a version of $\mathrm{E}^{\mathcal{F}}(1_\mathrm{G})$ for each $\mathrm{G} \in \mathcal{G}$. We say that G is $\mathrm{P}^{\mathcal{F}}$-*separable* if there is a countably generated sub-σ-algebra $\mathcal{H}$ of $\mathcal{G}$ such that $\mathcal{G} = \mathcal{H} \bmod \mathrm{P}_\omega^{\mathcal{F}}$ for P-almost all ω.

3. THE RESULT

THEOREM. — Let $(\Omega, \mathcal{A}, \mathrm{P})$ be a probability space. Let $\mathcal{F}$ and $\mathcal{G}_n$ $(n \in \mathbb{N})$ be sub-σ-algebras of $\mathcal{A}$, the $\mathcal{G}_n$ being decreasing with intersection $\mathcal{G}_\infty$. Let $\mathcal{G}^0$ be a generating system of $\mathcal{G}_1^b$ as a monotone class. Consider the following conditions where in *c*)-*e*) we assume a conditional probability kernel $(\mathrm{P}_\omega^{\mathcal{F}})_{\omega \in \Omega}$ on $\mathcal{G}_1$ given $\mathcal{F}$ to be fixed:

a) $$\bigcap_{n=1}^{\infty} \mathcal{F} \vee \mathcal{G}_n = \mathcal{F} \vee \mathcal{G}_\infty \mod \mathrm{P}$$

b) For every g in $\mathcal{G}^0$ and every $\varepsilon > 0$ there is a finite dimensional subset $\mathcal{R}$ of $\mathcal{F}^b$ and a uniformly bounded sequence (h_n) such that for each n

i) $$h_n \in (\mathcal{F} \otimes \mathcal{G}_n)^b$$

ii) $$h_n(\cdot, \omega) \in \mathcal{R} \quad \text{for every} \quad \omega \in \Omega.$$

iii) $$\int_\Omega | h_n(\omega, \omega) - \mathrm{E}^{\mathcal{F} \vee \mathcal{G}_n}(g)(\omega) | \, d\mathrm{P}(\omega) < \varepsilon.$$

c) For every $g \in \mathscr{G}^0$ there is some h in $(\mathscr{F} \otimes \mathscr{G}_\infty)^b$ such that

$$h(\omega, .) = \mathrm{E}^{\mathscr{G}_\infty}_{\mathrm{P}^{\mathscr{F}}_\omega}(g) \quad \mathrm{P}^{\mathscr{F}}_\omega\text{-a. e.} \qquad \text{for P-almost all } \omega. \tag{1}$$

(In this case for every h in $(\mathscr{F} \otimes \mathscr{G}_\infty)^b$ the relation (1) is equivalent to $h(\omega, \omega) = \mathrm{E}^{\mathscr{F} \vee \mathscr{G}_\infty}(g)(\omega)$ P-a. e.).

d) $\mathscr{G}_\infty$ is $\mathrm{P}^{\mathscr{F}}$-separable.

e) $\mathrm{P}^{\mathscr{F}}_\omega$ is trivial on $\mathscr{G}_\infty$ for P-almost every ω.

Then *a)* und *b)* are equivalent. If $\mathscr{G}_n$ is $\mathrm{P}^{\mathscr{F}}$-separable (e. g. countably generated) for all n, then *a)*-*d)* are equivalent. If in addition P is trivial on $\mathscr{G}_\infty$ then all five conditions are equivalent.

Let us illustrate these statements by the following elementary (counter-) example.

Example. — Suppose $(\xi_n)_{n \geqq 1}$ is a sequence of independent nontrivial (e. g. coin tossing) random variables on $(\Omega, \mathscr{A}, \mathrm{P})$. Let $\mathscr{F}$ be the σ-algebra generated by the increments $\mathscr{F} = \sigma\{\xi_{m+1} - \xi_m : m \geqq 1\}$, and let $\mathscr{G}_n$ be the « future after time n »: $\mathscr{G}_n = \sigma\{\xi_m : m \geqq n\}$. Then ξ_1 is $\bigcap_{n=1}^{\infty} \mathscr{F} \vee \mathscr{G}_n$-measurable since for each n

$$\xi_1 = - \sum_{m=1}^{n-1} (\xi_{m+1} - \xi_m) + \xi_n. \tag{2}$$

But ξ_1 is not $\mathscr{F} \vee \mathscr{G}_\infty$-measurable even outside a P-nullset since $\mathscr{F} = \mathscr{F} \vee \mathscr{G}_\infty$ mod P by Kolmogorov's 0-1 law. Under the conditional law $\mathrm{P}^{\mathscr{F}}_\infty$ on $\mathscr{G}_1$ given the increments, the starting value of ξ_1 has a nontrivial distribution, but according to (2) given the increments all other ξ_n and hence the tail behaviour is in a 1-1 correspondence to the starting value ξ_1. Hence the conditional law is also nontrivial on $\mathscr{G}_\infty$, i. e. *e)* is violated. Further, for different realizations of the increment process these laws $\mathrm{P}^{\mathscr{F}}_\omega$ behave quite differently on $\mathscr{G}_\infty$ which makes at least plausible that *c)* and *d)* do not hold. Finally equation (2) shows also that in the representation of ξ_1 as a $\mathscr{F} \vee \mathscr{G}_n$-measurable function the dependence on the $\mathscr{F}$ part varies strongly with n. An approximation of ξ_1 $(= \mathrm{E}^{\mathscr{F} \vee \mathscr{G}_n}(\xi_1))$ as indicated in *b)* would contradict this variation, so *b)* fails.

4. THE PROOF

For the sake of expository convenience we assume that Ω is a product space $\mathrm{X} \times \mathrm{Y}$, considering products $\mathscr{F} \otimes \mathscr{G}_*$ rather than suprema $\mathscr{F} \vee \mathscr{G}_*$

(where $\mathscr{G}_* \subseteq \mathscr{G}_1$). This is a particular case of the situation in the theorem but in fact the general case may easily be deduced to this situation via the measurable mapping $\omega \mapsto (\omega, \omega)$ from $(\Omega, \mathscr{A})$ to $(\Omega \times \Omega, \mathscr{F} \otimes \mathscr{G}_1)$. The conditional kernel $(P_\omega^{\mathscr{F}})$ in the theorem may now be substituted by a kernel $(P_x)_{x \in X}$ from $(X, \mathscr{F})$ to $(Y, \mathscr{G}_1)$ satisfying the generalized Fubini formula

$$\int_X \left\{ \int_Y f(x, y) dP_x(y) \right\} dP^1(x) = \int_{X \times Y} f(x, y) dP(x, y) \qquad (3)$$

for all f in $(\mathscr{F} \otimes \mathscr{G}_1)^b$ where P^1 is the first marginal of P.

I. Proof of the equivalence *a*) $\Leftrightarrow$ *b*).

I.1. *a* $\Rightarrow$ *b*. Suppose $\bigcap_{n=1}^{\infty} \mathscr{F} \otimes \mathscr{G}_n = \mathscr{F} \otimes \mathscr{G}_\infty$ mod P. Fix g in $\mathscr{G}^0$ and $\varepsilon > 0$. Let $\tilde{g}$ denote the function $(x, y) \mapsto g(y)$. By decreasing martingale convergence we have

$$E^{\mathscr{F} \otimes \mathscr{G}_n}(\tilde{g}) \to E^{\bigcap_{n=1}^{\infty} \mathscr{F} \otimes \mathscr{G}_\infty}(\tilde{g}) = E^{\mathscr{F} \otimes \mathscr{G}_\infty}(\tilde{g}) \quad \text{in} \quad \mathscr{L}^1(P).$$

Choose n_0 such that $\| E^{\mathscr{F} \otimes \mathscr{G}_n}(\tilde{g}) - E^{\mathscr{F} \otimes \mathscr{G}_\infty}(\tilde{g}) \|_1 < \frac{\varepsilon}{2}$ for all $n > n_0$. For $n \leqq n_0$ choose a finite subalgebra $\mathscr{F}_n$ of $\mathscr{F}$ such that $\| h_n - E^{\mathscr{F} \otimes \mathscr{G}_n}(\tilde{g}) \|_1 < \varepsilon$ where $h_n = E^{\mathscr{F}_n \otimes \mathscr{G}_n}(\tilde{g})$. This is possible by increasing martingale convergence. Similarly choose a finite subalgebra $\mathscr{F}_\infty$ of $\mathscr{F}$ such that $\| h_\infty - E^{\mathscr{F} \otimes \mathscr{G}_\infty}(\tilde{g}) \|_1 < \frac{\varepsilon}{2}$ where $h_\infty = E^{\mathscr{F}_\infty \otimes \mathscr{G}_\infty}(\tilde{g})$. For $n > n_0$ let h_n be the function h_∞. Finally let $\mathscr{R}$ be the finite dimensional space $\left[\bigvee_{n=1}^{n_0} \mathscr{F}_n \vee \mathscr{F}_\infty \right]^b$. Then for each n we have $h_n \in (\mathscr{F} \otimes \mathscr{G}_n)^b$, $h_n(\,.\,, y) \in \mathscr{R}$ for every $y \in Y$, and $\| h_n - E^{\mathscr{F} \otimes \mathscr{G}_n}(\tilde{g}) \|_1 < \varepsilon$. This proves *b*) or rather its substitute in the setting of this proof.

I.2. *b*) $\Rightarrow$ *a*). We want to prove $E^{\mathscr{F} \otimes \mathscr{G}_\infty}(a) = E^{\bigcap_{n=1}^{\infty} \mathscr{F} \otimes \mathscr{G}_n}(a)$ for every element a of $\mathscr{A}^b$. However once this assertion has been proved under one of the following assumptions it may easily be extended to the next, more general one :

$a(x, y) = g(y)$ for some $g \in \mathscr{G}^0$; $a(x, y) = g(y)$ for some $g \in \mathscr{G}_1^b$;
$a(x, y) = f(x)g(y)$ for some $f \in \mathscr{F}^b$, $g \in \mathscr{G}_1^b$; $a \in (\mathscr{F} \otimes \mathscr{G}_1)^b$; $a \in \mathscr{A}^b$.

We may therefore assume that $a(x, y) = g(y)$ for some $g \in \mathcal{G}^0$ and hence by condition b) that for every $\varepsilon > 0$ there is a finite dimensional subspace $\mathcal{R}$ of $\mathcal{F}^b$ and a uniformly bounded sequence $(h_n)_{n \geqq 1}$ such that for each n, $h_n \in (\mathcal{F} \otimes \mathcal{G}_n)^b$, $h_n(.\,, y) \in \mathcal{R}$ for all y and

$$(4) \qquad \| h_n - \mathrm{E}^{\mathcal{F} \otimes \mathcal{G}_n}(a) \|_1 \leqq \varepsilon .$$

Fix ε, $\mathcal{R}$ and (h_n).

Let d be the dimension of $\mathcal{R}$ and choose $x_1, \ldots, x_d$ in X so that the evaluations at these points form a basis of the dual of the linear space $\mathcal{R}$. Let $(f^1, \ldots, f^d)$ be the dual basis of $\mathcal{R}$, i. e. $f^i(x_j) = \delta_{ij}$ for $i, j \in \{ 1, \ldots, d \}$. Then $h_n(.\,, y) = \sum_{i=1}^{d} h_n(x_i, y) f^i$ for every y, i. e. $h_n = \sum_{i=1}^{d} h_n^i \cdot (f^i \circ \pi_{\mathrm{X}})$ where $h_n^i(x, y) = h_n(x_i, y)$. The sequence $\langle h_n^i \rangle$, being uniformly bounded, has a $\sigma(\mathcal{L}^\infty(\mathrm{P}), \mathcal{L}^1(\mathrm{P}))$-limit point h_∞^i which may be chosen to be $\{ \emptyset, \mathrm{X} \} \otimes \mathcal{G}_\infty$-measurable. Then for every $g \in \mathcal{L}^\infty(\mathrm{P})$ $\mathrm{E}(h_n g) = \sum_{i=1}^{d} \mathrm{E}(h_n^i (f^i \circ \pi_{\mathrm{X}}) g)$ has $\sum_{i=1}^{d} \mathrm{E}(h_\infty^i (f^i \circ \pi_{\mathrm{X}}) g) = \mathrm{E}(h_\infty g)$ as limit point where

$$h_\infty = \sum_{i=1}^{d} h_\infty^i (f^i \circ \pi_{\mathrm{X}}) \in (\mathcal{F} \otimes \mathcal{G}_\infty)^b .$$

Passing in (4) to the limit we conclude $\| h_\infty - \mathrm{E}^{\bigcap_{n=1}^{\infty} \mathcal{F} \otimes \mathcal{G}_n}(a) \|_1 \leqq \varepsilon$. Such an h_∞ exists for every $\varepsilon > 0$. This implies $\mathrm{E}^{\bigcap_{n=1}^{\infty} \mathcal{F} \otimes \mathcal{G}_n}(a) = \mathrm{E}^{\mathcal{F} \otimes \mathcal{G}_\infty}(a)$, completing the proof.

II. **Proof of the equivalence** $a) \Leftrightarrow c) \Leftrightarrow d)$.

II.1. We need the following lemma which is concerned with the question of when $\mathrm{P}^{\mathcal{F}}$-separability is hereditary. The proof is an adaption of well known arguments.

LEMMA. — Let $\mathcal{G}_*$ be a sub-σ-algebra of the $\mathrm{P}^{\mathcal{F}}$-separable σ-algebra $\mathcal{G}_1$. Then for every generating system $\mathcal{G}^0$ of $\mathcal{G}_1^b$ the following are equivalent

i) $\mathcal{G}_*$ is $\mathrm{P}^{\mathcal{F}}$-separable.

ii) For every $g \in \mathscr{G}^0$ there is some k in $(\mathscr{F} \oplus \mathscr{G}_*)^b$ such that

(5) $$k(x, .) = \mathrm{E}^{\mathscr{G}_*}_{\mathrm{P}_x^{\mathscr{F}}}(g) \qquad \text{for } \mathrm{P}^1\text{-almost all } x.$$

(*iii*) For every $g \in \mathscr{G}^0$ and every k in $(\mathscr{F} \otimes \mathscr{G}_*)^b$ satisfying $k = \mathrm{E}^{\mathscr{F} \otimes \mathscr{G}_*}(g)$, (5) holds.

Proof. — 1. *i*) $\Rightarrow$ *iii*). Assume $\mathscr{G}_*$ to be $\mathrm{P}^{\mathscr{F}}$-separable. Let $\mathscr{H}_* \subset \mathscr{G}_*$ be countably generated such that $\mathscr{H}_* = \mathscr{G}_*$ mod $\mathrm{P}_x^{\mathscr{F}}$ for P^1-almost all x. Let $k \in (\mathscr{F} \otimes \mathscr{G}_*)^b$ satisfy $k = \mathrm{E}^{\mathscr{F} \otimes \mathscr{G}_*}(g)$. Then

$$\int_{\mathrm{F}} \int_{\mathrm{H}} k(x, y) d\mathrm{P}_x^{\mathscr{F}}(y) d\mathrm{P}^1(x) = \int_{\mathrm{F}} \int_{\mathrm{H}} g(y) d\mathrm{P}_x^{\mathscr{F}}(y) d\mathrm{P}^1(x)$$

for all $\mathrm{F} \in \mathscr{F}$, $\mathrm{H} \in \mathscr{H}_*$, i. e.

$$\int_{\mathrm{H}} k(x, y) d\mathrm{P}_x^{\mathscr{F}}(y) = \int_{\mathrm{H}} g(y) d\mathrm{P}_x^{\mathscr{F}}(y) \qquad \text{for } \mathrm{P}^1\text{-almost all } x$$

whenever $\mathrm{H} \in \mathscr{H}_*$. Since $\mathscr{H}_*$ is countably generated there is a P^1-nullset N such that $k(x, .) = \mathrm{E}^{\mathscr{H}_*}_{\mathrm{P}_x^{\mathscr{F}}}(g)$ for all $x \notin \mathrm{N}$. Because of $\mathscr{H}_* = \mathscr{G}_*$ mod $\mathrm{P}_x^{\mathscr{F}}$, k then satisfies the same relation for $\mathscr{G}_*$ instead of $\mathscr{H}_*$.

2. *iii*) $\Rightarrow$ *ii*) is obvious.

3. *ii*) $\Rightarrow$ *i*). First note that the assertion in *ii*) easily carries over to all $g \in \mathscr{G}_1$ by a straightforward monotone class argument. Let $\mathscr{H} \subset \mathscr{G}_1$ be countably generated such that $\mathscr{H} = \mathscr{G}_1$ mod $\mathrm{P}_x^{\mathscr{F}}$ for all $x \notin \mathrm{N}_1$ where N_1 is a P^1-nullset. Fix a countable subset $\{ h_m \}_{m \in \mathbb{N}}$ of $\mathscr{H}^b$ which generates $\mathscr{H}^b$ as a monotone class. For each m choose some k_m in $(\mathscr{F} \otimes \mathscr{G}_*)^b$ satisfying $k_m(x, .) = \mathrm{E}^{\mathscr{G}_*}_{\mathrm{P}_x^{\mathscr{F}}}(h_m)$ for P^1-almost all x. Then all functions $k_m(x, .)$, $(m \in \mathbb{N}, x \in \mathrm{X})$ are measurable with respect to a countably generated sub-σ-algebra $\mathscr{H}_*$ of $\mathscr{G}_*$, since every product σ-algebra is the union of its countably generated sub-product-σ-algebras. Then $\mathrm{E}^{\mathscr{G}_*}_{\mathrm{P}_x^{\mathscr{F}}}(h_m) = \mathrm{E}^{\mathscr{H}_*}_{\mathrm{P}_x^{\mathscr{F}}}(h_m)$ for all $m \in \mathbb{N}$ and $x \notin \mathrm{N}_2$ where N_2 is another P^1-nullset. If $x \notin \mathrm{N}_1 \cup \mathrm{N}_2$ it follows that $\mathrm{E}^{\mathscr{G}_*}_{\mathrm{P}_x^{\mathscr{F}}}(g) = \mathrm{E}^{\mathscr{H}_*}_{\mathrm{P}_x^{\mathscr{F}}}(g)$ for all $g \in \mathscr{H}_1^b$ and hence all $g \in \mathscr{G}_1^b$. Thus $\mathscr{H}_* = \mathscr{G}_*$ mod P_x since $\mathscr{G}_* \subset \mathscr{G}_1$.

II.2. If $\mathscr{G}_1$ is $\mathrm{P}^{\mathscr{F}}$-separable the equivalence of *c*) and *d*) is now obvious in view of the preceding lemma.

Proof of a) $\Leftrightarrow$ *c*). — Suppose that each $\mathscr{G}_n$ is $\mathrm{P}^{\mathscr{F}}$-separable. Fix $g \in \mathscr{G}^0$. Let k_n be such that $k_n = \mathrm{E}^{\mathscr{F} \otimes \mathscr{G}_n}(g)$ and define k by $k = \overline{\lim}_{n \to \infty} k_n$. Then according to the lemma $k_n(x, .) = \mathrm{E}^{\mathscr{G}_n}_{\mathrm{P}_x^{\mathscr{F}}}(g)$ and hence by decreasing martingale convergence $k(x, .) = E^{\mathscr{G}_\infty}_{P^{\mathscr{F}}_x}(g)$ for P^1-almost all x. According to (3) therefore for every $h \in (\mathscr{F} \otimes \mathscr{G}_\infty)^b$ the two conditions

$$h = k \bmod P \tag{6}$$

and

$$h(x, .) = E^{\mathscr{G}_\infty}_{P^{\mathscr{F}}_x}(g) \quad \text{for} \quad P^1\text{-almost all } x \tag{7}$$

are equivalent.

Now if *a*) holds, i. e. $\bigcap_{n=1}^{\infty} \mathscr{F} \otimes \mathscr{G}_n = F \otimes \mathscr{G}_\infty \bmod P$, there is some h in $(\mathscr{F} \oplus \mathscr{G}_\infty)^b$ with property (6). Then h satisfies (7) which implies *c*).

If conversely *c*) holds there is some $h \in (\mathscr{F} \otimes \mathscr{G}_\infty)^b$ satisfying (7) and hence (6). Thus $k = E^{\bigcap_{n=1}^{\infty} \mathscr{F} \otimes \mathscr{G}_n}(g)$ coincides P.-a. e. with a $\mathscr{F} \otimes \mathscr{G}_\infty$ measurable function, i. e. $E^{\mathscr{F} \otimes \mathscr{G}_\infty}(g) = E^{\bigcap_{n=1}^{\infty} \mathscr{F} \otimes \mathscr{G}_n}(g)$. Since this is true for every $g \in \mathscr{G}^0$, condition *a*) follows via the same argument as in the beginning of I.2.

III. **Proof of *d*) $\Leftrightarrow$ *e*).**

III.1. If *e*) holds then $\mathscr{G}_\infty$ is $P^{\mathscr{F}}$-separable since then $\mathscr{G}_\infty = \{\emptyset, \Omega\}$ mod $P^{\mathscr{F}}_\omega$ for P-almost all ω.

III.2. Suppose that P is trivial on $\mathscr{G}_\infty$ and thai $\mathscr{H} = \mathscr{G}_\infty$ mod $P^{\mathscr{F}}_\omega$ for P-almost all ω where $\mathscr{H} = \sigma\{H_m\}$ is a countably generated sub-σ-algebra of G_∞. Then P is also trivial on $\mathscr{H}$, i. e. P is concentrated on the atom $A = \bigcap_{m:P(H_m)=1} H_m \cap \bigcap_{m:P(H_m)=0} \Omega/H_m$ of $\mathscr{H}$. Then $P^{\mathscr{F}}_\omega(A) = 1$ for P-almost all ω. These $P^{\mathscr{F}}_\omega$ then are trivial on $\mathscr{H}$ and hence P-almost all of them also trivial on $\mathscr{G}_\infty$, i. e. *e*) holds. This completes the proof of the theorem.

5. **RELATED QUESTIONS**

In this section we first give references to some situations in which the problem under consideration is closely related to the question whether $\mathscr{F} \vee \mathscr{G}_\infty = \bigcap_{n=1}^{\infty} \mathscr{F} \vee \mathscr{G}_n$ mod P for some specified σ-algebras.

5.1. **Natural filtrations.**

In [*9*] a right continuous process $(X_t)_{t \geqq 0}$ is constructed such that $\mathscr{F}_1 \vee \mathscr{G}_{0+} \subsetneqq \mathscr{F}_{1+}$ mod P where

$$\mathscr{F}_t = \sigma\{ X_s : 0 \leqq s \leqq t \} \qquad \text{and} \qquad \mathscr{G}_t = \sigma\{ X_{1+s} : 0 \leqq s \leqq t \}.$$

Clearly this is a problem of our type with $\mathscr{F} = \mathscr{F}_1, \mathscr{G}_n = \mathscr{G}_{\frac{1}{n}}$. It is known that (X_t) cannot be Markovian. Here is a short proof of this fact using our theorem: Consider $\mathscr{F}' = \sigma\{X_1\}$. Since $\mathscr{F}' \subset \mathscr{G}_{0+} = \mathscr{G}_\infty$ we have

$$\mathscr{F}' \vee \mathscr{G}_{0+} = \bigcap_{n=1}^{\infty} \mathscr{F}' \vee \mathscr{G}_{\frac{1}{n}}.$$

On the other hand by the Markov property the conditional distributions $P^{\mathscr{F}_1}$ and $P^{\mathscr{F}}$ coincide on $\mathscr{G}_{0+}$. By the equivalence *a*) $\Leftrightarrow$ *d*) of the theorem we thus have also

$$\mathscr{F}_1 \vee \mathscr{G}_{0+} = \bigcap_{n=1}^{\infty} \mathscr{F}_1 \vee \mathscr{G}_{\frac{1}{n}} = \mathscr{F}_{1+} \text{ mod P.}$$

5.2. **Global Markov property of lattice fields.**

On $\Omega = \{0, 1\}^{\mathbb{Z}^d}$ consider the σ-fields $\mathscr{F}_\Lambda$ generated by the projections $\omega \mapsto \omega_{|\Lambda}$ where $\Lambda \subset \mathbb{Z}^d$. Let G_U be the convex set of all Gibbs states associated with a translation invariant nearest neighbour potential U. It is still unknown whether every extreme point P of $\mathscr{G}_U$ has the *global Markov property*: for a hyperplane Λ_0 in $\mathbb{Z}^d$ the field on one of the two corresponding halfspaces is independent of the behaviour on the other halspace, given $\mathscr{F}_{\Lambda_0}$. The answer is « yes », if and only if

$$\text{(1)} \qquad \mathscr{F}_{\Lambda_0} \vee \bigcap_{\Lambda \text{ finite}} \mathscr{F}_{\mathbb{Z}^d \setminus \Lambda} = \bigcap_{\Lambda \text{ finite}} \mathscr{F}_{\mathbb{Z}^d \setminus \Lambda} \text{ mod P}$$

The « only if » can be seen by the implication *e*) $\Leftrightarrow$ *a*) of our theorem. We would like to point out that sometimes the global Markov property for non-extreme states can be deduced from the corresponding statement for extreme states, namely if conditionally on $\mathscr{F}_{\Lambda_0}$ the field is extreme (like in the 2-dimensional Ising model). For examples without the global M. p. of locally Markov field with trivial tail σ-algebra see [*14*] and [*7*], p. 59. Concrete positive results are contained in [*1*], [*2*], [*3*].

5.3. **Generalized Markov fields.**

For an open set $\mathcal{U} \subset \mathbb{R}$ let $\mathcal{F}(\mathcal{U})$ be the σ-algebra on the space $\Omega = \mathcal{D}'$ of one dimensional distributions which is generated by $\{ \langle \cdot, \varphi \rangle : \varphi \in \mathcal{D}, \text{supp } \varphi \subset \mathcal{U} \}$. For arbitrary $C \subset \mathbb{R}$ let $\mathcal{F}(C)$ be the σ-algebra $\cap \{ \mathcal{F}(\mathcal{U}) : C \subset \mathcal{U}, \mathcal{U} \text{ open} \}$. It is shown in [6] that $\mathcal{F}((-\infty, 0]) \vee \mathcal{F}((0, \infty)) \subseteq \mathcal{F}(\mathbb{R})$ mod P if P is the tight probability on $\mathcal{D}'$ induced by the canonical Gaussian cylindrical measure of the Sobolev Hilbert space $W^{1,2}(\mathbb{R})$. This example shows that some natural definitions of Markov property do not coincide. It corrects some earlier statements concerning generalized Markov fields ([5], Theorems 1 and 8, [10], Lemma 2).

5.4. **The innovation problem.**

Let $(X_t)_{t \in T}$ be a realvalued process with $T \subset \mathbb{R}$ satisfying

$$X_{t+1} - X_t = a(t, X) + \eta_t \quad \text{(discrete time)}$$

or

$$dX_t = a(t, X)dt + dW_t \quad \text{(continuous time)}$$

where for each t the noise (« innovation ») η_t resp. dW_t is independent of $\mathcal{G}_t = \sigma \{ X_s : s \leq t \}$ and the functional $a(t, X)$ is $\mathcal{G}_t$-measurable.

Even if the tail field $\mathcal{G} = \bigcap_{t \in T} \mathcal{G}_t$ is trivial the answer may be negative to the *innovation problem* i. e. whether X is measurable with respect to the σ-field $\mathcal{F}$ generated by the innovation process. In the case $T = \mathbb{Z}$ this is shown e. g. by a straightforward modification of our introductory example (even with $a(t, X) \equiv 0$). For $T = (0, \infty)$ the problem is much harder, a corresponding counterexample has been given in [13]. In both cases X is $\mathcal{F} \vee \mathcal{G}_t$-measurable for each t. Similar examples had been studied already in [12] for finite state space Markov chains and for diffusions with $T = \mathbb{R}$ in [4], p. 72.

5.5. **The symmetric problem.**

We have not been able to find satisfactory analoga to our theorem for the question whether $(\cap \mathcal{F}_n) \vee (\cap \mathcal{G}_n) = \cap (\mathcal{F}_n \vee \mathcal{G}_n)$ mod P where $(\mathcal{F}_n)$ also is a decreasing sequence. Even for finite state space S there are ergodic stationary processes for which both the future and past tailfields are trivial but the two-sided tailfield is equal to the full σ-algebra modulo nullsets [11]. This cannot happen for (even nonstationary) Markov fields on $S^{\mathbb{Z}}$ [15].

5.6. An example of the phenomenon studied in this paper which in some sense is uniform but lives on an infinite measure space is given in [8].

ACKNOWLEDGEMENT

I am grateful for useful hints and discussions to the following friends and colleagues: M. Ershov, N. Falkner, H. Föllmer, H.-O. Georgii, R. Kotecky, H. Künsch, D. Maharam-Stone, D. Preiss, M. Scheutzow, G. Winkler, M. Yor.

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(Manuscrit reçu le 23 mars 1982)