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Probability estimates for symmetric simple exclusion random walks

by

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ABSTRACT. — Finitely many symmetric random walks in $\mathbb{Z}$ interacting by simple exclusion are considered. It is assumed that $\sum_{x \in \mathbb{Z}} x^2 \rho(x) < \infty$ where $\rho(x)$ is the probability that a single random walk jumps by x. A coupling Q of this process and the corresponding free one (where the exclusion condition is dropped) is established so that

$$Q(\{ | x_i(t) - x_i^{(0)}(t) | \leqslant t^\alpha \ i = 1, \dots, n \}) \geqslant 1 - At^{-\gamma}$$

$$\alpha > 1/4 \quad \gamma < \alpha\beta - 1/2 \quad \beta : \sum_{x \in \mathbb{Z}} | x |^\beta \rho(x) < \infty \qquad (\beta \geqslant 2)$$

where n is the number of particles, $x_i(t)$ $[x_i^{(0)}(t)]$ the position at time t of particle i in the interacting [free] process and A may depend on α, γ and n.

RÉSUMÉ. — On considère un nombre fini de marches aléatoires symétriques sur $\mathbb{Z}$ qui interagissent par exclusion simple. On suppose que $\sum_{x \in \mathbb{Z}} x^2 \rho(x) < \infty$ où $\rho(x)$ est la probabilité pour l'une quelconque des marches aléatoires de faire un saut de longueur x. On définit un couplage Q

entre ce processus et le processus indépendant qui lui correspond (c'est-à-dire sans condition d'exclusion) avec la propriété :

$$Q(\{\, | x_i(t) - x_i^{(0)}(t) | \leqslant t^\alpha \;\; i = 1, \ldots, n \,\}) \geqslant 1 - At^{-\gamma}$$

$$\alpha > 1/4 \quad \gamma < \alpha\beta - 1/2 \quad \beta : \sum_{x \in \mathbb{Z}} | x |^\beta \rho(x) < \infty \qquad (\beta \geqslant 2)$$

où n est le nombre de particules, $x_i(t)\,[x_i^{(0)}(t)]$ la position au temps t de la particule i dans le processus avec interaction [indépendant] et A peut dépendre de α, γ et n.

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1. INTRODUCTION AND RESULTS

A system of identical particles randomly moving in $\mathbb{Z}$ with simple exclusion interaction is self dual when the probability of jumps is symmetric [*6*]. In such a case the n-correlation functions of the time dependent measure describing a system with infinitely many particles can be computed in terms of a process of only n symmetric simple exclusion random walks. As a consequence questions of physical interest like approach to global equilibrium, see for instance [*6*], or hydrodynamical behavior (local equilibrium distributions), see [*4*], [*5*], [*10*], are reduced to the study of a process with finitely many particles. The relevant probability estimates often exploit the closeness between the interacting system and the free one [i. e. when the simple exclusion condition is dropped], see for instance [*1*], [*4*], [*5*], [*10*]. In these papers however the estimates were obtained under a short range condition on the length of the jumps, actually for symplicity only the case ± 1 was considered over there. In this paper we study the long range case in the assumption that the probability $\rho(x)$ of a jump by x, $x \in \mathbb{Z}$, is such that

$$\rho(x) = \rho(-x) \qquad \text{(symmetric walks)} \qquad (1.1\,a)$$

$$\sum_{x \in \mathbb{Z}} x^2 \rho(x) = \langle x^2 \rangle < \infty \qquad \text{(finite second moment)} \qquad (1.1\,b)$$

We prove that there exists a coupling between the free and the interacting processes, so that the typical distances between corresponding particles will grow less than $t^{1/4+\varepsilon}$, for any positive $\varepsilon > 0$, as t diverges. In the remaining of this section we quote precisely the results we obtained and we sketch the ideas of the proofs. In the nex section we will give the details.

The most natural way to compare the interacting and the free processes is may be to use a « invariance principle » argument. As in [5], we renormalize space and time:

$$\xi_i^{(\mathrm{T})}(\tau) = \frac{1}{\sqrt{\mathrm{T}}} x_i(\tau \mathrm{T}); \qquad i = 1, \ldots, n; \qquad 0 \leqslant \tau \leqslant 1; \qquad \mathrm{T} > 0 \quad (1.2)$$

Here $x_i(t)$, $i = 1, \ldots, n$ denote the positions of the interacting particles, T is the parameter which is going to diverge, τ is the « renormalized » time, $\xi_i^{(\mathrm{T})}(\tau)$ the rescaled process. $\vec{\xi}^{(\mathrm{T})}(\tau) = (\xi_i^{(\mathrm{T})}(\tau), \ldots, \xi_n^{(\mathrm{T})}(\tau))$ is a collection of processes in $\mathscr{D}([0, 1], \mathbb{R}^n)$, see for instance [2], and we prove the following theorem along the lines of the analogous result in [5]:

THEOREM 1.1. — $\vec{\xi}^{(\mathrm{T})}(\tau)$ converges weakly in $\mathscr{D}([0, 1], \mathbb{R}^n)$ to $\vec{b}(\tau)$ where $\vec{b}(\tau) = (b_1(\tau), \ldots, b_n(\tau))$ is the process of n independent brownian particles, its generator being $\frac{1}{2}\langle x^2 \rangle \Delta$, $\langle x^2 \rangle$ is defined in eq. (1.1 *b*).

Theorem 1.1 makes it plausible that a coupling exists between the free and the interacting processes so that corresponding particles are typically at distances less than $t^{1/2+\varepsilon}$ (here we use that the free particles process converges to the brownian one by the invariance principle). This estimate has been strengthened in [4] where in the short range case a coupling was introduced for which particles remain $t^{\frac{1}{4}+\varepsilon}$ close. Our main result is the following improvement of Theorem 1.1.

THEOREM 1.2. — There is a joint representation (coupling) Q of the simple exclusion process $\vec{x}(t)$ (its generator is written in eq. (1.6) below) and the free one $\vec{x}^{(0)}(t)$ (its generator is defined in eq. (1.8) below) such that

$$\mathrm{Q}(\{ \mid x_i(t) - x_i^{(0)}(t) \mid \leqslant t^\alpha, \ i = 1, \ldots, n \}) \geqslant 1 - \mathrm{A}t^{-\gamma} \qquad (1.3)$$

where

$$\sum |x|^\beta \rho(x) < \infty \qquad \beta \geqslant 2 \qquad (1.4)$$

$$\gamma < \alpha\beta - \frac{1}{2}, \qquad \alpha > \frac{1}{4} \qquad (1.5)$$

and A may depend on α, γ and n but is independent of t.

If there exists $\mathrm{G} > 0$ and G' such that

$$\rho(x) \leqslant \mathrm{G}' e^{-\mathrm{G}|x|}$$

then there is G″ so that

$$Q(\{|x_i(t)-x_i^{(0)}(t)| \leqslant t^\alpha \ i=1,\ldots,n\}) \geqslant 1-G''e^{-t^\varepsilon}; \ \varepsilon < \frac{1}{5}\left(\alpha - \frac{1}{4}\right) \quad (1.3b)$$

The ergodic and hydrodynamic properties of the « infinite system » are consequence of Theorem 1.2 via its

COROLLARY. — Let $P_{\vec{x},t}[P^{(0)}_{\vec{x},t}]$ denote the distribution probability for n particles at time t starting from $\vec{x} = x_1, \ldots, x_n$ according to the simple exclusion [independent] process, then

$$\lim_{t\to\infty} \| P_{\vec{x},t} - P^{(0)}_{\vec{x},t} \| = 0$$

where $\|.\|$ denotes the variational norm.

The proof of the Corollary is completely analogous to that in [4] and so we just sketch the main idea. This is to couple the independent and interacting processes as in Theorem 1.2 up to time T where $t - T = t^{\frac{1}{2}+4\varepsilon}$, $\varepsilon > 0$, small enough. Then Theorem 1.2 states that independent and interacting particles with same label are $\lesssim t^{\frac{1}{4}+\varepsilon}$ far apart, while the independent particles are at mutual distance of order $t^{\frac{1}{2}}$. In the remaining time, $t - T$, therefore the interacting particles behave as free (with probability going to 1 as t diverges) and it is then possible to couple the two processes by stating that particles with same label move independently until they meet, after that they remain together. The corollary follows then from classical estimates on the return time to the origin for independent random walks.

We remark that the main point in this argument is that particles with same label are much closer to each other than to those with different labels: this is ensured by the estimates of Theorem 1.2, could not be obtained just by use of Theorem 1.1. On the other hand the Corollary, by use of duality, entails an alternative proof of the ergodic properties for the infinite system [6] and it enables to prove the local equilibrium structure of the model: we refer to [10] for a more comprehensive description, here we simply outline the main definitions. Denote by $\mathscr{E}$ the set of the extremal invariant measures (i. e. Bernoulli) for the infinite system, let μ be an initial distribution for the infinite system and μ_t its time evolution at time t. Then

$$\lim_{t\to\infty} d(\mu_t, \mathscr{E}) = 0$$

where d is a metric on the probability measures given by

$$d(\mu, \nu) = \sup_{x\in\mathbb{Z}} \bar{d}(D_x\mu, D_x\nu)$$

D_x is the space translation operator acting on measures, $\bar{d}$ is a distance equivalent to the weak topology, for instance

$$\bar{d}(\mu, \nu) = \sum_{n=1}^{\infty} 2^{-n} \| \mu \mid \Lambda_n, \nu \mid \Lambda_n \|$$

where $\| . \|$ is the variational norm, $\Lambda_n = \{ -n, \ldots, n \}$ and $\mu \mid \Lambda_n$ is the relativization of μ to the algebra generated by the $\eta(x)$, $x \in \Lambda_n$ and $\eta(x)$ denotes the occupation random variable at site x, i. e. $\eta(x) = 1, 0$ if x is occupied, empty. The physical meaning of the above condition is linked to the hydrodynamical behavior of the system: it states the local equilibrium property that in each bounded region and at any (large enough) time the system looks like an equilibrium one, its parameter varying in general with space and time (according to the « hydrodynamical equations » for the system [*10*]). The assumptions on μ for the result to hold are that for each $n \in \mathbb{N}$ there is a positive decreasing function $\varphi_n(x)$, $x \in \mathbb{N}$, $\lim_{x \to \infty} \varphi_n(x) = 0$ such that for all $n \in \mathbb{N}$ $x_1, \ldots, x_n \in \mathbb{Z}^n$

$$\left| \mu(\{ \eta(x_i) = 1, i = 1, \ldots, n \}) - \prod_{i=1}^{n} \mu(\{ \eta(x_i) = 1 \}) \right| \leqslant \varphi_n(\min_{x_i \neq x_j} | x_i - x_j |)$$

The result is derived rather straightforwardly by using duality and the above Corollary to Theorem 1.2.

We conclude this section with a heuristic argument for the proof of Theorem 1.2.

The coupling Q is the long range version of the coupling defined in [*4*] for the short range case. The generator A of the simple exclusion process can be written in the following way. Let $f: \mathbb{Z}^n \to \mathbb{R}$ then

$$(Af)(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sum_{y : y \neq x_i} p(y - x_i)[f(x_1, \ldots, x_n)(x_i, y) - f(x_1, \ldots, x_n)] \tag{1.6}$$

where $(x_1, \ldots, x_n)(x_i, y) \in \mathbb{Z}^n$ is defined by

$$(x_1, \ldots, x_n)(x_i, y) = \begin{cases} (x_1, ., x_{i-1}, y, x_{i+1}, ., x_n) \text{ if } y \notin \{ x_1, ., x_n \} & (1.7\,a) \\ (x_1, ., x_n) \text{ if } y \in \{ x_1, ., x_{i-1} \} & (1.7\,b) \\ (x_1, ., x_{i-1}, y, ., x_{j-1}, x_i, .) \text{ if } y \in \{ x_{i+1}, ., x_n \} & (1.7\,c) \end{cases}$$

This is the usual definition if f is symmetric and this can be assumed because particles have been supposed to be identical. We have written A like in eq. (1.6) because this offers a very natural correspondence with the free generator B:

$$(\mathrm{B}f)(x_1,.,x_n)=\sum_{i=1}^{n}\sum_{y\neq x_i}\rho(y-x_i)[f(x_1,.,x_{i-1},y,x_{i+1},.,x_n)-f(x_1,.,x_n)] \tag{1.8}$$

The generator C of the coupled process Q is in fact

$$(\mathrm{C}f)(\vec{x};\vec{x}^{(0)})=\sum_{i=1}^{n}\sum_{z\neq 0}\rho(z)[f((x_1,.,x_n)(x_i,x_i+\mathrm{Z});(x_1^{(0)},.,x_i^{(0)}+z,.,x_n^{(0)})) - f(\vec{x};\vec{x}^{(0)})] \tag{1.9}$$

The heuristic meaning of eq. (1.9) is the following: if the free particle « i » jumps by z at a given time, the same occurs for interacting particle « i » if the new site is available, empty. It could also happen that it is occupied by particle j, $j < i$, in this case no displacement occurs. If otherwise $j > i$ then particles i and j exchange their positions.

The main features of the coupling Q are the following. A different displacement between free and interacting particles occurs only when a particle « tries to jump » on a occupied site. The differences of the displacements are symmetric random variables, which, after suitable conditioning, become independent. We will then be reduced to an estimate of sums of centered independent random variables. Differences due to long jumps will play a different role than those arising from the short ones. The latter will give a contribution like in the short range case, while the former will be responsable for the slow decay of our probability estimates of Theorem 1.2. As in [*4*] we will need an *a priori* estimate for the time spent by a pair of particle at a mutual given distance, up time t. Again, as in [*4*], this will be reduced, after a suitable coupling, to estimate the return time to the origin of a single random walk. After conditioning to these times the differences between interacting and free displacements become independent, so it only remains to estimate the number of forbidden jumps occurring at that distance (with their sign) and then to sum them up. This will be quite straightforward and the result is quoted in Theorem 1.2.

2. PROOFS

An important tool in the proofs of Theorem 1.1 and 1.2 is an *a priori* estimate on the time spent by a given pair of interacting particles at a given distance. This is given by:

PROPOSITION 2.1. — Let

$$T_{i,j}(a, t) = \int_0^t d\tau 1(\{ | x_i(\tau) - x_j(\tau) | = a \}) \tag{2.1}$$

then for every $u > \frac{1}{2}$ exist $c > 0$ and c' so that

$$P(\{ T_{i,j}(a, t) < t^u, \forall a \geqslant 1 \ \forall i, j \in (1, \ldots, n), i \neq j \}) \geqslant 1 - c' e^{-ct^{(u-\frac{1}{2})}} \tag{2.2}$$

Proof. — We fix i and j. We condition to the random variable $\tau_{i,j}(a)$ whose value is the first time particle i and j are at distance a. We will then prove that

$$P(\{ T_{i,j}(a, t) > t^u \} | \{ \tau_{i,j}(a) < t \}) \leqslant D' e^{-Dt^{(u-\frac{1}{2})}} \tag{2.3}$$

$$P(\{ \tau_{i,j}(a) < t \}) \leqslant \tilde{D}(tR(a) + ta^{-2}) \tag{2.4}$$

$$R(a) = \sum_{b : |b| \geqslant a/3} \rho(b)$$

From eqs. (2.3) and (2.4), eq. (2.2) follows straightforwardly (because of eq. (1.1)).

The probabilities in eqs. (2.3), (2.4) can be computed as if only i and j were present. From eq. (1.6) it is in fact easy to see that if f is chosen to depend only on x_i and x_j, then the same holds also for $(Af)(x_1, \ldots, x_n)$. This shows that the particular labelling given in eq. (1.6) and (1.7) does not destroy the additivity of the process which holds for the identical particles [*6*].

We first prove eq. (2.4). Let $\mathscr{A}$ be the set of trajectories such that up to time t neither particle i nor j jumps by more than $a/3$, then it is easy to see that $P(\mathscr{A}^{\text{compl.}}) \leqslant 2tR(a)$. In the set $\{ \tau_{i,j}(a) < t \} \cap \mathscr{A}$ there is a first time $\hat{\tau}$ for which particles i and j are at a distance which is in the interval $(a/3, 2a/3)$, at least for large values of a. Starting from this time the trajectories in $\{ \tau_{i,j}(a) < t \} \cap \mathscr{A}$ behave as free, at least up to the time when they have changed again their distance by more than $a/3$. In this interval of time $x_i(t) - x_j(t)$ behaves as a single random walk and so the probability that it moves by more than $a/3$ is given by the second term in the r. h. s. of eq. (2.4), by Kolmogorov inequality, see for instance [*7*] and [*3*]. To prove eq. (2.3) we introduce $\mathscr{N}$ as the random variable which counts the number of returns of i and j at a distance a up to time t. We are going to show that for every $u > \frac{1}{2}$ exist D'', $D''' > 0$ so that

$$P(\{ \mathscr{N} > t^u \} | \{ \tau_{i,j}(a) < t \}) \leqslant D'' e^{-D'''t^{(u-\frac{1}{2})}} \tag{2.5}$$

from this eq. (2.3) follows straightforwardly. Eq. (2.5) is a consequence of the following fact. Let $\mathcal{N}$ be a random variable whose distribution is the same as that of the first return time of particles i and j at distance a, starting from distance a. Let $\hat{\mathrm{P}}$ denotes the law of the process of countably many independent random variables distributed each one like η. Then

$$\mathrm{P}(\{\mathcal{N}>t^u\}\mid\{\tau_{i,j}(a)<t\})=\hat{\mathrm{P}}\left(\left\{\sum_{i=1}^{[t^u]+1}\eta_i<t\right\}\right)\leqslant\hat{\mathrm{P}}(\{\eta<t\})^{[t^u]+1} \tag{2.6}$$

Since η is stochastically larger than $\eta^{(0)}$, which is the first arrival time [for two free particles starting from distance a] either at zero distance or back at distance a, we have

$$\mathrm{P}(\{\mathcal{N}>t^u\}\mid\{\tau_{i,j}(a)<t\})\leqslant[\hat{\mathrm{P}}^{(0)}(\{\eta^{(0)}<t\})]^{t^u}$$

and from classical probability estimates, see for instance [9, VII.32.P3] eq. (2.5) follows.

Theorem 1.1 could be obtained as a consequence of Theorem 1.2, however its proof can also be derived rather directely, so we have chosen to give it explicitely in rather a sketchy way.

Proof of Theorem 1.1. — We follow very closely the analogous proof of Lemma 3.3 of [5]. The trajectory space is $\mathscr{D}([0,1],\mathbb{R}^n)$, see [2], we denote by Φ_s a bounded function measurable with respect to the σ-algebra generated up to time s, $0\leqslant s\leqslant 1$; f is a C^∞ symmetric real function on $\mathbb{R}^n$ with compact support. $\mathbb{E}_\mathrm{T}$ denotes the expectation for the process on $\mathscr{D}$ obtained from the simple exclusion one after the renormalization procedure of eq. (1.2). Theorem 1.1 is a consequence of eq. (2.7) below for all $0<s<t\leqslant 1$, Φ_s, f:

$$\lim\ \mathbb{E}_\mathrm{T}\left(\left|\Phi_s\int_s^t dt'(\mathrm{A}_\mathrm{T}f)(\vec{\xi}_{t'}^{(\mathrm{T})})-\Phi_s\int_s^t dt'(\hat{\mathrm{B}}f)(\vec{\xi}_{t'}^{(\mathrm{T})})\right|\right)=0 \tag{2.7}$$

A_T is the generator of the renormalized process, see eq. (2.9) below, $\hat{\mathrm{B}}$ is the generator for n independent Brownian motions

$$(\hat{\mathrm{B}}f)(\vec{x})=\frac{1}{2}\langle x^2\rangle\sum_{i=1}^n\left(\frac{d^2}{dx_i^2}f\right)(\vec{x})$$

Theorem 1.1 is a consequence of eq. (2.7) because the sequence of processes is tight in the Skorokhood topology [8], and so each weak limit satisfies

$$
\begin{aligned}
&\mathbb{E}(\Phi_s[f\vec{\xi}_t) - f(\vec{\xi}_s)]) \\
&\quad = \lim \mathbb{E}_{\mathrm{T}}(\Phi_s[f(\vec{\xi}_t^{(\mathrm{T})}) - f(\vec{\xi}_s^{(\mathrm{T})})]) \\
&\quad = \lim \mathbb{E}_{\mathrm{T}}\left(\Phi_s \int_s^t dt'(\mathrm{A_T} f)(\vec{\xi}_{t'}^{(\mathrm{T})})\right) = \lim \mathbb{E}_{\mathrm{T}}\left(\Phi_s \int_s^t dt'(\hat{\mathrm{B}} f)(\vec{\xi}_{t'}^{(\mathrm{T})})\right) \qquad (2.8) \\
&\quad = \mathbb{E}\left(\Phi_s \int_s^t dt'(\hat{\mathrm{B}} f)(\vec{\xi}_{t'})\right)
\end{aligned}
$$

where $\mathbb{E}$ is the expectation with respect to the limiting process. Eq. (2.8) is known to imply that the generator for the process $\mathbb{E}$ is $\hat{\mathrm{B}}$, namely that the limiting process is brownian.

We will give now a sketch of the proof of eq. (2.7). From eqs. (1.6) and (1.2)

$$
(\mathrm{A_T} f)(\vec{x}) = \mathrm{T} \sum_{i=1}^{n} \sum_{a \in \mathbb{Z}}{}' \rho(a)\left[f\left(x_1, .., x_i + \frac{a}{\sqrt{\mathrm{T}}}, .\right) - f(x_1, ., x_n)\right] \qquad (2.9)
$$

where $\sum'$ is a sum restricted to those values of a such that

$$
x_i + \frac{a}{\sqrt{\mathrm{T}}} \notin \{ x_1, ., x_n \}.
$$

We now expand in the « small » parameter $\frac{a}{\sqrt{\mathrm{T}}}$, to do that we need to control the large values of a. From eq. (1.1b) it is possible to prove that there exists a non increasing function $\varphi(\mathrm{T})$ such that

$$
\lim \varphi(\mathrm{T}) = 0 \, ; \qquad \lim \mathrm{T} \sum_{|a| \geqslant \varphi(\mathrm{T})\sqrt{\mathrm{T}}} \rho(a) = 0 \qquad (2.10)
$$

We have

$$
\begin{aligned}
|(\mathrm{A_T} f)(\vec{x}) - (\hat{\mathrm{B}} f)(\vec{x})| \leqslant & \sum_{|a| \geqslant \varphi(\mathrm{T})\sqrt{\mathrm{T}}} (2n\rho(a) \, \| f \| \, \mathrm{T} + na^2 \rho(a) \, \| f \|) \\
+ \mathrm{T} & \sum_{|a| < \varphi(\mathrm{T})\sqrt{\mathrm{T}}} \left(\frac{a}{\sqrt{\mathrm{T}}}\right)^3 \rho(a) n! \, \| f \| + \sum_{i \neq j} \| f \| \, \rho(\sqrt{\mathrm{T}} \, | x_i - x_j |) \qquad (2.11)
\end{aligned}
$$

The first sum estimates the « large values of a » both for $\mathrm{A_T}$ and the term $\langle x^2 \rangle$ in B. The second sum bounds the remainder in the expansion of the r. h. s. of eq. (2.9) in the parameter $\left(\frac{a}{\sqrt{T}}\right)$. The second order terms cancel out with the corresponding ones in $\hat{B}f$ except for those which are taken into account in the third sum of eq. (2.11).

By eq. (2.10) the first sum in the r. h. s. of eq. (2.11) vanishes as $T \to \infty$: by eq. (2.10) in fact $\varphi(T)\sqrt{T}$ diverges and by eq. (1.1) $\sum a^2\rho(a) < \infty$. The second sum vanishes as $\varphi(T)$ when T diverges. T third one contributes in eq. (2.7) as

$$\lim \sum_{i,j} \frac{1}{T} \mathbb{E}\left(\int_{Ts}^{Tt} dt' \rho(\,|\,x_i(t') - x_j(t')\,|\,) \,\|\, \Phi_s \,\|\, \|\, f \,\|\right)$$

where $\mathbb{E}$ is the expectation with respect to the unrenormalized process. By Proposition 2.1 the limit can be estimated as

$$\lim n^2 \left(\sum_a \rho(a) \frac{T^u}{T} + c' e^{-cT^{(u-\frac{1}{2})}} \right) \|\, \Phi_s \,\|\, \|\, f \,\| = 0$$

if $\frac{1}{2} < u < 1$.

The following properties will be often used in the proof of Theorem 1.2:

P.2.1. is Proposition 2.1.

P.2.2. Let $(c_n)_{n\in\mathbb{N}}$ be a l_1 sequence, $(\varepsilon_n)_{n\in\mathbb{N}}$, $\varepsilon_n = \pm 1$ centered independent random variables, then

$$P\left(\left\{ \left| \sum c_n \varepsilon_n \right| > k \left(\sum c_n^2 \right)^{1/2} \right\} \right) \leqslant e^{-\frac{k^2}{4}}.$$

P.2.3. For $\xi > 0$, π_ξ is the probability on $\mathbb{N}$ defined as

$$\pi_\xi(n) = e^{-\xi} \frac{\xi^n}{n\,!}$$

If $\xi \geqslant \frac{1}{2}$, there exists B' such that

$$\pi_\xi(\{\, n > k\xi \,\}) \leqslant B' e^{-k \log k}$$

P.2.4. For $\xi < \frac{1}{2}$,

$$\pi_\xi(.\,\{\, n > k \,\}) \leqslant 2^{-k}.$$

The properties P.2.2., P.2.3., P.2.4. can be proven easily. We will also use

P.2.5. Let

$$\mathscr{A}_t = \left\{ a > 0 : 2\rho(a)t^u \geqslant \frac{1}{2} \right\} \qquad 1 > u > \frac{1}{2}.$$

There exists $\tilde{C}$ such that

$$|\mathscr{A}_t| \leqslant \tilde{C} + t^{1/2}$$

In fact assume the opposite: then there exist sequences t_k and a_k such that

$$2\rho(a_k)t_k^u \geqslant \frac{1}{2} \qquad a_k > t_k^{1/2} + k$$

Then t_K diverges and

$$a_k^2\,\rho(a_k) \geqslant t_k t_k^{-u} 1/4$$

which diverges as $k \to \infty$, against the assumption of eq. (1.1 *b*).

Proof of Theorem 1.2. — The variable $x_i(t) - x_i^{(0)}(t)$ is defined on the coupled process as follows (see eq. (1.9)). For $j < i$ and m positive integer $\tau_{m,j}$ are the times when *a*) $x_i^{(0)}(t)$ changes by z and $x_i(t) + z = x_j(t)$; *b*) $x_j^{(0)}(t)$ changes by z and $x_j(t) + z = x_i(t)$. Define $u_{m,j}$ to be $-z$ in case *a*) and z in case *b*). Let finally, $n_j(t)$ be such that $\tau_{n_j(t)+1,j} > t \geqslant \tau_{n_j(t),j}$. Then

$$x_i(t) - x_i^{(0)}(t) = \sum_{j=1}^{i-1} \sum_{m=1}^{n_j(t)} u_{m,j} \tag{2.12 a}$$

$$|x_i(t) - x_i^{(0)}(t)| \leqslant \sum_{j=1}^{i-1} d_j(t); \qquad d_j(t) = \left| \sum_{m=1}^{n_j(t)} u_{m,j} \right| \tag{2.12 b}$$

The law of $d_j(t)$ is the same as when only particles i and j are present. We introduce the following notation:

$$\mathrm{T}(a, t) = \mathrm{T}_{i,j}(a, t) \qquad \text{as defined in eq. (2.1)} \tag{2.13 a}$$

$$\mathrm{N}(a, t) = \mathrm{Card}\,(\{ n \leqslant n_j(t) : |u_{n,j}| = a \}) \tag{2.13 b}$$

$$\mathrm{S}(a, t) = \left| \sum_{\substack{|u_{n,j}| = a \\ n \leqslant n_j(t)}} u_{n,j} a^{-1} \right| \tag{2.13 c}$$

It is easy to see that conditioned to $\mathrm{T}(a, t)$ the law of $\mathrm{N}(a, t)$ is π_ξ, see P.2.3., with $\xi = 2\rho(a)\mathrm{T}(a, t)$ and that the law of $\mathrm{S}(a, t)$ conditioned to

$N(a, t)$ is the distribution of the absolute value of the sum of $N(a, t)$ independent symmetric random variables with values ± 1.

We distinguish between large and small values of a. The latter are those in $\mathscr{A}_t$, see P.2.5., and those such that $a \leqslant t^{1/6}$. Their contribution is like in the short range case. For $t^{1/6} \leqslant a \leqslant t^{\alpha'}, \alpha' < \alpha$, the sum does not exceed t^α with probability going to zero faster than any inverse power of t. The large values of a will then be estimated and will determine the slow decay of eq. (1.3).

From éq. (2.12 b)

$$d_j(t) \leqslant \left| \sum_{a \in \mathbb{N}} a\varepsilon(a) S(a, t) \right| \tag{2.14}$$

where $\varepsilon(a) (= \pm 1)$ are independent centered random variables. We fix $\varepsilon > 0$ and u in P.2.1. as

$$u = \frac{1}{2} + \varepsilon \quad \varepsilon < \frac{1}{5}\left(\alpha - \frac{1}{4}\right) \quad \varepsilon < \left(\alpha\beta - \frac{1}{2} - \gamma\right)(1 + 4\beta)^{-1} \tag{2.15}$$

By conditioning to $\{ N(a, t), S(a, t), T(a, t) \}$ and using P.2.2. we have

$$Q(\{ d_j(t) > t^\alpha \}) \leqslant e^{-\frac{2\varepsilon}{t^4}} + Q\left(\left\{ t^{2\varepsilon} \sum a^2 S(a, t)^2 > t^{2\alpha} \right\} \right). \tag{2.16}$$

By conditioning to $\{ N(a, t) \}$ and using again P.2.2. we have

$$Q(\{ S(a, t) > t^\varepsilon \sqrt{N(a, t)} \}) \leqslant e^{-\frac{t^{2\varepsilon}}{4}} \tag{2.17}$$

and using P.2.1., eqs. (2.16), (2.17)

$$\begin{aligned} &Q(\{ d_j(t) > t^\alpha \}) \leqslant \\ &\leqslant 2e^{-\frac{t^{2\varepsilon}}{4}} + Q\left(\left\{ t^{2\varepsilon} \sum a^2 t^{2\varepsilon} N(a, t) > t^{2\alpha} \right\} \bigcap \{ T(a, t) \leqslant t^{\frac{1}{2}+\varepsilon}, \forall a \in \mathbb{N} \} \right) \\ &+ c' e^{-ct^\varepsilon} \\ &\leqslant 2e^{-\frac{t^{2\varepsilon}}{4}} + c'e^{-ct^\varepsilon} + Q\left(\left\{ t^{4\varepsilon} \sum a^2 N(a, t) > t^{2\alpha} \right\} \middle| \{ T(a, t) = t^{\frac{1}{2}+\varepsilon}, \forall a \in \mathbb{N} \} \right) \end{aligned} \tag{2.18}$$

For $a \in \mathscr{A}_t$ we can use P.2.3. with $k = t^\varepsilon$ in the estimate of eq. (2.18) and using P.2.5, we have

$$Q\left(\left\{ t^{4\varepsilon} \sum_{a \in \mathscr{A}_t} a^2 N(a, t) > \frac{t^{2\alpha}}{4} \right\} \middle| \{ T(a, t) = t^{\frac{1}{2}+\varepsilon} \forall a \in \mathbb{N} \} \right) \leqslant (\tilde{C} + t^{1/2}) B' e^{-t^\varepsilon \log t^\varepsilon} \tag{2.19}$$

for t so large that (see eq. (2.15)),

$$\sum a^2\rho(a) < \frac{1}{8}t^{2\alpha-\frac{1}{2}-6\varepsilon}$$

Let

$$\mathscr{B}_t = \{\, a \notin \mathscr{A}_t : a \leqslant t^{\frac{1}{6}} \,\}$$

then using P.2.4. with $k = t^{\varepsilon}$

$$\mathrm{Q}\left(\left\{ t^{4\varepsilon}\sum_{a\in\mathscr{B}_t} a^2\mathrm{N}(a, t) > \frac{t^{2\alpha}}{4}\right\}\middle|\, \{\, \mathrm{T}(a, t) = t^{\frac{1}{2}+\varepsilon} \,\}\right) \leqslant t^{1/6}2^{-t^{\varepsilon}} \tag{2.20}$$

because for t large enough (see eq. (2.15)),

$$t^{5\varepsilon+\frac{1}{2}-2\alpha} < \frac{1}{2}$$

We shall now distinguish the case when $\rho(x)$ decays exponentially from the general case. If there is $\mathrm{G} > 0$ and G' so that

$$\rho(x) < \mathrm{G}'e^{-\mathrm{G}|x|}$$

we have

$$\mathrm{Q}\left(\left\{ t^{4\varepsilon}\sum_{a\geqslant t^{1/6}} a^2\mathrm{N}(a, t) > \frac{t^{\alpha}}{4}\right\}\middle|\, \{\, \mathrm{T}(a, t) \leqslant t^{\frac{1}{2}+\varepsilon} \,\}\right) \leqslant$$

$$\leqslant 1 - \mathrm{Q}(\{\, \mathrm{N}(a, t) = 0,\ \forall a \geqslant t^{1/6} \,\} \mid \{\, \mathrm{T}(a, t) \leqslant t^{\frac{1}{2}+\varepsilon} \,\})$$

$$\leqslant \sum_{a\geqslant t^{1/6}} 2\rho(a)t^{\frac{1}{2}+\varepsilon} \leqslant \frac{2\mathrm{G}'}{\mathrm{G}}e^{-\mathrm{G}\, t^{\frac{1}{6}}t^{\frac{1}{2}+\varepsilon}}$$

which together with eq. (2.18), (2.19), (2.20) proves the theorem in the exponential case.

In the general case it is important to distinguish values of a in the interval $(t^{1/6}, t^{\alpha'})$, $\alpha' < \alpha$ and those with $a \geqslant t^{\alpha'}$ which will be treated as above. Let

$$\mathscr{C}_t = \{\, a \in \mathscr{A}_t : t^{1/6} \leqslant a \leqslant t^{\alpha'},\ \alpha' = \alpha - 4\varepsilon \,\} \tag{2.21}$$

By P.2.4. we have

$$\mathrm{Q}\left(\left\{ t^{4\varepsilon}\sum_{a\in\mathscr{C}_t} a^2\mathrm{N}(a, t) > \frac{t^{2\alpha}}{4}\right\}\middle|\, \{\, \mathrm{T}(a, t) = t^{\frac{1}{2}+\varepsilon} \,\}\right)$$

$$\leqslant t^{\alpha'}2^{-t^{\varepsilon}} + \hat{\mathrm{P}}\left(\left\{ \mathrm{Y} > \frac{1}{4}t^{2\alpha-5\varepsilon}\right\}\right) \tag{2.22 a}$$

$$\mathrm{Y} = \sum_{a\in\mathscr{C}_t} a^2x(a) \tag{2.22 b}$$

where $x(a)$ are independent random variables which take value 0 with probability $2\rho(a)t^{\frac{1}{2}+\varepsilon}$ and value 1 with probability $1 - 2\rho(a)t^{\frac{1}{2}+\varepsilon}$. It is easy to see that

$$\hat{\mathbb{E}}(y^n) \leqslant \sum_{K=0}^{n} \hat{\mathbb{E}}\Big(\sum_{a\in\mathscr{C}_t} a^{2(n-k)}x(a)\Big)\,\hat{\mathbb{E}}(\mathrm{Y})^k$$
$$\leqslant \Big(\sum a^2\rho(a)\Big)^n nt^{2n\alpha'}, \quad \Big(\sum a^2\rho(a) \geqslant 1\Big) \tag{2.23}$$

because

$$2\alpha' > \frac{1}{2} + \varepsilon$$

Then

$$\hat{\mathrm{P}}\Big(\Big\{\mathrm{Y} > \frac{1}{4}t^{2\alpha-5\varepsilon}\Big\}\Big) \leqslant n\Big(\sum a^2\rho(a)\Big)^n \frac{1}{4}t^{-2(\alpha-\alpha')n+4\varepsilon n} \tag{2.24}$$

For n large enough the r. h. s. of eq. (2.24) is smaller than any negative power of t, by eq. (2.21).

Since

$$\mathrm{Q}\Big(\Big\{t^{4\varepsilon}\sum_{a\geqslant t^{\alpha'}} a^2\mathrm{N}(a,t) > \frac{1}{4}t^{\alpha}\Big\}\,\Big|\,\{\mathrm{T}(a,t) \leqslant t^{\frac{1}{2}+\varepsilon}\}\Big)$$
$$\leqslant 1 - \mathrm{Q}(\{\mathrm{N}(a,t) = 0, \forall a \geqslant t^{\alpha'}\}\,|\,\{\mathrm{T}(a,t) \leqslant t^{\frac{1}{2}+\varepsilon}\})$$
$$\leqslant \sum_{a\geqslant t^{\alpha'}} 2\rho(a)t^{\frac{1}{2}+\varepsilon} \leqslant t^{-\gamma}\Big(2\sum_a \rho(a)\,|\,a\,|^{\beta}\Big) \tag{2.25}$$

because of eq. (2.15). From eqs. (2.18), (2.19), (2.20), (2.22) and (2.24) and (2.25) we obtain the proof of the theorem.

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