ANNALES DE L'I. H. P., SECTION B

ANNIE MILLET

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Annales de l'I. H. P., section B, tome 19, nº 1 (1983), p. 25-42
<http://www.numdam.org/item?id=AIHPB_1983__19_1_25_0>

Ann. Inst. Henri Poincaré,
Vol. XIX, n° 1, 1983, p. 25-42.

Section B :
Calcul des Probabilités et Statistique.

On convergence and regularity of two-parameter (Δ1) submartingales

by

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RÉSUMÉ. — Une sous-martingale indicée par $\mathbb{N}^2$ ou $\mathbb{R}_+^2$ est une sous-martingale (Δ1) si l'espérance conditionnelle de l'accroissement rectangulaire $X(s, t]$ par rapport à la tribu $\mathscr{F}_s^1$ est positive. Nous montrons que les sous-martingales (Δ1) bornées dans L Log L convergent presque sûrement et admettent des modifications continues à droite limitées dans le quatrième quadrant.

ABSTRACT. — We introduce (Δ1) submartingales which are submartingales indexed by $\mathbb{N}^2$ or $\mathbb{R}_+^2$ such that the conditional expectation of the rectangular increment $X(s, t]$ given $\mathscr{F}_s^1$ is non negative. We show that L Log L-bounded (Δ1) submartingales converge almost surely, and have right-continuous modifications which have limits in the fourth quadrant.

It is known that under the conditional independence assumption (F4) L_∞-bounded two-parameter submartingales $(X_t, t \in \mathbb{N}^2)$ do not converge almost surely [*7*]; there is only equality between the upper limit of X_t and its L_1-limit [*11*], [*15*]. Thus in the continuous case the existence of regular modifications for two-parameter submartingales requires more stringent assumptions on the process. R. Cairoli [*6*] strengthened the definition of submartingale by adding the property (S): for all indices $s \ll t$, the conditional expectation of the rectangular increment $X(s, t]$ given $\mathcal{F}_s$ is non negative. He studied the existence of a Doob-Meyer decomposition of these processes, and their almost sure convergence under a boundedness assumption on the « quadratic variation ».

In this paper we relax the assumption (F4) on the σ-algebras, and define $(\Delta 1)$ submartingales as submartingales satisfying the property $(\Delta 1)$: for all indices $s \ll t$, the conditional expectation of $X(s, t]$ given the vertical σ-algebra $\mathcal{F}_s^1$ is non negative. This extends the notion of 1-martingale as defined in [*14*]. We show that L Log L-bounded positive $(\Delta 1)$ submartingales converge almost surely, and have right-continuous modifications with limits in the fourth quadrant. This generalizes theorems proved in [*14*], and the techniques are similar: the proof consists in showing that the processes are amarts with respect to the totally ordered family of σ-algebras $(\mathcal{F}_t^1)$, and applying the amart theorems on convergence and regularity. However our methods do not give the existence of left-limited versions proved by D. Bakry [*2*] for martingales with respect to product σ-algebras.

The first section states the precise definitions. The second section considers discrete parameter $(\Delta 1)$ submartingales. The existence of regular modifications for continuous parameter $(\Delta 1)$ submartingales is studied in the third section.

1. DEFINITIONS AND NOTATIONS

Let I denote $\mathbb{Z}^2$ or $\mathbb{R}_+^2$ with the usual order $s = (s_1, s_2) \leq (t_1, t_2) = t$ if $s_1 \leq t_1$ and $s_2 \leq t_2$; then I is filtering to the right. Set $s \ll t$ if $s_1 < t_1$ and $s_2 < t_2$. Let $(s, t]$ denote the rectangle $\{ u \in \mathbb{R}^2 : s \ll u \leq t \}$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, and let $(\mathcal{F}_t)$ be a *stochastic basis* indexed by I, i. e., an increasing family of complete sub-sigma-algebras of $\mathcal{F}$. For every $t = (t_1, t_2)$, set

$$\mathcal{F}_t^1 = \bigvee_u \mathcal{F}_{t_1,u} = \mathcal{F}_{t_1,\infty}, \quad \mathcal{F}_t^2 = \bigvee_u \mathcal{F}_{u,t_2} = \mathcal{F}_{\infty,t_2}, \quad \text{and} \quad \mathcal{F}_{\infty,\infty} = \bigvee_t \mathcal{F}_t.$$

A process (X_t) is *adapted* if X_t is $\mathcal{F}_t$-measurable for every $t \in I$. Given a process (X_t) and $s \ll t$, set

$$X(s, t] = X_{t_1,t_2} - X_{t_1,s_2} - X_{s_1,t_2} + X_{s_1,s_2}.$$

An integrable adapted process (X_t) is a *submartingale* [*supermartingale*] if $E(X_t \mid \mathcal{F}_s) \geq X_s$ $[E(X_t \mid \mathcal{F}_s) \leq X_s]$ whenever $s \leq t$. A *martingale* is both a submartingale and a supermartingale. An adapted integrable process (X_t) is a (Δ1) *submartingale* if it is a submartingale and has the property (Δ1):

$$(\Delta 1) \quad E[X(s,t] \mid \mathscr{F}_s^1] \geq 0 \quad \text{whenever} \quad s \ll t.$$

An adapted integrable process is a (Δ1) *supermartingale* if it is a supermartingale, and has the property (P1):

$$(P1) \quad E[X(s,t] \mid \mathscr{F}_s^1] \leq 0 \quad \text{whenever} \quad s \ll t.$$

Remark that the property (Δ1) [(P1)] may be interpreted as follows: for every fixed $b > a$ the map $u \mapsto E(X_{b,u} - X_{a,u} \mid \mathscr{F}_{a,\infty})$ is increasing [decreasing].

Suppose that the space is the product of two probability spaces, and that $X_{a,b}(\omega_1, \omega_2) = Y_a(\omega_1)Z_b(\omega_2)$. If $(Y_a, a \geq 0)$ is a one-parameter positive martingale and $(Z_b, b \geq 0)$ is a one-parameter submartingale, or if (Y_a) is a positive submartingale and (Z_b) is a positive increasing one-parameter process, then $(X_t, t \in I)$ is a (Δ1) submartingale. We give another example of a discrete (Δ1) submartingale. For $i \geq 0$ let $(M_{i,j}, j \geq 0)$ be a submartingale for the increasing family of σ-algebras $(\mathscr{A}_{i,j}, j \geq 0)$. For every $(i,j) \in \mathbb{N}^2$, set

$$X_{i,j} = \sum_{k \leq i} (M_{k,j} - M_{k,0}), \qquad \mathscr{F}_{i,j} = \bigvee_{k \leq i} \mathscr{A}_{k,j},$$

and suppose that $\mathscr{F}_{i,j}$ and $\bigvee_{k > i} \mathscr{A}_{k,j}$ are independent for every (i,j). The process $(X_t, \mathscr{F}_t, t \in \mathbb{N}^2)$ is a (Δ1) submartingale. Indeed the property (B1) is clearly satisfied, and to check the submartingale property, it suffices to verify that $X_{i,j} \leq E(X_{i,j+1} \mid \mathscr{F}_{i,j})$ for every (i,j). For every

$$A_1 \in \mathscr{A}_{1,j}, \ldots, A_1 \in \mathscr{A}_{1,i},$$

one has

$$E[1_{A_1 \cap \ldots \cap A_i}(X_{i,j+1} - X_{i,j})]$$
$$= \sum_{k \leq i} P(A_1 \cap \ldots \cap A_{k-1} \cap A_{k+1} \cap \ldots \cap A_i) E[1_{A_k}(M_{k,j+1} - M_{k,j})] \geq 0.$$

Set $\phi(x) = x \operatorname{Log}^+ x$; a random variable X *belongs to* L Log L if $E[\phi(|X|)] < \infty$, and a process (X_t) is *bounded in* L Log L if

$$\sup \{ E[\phi(|X_t|)] : t \in I \} < \infty .$$

Let J be a directed set filtering to the right, and let $(\mathscr{G}_t, t \in J)$ be a stochastic basis. A map $\tau: \Omega \to J$ is a *stopping time for* $(\mathscr{G}_t)$ if $\{\tau \leq t\} \in \mathscr{G}_t$ for every $t \in J$. A *1-stopping time* is a stopping time for $(\mathscr{F}_t^1, t \in I)$, where $I = \mathbb{Z}^2$ or $\mathbb{R}_+^2$. A stopping time is called *simple* if it takes on finitely many values. Let T^1 denote the set of simple 1-stopping times. If τ is a stopping time for $(\mathscr{G}_t)$, let

$$\mathscr{G}_\tau = \{ A \in \mathscr{F} : A \cap \{\tau \leq t\} \in \mathscr{G}_t \quad \text{for all } t \}.$$

An adapted integrable process $(X_t, \mathscr{F}_t, t \in \mathbb{Z}^2)$ is a *1-amart* [*descending 1-amart*] if the net $(EX_\tau, \tau \in T^1)$ converges when $\tau \to (\infty, \infty)$ $[\tau \to (-\infty, -\infty)]$.

We now give definitions relative to the continuous case. Given an index $t \in \mathbb{R}^2$, denote the quadrants determined by t by $Q_I(t) = \{s : s \geq t\}$, $Q_{II}(t) = \{s : s_1 \leq t_1, s_2 \geq t_2\}$, $Q_{III}(t) = \{s : s \leq t\}$, and $Q_{IV}(t) = \{s : s_1 \geq t_1, s_2 \leq t_2\}$. A stochastic process $(X_t : t \in \mathbb{R}^2)$ is *continuous in* Q_i, $i = I, \dots, IV$, if $X_t = \lim (X_s : s \to t, s \in Q_i(t))$ for every t. For every $i = I, \dots, IV$, denote $Q_i^0(t)$ the interior of $Q_i(t)$ for the euclidian topology. The process has *limits in* Q_i, $i = I, \dots, IV$, if $\lim (X_s : s \to t, s \in Q_i^0(t))$ exists for every t.

A sequence $\tau(n)$ of 1-stopping times *1-decreases to* τ *in* $Q_I [Q_{IV}]$ if $\lim \tau(n) = \tau$, the sequence $\tau(n)_1$ decreases, and $\tau(n) \geq \tau$ for every n

$$[\tau(n)_1 \geq \tau_1,\ \tau(n)_2 \leq \tau_2 \text{ for every } n].$$

A sequence $\tau(n)$ of 1-stopping times *1-recalls* τ *in* $Q_I [Q_{IV}]$ if $\tau(n)$ 1-decreases to τ in $Q_I [Q_{IV}]$, and $\tau(n) \gg \tau$ for every n $[\tau(n)_1 > \tau_1, \tau(n)_2 < \tau_2$ on the set $\{\tau_2 > 0\}$ for every $n]$. An integrable process $(X_n, \mathscr{G}_n, n \in \mathbb{N})$ $[(X_n, \mathscr{G}_n, n \in -\mathbb{N})]$ is an *ascending* [a *descending*] *amart* if the net $(EX_\tau, \tau \in T)$ $[(EX_\tau, \tau \in -T)]$ converges, where T denotes the set of simple stopping times for $(\mathscr{G}_n)$. If $\mathscr{G}_n$ decreases for $n \geq 0$, $Y_{-n} = X_n$, $\mathscr{H}_{-n} = \mathscr{G}_n$, then $(X_n, \mathscr{G}_n, n \in \mathbb{N})$ is a descending amart if $(Y_n, \mathscr{H}_n, n \in -\mathbb{N})$ is one. A one-parameter integrable process $(X_t, \mathscr{G}_t, t \geq 0)$ is an *ascending* [a *descending*] *amart* if for every stopping time τ for $(\mathscr{G}_t)$, and for every sequence $(\tau(n), n \in \mathbb{N})$ $[(\tau(n), n \in -\mathbb{N})]$ of simple stopping times for $(\mathscr{G}_t)$ that increases to τ, the process $(X_{t(n)}, \mathscr{G}_{\tau(n)}, n \in \mathbb{N})$ $[(X_{\tau(n)}, \mathscr{G}_{\tau(n)}, n \in -\mathbb{N})]$ is an ascending [a descending] amart. A process $(X_t, \mathscr{G}_t, t \geq 0)$ is of *class (AL)* if for every uniformly bounded sequence of simple stopping times $\tau(n)$, $\sup E|X_{\tau(n)}| < \infty$. An integrable process $(X_t, \mathscr{F}_t, t \in \mathbb{R}_+^2)$ is a *1-amart in* Q_I $[Q_{IV}]$ if for every bounded 1-stopping time τ, and for every uniformly bounded sequence $(\tau(n) : n \in \mathbb{N})$ in T^1 which 1-recalls τ in Q_I $[Q_{IV}]$, the process $(X_{\tau(n)}, \mathscr{F}_{\tau(n)}^1, n \in \mathbb{N})$ is a descending amart. The process $(X_t, \mathscr{F}_t, t \in \mathbb{R}_+^2)$ is a *descending 1-amart* if it is a 1-amart in Q_I and Q_{IV}, and if for every $b \geq 0$, the one-parameter process $(X_{t,b}, \mathscr{F}_{t,b}, t \geq 0)$ is a descending amart.

2. DISCRETE PARAMETER

In this section we prove that L Log L-bounded positive (Δ1) submartingales converge almost surely by showing that they are 1-amarts. This generalizes Theorem 1.1 [*14*].

THEOREM 2.1. — Let $(X_t, \mathscr{F}_t, t \in \mathbb{Z}^2)$ be an L Log L-bounded (Δ1) submartingale [submartingale with (P1)] such that $X_t \geq E(Y \mid \mathscr{F}_t)$ for some random variable $Y \in L \operatorname{Log} L$. Then $(X_t, t \in \mathbb{N}^2)$ is a 1-amart, and hence converges almost surely when $t \to (\infty, \infty)$, while $(X_t, t \in -\mathbb{N}^2)$ is a descending 1-amart, and converges almost surely when $t \to (-\infty, -\infty)$.

Proof. — We at first prove the 1-amart property of the (Δ1) submartingale $(X_t, t \in \mathbb{N}^2)$. Given an increasing sequence (t_n) of indices in $\mathbb{N}^2$, the process $(X_{t_n}, \mathscr{F}_{t_n}, n \geq 0)$ is an L Log L-bounded submartingale, and X_{t_n} converges in L_1. Hence the net (X_t) converges in L_1 to a random variable $X \in L \operatorname{Log} L$, such that $X_t \leq E(X \mid \mathscr{F}_t)$ for every t. Fix $j \in \mathbb{N}$; the one-parameter submartingale $(X_{n,j}, \mathscr{F}_{n,j}, n \geq 0)$ $[(X_{j,n}, \mathscr{F}_{j,n}, n \geq 0)]$ converges a. s. and in L_1 to an $\mathscr{F}_{\infty,j}$ $[\mathscr{F}_{j,\infty}]$ random variable $X_{\infty,j}$ $[X_{j,\infty}]$ that belongs to L Log L. The submartingale property and the L_1 convergence of the nets $(X_{n,j}, n \geq 0)$ and $(X_{j,n}, n \geq 0)$ show asymptotically in n that $(X_{\infty,j}, \mathscr{F}_{\infty,j}, j \geq 0)$ and $(X_{j,\infty}, \mathscr{F}_{j,\infty}, j \geq 0)$ are L Log L-bounded submartingales. Both sequences converge a. s. and in L_1 to X when $j \to +\infty$. Set $\bar{I} = (\mathbb{N} \cup \{+\infty\})^2$, $X_{\infty,\infty} = X$, and $\mathscr{F}_{\infty,\infty} = \bigvee_{t \in \mathbb{N}^2} \mathscr{F}_t$. It is easy to see that the process $(X_t, \mathscr{F}_t, t \in \bar{I})$ is a (Δ1) submartingale. By Doob's maximal inequality applied to the positive submartingales $X^+_{n,\infty}$, $X^+_{\infty,n}$, $X^+_{n,0}$, and to the positive martingales $E(Y^- \mid \mathscr{F}_{n,\infty})$, $E(Y^- \mid \mathscr{F}_{\infty,n})$ and $E(Y^- \mid \mathscr{F}_{n,0})$ (see e. g. [*16*], p. 69), one has $E(\sup |X_{n,\infty}| : n \geq 0) < \infty$, $E(\sup |X_{\infty,n}| : n \geq 0) < \infty$, and $E(\sup |X_{n,0}| : n \geq 0) < \infty$. Hence the sequences $(\sup_{j \geq n} |X_{j,\infty} - X| : n \geq 0)$, $(\sup_{j \geq n} |X_{\infty,j} - X| : n \geq 0)$, and $(\sup_{j \geq n} |X_{n,0} - X_{\infty,0}| : n \geq 0)$ are uniformly integrable, and they converge a. s. to zero. Fix $\varepsilon > 0$, and choose K such that

$$E[\sup\{ |X_{j,\infty} - X| : j \geq K \}] \leq \varepsilon,$$
$$E[\sup\{ |X_{\infty,j} - X| : j \geq K \}] \leq \varepsilon,$$

and
$$E[\sup\{ |X_{j,0} - X_{\infty,0}| : j \geq K \}] \leq \varepsilon.$$

Let $\tau \geq (K, K)$ be a simple 1-stopping time. The property ($\Delta 1$) applied to the rectangle $((t_1, 0), (\infty, t_2)]$ implies

$$\begin{aligned} EX_\tau &= \sum_t E[1_{\{\tau=t\}} X_t] \\ &\leq \sum_t E[1_{\{\tau=t\}}(X_{\infty,t_2} + X_{t_1,0} - X_{\infty,0})] \\ &\leq EX + E(\sup_{j \geq K} |X_{\infty,j} - X|) + E(\sup_{j \geq K} |X_{j,0} - X_{\infty,0}|) \\ &\leq EX + 2\varepsilon. \end{aligned}$$

Conversely, the property ($\Delta 1$) applied to the rectangle $(t, (\infty, \infty)]$ shows that

$$\begin{aligned} EX_\tau &\geq \sum_t E[1_{\{\tau=t\}}(X_{t_1,\infty} + X_{\infty,t_2} - X)] \\ &\geq EX - E(\sup_{j \geq K} |X_{j,\infty} - X|) - E(\sup_{j \geq K} |X_{\infty,j} - X|) \\ &\geq EX - 2\varepsilon. \end{aligned}$$

Hence (X_t) is an L_1-bounded 1-amart.

Consider now a submartingale $(X_t, \mathscr{F}_t, t \in \mathbb{N}^2)$ with the property (P1). Extend the process to a submartingale with (P1) $(X_t, t \in \bar{I})$ as in the first part of the argument. Fix $\varepsilon > 0$, and choose K as above. Let $\tau \geq (K, K)$ be a simple 1-stopping time. A similar argument shows that the property (P1) applied to the rectangle $((t_1, 0), (\infty, t_2)]$ leads to $EX_\tau \geq EX - 2\varepsilon$, while the property (P1) applied to the rectangle $(t, (\infty, \infty)]$ gives $EX_\tau \leq EX + 2\varepsilon$. This concludes the proof of the 1-amart property of (X_t). Finally similar proofs show the descending 1-amart property of the processes in the descending case.

The stochastic basis $(\mathscr{F}_t^1)$ is totally ordered, and hence satisfies the Vitali condition V. The almost sure convergence of (X_t) when $t \to (+\infty, +\infty)$ or $(-\infty, -\infty)$ follows from Astbury's theorem [*1*] □

Remark. — An analog of Theorem 2.1 can be proved by a similar technique for ($\Delta 1$) supermartingales [supermartingales having the property ($\Delta 1$)], say $(X_t, t \in (\mathbb{N} \cup \{+\infty\})^2)$, under the additional assumption:

$$E(\sup\{|X_{i,j}| : i \in \mathbb{N} \cup \{+\infty\}\}) < \infty,$$

and

$$E(\sup\{|X_{j,i}| : i \in \mathbb{N} \cup \{+\infty\}\}) < \infty \quad \text{for every} \quad j \in \mathbb{N} \cup \{+\infty\}.$$

We prove a Doob-Meyer decomposition of ($\Delta 1$) submartingales. The stochastic basis $(\mathscr{F}_t)$ has the *property* (F4) if $\mathscr{F}_t^1$ and $\mathscr{F}_t^2$ are conditionally independent given $\mathscr{F}_t$ for all t. A process (M_t) is a *martingale for* $(\mathscr{F}^1)$ if it has both properties (Δ1) and (P1) (but is not necessarily a martingale). Under the assumption (F4), every martingale is a martingale for $(\mathscr{F}^1)$. Given any process $(X_{i,j})$, set $\Delta X(i,j)=X((i-1,j-1),(i,j)]$ if $i\geq 1$ and $j\geq 1$, $\Delta X(0,j)=X_{0,j}-X_{0,j-1}$ if $j\geq 1$, $\Delta X(i,0)=X_{i,0}-X_{i-1,0}$ if $i\geq 1$, and $\Delta X(0,0)=0$. A process $(A_{i,j})$ is an increasing process if $\Delta A(i,j)\geq 0$ for every (i,j). The following proposition is an analog of [*6*] Theorem 2, and [*9*] Lemma 3.

PROPOSITION 2.2. — Suppose that the stochastic basis $(\mathscr{F}_t)$ has the property (F4).

(*i*) Let $(X_t,\mathscr{F}_t,t\in\mathbb{N}^2)$ satisfy (Δ1). Then (X_t) has a unique decomposition $X_t=M_t+A_t$, where (M_t) is a martingale for $(\mathscr{F}^1)$, and (A_t) is an increasing process such that $A_{m,n}$ is measurable with respect to $\mathscr{F}_{m-1,n}\vee\mathscr{F}_{m,n-1}$.

(*ii*) Let $(X_t,\mathscr{F}_t,t\in\mathbb{N}^2)$ be a (Δ1) submartingale.

Then (X_t) can be decomposed into $X_t=M_t+A_t-B_t$, where (M_t) is a martingale, (A_t) and (B_t) are increasing processes.

Proof. — (*i*) Set $a_{0,j}=a_{j,0}=0$ for $j\geq 0$, and $a_{i,j}=E[\Delta X(i,j)\mid\mathscr{F}_{i-1,\infty}]$ for $i\geq 1$ and $j\geq 1$. The conditional independence of $\mathscr{F}_t^1$ and $\mathscr{F}_t^2$ given $\mathscr{F}_t$ implies that $a_{m,n}$ is measurable with respect to $\mathscr{F}_{m-1,n}\vee\mathscr{F}_{m,n-1}$ (cf. [*9*]). Set

$$A_{m,n}=\sum_{i\leq m}\sum_{j\leq n}a_{i,j},\qquad\text{and}\qquad M_{m,n}=X_{m,n}-A_{m,n}.$$

It is easy to check that (A_t) and (M_t) have the required properties, and give the unique decomposition.

(*ii*) Set $m_{0,0}=X_{0,0}$, $a_{0,0}=b_{0,0}=0$,

$$\begin{cases} m_{0,j}=X_{0,j}-E(X_{0,j}\mid\mathscr{F}_{0,j-1}),\\ a_{0,j}=E(X_{0,j}\mid\mathscr{F}_{0,j-1})-X_{0,j-1},\\ b_{0,j}=0 \qquad\qquad \text{for } j\geq 1,\end{cases}$$

$$\begin{cases} m_{i,0}=X_{i,0}-E(X_{i,0}\mid\mathscr{F}_{i-1,0})\\ a_{i,0}=E(X_{i,0}\mid\mathscr{F}_{i-1,0})-X_{i-1,0},\\ b_{i,0}=0 \qquad\qquad \text{for } i\geq 1,\end{cases}$$

$$\begin{cases} m_{i,j}=\Delta X(i,j)-E(\Delta X(i,j)\mid\mathscr{F}_{i-1,j})-E(\Delta X(i,j)\mid\mathscr{F}_{i,j-1})\\ \qquad\qquad +E(\Delta X(i,j)\mid\mathscr{F}_{i-1,j-1}),\\ a_{i,j}=E(\Delta X(i,j)\mid\mathscr{F}_{i-1,j})+E(X_{i,j}-X_{i,j-1}\mid\mathscr{F}_{i,j-1}),\\ b_{i,j}=E(\Delta X(i,j)\mid\mathscr{F}_{i-1,j-1})+E(X_{i-1,j}-X_{i-1,j-1}\mid\mathscr{F}_{i,j-1})\\ \qquad\qquad \text{for every } (i,j)\geq(1,1).\end{cases}$$

The property (F4) and the submartingale property imply

$$E(X_{i-1,j} - X_{i-1,j-1} \mid \mathscr{F}_{i,j-1}) \geq 0.$$

Set

$$M_{m,n} = \sum_{i \leq m} \sum_{j \leq n} m_{i,j}, \quad A_{m,n} = \sum_{i \leq m} \sum_{j \leq n} a_{i,j}, \quad B_{m,n} = \sum_{i \leq m} \sum_{j \leq n} b_{i,j}.$$

The processes (A_t) and (B_t) are clearly increasing, and the process (M_t) is a martingale. □

3. CONTINUOUS PARAMETER

We prove that L Log L-bounded positive ($\Delta 1$) submartingales indexed by $\mathbb{R}_+^2$ have modifications which are well-behaved in the quadrants Q_I and Q_{IV}. This extends a result shown in [*14*] for 1-martingales. For every $n \geq 0$, set $D(n) = \{ i.2^{-n} : i \geq 0 \}$, and $D = \cup D(n)$. If S is a subset of $\mathbb{R}_+^2$, denote by $T^1(S)$ the set of simple 1-stopping times with all the values in S.

LEMMA 3.1. — Let $(X_t, \mathscr{F}_t, t \in \mathbb{R}_+^2)$ be an L Log L-bounded ($\Delta 1$) submartingale [submartingale with (P1)] such that $X_t \geq E(Y \mid \mathscr{F}_t)$ for some random variable $Y \in L \operatorname{Log} L$. Then for every $M > 0$, the net $\{ X_\tau : \tau \in T^1(D \times D), \tau \leq (M, M) \}$ is uniformly integrable.

Proof. — First consider the ($\Delta 1$) submartingale (X_t). Fix $a > 0$, and let $\tau \in T^1(D \times D)$ satisfy $\tau \ll (M, M)$. Then the property ($\Delta 1$) applied to $((t_1, 0), (M, t_2)]$ gives

$$\begin{aligned} E[1_{\{X_\tau > a\}} X_\tau] &= \sum_t E[1_{\{\tau = t\} \cap \{X_t > a\}} X_t] \\ &\leq \sum_t E[1_{\{\tau = t\} \cap \{X_t > a\}} (X_{M,t_2} + X_{t_1,0} - X_{M,0})] \\ &\leq E[1_{\{X_\tau > a\}} \sup \{ \mid X_{M,b} \mid : b \leq M, b \in D \}] \\ &+ E[1_{\{X_\tau > a\}} \sup \{ \mid X_{a,0} \mid : a \leq M, a \in D \}] \\ &+ E[1_{\{X_\tau > a\}} \mid X_{M,0} \mid]. \end{aligned}$$

Since the positive submartingales $(X_{M,u}^+ : u \geq 0)$ and $(X_{u,0}^+ : u \geq 0)$ are bounded in L Log L, and since $X^-_{M,u} \leq E(|Y| \mid \mathscr{F}_{M,u})$, and $X^-_{u,0} \leq E(|Y| \mid \mathscr{F}_{u,0})$, the random variables $S_1 = \sup(|X_{M,u}| : u \leq M,\ u \in D)$, and

$$S_2 = \sup(|X_{u,0}| : u \leq M,\ u \in D)$$

are integrable. Also $P(X_\tau > a) \leq a^{-1}E(X_\tau 1_{\{X_\tau > a\}}) \leq a^{-1}[ES_1 + ES_2 + E|X_{M,0}|]$. Given $\varepsilon > 0$, choose α such that $P(A) \leq \alpha$ implies $E[1_A(S_1 + S_2 + |X_{M,0}|)] \leq \varepsilon$, and choose a such that $a^{-1}[ES_1 + ES_2 + E|X_{M,0}|] \leq \alpha$. Then $E[1_{\{X_\tau > a\}}X_\tau] \leq \varepsilon$ for every $\tau \in T^1(D \times D)$ with $\tau \ll (M, M)$.

Apply the property ($\Delta 1$) to the rectangle $(t, (M, M)]$ to obtain

$$\begin{aligned} E[1_{\{X_\tau < -a\}} |X_\tau|] &= -\sum_t E[1_{\{\tau = t\} \cap \{X_t < -a\}} X_t] \\ &\leq \sum_t E[1_{\{\tau = t\} \cap \{X_t < -a\}}(X_{M,M} - X_{M,t_2} - X_{t_1,M})] \\ &\leq E[1_{\{X_\tau < -a\}} \sup(|X_{M,b}| : b \leq M,\ b \in D)] \\ &\quad + E[1_{\{X_\tau < -a\}} \sup(|X_{a,M}| : a \leq M,\ a \in D)] \\ &\quad + E[1_{\{X_\tau < -a\}} |X_{M,M}|]. \end{aligned}$$

The random variable $S_3 = \sup(|X_{u,M}| : u \leq M,\ u \in D)$ is integrable, and similarly it suffices to show that $\lim P(X_\tau < -a) = 0$ when $a \to +\infty$. The inequalities

$$P(X_\tau < -a) \leq a^{-1}E(X_\tau^- 1_{\{X_\tau^- > a\}}) \leq a^{-1}(ES_1 + ES_3 + E|X_{M,M}|)$$

conclude the proof in the case of a 1-submartingale.

Let (X_t) be a submartingale with (P1). Similarly the property (P1) applied to $(t, (M, M)]$ proves the uniform integrability of X_τ^+, and the property (P1) applied to $((t_1, 0), (M, t_2)]$ gives the uniform integrability of X_τ^-. □

The following lemma indicates perturbations of a sequence $\tau(n)$ which do not affect $EX_{\tau(n)}$ asymptotically.

LEMMA 3.2. — Let $(X_t, \mathscr{F}_t, t \in \mathbb{R}^2_+)$ be an L Log L-bounded ($\Delta 1$) submartingale [submartingale with (P1)] such that $X_t \geq E(Y \mid \mathscr{F}_t)$ for some random variable $Y \in$ L Log L. Let τ be a bounded 1-stopping time, and let $\tau(n)$ be a sequence of simple 1-stopping times taking on values in $D \times D$, bounded by (M, M), such that $\tau_1 < \tau(n)_1$, and $\lim \| \tau(n)_1 - \tau_1 \|_\infty = 0$. Then for every sequence of positive numbers α_n that converges to zero, one has $\lim EX_{\tau(n)_1 + \alpha_n, \tau(n)_2} - EX_{\tau(n)} = 0$.

Proof. — First study the case of a ($\Delta 1$) submartingale (X_t). The property ($\Delta 1$) applied to the rectangle $((t_1, 0), (t_1 + \alpha_n, t_2)]$ implies

$$\begin{aligned} EX_{\tau(n)} &\leq \sum_t E[1_{\{\tau(n)=t\}}(X_{t_1+\alpha_n,t_2} + X_{t_1,0} - X_{t_1+\alpha_n,0})] \\ &\leq E[\sup\{|X_{a,0} - X_{b,0}| : (a,b) \in D \times D, \tau_1 < a < b < \tau_1 + \alpha_n + \|\tau(n)_1 - \tau_1\|_\infty\}] \\ &\quad + EX_{\tau(n)_1+\alpha_n,\tau(n)_2} \\ &= EX_{\tau(n)_1+\alpha_n,\tau(n)_2} + \beta_n . \end{aligned}$$

Conversely the property ($\Delta 1$) applied to the rectangle $(t, (t_1 + \alpha_n, M)]$ implies

$$\begin{aligned} EX_{\tau(n)_1+\alpha_n,\tau(n)_2} &= \sum_t E[1_{\{\tau(n)=t\}} X_{t_1+\alpha_n,t_2}] \\ &\leq E[\sup\{|X_{a,M} - X_{b,M}| : (a,b) \in D \times D, \tau_1 < a < b < \tau_1 + \alpha_n + \|\tau(n)_1 - \tau_1\|_\infty\}] \\ &\quad + EX_{\tau(n)} \\ &= EX_{\tau(n)} + \delta_n . \end{aligned}$$

The one-parameter submartingales $(X_{a,0}, \mathscr{F}_{a,0}, a \geq 0)$, and $(X_{a,M}, \mathscr{F}_{a,M}, a \geq 0)$ have right limits almost surely along the elements of D. Also $\sup(|X_{a,0}| : a \in D, a \leq K)$ and $\sup(|X_{a,M}| : a \in D, a \leq K)$ are integrable for every K. Hence the sequences β_n and δ_n converge to zero. A similar argument concludes the proof in the case of submartingales satisfying the condition (P1). □

We now prove the amart property of positive ($\Delta 1$) submartingales.

THEOREM 3.3. — Let $(X_t, \mathscr{F}_t, t \in \mathbb{R}_+^2)$ be an L Log L-bounded ($\Delta 1$) submartingale [submartingale satisfying (P1)] such that $X_t \geq E(Y \mid \mathscr{F}_t)$ for some random variable $Y \in L \operatorname{Log} L$. Then (X_t) is a descending 1-amart.

Proof. — Suppose (X_t) is a ($\Delta 1$) submartingale. Fix $b \geq 0$; the amart property of the one-parameter submartingale $(X_{a,b}, \mathscr{F}_{a,b}, a \geq 0)$ has been proved in [*10*]. Let τ be a 1-stopping time bounded by (M, M). Let ε_n be a sequence of positive numbers which decreases to zero. For every $a \geq 0$, the one-parameter submartingale $(X_{a,b}, \mathscr{F}_{a,b}, b \geq 0)$ has left and right limits almost surely along the elements of D, and $\sup(|X_{a,b}| : b \in D, b \leq M)$ is integrable. Choose $\alpha_n > 0$ such that $P(A) \leq \alpha_n$ implies.

$E[1_A \sup\{|X_{a,b}| : a \in D(n), b \in D, (a,b) \leq (M+4, M+4)\}] \leq \varepsilon_n$. Choose an integer k_n such that

$$P\Big[\bigcup_{a \in D(n), a \leq M+4} \{\sup(|X_{a,b} - X_{a,c}| : (b,c) \in D \times D, \tau_2 < b < c < \tau_2 + 4.2^{-k_n}) \geq \varepsilon_n\}\Big] \leq \alpha_n,$$

and

$$P\Big[\bigcup_{a\in D(n), a\le M+4} \{ \sup(|X_{a,b} - X_{a,c}| : (b,c)\in D\times D, \tau_2 - 4.2^{-k_n} < b < c < \tau_2) \ge \varepsilon_n \}\Big] \le \alpha_n,$$

and $P[0 < \tau_2 \le 4.2^{-k_n}] \le \alpha_n$. Finally by Lemma 3.1 choose $\beta_n > 0$ such that $P(A) \le \beta_n$ implies

$$\sup \{ E(1_A | X_\tau |) : \tau \in T^1(D \times D), \tau \le (M, M) \} \le \varepsilon_n .$$

Changing α_n if necessary, we may and do assume that $\alpha_n \le \beta_n/2$. Set $a_n = E[\sup \{ | X_{a,0} - X_{b,0} | : (a, b) \in D \times D, \tau_1 < a < b < \tau_1 + 4.2^{-n} \}]$, and $b_n = E[\sup \{ | X_{a,M+4} - X_{b,M+4} | : (a, b) \in D \times D, \tau_1 < a < b < \tau_1 + 4.2^{-n} \}]$. The one-parameter submartingales $(X_{a,0}, \mathscr{F}_{a,0}, a \ge 0)$, and $(X_{a,M+4}, \mathscr{F}_{a,M+4}, a \ge 0)$ have right limits almost surely along the elements of D. Since $\sup(| X_{a,0} | : a \in D, a \le M+4)$ and $\sup(| X_{a,M+4} | : a \in D, a \le M+4)$ are integrable, $\lim a_n = \lim b_n = 0$. Finally set

$$c_n = E[\sup \{ | X_{a,b} - X_{a,c} | : a \in D(n), a \le M + 4, (b, c) \in D \times D, \tau_2 < b < c < \tau_2 + 4.2^{-k_n} \}],$$
$$d_n = E[\sup \{ | X_{a,b} - X_{a,c} | : a \in D(n), a \le M + 4, (b, c) \in D \times D, \tau_2 - 4.2^{-k_n} < b < c < \tau_2 \}].$$

Then $c_n \le 3\varepsilon_n$, and $d_n \le 3\varepsilon_n$.

We prove first the 1-amart property in Q_I at the 1-stopping time τ. Let $\tau(n)$ be a bounded sequence of simple 1-stopping times which 1-recalls τ in Q_I. Changing M if necessary in the conditions above, we may and do assume that the sequence $\tau(n)$ is bounded by (M, M). To lighten the notations we will assume that the $\tau(n)$ take on dyadic values, and it will be clear in the proof that this is no loss of generality. We define a sequence $T(n)$ which is « universal » for τ, compare $EX_{\tau(n)}$ with $EX_{T(n)}$, and show that $EX_{T(n)}$ converges.

For every $n \ge 0$ let $\nu(n)$ be the dyadic approximation of τ defined by $\nu(n) = ((i+4).2^{-n}, (j+4).2^{-k_n})$ on $\{ (i.2^{-n}, j.2^{-k_n}) \le \tau < ((i+1).2^{-n}, (j+1).2^{-k_n} \}$. Choose p_n such that $p \ge p_n$ implies that

$$P[\tau(p)_1 \ge \tau_1 + 2^{-n}] + P[\tau(p)_2 \ge \tau_2 + 2^{-k_n}] \le \beta_n .$$

We may and do assume that the sequence of integers p_n is strictly increasing. Fix p with $p_n \le p < p_{n+1}$, and set

$$T(p) = \nu(n),$$
$$\sigma(p) = \tau(p) \wedge [T(p) - (2^{-n+1}, 2^{-k_n})].$$

Then $T(p)$ and $\sigma(p)$ belong to $T^1(D \times D)$, and $P[\sigma(p) \neq \tau(p)] \leq \beta_n$. Hence $|EX_{\sigma(p)} - EX_{\tau(p)}| \leq 2\varepsilon_n$. Furthermore, for every (i, j),

$$T(p) = (i.2^{-n}, j.2^{-k_n}) \in \mathscr{F}_{\tau_1 + 2^{-n}, \infty} \subset \mathscr{F}_{\sigma(p)_1 + 2^{-n}, \infty}.$$

Set $S(p) = (\sigma(p)_1 + 2^{-n}, \sigma(p)_2)$ for $p_n \leq p < p_{n+1}$. Clearly $\lim \| \sigma(p)_1 - \tau_1 \|_\infty = 0$; Lemma 2.2 implies that $\lim EX_{\sigma(p)} - EX_{S(p)} = 0$. Fix p with $p_n \leq p < p_{n+1}$, and to lighten the notations set $S = S(p)$ and $T = T(p)$. One has

(*i*) $\tau \ll S \ll T \leq \tau + (4.2^{-n}, 4.2^{-k_n})$, $T \in T^1(D(n) \times D(k_n))$.

(*ii*) T is measurable with respect to $\mathscr{F}_{S_1,\infty}$.

Since $\{ S = s \} \cap \{ T = t \} \in \mathscr{F}^1_s$, the property ($\Delta$1) applied to the rectangle $((s_1, 0), (t_1, s_2)]$ gives

$$\begin{aligned} EX_S &\leq \sum_s \sum_t E[1_{\{S=s\}\cap\{T=t\}}(X_{t_1,s_2} + X_{s_1,0} - X_{t_1,0})] \\ &\leq EX_T + a_n + c_n \leq EX_T + a_n + 3\varepsilon_n. \end{aligned}$$

Conversely the property (Δ1) applied to the rectangle $(s, t]$ shows that

$$\begin{aligned} EX_S &\geq \sum_s \sum_t E[1_{\{S=s\}\cap\{T=t\}}(X_{s_1,t_2} + X_{t_1,s_2} - X_t)] \\ &= EX_T - \alpha - \beta, \end{aligned}$$

where

$$\alpha = \sum_s \sum_t E[1_{\{S=s\}\cap\{T=t\}}(X_t - X_{s_1,t_2})],$$

$$\beta = \sum_s \sum_t E[1_{\{S=s\}\cap\{T=t\}}(X_t - X_{t_1,s_2})].$$

Applying the property (Δ1) to the rectangle $((s_1, t_2), (t_1, M + 4)]$, one obtains

$$\alpha \leq \sum_s \sum_t E[1_{\{S=s\}\cap\{T=t\}}(X_{t_1,M+4} - X_{s_1,M+4})] \leq b_n.$$

The property (Δ1) applied to the rectangle $((t_1, s_2), (M + 4, t_2)]$ shows that

$$\beta \leq \sum_s \sum_t E[1_{\{S=s\}\cap\{T=t\}}(X_{M+4,t_2} - X_{M+4,s_2})] \leq c_n \leq 3\varepsilon_n.$$

Hence $\lim EX_{S(p)} - EX_{T(p)} = 0$. The argument showing that EX_S and EX_T are close depends only on the properties (*i*) and (*ii*) of S and T. Fix $n < m$, p and q with $p_n \leq p < p_{n+1}$, $p_m \leq q < p_{m+1}$. This argument applied to

$S = T(q)$ and $T = T(p)$ shows that the sequence $EX_{T(p)}$ converges, which completes the proof of the 1-amart property in Q_I.

We show that (X_t) is a 1-amart in Q_{IV}. Let $\tau(n)$ be a sequence of simple 1-stopping times taking on values in $D \times D$, which 1-recalls τ in Q_{IV} and is bounded by (M, M). For every $n \geq 0$, let $\rho(n)$ be the dyadic approximation of τ defined by

$$\rho(n) = ((i+4).2^{-n}, (j-4).2^{-k_n})$$

on

$$\{ (i.2^{-n}, j.2^{-k_n}) \ll \tau \leq ((i+1).2^{-n}, (j+1).2^{-k_n})) \} \quad \text{for} \quad j \geq 4,$$

$$\rho(n) = ((i+4).2^{-n}, 0) \quad \text{on} \quad \{ i.2^{-n} < \tau_1 \leq (i+1).2^{-n} \} \cap \{ \tau_2 \leq 4.2^{-k_n} \}.$$

Choose an integer q_n such that $p \geq q_n$ implies

$$P[\tau(p)_1 \geq \tau_1 + 2^{-n}] + P[\tau(p)_2 < \tau_2 - 2^{-k_n}] \leq \beta_n/2.$$

We may and do assume that the sequence q_n is strictly increasing. Fix p with $q_n \leq p < q_{n+1}$, and set

$$T(p) = \rho(n),$$

$$\sigma(p) = (\tau(p)_1 \wedge [T(p)_1 - 2^{-n+1}], \tau(p)_2 \vee [T(p)_2 + 2^{-k_n}]) \quad \text{on} \quad \{ \tau_2 > 4.2^{-k_n} \},$$

$$\sigma(p) = (\tau(p)_1 \wedge [T(p)_1 - 2^{-n+1}], 0) \quad \text{on} \quad \{ \tau_2 \leq 4.2^{-k_n} \}.$$

Then

$$\begin{aligned} P[\sigma(p) \neq \tau(p)] &\leq P[\tau(p)_1 \geq \tau_1 + 2^{-n}] + P[0 < \tau_2 \leq 4.2^{-k_n}] \\ &\quad + P[\tau(p)_2 < \tau_2 - 2^{-k_n}] \\ &\leq \alpha_n + \beta_n/2 \leq \beta_n. \end{aligned}$$

Set $S(p) = (\sigma(p)_1 + 2^{-n}, \sigma(p)_2)$. By Lemma 3.2 one has $\lim EX_{\sigma(p)} - EX_{S(p)} = 0$. By Lemma 3.1 one has $\lim EX_{\tau(p)} - EX_{\sigma(p)} = 0$. We compare the sequence $EX_{S(p)}$ to the « universal » sequence $EX_{T(p)}$, and show the convergence of $EX_{T(p)}$. Fix p with $q_n \leq p < q_{n+1}$, and set $S = S(p)$ and $T = T(p)$; one has

(*i'*) $\tau_1 < S_1 < T_1 < \tau_1 + 4.2^{-n}$, $T \in T^1(D(n) \times D(k_n))$,
(*ii'*) $\tau_2 - 4.2^{-k_n} < T_2 < S_2 < \tau_2$ on $\{ \tau_2 > 4.2^{-k_n} \}$,
(*iii'*) $S_2 = T_2 = 0$ on $\{ \tau_2 = 0 \}$,
(*iv'*) T is measurable with respect to $\mathscr{F}_{S_1, \infty}$.

The random variable τ_2 is measurable with respect to $\mathscr{F}_{S_1, \infty}$; hence

$$\{ \tau_2 = 0 \} \cap \{ S = s \} \cap \{ T = t \} \in \mathscr{F}_s^1$$

and

$$\{ \tau_2 > 4.2^{-k_n} \} \cap \{ S = s \} \cap \{ T = t \} \in \mathscr{F}_s^1.$$

The property ($\Delta 1$) applied to the rectangle $((s_1, t_2), (t_1, s_2)]$ implies

$$\begin{aligned}
\mathrm{EX_S} &\le \mathrm{E}[1_{\{\tau_2=0\}}\mathrm{X_T}] + a_n \\
&\quad + \mathrm{E}[1_{\{0<\tau_2<4.2^{-k_n}\}}(\mathrm{X_T} + |\mathrm{X_S}| + |\mathrm{X_T}|)] \\
&\quad + \sum_s \sum_t \mathrm{E}[1_{\{\tau_2>4.2^{-k_n}\}\cap\{\mathrm{S}=s\}\cap\{\mathrm{T}=t\}}(\mathrm{X}_{s_1,t_2} + \mathrm{X}_{t_1,s_2} - \mathrm{X}_t)] \\
&\le \mathrm{EX_T} + a_n + 2\varepsilon_n + \alpha' + \beta',
\end{aligned}$$

where

$$\alpha' = \sum_s \sum_t \mathrm{E}[1_{\{\mathrm{S}=s\}\cap\{\mathrm{T}=t\}\cap\{\tau_2>4.2^{-k_n}\}}(\mathrm{X}_{s_1,t_2} - \mathrm{X}_t)],$$

$$\beta' = \sum_s \sum_t \mathrm{E}[1_{\{\mathrm{S}=s\}\cap\{\mathrm{T}=t\}\cap\{\tau_2>4.2^{-k_n}\}}(\mathrm{X}_{t_1,s_2} - \mathrm{X}_t)].$$

Applying the property ($\Delta 1$) to the rectangle $((s_1, 0), t]$ one obtains

$$\alpha' \le \sum_s \sum_t \mathrm{E}[1_{\{\mathrm{S}=s\}\cap\{\mathrm{T}=t\}\cap\{\tau_2>4.2^{-k_n}\}}(\mathrm{X}_{s_1,0} - \mathrm{X}_{t_1,0})] \le a_n.$$

On the other hand $\beta' \le d_n \le 3\varepsilon_n$. Conversely,

$$\begin{aligned}
\mathrm{EX_S} &\ge \sum_s \sum_t \mathrm{E}[1_{\{\tau_2=0\}\cap\{\mathrm{S}=s\}\cap\{\mathrm{T}=t\}}\mathrm{X}_{s_1,0}] \\
&\quad + \sum_s \sum_t \mathrm{E}[1_{\{\tau_2>4.2^{-k_n}\}\cap\{\mathrm{S}=s\}\cap\{\mathrm{T}=t\}}\mathrm{X}_s] - \mathrm{E}[1_{\{0<\tau_2\le 4.2^{-k_n}\}}|\mathrm{X_S}|] \\
&\ge \mathrm{E}[\mathrm{X_T}1_{\{\tau_2=0\}}] - a_n + \sum_s \sum_t \mathrm{E}[1_{\{\tau_2>4.2^{-k_n}\}\cap\{\mathrm{S}=s\}\cap\{\mathrm{T}=t\}}\mathrm{X}_s] - \varepsilon_n.
\end{aligned}$$

For every s in the range of S, choose $s'_2 \in \mathrm{D}$, $s'_2 > s_2$, such that setting $\mathrm{A} = \bigcup_s (\{\mathrm{S} = s\} \cap \{0 < \tau_2 \le s'_2\})$, one has $\mathrm{P(A)} \le \alpha_n/2$. Apply the property ($\Delta 1$) to the rectangle $((s_1, s_2), (t_1, s'_2)]$. Then

$$\begin{aligned}
\mathrm{EX_S} &\ge \mathrm{E}[\mathrm{X_T}1_{\{\tau_2=0\}}] - a_n - \varepsilon_n - \mathrm{E}[1_\mathrm{A}|\mathrm{X_S}|] \\
&\quad + \sum_s \sum_t \mathrm{E}[1_{\{\tau_2>4.2^{-k_n}\}\cap\{\tau_2>s'_2\}\cap\{\mathrm{S}=s\}\cap\{\mathrm{T}=t\}}(\mathrm{X}_{s_1,s'_2} + \mathrm{X}_{t_1,s_2} - \mathrm{X}_{t_1,s'_2})] \\
&\ge \mathrm{E}[\mathrm{X_T}1_{\{\tau_2=0\}}] - a_n - 2\varepsilon_n + \mathrm{E}[\mathrm{X_T}1_{\{\tau_2>4.2^{-k_n}\}\cap\mathrm{A}^c}] \\
&\quad - \sum_s \sum_t \mathrm{E}[1_{\{\tau_2>4.2^{-k_n}\}\cap\{\tau_2>s'_2\}\cap\{\mathrm{S}=s\}\cap\{\mathrm{T}=t\}}(\mathrm{X}_{t_1,t_2} - \mathrm{X}_{t_1,s_2} + \mathrm{X}_{t_1,s'_2} - \mathrm{X}_{s_1,s'_2})] \\
&\ge \mathrm{EX_T} - a_n - 2\varepsilon_n - \mathrm{E}[|\mathrm{X_T}|1_{\mathrm{A}\cup\{0<\tau_2\le 4.2^{-k_n}\}}] - d_n - b_n \\
&\ge \mathrm{EX_T} - a_n - 2\varepsilon_n - 2\varepsilon_n - 3\varepsilon_n - b_n.
\end{aligned}$$

Hence $\lim EX_{S(p)} - EX_{T(p)} = 0$ when $p \to \infty$. This argument also shows that the sequence $EX_{T(p)}$ converges, and hence that the sequence $EX_{\tau(p)}$ converges too. This completes the proof of the 1-amart property in Q_{IV} in the case of a ($\Delta 1$) submartingale. A similar argument shows that submartingales with (P1) are 1-amarts in Q_I and Q_{IV}, which concludes the proof. □

The following theorem proves the existence of regular modifications of positive ($\Delta 1$) submartingales.

THEOREM 3.4. — Suppose that $(X_t, \mathcal{F}_t, t \in \mathbb{R}_+^2)$ is an L Log L-bounded ($\Delta 1$) submartingale [submartingale with (P1)] such that $X_t \geq E(Y \mid \mathcal{F}_t)$ for some random variable $Y \in L \operatorname{Log} L$. Assume that $(\mathcal{F}_t^1)$ is right-continuous, and that for every $a \geq 0$ the one-parameter family $(\mathcal{F}_{a,b}, b \geq 0)$ is right-continuous.

(*i*) If for every $a \geq 0$ the map $b \to EX_{a,b}$ is right-continuous, then (X_t) has a modification almost every trajectory of which has right limits.

(*ii*) If for every $b \geq 0$ the map $a \to EX_{a,b}$ is right-continuous, then (X_t) has a modification almost every trajectory of which has limits in Q_I and Q_{IV}.

(*iii*) If for every $a \geq 0$ the maps $b \to EX_{a,b}$ and $b \to EX_{b,a}$ are right-continuous, then (X_t) has a right-continuous modification almost every trajectory of which has limits in Q_{IV}.

Proof. — Our definition of descending 1-amart is slightly different from the one introduced in [*14*]. The difference lies in the fact that we only require the horizontal processes $(X_{a,b}, \mathcal{F}_{a,b}, a \geq 0)$ [and *not* $(X_{a,b}, \mathcal{F}^1_{a,b}, a \geq 0)$] to be descending amarts for all $b \geq 0$. However it is clear from the proofs of Proposition 2.2, Theorems 2.4, 2.5, and Corollaries 2.6, 2.7 [*14*] that the statements made there remain true for our notion of descending 1-amart.

(*i*) For every $a \geq 0$ the one-parameter submartingale $(X_{a,b}, \mathcal{F}_{a,b}, b \geq 0)$ is a descending and an ascending amart of class (AL) [*9*]. The right-continuity of the map $b \to EX_{a,b}$ insures the existence of a right-continuous modification of this process. Hence for every sequence $\tau(n)$ of simple one-dimensional stopping times for $(\mathcal{F}_{a,b}, b \geq 0)$, $b = \lim \searrow \tau(n)$ implies $EX_{a,b} = \lim EX_{a,\tau(n)}$. The existence of right limits follows from Theorem 3.3, and from [*14*], Theorem 2.4.

(*ii*) A similar argument shows that Theorem 3.3 together with [*14*] Theorem 2.4 imply the existence of a modification having limits in Q_I and Q_{IV}.

(*iii*) The argument is similar to the one given in [*14*], Theorem 2.5. By (*ii*) the process (X_t) has a modification (Y_t) having a. s. limits in Q_I.

Set $Z_t = \lim (Y_s : s \gg t)$; it is easy to see that (Z_t) is right-continuous. To prove that (Z_t) is a modification of (X_t), it suffices to prove that for every t, $Z_t = Y_t = X_t$ a. s. Fix $a \geq 0$; the right-continuity of the maps $b \to EX_{a,b}$ and $b \to EX_{b,a}$ insures the existence of right-continuous modifications for the one-parameter submartingales $(Y_{a,b}, b \geq 0)$ and $(Y_{b,a}, b \geq 0)$. Fix t; we may and do assume that the processes $(Y_{t_1,b}, b \geq 0)$, $(Y_{b,t_2}, b \geq 0)$, and all the processes $(Y_{t_1+1/n,b}, b \geq 0)$ and $(Y_{b,t_2+1/n}, b \geq 0)$ are right-continuous. Let $\varepsilon_n \searrow 0$, and for every fixed $n > 0$ let k_n be an integer such that

$$E[|Y_{t_1+1/n,t_2+1/k_n} - Y_{t_1+1/n,t_2}|] \leq \varepsilon_n,$$
$$E[|Y_{t_1,t_2+1/k_n} - Y_t|] \leq \varepsilon_n. \quad \text{Set} \quad E[|Y_{t_1+1/n} - Y_t|] = \alpha_n.$$

Fix $A \in \mathscr{F}_t^1$, set $\tau(n) = t$ on A^c, and $\tau(n) = (t_1 + 1/n, t_2 + 1/k_n)$ on A. Suppose that (X_t) is a 1-submartingale; then

$$\begin{aligned} EY_{\tau(n)} &\geq E[1_{A^c}Y_t] + E[1_A(Y_{t_1+1/n,t_2} + Y_{t_1,t_2+1/k_n} - Y_t)] \\ &\geq EY_t - \varepsilon_n - \alpha_n. \end{aligned}$$

Conversely

$$\begin{aligned} EY_{\tau(n)} &\leq E[1_{A^c}Y_{t_1+1/n,t_2}] + E[1_A Y_{t_1+1/n,t_2+1/k_n}] \\ &\leq E[Y_{t_1+1/n,t_2+1/k_n}] + \varepsilon_n. \end{aligned}$$

The map $t \to EX_t = EY_t$ is right-continuous by assumption. Hence

$$\lim E[Y_{t_1+1/n,t_2+1/k_n}] = EY_t, \qquad \text{and} \qquad \lim EY_{\tau(n)} = EY_t.$$

Lemma 3.1 implies the uniform integrability of $Y_{\tau(n)}$; clearly

$$\lim Y_{\tau(n)} = 1_{A^c}Y_t + 1_A Z_t \text{ a. s..}$$

Hence $E(1_A Z_t) = E(1_A Y_t)$ for every $A \in \mathscr{F}_t^1$. Given any index t the $\mathscr{F}_t^1$-measurable random variables Z_t and Y_t agree almost surely.

A similar argument concludes the proof for submartingales with (P1). □

Remark. — A theorem analogous to Theorem 3.4 can be proved if (X_t) is a (Δ1) supermartingale [a supermartingale satisfying (Δ1)] under the additional assumption that for every $b \geq 0$, and for every $M > 0$, one has $E[\sup |X_{a,b}| : a \in D, a \leq M] < \infty$ and $E[\sup |X_{b,a}| : a \in D, a \leq M] < \infty$.

Finally we state a Doob-Meyer decomposition of (Δ1) submartingales. The proof, similar to the argument given in [*6*], [*4*] and [*9*], is omitted. An adapted integrable process $(A_t, \mathscr{F}_t, t \in \mathbb{R}_+^2)$ is a 1-increasing process if A_t is a. s. right-continuous, null on the y-axis, and satisfies $A(s, t] \geq 0$ for every $s \ll t$. Recall that an adapted integrable process $(M_t, \mathscr{F}_t, t \in \mathbb{R}_+^2)$

is a martingale for $(\mathscr{F}^1)$ if it satisfies the conditions (Δ1) and (P1). A process (X_t) is of *class (D1)* if for every $t \gg (0, 0)$, the sequence

$$\left(\sum_i \sum_j \mathrm{E}\left[X\left(\left(\frac{i}{2^n}, \frac{j}{2^n}\right) \wedge t, \left(\frac{i+1}{2^n}, \frac{j+1}{2^n}\right) \wedge t\right] \mid \mathscr{F}^1_{i/2^n, j/2^n}\right], \quad n \geq 0\right)$$

is uniformly integrable. For every t set $\mathscr{A}(t) = \{(u, v] \times A : u \ll v \leq t, A \in \mathscr{F}^1_u\}$, and let $\mathscr{P}(t)$ be the σ-algebra generated by $\mathscr{A}(t)$. Set

$$\mu_X((u, v] \times A) = \mathrm{E}[1_A X(u, v]].$$

THEOREM 3.5. — Let $(X_t, \mathscr{F}_t, t \in \mathbb{R}^2_+)$ be a (Δ1) submartingale right-continuous in L_1, and let $(\mathscr{F}_t)$ satisfy (F4). The following are equivalent:

(*i*) (X_t) is of class (D1).

(*ii*) μ_X has a unique countably additive extension to $\mathscr{P}(t)$ for all t.

(*iii*) There exists a decomposition $X_t = M_t + A_t$, where (M_t) is a martingale for $(\mathscr{F}^1)$, (A_t) is a 1-increasing process, and both processes are adapted.

REFERENCES

[*1*] K. ASTBURY, Amarts indexed by directed sets, *Ann. Prob.*, t. **6**, 1978, p. 267-278.

[*2*] D. BAKRY, Sur la régularité des trajectoires des martingales à deux indices, *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, t. **50**, 1979, p. 149-157.

[*3*] D. BAKRY, Limites « quadrantales » des martingales, Colloque ENST CNET, Paris, 1980, *Lecture Notes in Math.*, t. **863**, 1981, p. 40-49.

[*4*] M. D. BRENNAN, Planar Semimartingales, *J. Multivariate Analysis*, t. **9**, 1979, p. 465-486.

[*5*] R. CAIROLI, Une inégalité pour martingales à indices multiples, Séminaire de Probabilité IV, Université de Strasbourg, *Lecture Notes in Math.*, t. **124**, 1970, p. 1-27.

[*6*] R. CAIROLI, Décomposition de processus à indices doubles, Séminaire de Probabilité V, Université de Strasbourg, *Lecture Notes in Math.*, t. **191**, 1971, p. 37-57.

[*7*] R. CAIROLI, J. B. WALSH, Stochastic integrals in the plane, *Acta M.*, t. **134**, 1975, p. 111-183.

[*8*] C. DELLACHERIE, P. A. MEYER, *Probabilité et Potentiel*, Herman 1975, 1980.

[*9*] M. DOZZI, On the decomposition and integration of two-parameter stochastic processes, Colloque ENST-CNET, Paris 1980, *Lecture Notes in Math.*, t. **863**, 1981, p. 162-171.

[*10*] G. A. EDGAR, L. SUCHESTON, Amarts: A class of asymptotic martingales, A: Discrete parameter, B: Continuous parameter, *J. Multivariate Analysis*, t. **5**, 1976, p. 193-221; p. 572-591.

[*11*] G. A. EDGAR, L. SUCHESTON, Démonstration de lois des grands nombres par les sous-martingales descendantes, *C. R. Acad. Sci. Paris*, Série I, t. 292, 1981, p. 967-969.

[*12*] J. P. FOUQUE, A. MILLET, Régularité des martingales fortes à plusieurs indices, *C. R. Acad. Sci. Paris*, Série A, t. **290**, 1980, p. 773-776.

[*13*] A. MILLET, Convergence and regularity of strong submartingales, Colloque ENST-CNET, Paris 1980, *Lecture Notes in Math.*, t. **863**, 1981, p. 50-58.

[14] A. Millet, L. Sucheston, On regularity of multiparameter amarts and martingales, *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, t. **56**, 1981, p. 21-45.
[15] A. Millet, L. Sucheston, Demi convergence des processus à deux indices, *Ann. Inst. Henri Poincaré*, to appear.
[16] J. Neveu, *Discrete parameter martingales*, North-Holland, Amsterdam, 1975.
[17] J. B. Walsh, Convergence and regularity of multiparameter strong martingales, *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, t. **46**, 1979, p. 177-192.

(Manuscrit reçu le 21 septembre 1981)