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Stochastic integration on nuclear spaces and its applications

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ABSTRACT. — This work is devoted to the study of stochastic processes with values in the dual of a nuclear space and to the construction of stochastic integration with respect to some classes of these processes. This permits us to establish a stochastic calculus and we give some applications to Physics and stochastic partial differential equations.

INTRODUCTION

In the recent years, because of the great number of the problems coming from Physics and Applied Mathematics, the theory of the stochastic processes with values in the infinite dimensional normed vector spaces has been considerably developped (cf. [*7*] [*8*] and the references there in). There exists another class of locally convex spaces which are often encountered in practice, that is the nuclear spaces. For instance, an infinite particle Brownian motion branching process converges in law (cf. [*6*]) to a Markov process with values in the space of the tempered distributions when one accelerates the time scale of this process. This kind of problems have led us to study systematically the « stochastic processes » with values in the nuclear spaces. Between these « processes », the most interesting ones

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seem to be the « semimartingales »; since one can construct a stochastic calculus based on this class.

The absence of a single semi-norm defining the topology of the nuclear spaces obliged us to extend the concept of the stochastic process, this extension is made by defining the projective systems of stochastic processes and it seems to be the optimal one to construct a stochastic calculus not only on the nuclear spaces but also on the general locally convex spaces. The interest of the nuclear spaces is that the powerful tools of the theory of cylindrical measures are in their simplest form on these spaces, as the theorem of Minlos-Sazonov-Badrikian (cf. [*1*][*4*] [*14*] and [*15*]). Let us also indicate that the cylindrical processes defined on the normed spaces can be regarded as the processes in a nuclear space if there is a nuclear rigging of the normed space in the sense of [*4*].

We have supposed that the theory of the stochastic integration on the Hilbert spaces is known by the reader (cf. [*7*] [*8*]), for the nuclear spaces he is referred to [*5*] and [*12*]. In order to simplify the proofs we have made a hypothesis of bornology but most of the results obtained remain true under some minor modifications, without this hypothesis.

In the first section we give the basic definitions and some technical results. The second section is devoted to the construction of dual projections of the Radon-Nikodym derivatives of some vector measures with values in the dual of a nuclear space which are absolutely continuous in a certain sense, with respect to a given probability measure. Also, the definitions of the martingales, local martingales and the construction of the stochastic integrals using these « processes » are the contents of the second section. In the third section we define and study the semimartingales, fourth section is devoted to the integration by parts formula. In the fifth section, we extend Ito's formula to the distributions and the last section deals with the weak form of Feynman-Kac formula on the distributions. The representation that we obtain suggests in particular that the Feynman's path integrals should be regarded rather as the stochastic integrals than the deterministic ones.

Further applications to the stochastic flows, evolution equations and to Physics will be given in the forthcoming papers.

I. NOTATIONS AND PRELIMINAIRES

Φ denotes a locally convex, reflexive, complete bornological nuclear space whose topological dual Φ' is complete and nuclear under its strong topology, denoted by Φ'_β. Let us recall that Φ is called nuclear if there exists a neighbourhood base (of zero), say $\mathscr{U}$, such that for any $U \in \mathscr{U}$, there exists $V \in \mathscr{U}$, $V \subset U$, for which the canonical mapping $k(U, V)$: $\Phi(V) \to \Phi(U)$ is a nuclear mapping, where $\Phi(U)$ denotes the set of equivalence classes with respect to mod $p_U^{-1}(0)$, p_U being the gauge functional of U, completed under the norm topology induced by p_U (cf. [5] [12]). If B is a bounded, absolutely convex (i. e. convex and balanced) subset of Φ, we note by $\Phi[B]$ the completion of the subspace spanned by B with respect to the topology generated by the norm p_B i. e. the gauge functional of B. It is well known that in each nuclear space there exists a neighbourhood base $\mathscr{U}_h(\Phi)$ such that, for any $U \in \mathscr{U}_h(\Phi)$, $\Phi(U)$ is a separable Hilbert space whose dual can be identified by $\Phi'[U^0]$, U^0 being the polar of U, and Φ is (a subspace of) the projective limit of

$$\{ (\Phi(U), k(U)) : U \in \mathscr{U}_h(\Phi) \}$$

(cf. [12], p. 102), where $k(U)$ represents the canonical mapping from Φ onto $\Phi(U)$. Let us note also that any complete nuclear space is a Montel space (i.e. every bounded, closed set is compact). If $p \in [1, \infty)$, we denote by $l^p[\Phi']$ the space of the weakly p-summable sequences in Φ', i. e. $(u_n) \in l^p[\Phi']$ if

$$\sum_{n=1}^{\infty} | u_n(\phi) |^p < +\infty \quad \text{for any } \phi \in \Phi .$$

If $U \in \mathscr{U}_h(\Phi'_\beta)$, let

$$\varepsilon_U[(u_n)] = \sup_{\phi \in U^0} \left(\sum_{n=1}^{\infty} | u_n(\phi) |^p \right)^{1/p},$$

then, equipped with the coarsest topology making the seminorms $\{ \varepsilon_U : U \in \mathscr{U}_h(\Phi'_\beta) \}$ continuous $l^p[\Phi']$ is a locally convex space. Let us note by $l^p\{ \Phi' \}$ the space of absolutely p-summable series in Φ', i. e. $(u_n) \in l^p\{ \Phi' \}$ if

$$\pi_U[(u_n)] = \left(\sum_{n=1}^{\infty} (p_U(u_n))^p \right)^{1/p} < +\infty , \qquad \text{for any } U \in \mathscr{U}_h(\Phi'_\beta) .$$

$l^p\{ \Phi' \}$, equipped with the topology induced by the seminorms $\{ \pi_U : U \in \mathscr{U}_h(\Phi'_\beta) \}$ is also a locally convex space. If Φ'_β is nuclear then $l^p[\Phi']$ and $l^p\{ \Phi' \}$ are topologically isomorphic, moreover this is a sufficient condition for the nuclearity of Φ'_β (cf. [5] [12]).

By $(\Omega, \mathscr{F}, P)$, we denote a completed probability space and $\{ \mathscr{F}_t ; t \geq 0 \}$

represents an increasing family of the sub-σ-algebras of $\mathcal{F}$ which is right continuous. We suppose that $\mathcal{F}_0$ contains all the P-negligeable subsets of Ω.

The concept of stochastic process will be generalized in the following manner:

DEFINITION I.1. — Let X be the set

$$\{ X^U : U \in \mathcal{U}_h(\Phi'_\beta) \},$$

where, for any $U \in \mathcal{U}_h(\Phi'_\beta)$, X^U is a stochastic process with values in the separable Hilbert space $\Phi'(U)$. X will be called a projective system (of stochastic process) if for any $V \subset U$, $V \in \mathcal{U}_h(\Phi'_\beta)$, the stochastic processes $k(U, V) \circ X^V$ and X^U are undistinguishable.

DEFINITION I.2. — Let X be a projective system of stochastic processes as above. We say that X has a limit in Φ' if there exists a mapping $X' : \mathbb{R}_+ \times \Omega \to \Phi'$ such that for any $t \geq 0$, $\phi \in \Phi$, the mapping $\omega \to \langle \phi, X'_t(\omega) \rangle$ is measurable and if for any $X^U \in X$, $k(U) \circ X'$ is a modification of X^U.

DEFINITION I.3. — *i*) Let X and Y be two projective systems of stochastic processes. X and Y are called undistinguishable (respectively, equivalent, etc.) if the stochastic processes X^U and Y^U are undistinguishable (respect. equivalent, etc.) for any $U \in \mathcal{U}_h(\Phi'_\beta)$.

ii) A projective system X is called right continuous (respectively left continuous, continuous, with left limits, etc.) if, for any $U \in \mathcal{U}_h(\Phi'_\beta)$, the stochastic process X^U is right continuous (resp. left continuous, continuous, with left limits, etc.) for almost all $\omega \in \Omega$ (in the strong topology of $\Phi'(U)$).

The following result will be useful in the sequel:

LEMMA I.1. — *i*) Suppose that X is a projective system of right continuous processes having left limits. Define X as

$$X_- = \{ (X^U_{t-}) : U \in \mathcal{U}_h(\Phi'_\beta) \}$$

where (X^U_{t-}) denotes the stochastic process obtained by taking the left hand side limits. Then X_- is a left continuous projective system.

ii) Suppose that Φ is a nuclear Fréchet space or strict inductive limit of a sequence of such spaces. Then any projective system X in Φ' has a limit (in Φ').

Proof. — *i*) is obvious. Suppose that Φ is a nuclear Fréchet space. Then, for any $t \geq 0$, $X_t = \{ X_t^U ; U \in \mathscr{U}_h(\Phi'_\beta) \}$ induces a cylindrical measure on Φ'. If (ϕ_n) converges to ϕ in Φ, then there exists a compact, absolutely convex subset of Φ for which $\Phi[B]$ is a separable Hilbert space such that (ϕ_n) converges to ϕ in the weak topology of $\Phi[B]$. Hence $X_t^{B^0}(\phi_n)$ converges to $X_t^{B^0}(\phi)$ in probability. This result implies that the mapping $\phi \to X_t(\phi)$ defined by

$$(I.1) \qquad X_t(\phi) = X_t^U(\phi) \qquad \text{if} \quad \phi \in \Phi[U^0]$$

is continuous on Φ with values in $L^0(\Omega, \mathscr{F}, P)$. Consequently it is o-decomposable, i. e., there exists a random variable X'_t with values in Φ' such that

$$X'_t(\phi) = X_t(\phi) \qquad a.\ e.$$

for any $\phi \in \Phi$ (cf. [*1*], p. 214). Hence

$$k(U)(X'_t) = X_t^U \qquad a.\ e.$$

for any $U \in \mathscr{U}_h(\Phi'_\beta)$.

If Φ is the strict inductive limit of (Φ_n), by Proposition 28, p. 94 of [*11*], Φ is topologically isomorphic to a quotient of the direct sum $\Sigma\Phi_n$. Denote by k the corresponding canonical mapping. By what we have shown above and by the Theorem of Minlos-Sazonov-Badrikian (cf. [*14*]), the restriction of X_t to Φ_n induces a Radon measure μ_n on Φ'_n. Then the product measure μ is a Radon measure on $\prod_{n=1}^{\infty} \Phi'_n$, hence its characteristic function g is continuous on $\Sigma\Phi_n$. Let f be the function defined by

$$\phi \to E(\exp iX_t(\phi)), \qquad i = \sqrt{-1}.$$

Then it is easy to see that

$$g = f \circ k.$$

Since k is an open mapping, f is also continuous, hence the linear mapping $\phi \mapsto X_t(\phi)$ is continuous (cf. [*1*]), Φ being nuclear, X_t is o-decomposable.

Q. E. D.

DEFINITION I.4. — Let X be a projective system of stochastic processes with a limit X′ in Φ'. We call a *g*-process the pair (X, X′). If there is no limit in Φ' then the projective system X will be called a *w*-process.

Remark. — In the following, if there is no confusion X and X′ will be denoted by the same letters. Note that if a projective system has a limit, then it is unique upto a modification, but the converse is not always true,

i. e. two different projective systems may have the limits one being the modification of the other. Of course, when the projective systems are right continuous then they are undistinguishable.

II. STOCHASTIC INTEGRALS AND DUAL PROJECTIONS

If μ is a vector measure on $(\mathbb{R}_+ \times \Omega, \mathscr{B}(\mathbb{R}_+) \otimes \mathscr{F})$ with values in Φ', μ is called of finite weak total variation if the measure μ_ϕ defined as $\mu_\phi(A) = \langle \phi, \mu(A) \rangle$ has a finite total variation for any $\phi \in \Phi$. μ is called of finite variation if

$$\mathrm{Var}_U(\mu) = \sup_I \sum_{i \in I} p_U(\mu(A_i)) < +\infty ,$$

for any $U \in \mathscr{U}_h(\Phi'_\beta)$, where the supremum is taken over all the measurable, countable partition of $\mathbb{R}_+ \times \Omega$. The following result is due to the special structure of the nuclear spaces:

LEMMA II.1. — Is μ is of finite weak total variation, then it is of finite variation.

Proof. — Let $\mathscr{P}$ be the set of all countable, measurable partitions of $\mathbb{R}_+ \times \Omega$. Since μ_ϕ is of finite total variation for any $\phi \in \Phi$, the set

$$\{ (\mu(A_i) : i \in I) : I \in \mathscr{P} \}$$

is bounded in $l^1[\Phi']$, hence it is bounded in $l^1\{\Phi'\}$ for the π-topology.

Q. E. D.

The following result will be used frequently in sequel:

THEOREM II.1. — Let μ be a vector measure on $(\mathbb{R}_+ \times \Omega, \mathscr{B}(\mathbb{R}_+) \otimes \mathscr{F})$ with values on Φ' such that it does not charge the evanescent sets and it is of finite total variation. Then there exists a unique right continuous g-process A (i. e. the corresponding projective system is right continuous) such that it is of integrable variation and

$$\langle \phi, \mu(X) \rangle = E \int_{0^-}^{\infty} X_s dA_s(\phi)$$

for any bounded, measurable, scalar process X. Moreover, there exists an ordinary sense right continuous stochastic process B with values in Φ' satisfying the above properties such that $k(U) \circ B$ and A^U are undistinguishable for any $U \in \mathscr{U}_h(\Phi'_\beta)$.

Proof. — If $U \in \mathscr{U}_h(\Phi'_\beta)$, let $\mu^U = k(U) \circ \mu$, then μ^U is with values in $\Phi'(U)$, it is of finite total variation and it does not charge the evanescent sets. Hence there exists a right continuous stochastic process A^U with values in $\Phi'(U)$, which is of integrable variation such that

$$\mu^U(X) = E\int_{0-}^{\infty} X_s dA_s^U \quad \text{and} \quad \text{Var}(\mu^U) = E\int_{0-}^{\infty} \| dA_s^U \|, \quad (\text{cf. } [7] \text{ and } [8]).$$

Then $A = \{ A^U : U \in \mathscr{U}_h(\Phi'_\beta) \}$ is the projective system for which we are looking. Denote by W the set of measures on $(\mathbb{R}_+ \times \Omega, \mathscr{B}(\mathbb{R}_+) \otimes \mathscr{F})$ normed by the total variation, which does not charge the evanescent sets. Then W is a Banach space and the closed graph theorem and the fact that Φ is bornological implies the continuity of the linear mapping

$$\phi \to \mu_\phi$$

on Φ with values in W. Since Φ is nuclear, this mapping can be represented as

$$\sum_{i=1}^{\infty} \lambda_i F_i \otimes v^i$$

(cf. [*12*]) where $(\lambda_i) \in l^1$ (i. e. the space of summable, scalar sequences), $(F_i) \subset \Phi'$ is equicontinuous and $(v^i) \subset W$ is bounded. Let K be a absolutely convex compact subset of Φ'_β containing (F_i) such that $\Phi'[K]$ is a separable Hilbert space. Then μ takes its values in $\Phi'[K]$ and it is of finite total variation. Hence there exists a $\Phi'[K]$-valued right continuous stochastic process $\tilde{B}$ (cf. [*7*] and [*8*]) such that

$$\mu(X) = E\int_{0-}^{\infty} X_s d\tilde{B}_s,$$

and the total variation of μ in $\Phi'[K]$ is equal to

$$E\int_{0-}^{\infty} \| d\tilde{B}_s \|_{\Phi'[K]}.$$

To complete the proof it is sufficient to take $B = i(U^0) \circ \tilde{B}$, where $i(U^0)$ is the injection $\Phi'[K^0] \hookrightarrow \Phi'_\beta$. Q. E. D.

The converse of this result is also true:

THEOREM II.2. — Let A be a weakly measurable mapping on $\mathbb{R}_+ \times \Omega$ in Φ' such that for any $\phi \in \Phi$, $(t, \omega) \to \langle \phi, A_t(\omega) \rangle$ has a modification $A(\phi)$ which is right continuous and of integrable variation. Then there exists a unique vector measure μ on $(\mathbb{R}_+ \times \Omega, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F})$ which is of finite total variation with values in Φ', which does not charge the evanescent sets and it satisfies the following relation:

$$\langle \phi, \mu(X) \rangle = E\int_{0-}^{\infty} X_s dA_s(\phi)$$

for any $\phi \in \Phi$ and for any bounded, measurable, scalar process X.

Proof. — If $\phi \in \Phi$, define μ_ϕ as

$$\mu_\phi(X) = E\int_{0-}^{\infty} X_s dA_s(\phi).$$

If $\tilde{A}(\phi)$ is another right continuous measurable modification of $\{ \langle \phi, A_t \rangle ; t \geq 0 \}$ then

$$E\int_{0-}^{\infty} X_s dA_s(\phi) = E\int_{0-}^{\infty} X_s d\tilde{A}_s(\phi),$$

hence μ_ϕ is well defined. The mapping $\phi \to \mu_\phi$ is linear on Φ with values in W (cf. the proof of Theorem II.1 for the notation). Suppose that B is an absolutely convex, compact subset of Φ such that $B^0 \in \mathcal{U}_h(\Phi'_\beta)$. If (ϕ_n) converges to ϕ in $\Phi[B]$ and (μ_{ϕ_n}) to ν in W, ν can be represented as (cf. [*3*] [*9*])

$$\nu(X) = E\int_{0-}^{\infty} X_s db_s$$

where b is a right continuous scalar process of integrable variation. If $X = 1_F 1_{]s,t]}$ with $F \in \mathcal{F}$, $s \leq t$, then

$$E[1_F(A_t(\phi_n) - A_s(\phi_n))] \underset{n\to\infty}{\longrightarrow} E[1_F(b_t - b_s)].$$

Moreover, for almost all $\omega \in \Omega$, we have

$$1_F(A_t(\phi_n) - A_s(\phi_n)) = 1_F \langle \phi_n, A_t - A_s \rangle \underset{n\to\infty}{\longrightarrow} 1_F \langle \phi, A_t - A_s \rangle.$$

Hence

$$E[1_F(b_t - b_s)] = E[1_F(A_t(\phi) - A_s(\phi))],$$

and $\nu(X) = \mu_\phi(X)$ for all simple processes. Consequently $\nu = \mu_\phi$ and the restriction of $\phi \to \mu_\phi$ to $\Phi[B]$ is continuous. This implies that $\phi \to \mu_\phi$ is a bounded mapping on Φ and Φ being bornological, it is continuous. Then the nuclear mapping $\phi \to \mu_\phi$ can be represented as

$$\phi \to \mu_\phi = \sum_{i=1}^{\infty} \lambda_i F_i(\phi) \nu^i$$

where $(\lambda_i) \in l^1$, $\{ F_i \} \subset \Phi'$ is equicontinuous and $(v^i) \subset W$ is bounded. Define μ as

$$\mu = \sum_{i=1}^{\infty} \lambda_i F_i \otimes v^i ,$$

then μ satisfies the properties announced in the theorem. If $\tilde{\mu}$ is another such measure, then it is obvious that

$$\langle \phi, \mu(X) \rangle = \langle \phi, \tilde{\mu}(X) \rangle$$

for any $\phi \in \Phi$ and bounded scalar measurable process X. Q. E. D.

COROLLARY II.1. — Under the hypothesis of Theorem II.2, the mapping A induces a right continuous g-process of integrable variation. Hence there exists a one-to-one correspondence between the measures of the type described above and the right continuous g-processes of integrable variation.

COROLLARY II.2. — Suppose that A is a projective system of right continuous processes which are of integrable variation. Then A has a limit in Φ'.

Proof. — Using the closed graph theorem and the fact that Φ is bornological, one can show that the mapping

$$\phi \to A_t(\phi) = A_t^U(\phi) \qquad \text{if} \qquad \phi \in \Phi[U^0] ,$$

is continuous on Φ with values in $L^1(\Omega, \mathscr{F}, P)$. Then the result follows from the theorem of Minlos-Sazanov-Badrikian. Q. E. D.

COROLLARY II.3. — Let μ be a vector measure on $(\mathbb{R}_+ \times \Omega, \mathscr{B}(\mathbb{R}_+) \otimes \mathscr{F})$ which does not charge the evanescent sets. Then the corresponding g-process A is previsible (respectively adapted) if and only if μ commutes with the previsible (resp. optional) projections of the bounded, scalar, measurable processes.

COROLLARY II.4. — Let A be a previsible, right continuous g-process of integrable variation. Then there exists two g-processes A^c and A^d such that A^c is continuous and A^d is purely discontinuous with $A = A^c + A^d$.

Proof. — Let $\tilde{B}$ be the ordinary sense process which we have constructed in the proof of Theorem II.1. Since $\tilde{B}$ is with values in some separable Hilbert space $\Phi'[K]$, it can be decomposed as

$$\tilde{B}_t = \tilde{B}_t^c + \sum_{n=1}^{\infty} F_n 1_{\{T_n \leq t\}} = \tilde{B}_t^c + \tilde{B}_t^d ,$$

where (T_n) is a sequence of previsible stopping times. Injecting $\tilde{B}^c$ and $\tilde{B}^d$ in Φ' completes the proof. Q. E. D.

The correspondence between the vector measures and the g-processes of integrable variation permits us to give the following

DEFINITION II.1. — Let μ be a vector measure of finite total variation on $(\mathbb{R}_+ \times \Omega, \mathscr{B}(\mathbb{R}_+) \otimes \mathscr{F})$ with values in Φ', which does not charge the evanescent sets. The vector measure defined by

$$\mu^0(X) = \mu(X^0) \qquad \text{(respectively } \mu^p(X) = \mu(X^p)),$$

when X runs in the set of bounded, measurable scalar stochastic-processes, is-called the optional (resp. previsible) projection of μ and the corresponding g-process denoted by A^1 (resp. A^3) is called the dual optional (resp. previsible) projection of A where A is the g-process corresponding to μ and X^0 (resp. X^p) represents the optional (resp. previsible) projection of X.

Remark. — By Theorem II.1 A^3 and A^1 are two right continuous g-processes of integrable variation.

DEFINITION II.2. — A right continuous g-process M with the projective system $\{ M^U : U \in \mathscr{U}_h(\Phi'_\beta) \}$ is called a martingale if M^U is a $\Phi'(U)$-valued martingale for any $U \in \mathscr{U}_h(\Phi'_\beta)$.

Now we can give a characterization of martingales which are of integrable variation:

PROPOSITION II.1. — In order that a right continuous g-process M of integrable variation to be a martingale it is necessary and sufficient that it is of the form

$$M = M_0 + A - A^3$$

where M_0 is $\{ M_0^U : U \in \mathscr{U}_h(\Phi'_\beta) \}$ and A is a g-process of integrable variation.

Proof. — Let us first note that the above equality should be understood as

$$M^U = M_0^U + A^U - A^{U,3} \qquad \text{for any } U \in \mathscr{U}_h(\Phi'_\beta).$$

For the proof, from the finite dimensional case (cf. [3]), for any $\varphi \in \Phi$, there exists a real-valued process of integrable variation A^φ such that

$$M(\varphi) = M_0(\varphi) + A^\varphi - A^{\varphi,3}.$$

Let μ_φ be the measure $dPxdA_s^\varphi(\omega)$ and ν_φ be the measure $dPxdA_s^{\varphi,3}(\omega)$. Then $\varphi \mapsto \mu_\varphi$ and $\varphi \mapsto \nu_\varphi$ define two vector measures μ and ν such that $\nu = \mu^P$. Hence there exists A and A^3 corresponding respectively to μ and ν.

If $A^{U,3}$ is the dual previsible projection of A^U, for $U \in \mathscr{U}_h(\Phi'_\beta)$, it is easy to see that $A^{U,3}$ and $A^{3,U}$ are undistinguishable. Q. E. D.

COROLLARY II.5. — Let M be a martingale of integrable variation in Φ' and h be a real valued previsible stochastic process such that

$$E\int_{0-}^{\infty} |h_s||dM_s(\phi)| < +\infty, \qquad \text{for} \quad \phi \in \Phi.$$

Then $(h.M)_t = \int_{0-}^{t} h_s dM_s$, defined as

$$(h.M)_t(\phi) = \int_{0-}^{t} h_s dM_s(\phi)$$

defines a martingale in Φ'.

Proof. — For any bounded, measurable real valued process X define

$$\mu_\phi(X) = E\int_{0-}^{\infty} X_s h_s dM_s(\phi), \qquad \phi \in \Phi.$$

Then $\phi \to \mu_\phi$ defines a vector measure μ with values in Φ' by Theorem II.2. Consequently, for any t, $\int_{0-}^{t} h_s dM_s$ has a version with values in Φ' as one can see using the theorem of Minlos-Sazonov-Badrikian. Moreover

$$\left\{ \int_{0-} h_s dM_s^U : U \in \mathscr{U}_h(\Phi'_\beta) \right\}$$

defines the corresponding projective system and the integrals are well defined since $k(U) \circ \mu = \mu(U)$ is of finite total variation in $\Phi'(U)$. Q. E. D.

The following class of g-processes is essential for the definition of the stochastic integrals defined with respect to the g-processes which are not of integrable variation:

DEFINITION II.3. — A g-process M is called a square integrable martingale (in Φ') if for any $U \in \mathscr{U}_h(\Phi'_\beta)$, M^U is a square integrable martingale in $\Phi'(U)$.

In the following we shall denote by $\mathscr{M}(\Phi')$ the set of square integrable martingales in Φ'. Let us recall that, when we speak of M^U, we understand that it has right continuous trajectories with left limits in $\Phi'(U)$. Hence, the projective system corresponding to $M \in \mathscr{M}(\Phi')$ is uniquely defined.

THEOREM II.3. — $\mathscr{M}(\Phi')$ is a complete, reflexive locally convex space under the topology induced by the seminorms

$$g_U(M) = (E[\,\|M_\infty^U\|^2\,])^{1/2}, \qquad U \in \mathscr{U}_h(\Phi'_\beta),$$

where $M_\infty^U = \lim_{t\uparrow\infty} M_t^U$ and $\|M_\infty^U\|$ means the norm of M_∞^U in $\Phi'(U)$. If $\mathscr{M}(U)$ denotes the Hilbert space of square integrable martingales with values in $\Phi'(U)$, $U \in \mathscr{U}_h(\Phi'_\beta)$, then $\mathscr{M}(\Phi')$, under the topology defined above, is isomorphic to the projective limit of the Hilbert spaces

$$\{\, \mathscr{M}(U) : U \in \mathscr{U}_h(\Phi'_\beta) \,\}.$$

Proof. — The last part of the theorem is obvious by the choice of the topology of $\mathscr{M}(\Phi')$, hence $\mathscr{M}(\Phi')$ is complete. If B is a bounded set in $\mathscr{M}(\Phi')$, the image of B in $\mathscr{M}(U)$ is weakly weakly relatively compact. Since $\mathscr{M}(\Phi')$ is isomorphic to a closed subspace of $\prod_{U \in \mathscr{U}_h(\Phi'_\beta)} \mathscr{M}(U)$ (cf. [*11*]), B is also weakly relatively compact hence $\mathscr{M}(\Phi')$ is reflexive (cf. [*11*], p. 73, Corollary 1). Q. E. D.

The following result is useful for the identification of the square integrable martingales:

THEOREM II.4. — Suppose that Z is a weakly measurable mapping on $\mathbb{R}_+ \times \Omega$ with values in Φ' such that, for any $\phi \in \Phi$, $(t, \omega) \to \langle \phi, Z_t(\omega) \rangle$ has a modification which is a square integrable martingale. Then there exists a unique projective system of square integrable martingales whose projective limit is Z.

Proof. — Denote by $Z(\phi)$ the modification of $\langle \phi, Z \rangle$ which is a square integrable martingale. Then $\phi \to Z(\phi)$ is a linear mapping on Φ with values in $\mathscr{M}$ (i. e. the Hilbert space of the square integrable real-valued martingales). An application of the closed graph theorem shows that this mapping is sequentially continuous hence it is bounded, Φ being a bornological nuclear space, it is a nuclear mapping. We can represent it as

$$\phi \to Z(\phi) = \sum_{i=1}^{\infty} \lambda_i F_i(\phi) m^i$$

where $(\lambda_i) \in l^1$, $(F_i) \subset \Phi'$ is equicontinuous and $(m^i) \subset \mathscr{M}$ is bounded.

Let B be a compact subset of Φ' including (F_i) such that $B^0 \in \mathscr{U}_h(\Phi)$ and $B = B^{00}$ (i. e. the bi-polar of B). Define $M_t^U(\omega)$ as

$$M_t^U(\omega) = \sum_{i=1}^{\infty} \lambda_i k(U)(F_i) m_t^i(\omega), \qquad U \in \mathscr{U}_h(\Phi'_\beta).$$

Then

$$E[\sup_t \| M_t^U \|] \leq \sum_i |\lambda_i| E[\sup_t |m_t^i|] < +\infty,$$

i. e. (M_t^U) is a square integrable martingale with values in $\Phi'(U)$ (after taking off a set of measure zero), hence it has a right continuous modification which we denote again by M^U. Obviously, we have $M_t^U(\phi) = Z_t(\phi)$ *a. e.* for $\phi \in \Phi[U^0]$ and $k(U, V)(M_t^V) = M_t^U$ *a. e.* for $V \in \mathscr{U}_h(\Phi'_\beta)$, $V \subset U$, but M^V and M^U are right continuous hence they are undistinguishable and $\{ M^U, U \in \mathscr{U}_h(\Phi'_\beta) \}$ is the projective system announced above. Its uniqueness is obvious. Q. E. D.

Remark. — Let $B \subset \Phi'$ be a compact absolutely convex set such that $(F_i) \subset B$ and $B^0 \in \mathscr{U}_h(\Phi)$ (with the notations of the proof). Then, $M_t(\omega)$ defined by

$$M_t(\omega) = \sum_{i=1}^{\infty} \lambda_i F_i m_t^i(\omega)$$

is a square integrable martingale in $\Phi'[B]$. If we inject it in Φ', we obtain a modification of Z.

EXAMPLE II.1. — Let (B_t) be the one dimensional standard Wiener process. Extend it to whole $\mathbb{R}$ by letting $B_t = 0$ for $t \leq 0$. Let W be its derivative in $\mathscr{D}'(\mathbb{R})$ (i. e. the space of the distributions on $\mathbb{R}$) and define $W_t(\phi)$ as

$$W_t(\phi) = E[W(\phi) \mid \mathscr{F}_t], \qquad \phi \in \mathscr{D}(\mathbb{R}).$$

Using Ito's formula, one sees that

$$W_t(\phi) = \int_0^t \phi(s) dB_s \qquad a.\ e.,$$

hence (W_t) determines a unique continuous square integrable martingale with values in $\mathscr{D}'(\mathbb{R})$.

In the following we shall denote by $\mathscr{K}_h(\Phi)$ (respectively $\mathscr{K}_h(\Phi')$) the set $\{ U^0 : U \in \mathscr{U}_h(\Phi'_\beta) \}$ (resp. $\{ U^0 : U \in \mathscr{U}_h(\Phi) \}$). Let $B \subset \Phi$ be in $\mathscr{K}_h(\Phi)$ and suppose that H is a bounded previsible stochastic process with values in $\Phi[B]$. Without loss of generality, we may suppose that H is absorbed by B for almost all $\omega \in \Omega$. If $M \in \mathcal{M}(\Phi')$, then M^{B^0} is a square integrable martingale and we can define the stochastic integral of H with respect to M^{B^0} with respect to the dual pair $(\Phi[B], \Phi'(B^0))$ (cf. [7] [8] and [10]), such that

$$E[(H.M^{B^0})^2_\infty] = E\left(\int_0^\infty (H_s \mid dM_s^{B^0})\right)^2 = E\int_{0-}^\infty \sigma_B(s, H_s, H_s)d\{\!\!\{ M^{B^0}, M^{B^0} \}\!\!\}_s$$

where σ_B is the strongly previsible process with values in the set of bilinear forms on $\Phi[B]$, of trace one such that

$$\int_{0-}^t \sigma_B(s, \phi, \psi)d\{\!\!\{ M^{B^0}, M^{B^0} \}\!\!\} = \langle M(\phi), M(\psi) \rangle_t, \qquad \phi, \psi \in \Phi[B^0],$$

$\{\!\!\{ M^{B^0}, M^{B^0} \}\!\!\}_t$ denotes the trace of the bilinear form $(\phi, \psi) \to \langle M(\phi), M(\psi)\rangle_t$ with respect to the dual pair $(\Phi[B], \Phi'(B^0))$ (cf. [10] for the construction of these processes) and $\langle M(\phi), M(\psi) \rangle$ is the unique previsible process of integrable variation such that

$$M(\phi)M(\psi) - \langle M(\phi), M(\psi) \rangle$$

is a martingale. We pretend that $H.M^{B^0}$, constructed above, is independent of the particular choice of B. In fact, let B_1 and B_2 be in $\mathcal{K}_h(\Phi)$, absorbing H. Then

$$E\left[\int_0^\infty \sigma_{B_1}(s, H_s, H_s)d\{\!\!\{ M^1, M^1 \}\!\!\}_s + \int_0^\infty \sigma_{B_2}(s, H_s, H_s)d\{\!\!\{ M^2, M^2 \}\!\!\}_s\right] < +\infty,$$

where $M^i = M^{B_i^0}$ for $i = 1,2$. Since $\Phi[B_1] \cap \Phi[B_2]$, under the topology induced by the norm $\|\,.\,\| = p_{B_1} + p_{B_2}$, is separable, we can choose a sequence of simple previsible processes (H^n) with values in $\Phi[B_1] \cap \Phi[B_2]$ such that

$$E\left[\int_0^\infty \sigma_{B_1}(s, H_s^n - H_s, H_s^n - H_s)d\{\!\!\{ M^1, M^1 \}\!\!\}_s \right.$$
$$\left. + \int_0^\infty \sigma_{B_2}(s, H_s^n - H_s, H_s^n - H_s)d\{\!\!\{ M^2, M^2 \}\!\!\}_s\right] \underset{n\to\infty}{\longrightarrow} 0,$$

which implies that

$$(H^n.M^1)_\infty \to (H.M^1)_\infty \qquad \text{and} \qquad (H^n.M^2)_\infty \to (H.M^2)_\infty$$

in $L^2(\Omega, \mathcal{F}, P)$. Howewer, it is trivial to see that for any simple, previsible process with values in $\Phi[B_1] \cap \Phi[B_2]$, one has

$$(H^n.M^1)_\infty = (H^n.M^2)_\infty \qquad a.\ e.$$

hence $H.M^1$ and $H.M^2$ are undistinguishable. We summarize this construction as a theorem:

THEOREM II.5. — Let H be a Φ-valued, bounded, weakly previsible mapping on $\mathbb{R}_+ \times \Omega$ and M be a square integrable martingale in Φ'. Then, there exists a unique scalar square integrable martingale l, called the stochastic integral of H with respect to M, denoted as

$$l_t = (H.M)_t = \int_0^t \langle H_s, dM_s \rangle ,$$

such that, for any $B \in \mathscr{K}_h(\Phi)$ absorbing H, we have

$$E[l_\infty^2] = E\int_{0^-}^{\infty} \sigma_B(s, H_s, H_s)d \langle\!\langle M^{B^0}, M^{B^0} \rangle\!\rangle_s .$$

If n is a real valued square integrable martingale then

$$E[l_\infty n_\infty] = E[\langle l, n \rangle_\infty] = E\int_{0^-}^{\infty} (H_s \mid dA_s^B)$$

where $A = (A_t)$ is Φ'-valued g-process of integrable variation defined by

$$A_t(\phi) = \langle n, M(\phi) \rangle_t , \qquad \phi \in \Phi ,$$

and (A_t^B) is the element of the projective system, corresponding to $B^0 \in \mathscr{U}_h(\Phi'_\beta)$. Moreover, the last relation characterizes l in a unique manner.

DEFINITION II.4. — Suppose that H is a weakly measurable mapping on $(\mathbb{R}_+ \times \Omega, \mathscr{B}(\mathbb{R}_+) \otimes \mathscr{F})$ with values in Φ. H will be called (locally bounded) if there exists a sequence of stopping times (T_n), increasing to infinity, such that for any $n \in \mathbb{N}$, the mapping $(t, \omega) \to H_{t \wedge T_n}(\omega)$ takes its values in a bounded subset of Φ for almost all $\omega \in \Omega$. One says that (T_n) reduces H.

Remark. — If H is locally bounded and weakly previsible, then one can integrate H with respect to $M \in \mathscr{M}(\Phi')$ since

$$(H^{T_n}.M)_{t \wedge T_m} = (H^{T_m}.M)_t \qquad \text{for any} \quad m \leq n$$

as one can show using the theory of the stochastic integration on the Hilbert spaces (cf. [*7*] [*8*]).

EXAMPLE II.2. — With the notations of Example II.1, if $\phi \in \mathscr{D}(\mathbb{R})$, then the mapping

$$(t, \omega) \to \phi(. + B_t(\omega))$$

is locally bounded and weakly previsible for almost all $\omega \in \Omega$. Consequently the stochastic integral

$$\int_0^t \langle \phi(. + B_s), dW_s \rangle$$

is well defined.

Suppose that $M \in \mathscr{M}(\Phi')$. Then there exists a unique projective system $\{ M^U : U \in \mathscr{U}_h(\Phi'_\beta) \}$ such that M^U is in $\mathscr{M}(U)$ and

$$k(U)(M_t(\omega)) = M_t^U \quad a.\,e. \qquad \text{for any} \qquad t \geq 0 .$$

Since M^U is a square integrable martingale with values in $\Phi'(U)$, it can be decomposed as (cf. [*8*])

$$M^{U,c} + M^{U,d},$$

where $M^{U,c}$ represents the projection of M^U on $\mathscr{M}(U)^c$, i. e. the stable subspace of $\mathscr{M}(U)$ consisting of the continuous square integrable martingales and $M^{U,d}$ belongs to $\mathscr{M}(U)^d$, the orthogonal complement of $\mathscr{M}(U)^c$ in $\mathscr{M}(U)$ (called also topological supplement of $\mathscr{M}(U)^c$). If $V \subset U$ with $V \in \mathscr{U}_h(\Phi'_\beta)$, then the canonical mapping $k(U, V) : \Phi'(V) \to \Phi'(U)$ is continuous and

$$k(U, V)(M^V - M^{V,d}) = M^U - k(U, V)(M^{V,d})$$

is a continuous square integrable martingale. $M^U - M^{U,d}$ belongs also to $\mathscr{M}(U)^c$, taking the difference, we see that

$$M^{U,d} - k(U, V)(M^{V,d}) \in \mathscr{M}(U)^c \cap \mathscr{M}(U)^d = \{ 0 \},$$

therefore $k(U, V)(M^{V,c}) = M^{U,c}$ and $k(U, V)(M^{V,d}) = M^{U,d}$, i. e. there exists two projective systems $M^c = \{ M^{U,c} : U \in \mathscr{U}_h(\Phi'_\beta) \}$ and

$$M^d = \{ M^{U,d} : U \in \mathscr{U}_h(\Phi'_\beta) \}$$

corresponding to M. Using the closed graph theorem, one can see that M^c and M^d have their limits in Φ'. The subset of $\mathscr{M}(\Phi')$ formed by the martingales such that $M^U = M^{U,c}$, $U \in \mathscr{U}_h(\Phi'_\beta)$, is denoted by $\mathscr{M}^c(\Phi')$ and its elements are called continuous martingales. We denote by $\mathscr{M}^d(\Phi')$ the algebraic complement of $\mathscr{M}^c(\Phi')$ in $\mathscr{M}(\Phi')$. Since the projection $j : \mathscr{M}(\Phi') \to \mathscr{M}^c(\Phi')$ is continuous (cf. [*11*], p. 95, Prop. 29), $\mathscr{M}(\Phi')$ is the topological direct sum of $\mathscr{M}^c(\Phi')$ and $\mathscr{M}^d(\Phi')$.

COROLLARY II.6. — Let H and M be as in Theorem II.5. Then one has the following properties:

i) $(H.M)^c = H.M^c, \qquad (H.M)^d = H.M^d$

ii) $\Delta(H.M)_t = M_t(H_t) - M_{t^-}(H_t) = \Delta M_t(H_t)$

where M_{t^-} is defined by $M_- = \{ (M_{t^-}^U) : U \in \mathscr{U}_h(\Phi'_\beta) \}$ and its limit in Φ' (cf. Lemma I.1).

Proof. — All follows from the stochastic integration theory in the Hilbert spaces. Note that, M_- has a limit in Φ' since the mapping

$$\phi \to E[\exp iM_{t^-}(\phi)]$$

is continuous on Φ as one can show by the closed graph theorem. Q. E. D.

COROLLARY II.7. — If $M \in \mathscr{M}(\Phi')$ is of integrable variation, then H.M coincides with the Stieltjes integral.

Remark. — If $B \in \mathscr{K}_h(\Phi)$ absorbs H then the Stieltjes integral is defined as

$$\int_0^t \langle H_s, dM_s \rangle = \int_0^t (H_s \mid dM_s^{B^0})$$

i. e. relative to $(\Phi[B], \Phi'(B^0))$.

DEFINITION II.5. — Let M be a g-process in Φ'. M is called a local martingale if it is right continuous and if M^U is a $\Phi'(U)$-valued local martingale for any $U \in \mathscr{U}_h(\Phi'_\beta)$.

The following result gives some information about the structure of the local martingales:

PROPOSITION II.2. — Let M be a local martingale in Φ'. Then there exists a unique projective system of stochastic processes

$$M^c = \{ M^{U,c} : U \in \mathscr{U}_h(\Phi'_\beta) \}$$

such that $M^{U,c}$ is the continuous local martingale part of M^U for any $U \in \mathscr{U}_h(\Phi'_\beta)$. If $\mathscr{K}$ is a nuclear Fréchet space or strict inductive limit of a sequence of such spaces then this projective system has a limit in Φ'.

Proof. — Without loss of generality, we may suppose $M_0^U \equiv 0$ for any $U \in \mathscr{U}_h(\Phi'_\beta)$. As in the finite dimensional case, for any $U \in \mathscr{U}_h(\Phi'_\beta)$, M^U has a unique continuous local martingale part (cf. [*8*]) $M^{U,c}$. By the uniqueness, we have

$$k(U, V) \circ M^{V,c} = M^{U,c}$$

for any $V \in \mathscr{U}_h(\Phi'_\beta)$, $V \subset U$ and this shows the existence and the uniqueness of the projective system M^c. If Φ is Fréchet space or the strict inductive limit of a sequence of such spaces, then as in the proof of Lemma I.1, the mapping

$$\phi \to E[\exp iM_t^c(\phi)]$$

is continuous, where $M^c_t(\phi)$ is defined as

$$M^c_t(\Phi) = M^{U,c}_t(\phi) \qquad \text{if} \qquad \phi \in \Phi[U^0],$$

and this result implies the existence of the limit of the projective system M^c.

Q. E. D.

Now we can extend stochastic integration to the local martingales:

THEOREM II.6. — Let M be a local martingale in Φ' and H be a weakly previsible, locally bounded mapping on $\mathbb{R}_+ \times \Omega$ with values in Φ. Then, there exists a real valued local martingale l, called the stochastic integral of H with respect to M, denoted by $(H.M)_t$ such that

i) $\Delta l_t \equiv l_t - l_{t^-} = M_t(H_t) - M_{t^-}(H_t)$

ii) For any sequence of stopping times (T_n) reducing H and $(B_n) \subset \mathscr{K}_h(\Phi)$ such that B_n absorbs $H^{T_n} = (H_{t\wedge T_n};\, t \geq 0)$, there exists a sequence of strongly optional, positive, symmetric bilinear forms (β_n) on $\Phi[B_n]$, of trace one, such that

$$[l^{T_n}, l^{T_n}]_\infty = \int_{0^-}^{\infty} \beta_n(H^{T_n}_s, H^{T_n}_s, s) d[\![M^{B^0_n}, M^{B^0_n}]\!]_s \qquad a.\ e.$$

iii) If m is any bounded, real martingale we have

$$[l^{T_n}, m]_t = \int_{0^-}^{t} (H^{T_n}_s \mid d[m, (.\mid M(B^0_n))]_s).$$

Proof. — Let us explain first some notations: If l is a real valued local martingale, $[l, l]_t$ denotes $\langle l^c, l^c \rangle_t + \sum_{0\leq s\leq t} (\Delta l_s)^2$ (cf. [*3*]). If L is a local martingale with values in a Hilbert space F, then we denote by $[\![L, L]\!]_t$ the trace of the bilinear form $(x, y) \to [L(x), L(y)]_t$. Then one can show that as a random measure $d[L(x), L(y)]_t$ is absolutely continuous with respect to $d[\![L, L]\!]_t$, with the density $\beta(x, y, s)$ as described in *ii*) (cf. [*8*]).

For the proof, we may suppose that H is bounded and $B \in \mathscr{K}_h(\Phi)$, absorbing H. Then M^{B^0} being a local martingale in $\Phi'(B^0)$, there exists a sequence of stopping times (S_n) increasing to infinity such that M^{B^0} stopped at each S_n can be written as a sum of a square integrable martingale and a martingale of integrable variation. Since H is bounded and previsible in $\Phi[B]$, the integral of H with respect to M^{B^0} is well defined. We pretend that the integral is independent of the particular choice of B. To prove this we need the following

LEMMA II.1. — Suppose that χ is a separable Hilbert space, H is a previsible, locally bounded stochastic process with values in χ, M is a local martingale in χ and m is a real-valued local martingale. Then one has the following identity:

$$\left[m, \int_0^{\cdot} (H_s \mid dM_s) \right]_t = \left[\!\!\left[\int_0^{\cdot} H_s dm_s, M \right]\!\!\right]_t \qquad a.\ e.$$

(hence the two processes are undistinguishable).

Proof. — By definition, we have

$$\begin{aligned} \left[\!\!\left[\int_0^{\cdot} H_s dm_s, M \right]\!\!\right]_t &= \left\langle\!\!\left\langle \int_0^{\cdot} H_s dm_s^c, M^c \right\rangle\!\!\right\rangle_t + \sum_{s \leq t} (H_s \Delta m_s \mid \Delta M_s) \\ &= \left\langle\!\!\left\langle \int_0^{\cdot} H_s dm_s^c, M^c \right\rangle\!\!\right\rangle_t + \sum_{s \leq t} \Delta m_s . \Delta\left(\int_0^s (H_s \mid dM_s) \right), \end{aligned}$$

hence it is sufficient to prove the lemma for the continuous, square integrable martingales. If H is a simple process, we have

$$\begin{aligned} E\left[\left(M_\infty \,\middle|\, \int_0^\infty H_s dm_s \right) \right] &= E\left[\sum_k (H_k \mid M_\infty)(m_{t_{k+1}} - m_{t_k}) \right] \\ &= E\left[\sum_k (H_k \mid M_{t_{k+1}} - M_{t_k})(m_{t_{k+1}} - m_{t_k}) \right] \\ &= E\left(\left[\int_0^{\cdot} (H_s \mid dM_s), m \right]_\infty \right). \end{aligned}$$

If H is bounded and previsible in χ, there exists a sequence of simple, previsible processes with values in χ, (H^n), such that

$$E \int_{0-}^\infty \sigma_M(s, H_s^n - H_s, H_s^n - H_s) d\langle\!\langle M, M \rangle\!\rangle_s + E \int_{0-}^\infty \| H_s^n - H_s \|^2 d\langle m, m \rangle_s \underset{n \to \infty}{\longrightarrow} 0.$$

Hence, we can pass to the limit and obtain

$$E\left[\left(M_\infty \int_0^\infty H_s dm_s \right) \right] = E\left(\left[\int_0^{\cdot} (H_s \mid dM_s), m \right]_\infty \right),$$

moreover, by definition

$$E\left[\left(M_\infty \,\middle|\, \int_0^\infty H_s dm_s \right) \right] = E\left(\left[\!\!\left[M, \int_0^{\cdot} H_s dm_s \right]\!\!\right]_\infty \right) = E\left(\left\langle\!\!\left\langle M, \int_0^{\cdot} H_s dm_s \right\rangle\!\!\right\rangle_\infty \right).$$

Therefore, for any stopping time T, we have

$$E\left[\left[\!\left[M, \int_0^{\cdot} H_s dm_s\right]\!\right]_T\right] = E\left(\left[\int_0^{\cdot} (H_s \mid dM_s), m\right]_T\right),$$

hence the two processes are undistinguishable. Q. E. D.

Let us complete the proof of Theorem II.6: Suppose that there exists B_1 and B_2 in $\mathcal{K}_h(\Phi)$, absorbing H. Then for any bounded, real valued martingale m, we have

$$[m, H.M^{B_1^0}]_t = \langle m^c, H.M^{c,B_1^0} \rangle_t + \sum_{s \leq t} \Delta m_s (H_s \mid \Delta M_s^{B_1^0}).$$

Since $\Delta M = \{ (\Delta M_t^U) : U \in \mathcal{U}_h(\Phi'_\beta) \}$ is a projective system, one has

$$(H_s \mid \Delta M_s^{B_1^0}) = (H_s \mid \Delta M_s^{B_2^0}) \qquad a.\ e.$$

(cf. the Remark following Lemma I.1). By Proposition II.2, M^c is also a projective system, hence

$$\left(\int_0^t H_s dm_s^c \mid M_t^{c,B_1^0}\right) = \left(\int_0^t H_s dm_s^c \mid M_t^{c,B_2^0}\right) \qquad a.\ e.$$

since $\int_0^t H_s dm_s^c$ takes its values in $\Phi[B_1] \cap \Phi[B_2]$, and we have

$$[m, H.M^{B_1^0}] = [m, H.M^{B_2^0}]$$

for any bounded scalar martingale m and this property defines uniquely H . M. The rest of the proof follows from the theory of stochastic integration on the Hilbert spaces. Q. E. D.

Remark. — It is trivial to check up that Lemma II.1 remains true when we replace M and m by the semimartingales.

III. SEMIMARTINGALES

We begin by

DEFINITION III.1. — Let X be a right continuous g-process in Φ'. X is called a semimartingale if, for any $U \in \mathcal{U}_h(\Phi'_\beta)$, the stochastic process X^U is a semimartingale with values in $\Phi'(U)$.

Remark. — If χ is a separable Hilbert space, then by a semimartingale K with values in χ we understand that K can be written as

$$K_t = M_t + A_t + K_0, \qquad M_0 = A_0 = 0, \qquad t \geq 0,$$

where (M_t) is a local martingale in χ and (A_t) is a right continuous process of finite variation (supposed always adapted unless the contrary is indicated).

Before constructing the stochastic integrals, we study some interesting subclasses of the semimartingales:

DEFINITION III.2. — A semimartingale X in Φ' is called a special semimartingale if for any $U \in \mathcal{U}_h(\Phi'_\beta)$, X^U has a decomposition

$$X_0^U + M^U + A^U$$

such that A^U is previsible, of finite variation, right continuous in $\Phi'(U)$ and M^U is a $\Phi'(U)$-valued local martingale with $A_0^U = M_0^U = 0$.

PROPOSITION III.1. — Suppose that X is a special semimartingale in Φ'. Then there exists two projective systems of right continuous stochastic processes

$$A = \{ A^U : U \in \mathcal{U}_h(\Phi'_\beta) \}, \qquad M = \{ M^U : U \in \mathcal{U}_h(\Phi'_\beta) \}$$

such that

$$X^U = X_0^U + A^U + M^U, \qquad U \in \mathcal{U}_h(\Phi'_\beta)$$

where A^U is previsible, of finite variation, right continuous in $\Phi'(U)$ and M^U is a $\Phi'(U)$-valued local martingale. If Φ is a nuclear Fréchet space or strict inductive limit of a sequence of such spaces then A and M have their limits in Φ'.

Proof. — Let V, $U \in \mathcal{U}_h(\Phi'_\beta)$ with $V \subset U$. X^V and X^U are special semimartingales hence there exists previsible, right continuous processes of finite variation A^V and A^U and local martingales M^V and M^U with values respectively in $\Phi'(V)$ and $\Phi'(U)$, decomposing X^V and X^U. If $k(U, V)$ denotes the canonical mapping from $\Phi'(V)$ onto $\Phi'(U)$, then

$$M^U - k(U, V)(M^V) = k(U, V)(A^V) - A^U,$$

but a previsible local martingale of finite variation is constant, hence $k(U, V)(M^V) = M^U$ and $k(U, V)(A^V) = A^U$ and this shows the existence and the uniqueness of the projective systems. The rest of the theorem can be proved as in Lemma I.1. Q. E. D.

Is x is a real valued semimartingale, we denote by $\| x \|_1$ the following number:

$$\| x \|_1 = \inf \left\{ E\left([m, m]_\infty^{1/2} + \int_0^\infty | da_s | \right) : x = m + a \right\}$$

where the infimum is taken over all the decompositions of x (as a sum of local martingale and a process of finite variation). The set of the semimartingales $\{ x : \| x \|_1 < +\infty \}$ is a Banach space under the norm $\| \, . \, \|_1$ and it will be denoted by S^1 (cf. [3]).

The following result gives a practical method to identify the semimartingales in Φ':

THEOREM III.1. — Suppose that X is a weakly measurable mapping on $(\mathbb{R}_+ \times \Omega, \mathscr{B}(\mathbb{R}_+) \otimes \mathscr{F})$ with values in Φ' such that, for any $\phi \in \Phi$, the stochastic process $(t, \omega) \to \langle \phi, X_t(\omega) \rangle$ has a modification $(t, \omega) \to \tilde{X}_t(\phi)$ which is an S^1-semimartingale. Then there exists a projective system of semimartingale $\{ X^U : U \in \mathscr{U}_h(\Phi'_\beta) \}$ accepting X as its limit in Φ'.

Proof. — $\phi \to \tilde{X}(\phi)$ induces a linear mapping on Φ with values in S^1. If $U \in \mathscr{U}_h(\Phi'_\beta)$, suppose that (ϕ_n) converges to ϕ in $\Phi[U^0]$ and $\tilde{X}(\phi_n)$ to y in S^1. Since $\tilde{X}_t(\phi_n) = \langle \phi_n, X_t \rangle$ *a. e.*, and $X_t(\omega) \in \Phi'$, $\langle \phi_n, X_t \rangle$ converges to $\langle \phi, X_t \rangle$ for all $\omega \in \Omega$. Hence $\tilde{X}_t(\phi_n)$ converges to $\tilde{X}_t(\phi)$ in probability and this implies that $y_t = \tilde{X}_t(\phi)$, y and $\tilde{X}(\phi)$ being right continuous processes, they are undistinguishable. We have proved the fact that the restriction of $\tilde{X}$ to $\Phi[U^0]$ is continuous, for any $U \in \mathscr{U}_h(\Phi'_\beta)$. Choose any $V \in \mathscr{U}_h(\Phi'_\beta)$, $V \subset U$ such that the canonical mapping $k(U, V)$ is nuclear. Then

$$\tilde{X}\,|_{\Phi[U^0]} = \tilde{X}\,|_{\Phi[V^0]} \circ i(V^0, U^0)$$

where $i(V^0, U^0)$ denotes the adjoint of $k(U, V)$. Since $i(V^0, U^0)$ is nuclear, the restriction of $\tilde{X}$ to $\Phi[U^0]$ induces a nuclear mapping. Let us choose any representation of it:

$$\sum_{i=1}^{\infty} \lambda_i F_i(U) \otimes x^i$$

where $(\lambda_i) \in l^1$, $(F_i(U)) \subset \Phi'(U)$ is equicontinuous and $(x^i) \subset S^1$ is bounded. For any $\varepsilon > 0$, choose m^i and a^i with $x^i = m^i + a^i$, m^i being a local martingale and a^i a process of integrable variation such that

$$E\left[[m^i, m^i]_\infty^{1/2} + \int_{0^-}^{\infty} | da_s^i | \right] < \| x^i \|_1 + \varepsilon$$

Define M_t^U as

$$M_t^U(\omega) = \sum_{i=1}^{\infty} \lambda_i F_i(U) m_t^i(\omega)$$

we have

$$\sum_{i=1}^{\infty} | \lambda_i | \, E[\sup_t | m_t^i |] < +\infty$$

from the inequality of Davis (cf. [3]). This result means that the series which defines $M_t^U(\omega)$ converges almost surely uniformly in t, hence (M_t^U) has right continuous trajectories with left limits in $\Phi'(U)$ for almost all $\omega \in \Omega$. Moreover, we have

$$E[\sup_t \| M_t^U \|] < +\infty$$

and this means that M^U is a local martingale (it is even an $H^1(\Phi'(U))$-semimartingale cf. [*16*]). Similarly, define A_t^U as

$$A_t^U(\omega) = \sum_{i=1}^{\infty} \lambda_i F_i(U) a_t^i(\omega).$$

Then A^U is a right continuous process of integrable variation in $\Phi'(U)$:

$$E\int_{0^-}^{\infty} \| dA_s^U \| \leq E\sum_{i=1}^{\infty} | \lambda_i | \int_{0^-}^{\infty} | da_s^i | < +\infty$$

and

$$\sum_i | \lambda_i | E[\sup_{t \in \mathbb{R}_+} | a_t^i |] < +\infty.$$

Let X^U be $M^U + A^U$, then for any $\phi \in \Phi[U^0]$, we have $(\phi \mid X_t^U) = \tilde{X}_t(\phi)$, hence $\{ X^U : U \in \mathscr{U}_h(\Phi'_\beta) \}$ is a projective system of semimartingales whose limit is X. Q. E. D.

Remark. — In the proof of the theorem we did not use the fact that Φ is bornological. When Φ is bornological, then $\tilde{X} ; \Phi \to S^1$ is continuous (since it is bounded), hence it is nuclear. Take any representation of it:

$$\phi \to \tilde{X}(\phi) = \sum_{i=1}^{\infty} \lambda_i F_i \otimes x^i$$

with $(F_i) \subset \Phi'$ is equicontinuous, $(x^i) \subset S^1$ bounded and $(\lambda_i) \in l^1$. Choose any $B \in \mathscr{K}_h(\Phi')$ such that B absorbs (F_i) and m^i and a^i as above. Define

$$W_t(\omega) = \sum_{i=1}^{\infty} \lambda_i F_i(m_t^i(\omega) + a_t^i(\omega)).$$

Then (W_t) is a semimartingale with values in $\Phi'[B]$ and its injection into Φ' gives also the projective system and its limit. We shall use this fact in the proof of the integration by parts formula.

This result can be expressed also in the following form:

COROLLARY III.1. — Suppose that $\tilde{X}$ is a linear mapping on Φ with values in S^1 such that for any $t \geq 0$, the mapping

$$\phi \rightarrow E[\exp i\tilde{X}_t(\phi)]$$

is continuous. Then, there exists a semimartingale X in Φ' such that for any $t \in \mathbb{R}_+$, $\phi \in \Phi$, $\langle \phi, X_t \rangle$ is in the equivalence class $\tilde{X}_t(\phi)$ i. e. $\tilde{X}$ can be « lifted » to a semimartingale in Φ'.

EXAMPLE III.1. — Let (B_t) be the standard Wiener process with values in $\mathbb{R}^d$. For $\phi \in \mathscr{D}(\mathbb{R}^d)$, define $X_t(\phi)$ as

$$X_t(\phi) = \delta_{B_t}(\phi) = \phi(B_t)$$

where δ_x denotes the Dirac measure at $x \in \mathbb{R}^d$. Then (X_t) and its derivatives of all orders are the semimartingales in $\mathscr{D}'(\mathbb{R}^d)$. Note that (X_t) is not even a measure valued weak semimartingale.

We shall need also the following type of the stochastic integrals:

THEOREM III.2. — Let H be a Φ'-valued, locally bounded, weakly previsible mapping on $\mathbb{R}_+ \times \Omega$. If x is a real valued semimartingale, define $H.x$ as

$$(H.x)_t(\phi) = \int_0^t H_s(\phi)dx_s\,, \qquad t \geq 0\,, \qquad \phi \in \Phi\,.$$

Then $H.x$ defines a semimartingale in Φ'.

Proof. — Without loss of generality, we may suppose that H is bounded. Let $B \in \mathscr{K}_h(\Phi')$ absorbing H. Then H, as a $\Phi'[B]$-valued mapping, is a bounded, previsible stochastic process and $H.x$ is well defined in $\Phi'[B]$. Injecting it into Φ' we obtain the limit of the following projective system:

$$\left\{ \left(\int_0^t k(U)(H_s)dx_s \right) : U \in \mathscr{U}_h(\Phi'_\beta) \right\}. \qquad \text{Q. E. D.}$$

EXAMPLE III.2. — With the notations of Example III.1, let $T \in \mathscr{D}'(\mathbb{R}^d)$. Stopping (B_t) on the increasing, compact subsets of $\mathbb{R}^d$, it is easy to see that the mapping $(t, \omega) \rightarrow T * \delta_{B_t(\omega)}$ is locally bounded, weakly previsible. Hence the stochastic integral

$$\int_0^t T * \delta_{B_s} dx_s$$

defines a semimartingale in $\mathscr{D}'(\mathbb{R}^d)$, for any real valued-semimartingale x, where « $*$ » denotes the convolution.

THEOREM III.3. — Let h be a real valued, locally bounded, previsible stochastic process and X be a semimartingale in Φ'. Define L_t as

$$L_t(\phi) = \int_0^t h_s dX_s(\phi), \quad t \geq 0, \quad \phi \in \Phi.$$

Then there exists a projective system of semimartingales

$$\{ L^U : U \in \mathscr{U}_h(\Phi'_\beta) \}$$

such that

$$L_t^U(\phi) = L_t(\phi) \quad \text{if} \quad \phi \in \Phi[U^0].$$

If Φ is a nuclear Fréchet space or strict inductive limit of a sequence of such spaces, then (L_t) has a modification in Φ', i. e. $\{ L^U : U \in \mathscr{U}_h(\Phi'_\beta) \}$ has a projective limit in Φ'.

Proof. — If $U \in \mathscr{U}_h(\Phi'_\beta)$, then

$$L_t^U = \int_0^t h_s dX_s^U$$

is well defined and it is a semimartingale in $\Phi'(U)$. If $V \subset U$, $V \in \mathscr{U}_h(\Phi'_\beta)$, then

$$k(U, V)(L_t^V) = L_t^U \quad a.\ e.,$$

both sides being right continuous, they are undistinguishable. The proof of the last statement is same as the proof of Lemma I.1. Q. E. D.

Now we can prove the following:

THEOREM III.4. — Suppose that H is a mapping on $\mathbb{R}_+ \times \Omega$ with values in Φ which is weakly previsible and locally bounded. If X is a semimartingale in Φ', then there exists a unique real valued semimartingale H.X such that:

i) $(H.X^c) = H.X^c$,

ii) $\Delta(H.X) = \Delta X(H)$,

iii) $(H.X)^T = H.X^T$ for any stopping time T, where X^T denotes the projective system stopped at T.

Remark. — X^c, X^T and ΔX are not in general g-processes but the projective systems, however in the case which Φ is the strict inductive limit of a sequence of nuclear Fréchet spaces, these systems have the limits in Φ'.

Proof. — Without loss of generality, we may suppose that H is bounded. Let $B \in \mathscr{K}_h(\Phi)$ absorbing H then $H.X^{B^0}$ is well defined with respect to the pair $(\Phi[B], \Phi'(B^0))$. We shall show that the real valued semimartingale obtained this way is independent of the particular choice of B. Suppose that B_1 and B_2 are in and $X^{B_2^0}$ will be denoted respectively by Y and Z. Choose any two decompositions of Y and Z as

$$\begin{aligned} Y &= Y_0 + M + A \\ Z &= Z_0 + N + B \end{aligned}$$

Where M and N are local martingales with values respectively in $\Phi'(B_1^0)$ and $\Phi'(B_2^0)$ and A and B are of finite variation with values respectively in $\Phi'(B_1^0)$ and $\Phi'(B_2^0)$. Let

$$\begin{aligned} T_n^1 &= \inf\left\{ t : \int_0^t \| dA_s \|_{\Phi'(B_1^0)} > n \right\}, \\ T_n^2 &= \inf\left\{ t : \int_0^t \| dB_s \|_{\Phi'(B_2^0)} > n \right\}, \end{aligned}$$

since the variations are right continuous, the sequence of stopping times (T_n), defined by $T_n = \inf(T_n^1, T_n^2)$, increases to infinity by n. Define Y^n and Z^n in the following manner:

$$\begin{aligned} Y_t^n &= Y_t 1_{\{t < T_n\}} + Y_{T_n -} 1_{\{t \geq T_n\}} \\ Z_t^n &= Z_t 1_{\{t < T_n\}} + Z_{T_n -} 1_{\{t \geq T_n\}}. \end{aligned}$$

Then Y^n and Z^n are the special semimartingales since M^n and N^n (defined similarly) are special semimartingales and A^n and B^n (defined similarly) are of integrable variation. Moreover we have

$$\begin{aligned} H.Y^n &= H.Y^{T_n} - (H_{T_n} \mid \Delta Y_{T_n})_1 1_{[\![T_n, \infty[\![}, \\ H.Z^n &= H.Z^{T_n} - (H_{T_n} \mid \Delta Z_{T_n})_2 1_{[\![T_n, \infty[\![} \end{aligned}$$

where Y^{T_n} (respectively Z^{T_n}) denotes the semimartingale Y (resp. Z) stopped at T_n, $[\![T_n, \infty[\![$ is the stochastic interval defined by $\{(t, \omega) : T_n(\omega) \leq t < +\infty\}$ and $(. \mid .)_1$ (respectively $(. \mid .)_2$) is the bilinear form corresponding to the dual pair $(\Phi[B_1], \Phi'(B_1^0))$ (respectively $(\Phi[B_2], \Phi'(B_2^0))$. Since Y^n and Z^n are the special semimartingales, they can be decomposed as

$$\begin{aligned} Y^n &= Y_0 + K + C \\ Z^n &= Z_0 + L + D \end{aligned}$$

where K (respectively L) is a local martingale in $\Phi'(B_1^0)$ (resp. $\Phi'(B_2^0)$) and C (respectively D) is a right continuous previsible process of finite variation with values in $\Phi'(B_2^0)$ (resp. $\Phi'(B_2^0)$). If $\phi \in \Phi[B_1] \cap \Phi[B_2]$, we have $(\phi \mid Y^n)_1 = (\phi \mid Z^n)_2$, hence

$$(\phi \mid K)_1 - (\phi \mid L)_2 = (\phi \mid D)_2 - (\phi \mid C)_1$$

and this implies that the right hand side is a previsible local martingale of finite variation, i. e. it is an evanescent process (cf. [3] and [9]). Denote by χ the Banach space $\Phi[B_1] \cap \Phi[B_2]$ under the norm $p_{B_1} + p_{B_2}$ and by χ' its continuous dual. Then K, L, C and D are right continuous stochastic processes with values in χ' (i. e. χ' with its strong topology). χ being separable, K and L and C and D are undistinguishable as the stochastic processes with values in χ'. Same argument works also for Y and Z. Consequently we have

$$(H_T \mid \Delta Y_T)_1 = (H_T \mid \Delta Z_T)_2 \qquad a.\ e.$$
$$(H_T \mid \Delta K_T)_1 = (H_T \mid \Delta L_T)_2 \qquad a.\ e.$$

for any stopping time T and this implies that they are undistinguishable. Moreover, the following relations hold up to an evanescent process:

$$Y^{n,c} = K^c = Y^{T_n,c} = Y^{c,T_n},$$
$$Z^{n,c} = L^c = Z^{T_n,c} = Z^{c,T_n}.$$

If l is any real valued, bounded martingale, we have

$$[l, H.K]_t = [\![H.l, K]\!]_t^1 = \langle\!\langle H.l^c, K^c \rangle\!\rangle_t^1 + \sum_{s \leq t} (H_s \Delta l_s \mid \Delta K_s)_1$$

but

$$\langle\!\langle H.l^c, K^c \rangle\!\rangle_t^1 = \langle\!\langle H.l^c, Y^{c,T_n} \rangle\!\rangle_t^1 = \langle\!\langle H.l^c, Z^{c,T_n} \rangle\!\rangle_t^2 = \langle\!\langle H.l^c, L^c \rangle\!\rangle_t^2$$

from Lemma II.1 and Theorem II.6. By what we have shown above, we have

$$[l, H.K]_t = [l, H.L]_t \qquad a.\ e.$$

and this relation characterizes uniquely H.K, i. e. H.K = H.L up to an evanescent process. Since χ is separable under the norm topology induced by $p_{B_1} + p_{B_2}$ (p_{B_i} is the gauge function of B_i, for $i = 1, 2$), there exists a sequence of simple functions H^k with values in χ such that

$$\int_0^t \| H_s(\omega) - H_s^k \|_1 \, \| dC_s(\omega) \|_1 + \int_0^t \| H_s(\omega) - H_s^k \|_2 \, \| dD_s(\omega) \|_2 \xrightarrow[k \to \infty]{} 0,$$

hence the Stieltjes integrals H.C and H.D are undistinguishable. We have

$$H.Y^n = H.Z^n$$

hence

$$H.Y^{T_n} = H.Z^{T_n}, \qquad n \in \mathbb{N}$$

up to evanescent process. Since T_n increases to infinity for almost all $\omega \in \Omega$, $H.X^1$ and $H.X^2$ are also undistinguishable, i. e. $H.X$ defined by $H.X^{B^0}$ is independant of the particular choice of $B \in \mathscr{K}_h(\Phi)$. The rest of the theorem follows from the theory of stochastic integration on the Hilbert spaces (cf. [7] [8]) and from the fact that X^T, X^c and ΔX are the projective systems.

Q. E. D.

IV. INTEGRATION BY PARTS FORMULA

Since the duality form on $\Phi \times \Phi'_\beta$ is not in general continuous the analogous of the integration by parts formula of the finite dimensional case is not in general true for the infinite dimensional nuclear spaces. However, for certain classes of the semimartingales or certain classes of the nuclear spaces we can show that this formula holds.

Suppose that Z is a semimartingale in Φ' such that for any $\phi \in \Phi$, the stochastic process defined by

$$Z(\phi) = Z^U(\phi) \qquad \text{if} \qquad \phi \in \Phi[U^0],$$

is a semimartingale in S^1 (cf. Section III). Since Φ is supposed to be bornological, from the remark following Theorem III.1, there exists a set $B \in \mathscr{K}_h(\Phi')$ and a semimartingale with values in $\Phi'[B]$, say $\tilde{Z}$ such that, for any $U \in \mathscr{U}_h(\Phi'_\beta)$, $k(U)(i_B(\tilde{Z}))$ and Z^U are undistinguishable, where i_B denotes the injection of $\Phi'[B]$ into Φ' and $k(U)$ is the canonical mapping from Φ' onto $\Phi'(U)$. Suppose now that X is a semimartingale in Φ and that Φ is separable (this is not an important restriction; for instance all the distribution spaces on $\mathbb{R}^d$, $d \geq 1$ and their strong duals are separable). Then, for any fixed $t \in \mathbb{R}_+$ we have

$$\begin{aligned} \langle X_t(\omega), Z_t(\omega) \rangle &= \langle X_t(\omega), i_B(\tilde{Z}_t(\omega)) \rangle \\ &= (X_t^{B^0}(\omega) \mid \tilde{Z}_t(\omega)) \qquad a.\ e. \end{aligned}$$

but, from the integration by parts formula for the Hilbert space valued semimartingales, we have

$$\text{(IV.1)} \quad (X_t^{B^0} \mid \tilde{Z}_t) = \int_0^t (X_{s-}^{B^0} \mid d\tilde{Z}_s) + \int_0^t (dX_s^{B^0} \mid \tilde{Z}_{s-}) + [\![X^{B^0}, \tilde{Z}]\!]_t \quad a.\ e.$$

We will show that the right hand side of this expression is independent of the particular choice of B. For this, we need the following result, whose proof, being very similar to the finite dimensional case (cf. [*3*], p. 340), will be omitted.

LEMMA IV.1. — Suppose that χ is a separable Hilbert space, U is a right continuous adapted stochastic process, having its left limits, with values in χ and W is a semimartingale in χ. Then the stochastic integral

$$\int_0^t (U_{s-} \mid dW_s)$$

is the limit in probability of following Riemann sums:

$$\sum_{0 \leq k \leq 2^n} (U_{\frac{tk}{2^n}} \mid W_{\frac{t(k+1)}{2^n}} - W_{\frac{tk}{2^n}})$$

when n tends to infinity.

Now, suppose that $B_1 \in \mathscr{K}_h(\Phi')$ is another set as B. Then we have

$$\int_0^t (X_{s-}^{B^0} \mid d\tilde{Z}_s) = \lim_{|\pi_n| \to 0} \sum_{t_i \in \pi_n} (X_{t_i}^{B^0} \mid \tilde{Z}_{t_{i+1}} - \tilde{Z}_{t_i})$$

but, if Z' is the representation of Z for B_1, we have

$$(X_{t_i}^{B^0} \mid \tilde{Z}_{t_{i+1}} - \tilde{Z}_{t_i}) = (X_{t_i}^{B_1^0} \mid Z'_{t_{i+1}} - Z'_{t_i})$$

where π_n denotes the dyadic partition of $[0, t]$, of order n and $|\pi_n|$ is $\sup(|t_i - t_{i+1}| : t_i, \ t_{i+1} \in \pi_n)$. Consequently the stochastic integral $\int_0^t (X_{s-}^{B^0} \mid d\tilde{Z}_s)$ is independent of any particular choice of B and the same result is true for $\int_0^t (dX_s^{B^0} \mid \tilde{Z}_{s-})$ hence we represent them respectively as $\int_0^t \langle X_{s-}, dZ_s \rangle$ and $\int_0^t \langle dX_s, Z_{s-} \rangle$. Since $(X^{B^0} \mid \tilde{Z})$ and $(X^{B_1^0} \mid Z')$ are undistinguishable stochastic processes, $[\![X^{B^0}, \tilde{Z}]\!]$ is also independent of any particular choice of B and we shall denote it by $[\![X, Z]\!]$. Let us note that, since

$$[\![X^{B^0}, \tilde{Z}]\!]_t = \langle\!\langle X^{c,B^0}, \tilde{Z}^c \rangle\!\rangle_t + \sum_{s \leq t} (\Delta X_s^{B^0} \mid \Delta \tilde{Z}_s)$$

and since $(\Delta X^{B^0} \mid \Delta \tilde{Z}) = (\Delta X^{B_1^0} \mid \Delta Z')$ up to an evanescent process, $\langle\!\langle X^{c,B^0}, \tilde{Z}^c \rangle\!\rangle$ is also independent of B and we shall denote it by $\langle\!\langle X^c, Z^c \rangle\!\rangle$.

We have proved the following theorem:

THEOREM IV.1. — Suppose that either Φ or Φ'_β is separable and X be a semimartingale in Φ and Z a semimartingale as described above. Then the mapping

$$(t, \omega) \to \langle X_t(\omega), Z_t(\omega) \rangle$$

has a modification which is a semimartingale (denoted again by the same notation) and it can be expressed in the following form:

$$\text{(IV.1)} \quad \langle X_t, Z_t \rangle = \int_0^t \langle X_{s^-}, dZ_s \rangle + \int_0^t \langle dX_s, Z_{s^-} \rangle + [\![X, Z]\!]_t$$

where all the stochastic integrals are well defined, $[\![X, Z]\!]$ is an adapted, right continuous stochastic process of finite variation.

Remark. — Suppose that Φ'_β is bornological (instead of Φ) and X is a semimartingale in Φ such that, for any $F \in \Phi'$, the mapping

$$(t, \omega) \to \langle X_t(\omega), F \rangle$$

has a modification which is an S^1-semimartingale. If Z is a semimartingale in Φ' and if either Φ or Φ'_β is separable, then the theorem is again true when Φ and Φ' are interchanged.

Remark. — In fact we have proved a result stronger than the one which is announced in Theorem IV.1: the mapping $(t, \omega) \to \langle X_t(\omega), Z_t(\omega) \rangle$ has at least one modification which is right continuous with left limits and any such modification is a semimartingale.

V. ON ITO'S FORMULA

In this section we give some applications of the theory constructed in the preceeding sections. For the sake of simplicity we shall work in the one-dimensional case but, the results extend trivially to higher dimensions.

Let $\mathcal{D}$ be the space of the infinitely differentiable functions of compact support on $\mathbb{R}$ and $\mathcal{D}'$ its dual equipped with the strong topology. We denote by $B = (B_t)$ a standard Wiener process in $\mathbb{R}$. Define T_n as

$$T_n = \inf \{ t : | B_t | > n \} \wedge n ; \qquad n \in \mathbb{N} ,$$

T_n is a stopping time and it increases to infinity with n. Continuity of the trajectories of B implies that

$$| B_{t \wedge T_n} | \leq n \qquad a.\ e.$$

for any $t \geq 0$. If $\phi \in \mathcal{D}$, the mapping

$$(t, \omega) \to \phi(. + B_{t \wedge T_n}(\omega))$$

is with values in a bounded subset of $\mathscr{D}$ for almost all $\omega \in \Omega$ since the translation $\phi \to \phi(. + y)$ is continuous on $\mathscr{D}$ for any $y \in \mathbb{R}$. For the same reason, for any $S \in \mathscr{D}'$, the mapping

$$t \to \langle \phi(. + B_{t \wedge T_n}(\omega)), S \rangle$$

is also continuous for almost all $\omega \in \Omega$. By Ito's formula we have

$$\phi(x + B_t) = \phi(x) + \int_0^t \phi'(x + B_s)dB_s + \frac{1}{2}\int_0^t \phi''(x + B_s)ds \quad a.\,e.$$

Let K be in $\mathscr{K}_h(\mathscr{D})$ absorbing $\{ \phi'(. + B_{t \wedge T_n}(\omega)) : (t, \omega) \in \mathbb{R}_+ \times \Omega \}$ and $\{ \phi''(. + B_{t \wedge T_n}(\omega)) : (t, \omega) \in \mathbb{R}_+ \times \Omega \}$ for almost all $\omega \in \Omega$. Then these two mappings can be regarded as bounded, previsible stochastic processes with values in the separable Hilbert space $\mathscr{D}[K]$ (up to an evanescent process) hence the integrals converge as the integrals of the Hilbert space-valued process with respect to scalar semimartingales. Injecting them in $\mathscr{D}$ and denoting their images by the same notations, for any $S \in \mathscr{D}'$, we have

$$\begin{aligned} &\left\langle \int_0^{t \wedge T_n} \phi'(. + B_s)dB_s + \frac{1}{2}\int_0^{t \wedge T_n} \phi''(. + B_s)ds, S \right\rangle \\ &= \left(\int_0^{t \wedge T_n} \phi'(. + B_s)dB_s + \frac{1}{2}\int_0^{t \wedge T_n} \phi''(. + B_s)ds \mid S(K^0) \right) \\ &= \int_0^{t \wedge T_n} (\phi'(. + B_s) \mid S(K^0))dB_s + \frac{1}{2}\int_0^{t \wedge T_n} (\phi''(. + B_s) \mid S(K^0))ds \\ &= \int_0^{t \wedge T_n} \langle \phi'(. + B_s), S \rangle dB_s + \frac{1}{2}\int_0^{t \wedge T_n} \langle \phi''(. + B_s), S \rangle ds \end{aligned}$$

where $S(K^0) = k(K^0)(S)$ and $k(K^0)$ denotes the canonical mapping from $\mathscr{D}'$ onto $\mathscr{D}'(K^0)$, K^0 being the polar of K. The integrals that we have constructed are the modifications of the integrals of the Ito's formula and the negligeable set on which it fails is independent of $x \in \mathbb{R}$ if we replace the original integrals with their modifications. Consequently we have

THEOREM V.1. — If $S \in \mathscr{D}'$, denote by X the mapping $(t, \omega) \to S * \delta_{B_t}(\omega)$. Then X generates a continuous g-process which is a semimartingale in $\mathscr{D}'$. Denote by $\tilde{X}$ the linear mapping on $\mathscr{D}$ with values in the set of the real-valued semimartingales defined by

$$\tilde{X}(\phi) = X^U(\phi) \quad \text{if} \quad \phi \in \mathscr{D}[U^0], \; U \in \mathscr{U}_h(\mathscr{D}'_\beta)$$

Then $\tilde{X}$ satisfies the following relation:

$$(V.1) \qquad \tilde{X}_t(\phi) = S(\phi) + \int_0^t \tilde{X}_s(\phi')dB_s + \frac{1}{2}\int_0^t \tilde{X}_s(\Delta\phi)ds\,.$$

Remark. — In (V.1), the integrals are well defined since for any two semimartingales, belonging to the same equivalence class the corresponding integrals are undistinguishable, hence they belong to the same equivalence class.

Proof. — For any $\phi \in \mathscr{D}$, the mapping $(t, \omega) \to \langle \phi, S * \delta_{B_{t \wedge T_n(\omega)}} \rangle$ has modification which is an S^1-semimartingale. Hence, by Theorem II.1, for any $U \in \mathscr{U}_h(\mathscr{D}')$, $k(U) \circ X^{T_n}$ has a modification, say $X^{U,n}$ which is a semimartingale in $\mathscr{D}'(U)$. If $m < n$, then

$$X^{U,n}_{t \wedge T_m} = X^{U,m}_t \qquad a.\ e.$$

consequently there exists $X^U = (X^U_t)$ which is a semimartingale with values in $\mathscr{D}'(U)$ such that

$$X^U_{t \wedge T_n} = X^{U,n}_t \qquad a.\ e.$$

Obviously $\{ X^U : U \in \mathscr{U}_h(\mathscr{D}') \}$ is a projective system of semimartingales whose projective limit is X. The rest of the theorem is now obvious. Q. E. D.

Remark. — The relation (V.1) can be read also in the following form: For any $\phi \in \mathscr{D}$, $(t, \omega) \to \langle \phi, X_t(\omega) \rangle$ has a modification $X(\phi)$ which is a continuous semimartingale satisfying (V.1) for almost all $\omega \in \Omega$.

Remark. — Regarding $\tilde{X}$ as a mapping, define $\frac{d}{dx}\tilde{X}$ as $-\tilde{X} \circ \frac{d}{dx}$, $\Delta\tilde{X}$ as $\tilde{X} \circ \Delta$, then $\tilde{X}$ satisfies the following equation written in the differential form:

$$(V.2) \qquad d\tilde{X}_t = -\frac{d}{dx}\tilde{X}_t dB_t + \frac{1}{2}\Delta\tilde{X}_t dt,\ \tilde{X}_0 = S$$

Remark. — Using the same technique as in Lemma I.1, one can show that, for fixed $t \geq 0$, the mappings

$$\phi \to E\left[\exp\left(i\int_0^t \tilde{X}_s(\phi')dB_s\right)\right],$$

$$\phi \to E\left[\exp\left(i\int_0^t \tilde{X}_s(\Delta\phi)ds\right)\right]$$

are continuous, hence, for fixed t, the terms at the right of (V.1), can be lifted to the random variables with values in $\mathscr{D}'$.

Remark. — Choose S that $\left(\int_0^t \tilde{X}_s(\phi)dB_s\,;\, t \geq 0\right)$ is a martingale and define H_t as

$$H_t(\phi) = E[\tilde{X}_t(\phi)].$$

Then H_t is a solution of the heat equation

$$\frac{dH}{dt} = \frac{1}{2}\Delta H, \qquad t > 0,$$
$$H_0 = S$$

in $\mathscr{D}'$.

VI. ON FEYNMAN-KAC FORMULA

Let $B = (B_t)$ be the d-dimensional standard Wiener process and V be a real valued infinitely differentiable function on $\mathbb{R}^d$. If $T \in \mathscr{D}'(\mathbb{R}^d)$, we denote by $Z = (Z_t)$ the mapping defined by

$$Z_t(\omega) = \exp(-\,M_t(\omega, .)).T$$

where

$$M_t(\omega, x) = \int_0^t V(x + B_s(\omega))ds.$$

Stopping B on the increasing compact subsets of $\mathbb{R}^d$ as in the preceding section, we see that for any $\phi \in \mathscr{D}(\mathbb{R}^d)$, the mapping

$$(t, \omega) \to \langle \phi, Z_t(\omega) \rangle$$

has a modification which is continuous, adapted, of integrable variation. Hence, by Theorem II.1, it generates a semimartingale in $\mathscr{D}'(\mathbb{R}^d)$ satisfying the hypothesis of Theorem IV.1.

Again, as in the preceding section, $(t, \omega) \to \phi(. + B_{t \wedge T_n}(\omega))$ generates a semimartingale in $\mathscr{D}(\mathbb{R}^d)$. Therefore we may apply Theorem IV.1 to calculate $\langle \phi(. + B_t), Z_t \rangle$:

$$\text{(VI.1)}\quad \langle \phi(.+B_t), Z_t \rangle = \langle \phi, T \rangle + \int_0^t \langle \phi(.+B_s, dZ_s \rangle + \int_0^t \langle d\phi(.+B_s), Z_s \rangle$$
$$= \langle \phi, T \rangle - \int_0^t \langle \phi(. + B_s), V(. + B_s)Z_s \rangle\, ds$$
$$+ \int_0^t \langle D_i \phi \quad (. + B_s), Z_s \rangle\, dB_s^i$$
$$+ \frac{1}{2}\int_0^t \langle \Delta\phi(. + B_s), Z_s \rangle\, ds$$

where D_i is the operator $\frac{\partial}{\partial x_i}$ and we have used the habitual summation convention. Define $X = (X_t)$ as following:

$$\langle \phi, X_t \rangle = \langle \phi(. + B_t), Z_t \rangle, \qquad \phi \in \mathscr{D}(\mathbb{R}^d).$$

We have:

THEOREM VI.1. — X generates a semimartingales in $\mathscr{S}'(\mathbb{R}^d)$ whose projective system is continuous. Denote by $\tilde{X}$ the linear mapping induced on $\mathscr{D}(\mathbb{R}^d)$ with values in the linear space of continuous real-valued semimartingales. Then $\tilde{X}$ satisfies the following relation

$$\text{(VI.2)} \qquad \tilde{X}_t = T - \int_0^t V\tilde{X}_s ds - \int_0^t D_i\tilde{X}_s dB_s^i + \frac{1}{2}\int_0^t \Delta\tilde{X}_s ds$$

where $D_i\tilde{X}$ and $V\tilde{X}$ are defined respectively as

$$D_i\tilde{X} = -\tilde{X} \circ D_i \qquad \text{and} \qquad V\tilde{X} = \tilde{X} \circ V.$$

Proof. — By definition, we have $X_t = Z_t * \delta_{B_t}$. Stopping B on the increasing compact subsets of $\mathbb{R}^d$, we see that, for any $\phi \in \mathscr{D}(\mathbb{R}^d)$, $e^{-M_t}\phi(. + B_t)$ has a fixed compact support in $\mathbb{R}^d$ when (t, ω) belongs to the stochastic interval $[\![0, T_n]\!]$, for any $n \in \mathbb{N}$. Hence there exists a compact K_n in $\mathbb{R}^d$ and a continuous function g such that

$$T = D^\beta g \quad \text{on} \quad K_n$$

for some $\beta \in \mathbb{N}^d$ (g is not unique and it depends on K_n, cf. [*13*]), such that

$$T(e^{-M_t^n}\phi(. + B_t^n)) = (-1)^{|\beta|}\int_{K_n} g(x)D_x^\beta(e^{-M_t^n(x)}\phi(x + B_t^n))dx$$

where $B_t^n = B_{t \wedge T_n}$ and $M_t^n = M_{t \wedge T_n}$. It is not difficult to see that this mapping is a continuous S^1-semimartingale (by Ito's formula and interchanging the order of the integrals). The representation is obvious from the relation VI.1. Q. E. D.

Remark. — The integrals in the representation of $\tilde{X}$ are well defined since any two element of the equivalence class $\tilde{X}(\phi)$ are undistinguishable hence their integrals also.

Remark. — Suppose that $\left(\int_0^t D_i\tilde{X}_s dB_s^i(\phi)\right)$ is a martingale for any $\phi \in \mathscr{D}(\mathbb{R}^d)$, and define H_t as

$$H_t(\phi) = E[\tilde{X}_t(\phi)].$$

Then $H = (H_t ; t \geq 0)$ is a « curve » in $\mathscr{D}'(\mathbb{R}^d)$ and satisfies the following equation:

$$\frac{dH}{dt} = \frac{1}{2}\Delta H - VH, \qquad H_0 = T.$$

Hence we shall call (VI.2) the stochastic form of Feynman-Kac formula. Note that the semimartingale formalism gives directly the weak form of Feynman-Kac formula without passing by the operator theory. Let us also indicate that the above results remain valid when V is time dependent with the following additional hypothesis:

$$(t, x) \to D_x^\alpha V(t, x)$$

is continuous for any $\alpha \in \mathbb{N}^d$.

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