ANNALES DE L'I. H. P., SECTION B

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Annales de l'I. H. P., section B, tome 18, n° 3 (1982), p. 293-304
<http://www.numdam.org/item?id=AIHPB_1982__18_3_293_0>

Ann. Inst. Henri Poincaré, Section B :
Vol. XVIII, n° 3, 1982, p. 293-304. *Calcul des Probabilités et Statistique.*

A restricted form of the theorem of Maurey-Pisier for the cotype in p-Banach spaces

by

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SOMMAIRE. — Dans cet article nous exposons une version, non générale, du théorème de Maurey-Pisier pour le cotype dans le cas des espaces p-Banach ($0 < p \leqslant 1$). Le correspondant résultat pour le type a été obtenu par N. Kalton. Ici, nous obtenons le théorème suivant : soit X un espace p-Banach $0 < p \leqslant 1$ et $q_X = \inf \{ q > 0 ;$ l'identité est $(q, 1)$-sommant $\}$, alors, si $q_X < \infty$, l^{q_X} est finiment représentable dans X et

$$q_X = \sup \{ q ; \text{l'injection } l^q \to l^\infty \text{ est finiment factorisable dans X} \}.$$

ABSTRACT. — In this paper we give a restricted version of the theorem of Maurey-Pisier for the cotype in the general case of p-Banach spaces $0 < p \leqslant 1$. The result for the type has been obtained by Kalton in a unpublished paper [*2*]. Exactly, we prove the following theorem: « Let X be a p-Banach space $0 < p \leqslant 1$ and

$$q_X = \inf \{ q > 0 ; \text{the identity is } (q, 1)\text{-summing} \}.$$

If $q_X < \infty$, then

$$q_X = \sup \{ q ; \text{the embedding } l^q \to l^\infty \text{ is finitely factorizable through X} \}$$

and l^{q_X} is finitely representable in X ».

Let X be a real vector space and $0 < p \leqslant 1$. A p-convex norm on X is a mapping $x \to \|x\|$ of X into $\mathbb{R}_+$ satisfying the conditions:

i) $\|x\| > 0$ if $x \neq 0$, $x \in X$,

ii) $\|ax\| = |a| \, \|x\|$, $a \in \mathbb{R}$, $x \in X$,

iii) $\|x + y\|^p \leqslant \|x\|^p + \|y\|^p$, $x, y \in X$.

A p-convex norm induces a locally bounded topology on X; if X is complete with respect to this topology we say that X is a p-Banach space.

The identity in a Banach space X is $(q, 1)$-summing $0 < q \leqslant \infty$ if there exists a constant C such that for each finite sequence $x_1, \ldots, x_n \in X$ we have

$$\left(\sum_{i=1}^{n} \|x_i\|^q\right)^{1/q} \leqslant C \sup \left\{ \sum_{1}^{n} |\langle x_i, \xi \rangle| \, ; \, \xi \in X', \|\xi\| \leqslant 1 \right\}$$
$$= C \sup \left\{ \left\| \sum_{1}^{n} a_i x_i \right\| ; \, a_i \in R, \sup_{1 \leqslant i \leqslant n} |a_i| \leqslant 1 \right\}$$

($\sup\limits_{1 \leqslant i \leqslant n} \|x_i\|$, if $q = \infty$). There are p-Banach spaces with trivial dual and because of this, we must adopt the second expression as the definition in our case.

It is not difficult to see that the above definition is equivalent to the following

DÉFINITION 1. — *The identity in a p-Banach space* X *is* $(q, 1)$-*summing* $0 < q \leqslant \infty$, if there exists a constant C such that

$$\left(\sum_{i=1}^{n} \|x_i\|^q\right)^{1/q} \leqslant C \sup_{\varepsilon_i = \pm 1} \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\|$$

whenever $x_1, \ldots, x_n \in X$. It is clear that the identity is always $(\infty, 1)$-summing for every p-Banach space, because

$$\|x_i\| = \frac{1}{2} \left\| \sum_{j=1}^{n} x_j + x_i - \sum_{j \neq i} x_j \right\|$$
$$\leqslant \frac{1}{2} \left(\left\| \sum_{j=1}^{n} x_j \right\|^p + \left\| x_i - \sum_{j \neq i} x_j \right\|^p \right)^{1/p} \leqslant 2^{1/p - 1} \sup_{\varepsilon_j = \pm 1} \left\| \sum_{j=1}^{n} \varepsilon_j x_j \right\|$$

We denote by $q_X = \inf \{ q \, ; \text{ the identity in X is } (q, 1)\text{-summing} \}$.

Another concept which we shall use is that of finite factorization.

DÉFINITION 2. — *The embedding* $l^r \to l^q$ $(0 < r \leqslant q \leqslant \infty)$ *is finitely factorizable through a p-Banach space* X, (f. f. t. X), if, for each $\varepsilon > 0$ and for each $n \in \mathbb{N}$, there exist $x_1, \ldots, x_n$ in X, depending on ε and n, such that

$$(1-\varepsilon)\left(\sum_{i=1}^{n} |\lambda_i|^q\right)^{1/q} \leqslant \left\|\sum_{i=1}^{n} \lambda_i x_i\right\| \leqslant (1+\varepsilon)\left(\sum_{i=1}^{n} |\lambda_i|^r\right)^{1/r}$$

whenever $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, $\left(\text{if } q = \infty, \sup_{1 \leqslant i \leqslant n} |\lambda_i| \text{ instead of } \left(\sum_{i=1}^{n} |\lambda_i|^q\right)^{1/q}\right)$.

When the embedding $l^q \to l^q$ is f. f. t. X, we shall say that l^q *is finitely representable in* X. More generaly, *a p-Banach space* Y *is finitely representable in* X if for each $\varepsilon > 0$ and each finite sequence $y_1, \ldots, y_n \in Y$, there exist $x_1, \ldots, x_n \in X$ such that

$$(1-\varepsilon)\left\|\sum_{1}^{n} \lambda_i y_i\right\| \leqslant \left\|\sum_{1}^{n} \lambda_i x_i\right\| \leqslant (1+\varepsilon)\left\|\sum_{1}^{n} \lambda_i y_i\right\|$$

whenever $\lambda_1, \ldots, \lambda_n \in \mathbf{R}$. The theorem of Dvoretzky-Rogers proves that l^2 is finitely representable on X, if X is a Banach space. This is also true in the case of p-Banach spaces (Kalton [*2*]), and thus, necessarily we have $q_X \geqslant 2$ and the embedding $l^2 \to l^\infty$ is f. f. t. X. Moreover it is trivial that $\sup\{ q ; l^q \to l^\infty \text{ is f. f. t. X} \} \leqslant q_X$. Our main result in this paper is that l^{q_X} is finitely representable in X in the case $q(X) < \infty$ and, hence, the above inequality becomes an equality. This is a restricted version of the theorem of Maurey-Pisier for the cotype, that says that, for Banach spaces, $\sup\{ q ; l^q \to l^\infty \text{ is f. f. t. X} \} = q_X = \inf\{ q ; \text{X is of cotype } q\text{-Rademacher} \}$ and l^{q_X} is finitely representable in X.

A Banach space X is of cotype q, $2 \leqslant q \leqslant \infty$ if these exists a constant C such that for each finite sequence $x_1, \ldots, x_n \in X$

$$\left(\sum_{i=1}^{n} \|x_i\|^q\right)^{1/q} \leqslant C\int_0^1 \left\|\sum_{i=1}^{n} r_i(t) x_i\right\| dt$$

where $r_i(t)$, $1 \leqslant i \leqslant n$ are the Rademacher functions. This definition make sense for p-Banach spaces, and if a p-Banach space is of cotype q, necessarily the identity in X is $(q, 1)$-summing. We think that in our general case it is also true the general version of the theorem of Maurey-Pisier, i. e. $\inf\{ q ; \text{X is of cotype } q \} = q_x$, but, so far, we have not been able to prove this. The restriction $q_X < \infty$ in our theorem is based in the application of a theorem of Krivine that we must use. For Banach spaces, the case $q_X = \infty$ is very simple but in this general situation it is more complicated.

THEOREM 3. — *Let* X *be a real p-Banach space. If* $q_X < \infty$, *then* l^{q_X} *is finitely representable in* X.

Before proving this theorem, we need the notion of ultrapower of a p-Banach space and some other auxiliary results.

Let I be and infinite set and let $\mathcal{U}$ be a non-trivial ultrafilter in I, we denote by $l^\infty(\mathrm{I}, \mathrm{X}) = \{ f : \mathrm{I} \to \mathrm{X} \text{ bounded} \}$,

$$C_{0,\mathcal{U}}(\mathrm{I}, \mathrm{X}) = \{ f \in l^\infty(\mathrm{I}, \mathrm{X}) ; \lim_{\mathcal{U}} \| f(i) \| = 0 \} .$$

Then we define *the ultrapower* $\mathrm{X}^{\mathrm{I}}_{\mathcal{U}}$ as $l^\infty(\mathrm{I}, \mathrm{X})/C_{0,\mathcal{U}}(\mathrm{I}, \mathrm{X})$.

The mapping $\| f \| = \lim_{\mathcal{U}} \| f(i) \|$ is a p-convex-norme in $\mathrm{X}^{\mathrm{I}}_{\mathcal{U}}$, so that $\mathrm{X}^{\mathrm{I}}_{\mathcal{U}}$ is a p-Banach space wich contains X isometricaly. As in the case of Banach spaces, a p-Banach space Y is finitely representable in X if and only if Y is isometric to a closed subspace of some ultrapower of X -see [*4*]).

The deep result on which our theorem is based, is a theorem of Krivine, which can be adapted to p-Banach spaces. The result of Krivine is valid in Banach lattices and particulary in Banach spaces with a inconditional basis. In our situation, we have no inconditional basis or lattice sequences, but this turns out to be immaterial because we can use the sign-invariant sequences defined by Kalton ([*2*])

DÉFINITIONS 4. — A sequence $\{ e_n \}_1^\infty$ in a p-Banach space X is *sign-invariant* if $\left\| \sum_{i=1}^{n} \varepsilon_i t_i e_i \right\| = \left\| \sum_{i=1}^{n} t_i e_i \right\|$ when $t_1, \ldots, t_n \in \mathbb{R}$ and $\varepsilon_i = \pm 1$, $1 \leqslant i \leqslant n$.

If $\{ e_n \}_1^\infty$ is sign-invariant there exists a constant $C > 0$, depending only on p, such that

$$\left\| \sum_{i=1}^{n} \lambda_i t_i e_i \right\| \leqslant C \sup_{1 \leqslant i \leqslant n} | \lambda_i | \left\| \sum_{i=1}^{n} t_i e_i \right\|$$

when $\lambda_1, \ldots, \lambda_n, t_1, \ldots, t_n \in \mathbb{R}$.

A sequence $\{ e_n \}_1^\infty$ in a p-Banach space X is *invariant for spreading* or simply *spreading* if

$$\left\| \sum_{i=1}^{k} \lambda_i e_{m_i} \right\| = \left\| \sum_{i=1}^{k} \lambda_i e_i \right\|$$

when $m_1 < \ldots < m_k$, $k \in \mathrm{N}$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$.

If $\{e_n\}_1^\infty$ is spreading in X and $\|e_1 - e_2\| = \delta > 0$, necessarily the e_n's are linearly independent vectors. Indeed, if $\sum_{i=1}^{n} \lambda_i e_i = 0$, we have $\lambda_n = 0$. because

$$\delta |\lambda_n| = \|\lambda_n(e_n - e_{n+1})\| = \left\| \sum_{i=1}^{n} \lambda_i e_i - \sum_{i=1}^{n-1} \lambda_i e_i - \lambda_n e_{n+1} \right\|$$

$$\leqslant \left(\left\| \sum_{i=1}^{n} \lambda_i e_i \right\|^p + \left\| \sum_{i=1}^{n-1} \lambda_i e_i + \lambda_n e_{n+1} \right\|^p \right)^{1/p} = 2^{1/p} \left\| \sum_{i=1}^{n} \lambda_i e_i \right\| = 0$$

In this situation, we say that *two finite linear combinations* $x = \sum_{i=1}^{k} \lambda_i e_{m_i}$ *and* $y = \sum_{j=1}^{l} \mu_j e_{n_j}$ *are disjoint* if $m_i \neq n_j$ for all $1 \leqslant i \leqslant k$, and $1 \leqslant j \leqslant l$.

Now, we shall enunciate the following auxiliary lemma; its proof can be adapted without special difficulties from Krivine (cf. Th. II, 1 of [3]).

LEMMA 5. — *Let* X *be a p-Banach space with* $q_X < \infty$ *and let* $\{e_n\}_1^\infty$ *be a sign-invariant spreading sequence in* X. *Then there exist* $r > 0$ *and* $K > 0$ *such that*

$$\left(\sum_{i=1}^{m} \|x_i\|^r \right)^{1/r} \leqslant K \left\| \sum_{i=1}^{m} x_i \right\|$$

where the x_i's $(1 \leqslant i \leqslant m)$ *are pairwise disjoint finite linear combinations of* e_n's.

Now, we shall state the theorem of Krivine that we must use (Theorems II.2 and III.1 of [3]). The statement given here is a generalization of this theorem to the case of p-Banach spaces, but we omit the proof since it is the same as Krivine's with a few modifications which are necessary to replace Banach lattices by our more general situation.

THEOREM 6. — *Let* X *be a p-Banach space with* $q_X < \infty$, *and let* $\{e_n\}_1^\infty$ *be a sing-invariant spreading sequence in* X *such that* $\|e_1 - e_2\| > 0$, *and* $\|e_n\| = 1$, $n \in \mathbb{N}$. *Let* $f_n = \sum_{k=1}^{2^n} e_k$ *(necessarily* $\sup_n \|f_n\| = \infty$*), if*

$$2^{-1/q} = \inf\{\lambda > 0 ; \lim_{n\to\infty} \lambda^n \|f_n\| = \infty\}$$

l^q *is finitely representable in* X.

(If $\sup_n \| f_n \| = M < \infty$, for each n and $\varepsilon_i = \pm 1$, $1 \leqslant i \leqslant n$

$$\left\| \sum_1^n \varepsilon_i e_i \right\| = \left\| \sum_1^n e_i \right\| \leqslant 2^{1/p-1} M$$

and then, $q_X = \infty$).

Let X be a p-Banach space. If $2 \leqslant q < \infty$ and $n \in \mathbb{N}$, we define $c_q(n)$ as the smallest positive constant c such that for each n elements $x_1, \ldots, x_n$ of X

$$\left(\sum_{i=1}^n \| x_i \|^q \right)^{1/q} \leqslant c \sup_{\varepsilon_i = \pm 1} \left\| \sum_{i=1}^n \varepsilon_i x_i \right\|$$

It is trivial that $c_q(1) = 1$, $c_q(n) \leqslant 2^{1/p-1} n^{1/q}$ and that the sequence $c_q(n)$ is increasing.

The following lemma, with standard proof, replaces lemmas 1.2 and 1.3 of [5].

LEMMA 7.

i) The sequence $c_q(n)$ is submultiplicative.

ii) Let $N > 1$ be an integer; if $c_q(\mathrm{N}) = \mathrm{N}^{1/q - 1/t}$ $(t \geqslant q)$ then $q_X \leqslant t$.

COROLLARY 8. — *If $2 \leqslant q < q_X$ and $\alpha = \frac{1}{2}\left(1 - \frac{q}{q_X}\right) > 0$, then*

$$\lim_{n \to \infty} \frac{n^\alpha}{c_q(n)^q} = 0 .$$

Proof. — If $2 \leqslant q < q_X$, necessarily $c_q(n) \geqslant n^{1/q - 1/q_X}$. #

Now, we shall prove a Proposition which is essencially based in [6]. We must introduce a few changes so that the arguments work in our special situation. Moreover, we cannot use the extraction theorem of Brunel Sucheston; in particular, if $\{ e_n \}_1^\infty$ is a spreading sequence, then $\{ e_{2n-1} - e_{2n} \}_1^\infty$ is not necessarily an unconditional basic sequence.

PROPOSITION 9. — *Let X be a p-Banach space with $2 < q_X < \infty$. For each $2 \leqslant q < q_X$, for each $0 < \delta < 1$ and for each integer n, there exists a subset $\mathrm{B} \subseteq \mathrm{X}$ such that, $|\mathrm{B}| > n$, $1 - \delta \leqslant \| x \| \leqslant 1$ for every $x \in \mathrm{B}$* and

$$\sup_{\varepsilon_i = \pm 1} \left\| \sum_{x_i \in \mathrm{T}} \varepsilon_i x_i \right\| \leqslant 2^{1/p} | \mathrm{T} |^{1/q}$$

whenever $\mathrm{T} \subseteq \mathrm{B}$ and $|\mathrm{T}| \leqslant n$.

Proof. — We use the same notation of [*6*]. Let q, δ be given; put $\alpha = \frac{q}{2}\left(\frac{1}{q} - \frac{1}{q_X}\right)$ and choose $\varepsilon > 0$ such that $1 < 2^q(1-\varepsilon)^{q+1}$. By Corollary 8, there exists $N_0 \in \mathbb{N}$ such that if $N > N_0$, $N^\alpha < c_q(N)^q \varepsilon_1$, being $\varepsilon_1 = \varepsilon[1-(1-\delta)^q].(1-\varepsilon)^q.2^{-q(1/p-1)}$; we put $w = 1 - \delta > 0$.

By definition of $c_q(N)$, there are $x_1, \ldots, x_N \in X$ with $\max_{1 \leqslant i \leqslant N} \| x_i \| = 1$ such that

$$c_q(N)(1-\varepsilon) \sup_{\varepsilon_i = \pm 1} \left\| \sum_1^N \varepsilon_i x_i \right\| < \left(\sum_1^N \| x_i \|^q \right)^{1/q}$$

Define, for each $j \geqslant 1$ and $i \geqslant 1$

$$A_j = \{ i ; w^j < \| x_i \| \leqslant w^{j-1} \}, \qquad \hat{x}_i = \frac{x_i}{w^{j-1}} \qquad \text{for} \quad i \in A_j$$

If $| A_j | \leqslant N^\alpha$, we set $A_{j,0} = A_j$ and do nothing else with this set; if $| A_j | > N^\alpha$, then we choose a subset $A_{j,1} \subseteq A_j$ with $| A_{j,1} | \leqslant N^\alpha$ for which

$$\sup_{\varepsilon_i = \pm 1} \left\| \sum_{i \in A_{j,1}} \varepsilon_i \hat{x}_i \right\| / | A_{j,1} |^{1/q}$$

is maximal.

If $| A_j - A_{j,1} | \leqslant N^\alpha$, we set $A_{j,0} = A_j - A_{j,1}$, if not, there exists an other subset $A_{j,2} \subseteq A_j - A_{j,1}$ with the same properties, and so on. We can obtain a disjoint partition of $A_j = A_{j,1} \cup \ldots \cup A_{j,n_j} \cup A_{j,0}$ such that $| A_{j,p} | \leqslant N^\alpha$, $0 \leqslant p \leqslant m_j$ and if $p > 0$ $\sup_{\varepsilon_i = \pm 1} \left\| \sum_{i \in A_{j,p}} \varepsilon_i \hat{x}_i \right\| / | A_{j,p} |^{1/q}$ is maximal among all the subsets $T \subseteq A_j - (A_j \cup \ldots \cup A_{j,p-1})$ such that $| T | \leqslant N^\alpha$.

Now, the proof goes through almost verbation to [*6*], and then we obtain a subset $B_{s_0} \subseteq \{ 1, 2, \ldots, N \}$ such that $1 - \delta \leqslant \| \hat{x}_i \| \leqslant 1$ if $i \in B_{s_0}$, $| B_{s_0} | \leqslant N^\alpha$ and

$$\sup_{\varepsilon_i = \pm 1} \left\| \sum_{i \in B_{s_0}} \varepsilon_i \hat{x}_i \right\| \leqslant 2^{1/p} | B_{s_0} |^{1/q}$$

We call $B = \{ \hat{x}_i \mid i \in A_j - (A_{j,1} \cup \ldots \cup A_{j,p}) \}$ if $B_{s_0} = A_{j,p+1}$; clearly $| B | > N^\alpha$, $1 - \delta \leqslant \| \hat{x}_i \| \leqslant 1$ and if $T \subseteq B$ with $| T | \leqslant N^\alpha$

$$\sup_{\varepsilon_i = \pm 1} \left\| \sum_{\hat{x}_i \in T} \varepsilon_i \hat{x}_i \right\| \leqslant 2^{1/p}.| T |^{1/q}$$

by the maximality of B_{s_0}.

To conclude, for each integer n, we choose $N > \max\{N_0, n^{1/\alpha}\}$ apply the preceeding construccion to this N, and the assertion of the proposition follows.

Remark. — By the above proposition, for each $2 \leqslant q < q_X$, for each, $0 < \delta < 1$ and for each $n \in N$ we can choose n vectors $x_1^{q,n,\delta}, \ldots, x_n^{q,n,\delta}$ such that $1 - \delta \leqslant \| x_i^{q,n,\delta} \| \leqslant 1$, $1 \leqslant i \leqslant n$, and

$$\sup_{\varepsilon_i = \pm 1} \left\| \sum_{T} \varepsilon_i x_i^{q,n,\delta} \right\| \leqslant 2^{1/p} | T |^{1/q}$$

if $T \subseteq \{ x_1^{q,n,\delta}, \ldots, x_n^{q,n,\delta} \}$.

By passing to consecutive ultrapowers we obtain the

PROPOSITION 10. — *Let* X *be a Banach space with* $2 < q_X < \infty$. *There exist an ultrapower* Y *of* X *and a sequence* $\{ x_n \}_1^\infty$ *in* Y *such that* $\| x_n \| = 1\ \forall n$ *and*

$$\sup_{\varepsilon_i = \pm 1} \left\| \sum_{i \in \mathbb{N}_1} \varepsilon_i x_i \right\| \leqslant 2^{1/p} | \mathbb{N}_1 |^{1/q_X}$$

whenever $\mathbb{N}_1$ *is a finite subset of* $\mathbb{N}$.

Proof. — Let $\mathcal{U}$ a non trivial ultrafiltre in $\mathbb{N}$. We shall do the proof in three steps:

i) Let $X' = X_{\mathcal{U}}^{N}$, and let $\{ \delta_m \}_{m \in N}$ be an increasing sequence converging to 1. For each $2 \leqslant q < q_X$ and $n \in N$, we define n vectors of

$$X' : y_k^{q,n} = \left(x_k^{q,n,\delta_m} \right)_{m \in N}, \qquad 1 \leqslant k \leqslant n .$$

Trivially $\| y_k^{q,n} \| = 1$, $1 \leqslant k \leqslant n$ and if $T \subseteq \{ y_1^{q,n}, \ldots, y_n^{q,n} \}$

$$\sup_{\varepsilon_i = \pm 1} \left\| \sum_{T} \varepsilon_i y_i^{q,n} \right\| \leqslant 2^{1/p} . | T |^{1/q}$$

ii) Let $X'' = X_{\mathcal{U}}^{\prime N}$, for each q, $2 \leqslant q < q_X$ we define a sequence in X''

$$Z_n^q = (0, \overset{n-1}{\ldots}, 0, y_n^{q,n}, y_n^{q,n+1}, \ldots) \qquad n \in N$$

It is easy to prove that $\| Z_n^q \| = 1$, $n \in N$ and if $\mathbb{N}_1$ is a finite subset of $\mathbb{N}$

$$\sup_{\varepsilon_i = \pm 1} \left\| \sum_{i \in \mathbb{N}_1} \varepsilon_i Z_i^q \right\| \leqslant 2^{1/p} | \mathbb{N}_1 |^{1/q}$$

iii) Let $X''' = X''^{N}_{\mathscr{U}}$ and let $(q_m)_{m\in N}$ be an increasing sequence or real numbers converging to q_X. We define a sequence in X'''

$$x_n = (Z_n^{q_m})_{m\in\mathbb{N}} \qquad n\in\mathbb{N}$$

This sequence verifies the result, because $Y = X'''$ is an ultrapower of X. #

Now, we need a sign-invariant spreading sequence to use the result of Krivine (Theorem 6). This is done in the next proposition by applying Theorem 3.2 of [*2*] and Theorem I.1 of [*3*].

PROPOSITION 11. — *Let* X *be a p-Banach space with* $2 < q_X < \infty$. *Then, there exists a sign-invariant spreading sequence* $\{v_n\}_1^\infty$ *in some ultrapower of* X, *such that* $\| v_n \| = 1, n\in\mathbb{N}, \| v_1 - v_2 \| > 0, \left\| \sum_{i=1}^n v_i \right\| \leqslant 2^{1/p} n^{1/q_x}, n\in\mathbb{N}$.

Proof. — Let X_0 be the ultrapower obtained in the preceeding proposition and let $\{x_n\}_{n\in\mathbb{N}}$ be the corresponding sequence. We can apply theorem 3.2 of [*2*] to obtain a spreading sequence $\{y_n\}_{n\in N}$ in an ultrapower Y of X_0, such that

$$\left\| \sum_{i=1}^n \lambda_i y_i \right\| = \lim_{\substack{m_n \\ \mathscr{U}}} \dots \lim_{\substack{m_1 \\ \mathscr{U}}} \left\| \sum_{i=1}^n \lambda_i x_{m_i} \right\|$$

where $\mathscr{U}$ is a non trivial ultrafiltre in $\mathbb{N}$ and $\lambda_1 \dots \lambda_n \in \mathbb{R}$. The above equality easily proves that $\| y_n \| = 1$, $n\in N$ and

$$\sup_{\varepsilon_i=\pm 1} \left\| \sum_{i=1}^n \varepsilon_i y_i \right\| \leqslant 2^{1/p} n^{1/q_x}$$

Necessarily $\| y_1 - y_2 \| > 0$, because otherwise $y_1 = y_n$, $\forall n\in\mathbb{N}$, and then

$$2^{1/p} n^{1/q_x} \geqslant \sup_{\varepsilon_i=\pm 1} \left\| \sum_{i=1}^n \varepsilon_i y_i \right\| = n$$

which is not possible.

Moreover, $\sup_n \left\| \sum_{k=1}^n y_{2k-1} - y_{2k} \right\| = \infty$. Indeed, if this sup were equal to $M < \infty$, given $\xi_k = \pm 1$, $1 \leqslant k \leqslant n$, denoting by $A = \{ k ; \xi_k = 1 \}$ and $B = \{ k ; \xi_k = -1 \}$

$$\left\| \sum_{k=1}^{n} \xi_k (y_{2k-1} - y_{2k}) \right\|^p \leqslant \left\| \sum_{k \in A} (y_{2k-1} - y_{2k}) \right\|^p + \left\| \sum_{k \in B} (y_{2k-1} - y_{2k}) \right\|^p \leqslant 2M^p$$

because the sequence $\{ y_{2k-1} - y_{2k} \}_k$ is also spreading; then, as $\| y_{2k-1} - y_{2k} \| = \| y_1 - y_2 \| > 0$, $\forall k$ the space would have $q_Y = \infty$, but this is not possible since $q_X = q_Y$ (because Y contains X isometrically and is finitely representable in X). We put $M_n = \left\| \sum_{k=1}^{n} y_{2k-1} - y_{2k} \right\| \to \infty$ and, let $q < q_X$ be given. Since

$$M_n . n^{-1/q} \leqslant 2^{1/p + 1/q_X} . n^{1/q_X - 1/q}$$

$\lim_n M_n n^{-1/q} = 0$, and thus, there exists an increasing sequence of intergers $\{ n_r \}_{r \in \mathbb{N}}$ such that $M_{n_r} n_r^{-1/q} \geqslant M_n n^{-1/q}$ if $n_r \leqslant n$. Now, we define

$$u_k^r = \frac{1}{M_{n_r}} \sum_{i=1}^{n_r} (y_{2(k-1)n_r + 2i - 1} - y_{2(k-1)n_r + 2i}), \qquad r, k \in \mathbb{N}$$

Clearly $\| u_k^r \| = 1$ because $\{ y_n \}$ is spreading. We considere $Y' = Y_{\mathcal{U}}^{\mathbb{N}}$, where $\mathcal{U}$ is a non trivial ultrafilter of $\mathbb{N}$, and the sequence $u_k = (u_k^r)_{r \in \mathbb{N}}$ in Y'.

Repeating the arguments of Theorem I.1 of [3] it is possible to see that the sequence $(u_k)_{k \in \mathbb{N}}$ is spreading and sign-invariant. Moreover $\| u_k \| = 1$ and

$$\begin{aligned} \left\| \sum_{k=1}^{m} u_k \right\| &= \lim_{r, \mathcal{U}} \left\| \sum_{k=1}^{m} u_k^r \right\| \\ &= \lim_{r, \mathcal{U}} \frac{1}{M_{n_r}} \left\| \sum_{k=1}^{m} \sum_{i=1}^{n_r} (y_{2(k-1)n_r + 2i - 1} - y_{2(k-1)n_r + 2i}) \right\| \\ &= \lim_{r, \mathcal{U}} \frac{1}{M_{n_r}} \left\| \sum_{i=1}^{m.n_r} (y_{2i-1} - y_{2i}) \right\| \\ &= \lim_{r, \mathcal{U}} \frac{n_r^{-1/q}}{(m.n_r)^{-1/q}} = m^{1/q} \end{aligned}$$

Finally, by choosing a sequence of real numbers $q_m \nearrow q_X$ and passing to another ultrapower we obtain the result. #

Proof of the Theorem 3. — Let X be a p-Banach space with $q_X < \infty$. If $q_x = 2$, there is nothing to prove, because it is Dvoretzky-Rogers Theorem. We suppose now that $2 < q_X$, and let Y and $\{ v_n \}_1^\infty$ be the ultrapower of X and the corresponding sequence in Y that verify the preceeding proposition; we can apply theorem 6; if we put $f_n = \sum_{k=1}^{2^n} v_k$,

$$\| f_n \| \leqslant 2^{1/p} . 2^{n/q_X}$$

and then, when $q < q_X$, $2^{-n/q} \| f_n \| \leqslant 2^{1/p} . 2^{n\left(\frac{1}{q_X} - \frac{1}{q}\right)} \underset{n \to \infty}{\longrightarrow} 0$ and when $q > q_X$, as the identity is $(q, 1)$-summing in X so it is in Y (Y contains X isometrically and Y is finitely representable in X). There exists $C_1 > 0$ such that

$$2^{n/q} = \left(\sum_{k=1}^{2^n} \| v_k \|^q \right)^{1/q} \leqslant C_q \sup_{\varepsilon_k = \pm 1} \left\| \sum_{k=1}^{2^n} \varepsilon_k v_k \right\| = \| f_n \| .$$

Hence

$$2^{-n/q} \| f_n \| \geqslant C_q^{-1}$$

Moreover

$$\lim_n 2^{-n/q} \| f_n \| = \infty \qquad \text{if} \qquad q > q_X$$

Thus

$$2^{-1/q_X} = \inf \{ \lambda > 0 ; \lim_n \lambda^n \| f_n \| = \infty \}$$

and l^{q_X} is finitely representable in Y and in X. q. e. d. #

ACKNOWLEDGEMENTS

I am very grateful to J. L. Rubio de Francia for many fruitful conversations while this work was done. I wish to thank G. Pisier for making me be interested in this subject during his stay in the University of Zaragoza, and N. J. Kalton for sending his unpublished manuscript [2].

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(Manuscrit reçu le 3 décembre 1981)