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Section B :
Calcul des Probabilités et Statistique.

A biased roulette

by

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SUMMARY. — A biased roulette, i. e. a weightless disc with one heavy point or, equivalently, a mathematical pendulum rotates in a vertical position. Assuming that the initial velocity is a random variable with a density and that this random variable is dilated by a positive constant d, theorems concerning the limit distribution (as $d \to \infty$) of the stopping angles are derived for different kinds of stopping, namely an instantaneous stopping or stopping by a constant braking force or stopping by friction.

0. INTRODUCTION

Poincaré introduced the method of arbitrary functions as a tool for proving equi-distribution of certain events rather than postulating it with reference to symmetry. The following is the best known example. A roulette is divided into $2n$ equal sections painted alternately white and red. At time 0 the roulette is given a speed V, where V is a non-negative random variable with a density f, and then stopped at a fixed time s. It is possible to show that the probability that a white section stops opposite a mark tends to 1/2 as $n \to \infty$, whatever the density f is. The method of arbitrary functions was later studied extensively by Fréchet, Hostinský and others. For more references see [*1*] and [*2*]. To the author's best knowledge, all applications of the method of arbitrary functions concern « fair » problems. The present paper is an attempt to apply a similar method to a biased roulette. Under a biased roulette we understand a weightless roulette with one heavy point rotating in a vertical position, so that the motion is not uniform. Intuitively it is clear that if the problem were formulated as above, i. e. with the number of sections $n \to \infty$, the probabilities for white and red would be again 1/2 in the limit. Therefore we shall reformulate the problem. Mathematically it is irrelevant in the above example if $n \to \infty$ with the density f fixed or, alternatively, if n is kept fixed and the initial speed V is dilated by a constant d (i. e. replaced by dV), with $d \to \infty$. This can be interpreted as obtaining the probabilities 1/2 for the two colours by assuming that the velocity V is large. We shall adopt mainly this approach. It could be argued that this amounts to the same as assuming that the initial velocity has a very flat distribution, in fact uniform on $[0, \infty)$ in the limit. However, it is quite possible that an inexperienced person would have a very flat distribution of initial velocities. Leaving the number of sections on the roulette fixed amounts to the same as to measure the angle between a fixed point on the roulette and a fixed point on an outer scale. Therefore we can replace the roulette by a weightless pointer with a heavy tip (a mathematical pendulum) rotating in a vertical position. The zero position will be the upmost one (12 o'clock on a wall clock). Then the motion of the pointer will be controlled by the differential equation

$$\phi''(t) = c \sin (2\pi\phi(t)), \qquad c > 0$$

if there is no friction or any other braking force. We will call this model the classical dynamic model. The reason for the factor 2π in the above equation is that we shall consider more general models in which it is more convenient to attach number 1 to one full circle (instead of 2π or 360°). $\phi(t)$ — the total angular distance the pointer covered between the times 0 and t — will be called the accumulated angular displacement (a. a. d.) at time t, while the same angular distance measured modulo 1 will be called the angle at time t. Hence, a. a. d. is an arbitrary number while an angle is restricted to $(0, 1]$. Similarly, if s is the stopping time, $\phi(s)$ will be called the stopping a. a. d. and $\phi(s)$ mod 1 will be called the stopping angle. Velocity will always mean the angular velocity, i. e. $\phi'(t)$. In models without a braking force the stopping is artificial at a given time s. In models with a braking force the stopping time will be the time of the first stopping, i. e. $\min \{ s > 0 : \phi'(s) = 0 \}$. In the classical dynamic model, the pointer becomes a swinging pendulum after the first stopping, but we will disregard these subsequent motions. Instead of considering only the classical dynamic model, we will assume that the pointer is kept in motion by a mechanism with an adjustable initial velocity v and such that v determines uniquely the whole subsequent motion. The initial position of the pointer will always be 0. We will use the following symbols for several different approaches : $(dV, d \to \infty)$, $(dV, d \to \infty, s)$, $(V, s \to \infty)$, $(v, dS, d \to \infty)$ and $(v \to \infty, S)$. They are self-explanatory if we adopt the following conventions: v, or V, or dV respectively means: the initial velocity is deterministic, or a fixed random variable, or a dilated random variable respectively; s, or S, or dS respectively means: the stopping time is artificial and is deterministic, or a fixed random variable, or a dilated random variable respectively. In all cases the stopping angle is a random variable Z and all theorems of this paper concern the limit distribution of Z. Perhaps only the cases $(dV, d \to \infty, s)$, $(v \to \infty, S)$ and $(dV, d \to \infty)$ under a braking force would qualify as a method of arbitrary functions, as only in these cases are the limit distributions independent of initial data.

In the classical dynamic model, the limit distribution is uniform in the $(dV, d \to \infty, s)$ case inspite of the bias, while in the $(dV, d \to \infty)$ case under two different braking forces considered in this paper, the limit distribution is not uniform, although it would be if there were no bias.

1. A GENERAL THEOREM

All basic theorems of this paper will be deduced from the following Th. 1. We assume that to any positive parameter p, two sequences $x_{k,p}$, $y_{k,p}$ of real numbers are given such that

(A 1.1) $\quad 0 = x_{0,p}, \quad x_{k,p} \leq y_{k,p} \leq x_{k+1,p}, \quad x_{k,p} < x_{k+1,p}$ for all $k = 0, 1, \ldots$

(A 1.2) $\quad x_{k,p} \underset{k}{\to} \infty \quad$ for each p.

For any finite interval J on the real line, $|J|$ will denote its length. We shall write $I_{k,p} = [x_{k,p}, x_{k+1,p})$. (A 1.1) and (A 1.2) guarantee that to each $z \in [0, \infty)$ and p, there exists exactly one $I_{k,p}$ such that $z \in I_{k,p}$. We will call it $I_p(z)$ and we will assume that

(A 1.3) $\quad |I_p(z)| \underset{p \to \infty}{\longrightarrow} 0 \quad$ for each z.

Finally, R_p will denote a function on $[0, \infty)$ defined by

$$R_p(z) = \frac{y_{k,p} - x_{k,p}}{x_{k+1,p} - x_{k,p}} \qquad \text{if } z \in I_{k,p}.$$

THEOREM 1. — Let (A 1.1)-(A 1.3) hold and let there exist a function R on $[0, \infty)$ such that

$$R_p(z) \underset{p \to \infty}{\longrightarrow} R(z) \qquad \text{for almost all (Lebesgue measure) } z.$$

Then, for an arbitrary probability density function f on $[0, \infty)$

$$\Sigma_p = \sum_{k=0}^{\infty} \int_{x_{k,p}}^{y_{k,p}} f(z)dz \underset{p\to\infty}{\longrightarrow} \int^{\infty} R(z)f(z)dz .$$

Proof. — To an $\varepsilon > 0$, there exists $z_0 > 0$ and a non-negative continuous function f^* on $[0, \infty)$, vanishing outside $[0, z_0)$ and such that $\int_0^{\infty} | f(z) - f^*(z) | dz < \varepsilon$. Put $\Sigma_p^* = \sum_{k=0}^{\infty} \int_{x_{k,p}}^{y_{k,p}} f^*(z)dz$. Then

$$(1.1) \qquad | \Sigma_p - \Sigma_p^* | < \varepsilon \qquad \text{for all } p$$

and

$$\Sigma_p^* = \sum_{k=0}^{\infty} (y_{k,p} - x_{k,p}) f^*(z_{k,p})$$

where $x_{k,p} < z_{k,p} < x_{k+1,p}$. Define f_p^* on $[0, \infty)$ by

$$f_p^*(z) = f^*(z_{k,p}) \qquad \text{if } z \in I_{k,p} .$$

Then by (A 1.3) and the continuity of f^*, $f_p^*(z) \underset{p\to\infty}{\longrightarrow} f^*(z)$ for all z and

$$\Sigma_p^* = \int_0^{\infty} R_p(z) f_p^*(z)dz \underset{p\to\infty}{\longrightarrow} \int_0^{\infty} R(z)f^*(z)dz .$$

This together with (1.1) proves the theorem. The last limit passage is guaranteed by the Lebesgue boundedness theorem, as $0 \leq R_p(z) \leq 1$, $f^*(z) \leq C$ for all z and $f^*(z) = 0$ for all $z \geq z_0$, so that $f_p^*(z) \leq C$ for all z and $f_p^*(z) = 0$ for all $z \geq z_0 + 1$ if p is sufficiently large.

Remark. — The probability distribution represented by the density function f in the above theorem cannot be replaced generally by an arbitrary probability measure.

2. THE GENERAL (dV, $d \to \infty$) CASE

In this section, V is a random variable on $[0, \infty)$ representing the initial velocity with a density function f, $d > 0$. The nature of the stopping time is irrelevant in this section. Z_d will denote the stopping angle under the initial velocity dV. Our starting point is a function A defined on $(0, \infty)$ with the interpretation: $A(v)$ = the stopping a. a. d. under the initial velocity v. Hence $Z_d = A(dV) \bmod 1$. A is supposed to satisfy the following conditions:

$$\text{(A 2.1)} \qquad \begin{cases} A(v) > 0 \text{ for all } v > 0,\ A(\infty) = \infty, \\ \text{A is non-decreasing and left-continuous.} \end{cases}$$

Anticipating that not all values of A are possible as stopping a. a. d.'s in certain models, we do not assume A continuous everywhere — its values may jump over certain intervals. Similarly, anticipating that in certain models a whole interval of initial velocities may lead to the same stopping a. a. d., we allow A to be constant over some intervals. A function B, defined on $[0, \infty)$ will have the following interpretation: $B(\beta)$ is the initial velocity under which the stopping a. a. d. is β. If A is continuous and strictly increasing B is imply an inverse to A. Generally, we will define B by

$$\text{(2.1)} \qquad B(\beta) = \max \{ v : A(v) \leq \beta \} \qquad \text{if } \beta > 0,\ B(0) = 0.$$

Under (A 2.1), B has the following properties:

$$\text{(A 2.2)} \qquad \begin{cases} B(0) = 0,\ B(\infty) = \infty, \text{ B is non-decreasing} \\ \text{and right-continuous} \end{cases}$$

Notice that the values of B filling the jumps of A do not have the above mentioned interpretation and if A is constant over a certain interval of velocities, the definition (2.1) chooses the maximal velocity as the corresponding value of B. However the relation $\{ v : A(v) \leq \beta \} = \{ v : v \leq B(\beta) \}$ holds always so that for $d = 1$ and $0 < \alpha \leq 1$ we have

$$\mathscr{P}(Z_1 \leq \alpha) = \sum_{k=0}^{\infty} \mathscr{P}(k < A(V) \leq k + \alpha) = \sum_{k=0}^{\infty} \int_{B(k)}^{B(k+\alpha)} f(v)dv .$$

Replacing V by dV or, equivalently, replacing $f(v)$ by $\frac{1}{d} f\left(\frac{v}{d}\right)$, we obtain

$$\mathscr{P}(Z_d \leq \alpha) = \sum_{k=0}^{\infty} \int_{\frac{1}{d}B(k)}^{\frac{1}{d}B(k+a)} f(v)dv .$$

If

$$\text{(A 2.3)} \qquad B(k + 1) - B(k) > 0 \qquad \text{for all } k = 0, 1, \ldots,$$

then we may write for all $k = 0, 1, \ldots$ and all $0 \leq \alpha \leq 1$

$$G_k(\alpha) = \frac{B(k + \alpha) - B(k)}{B(k + 1) - B(k)} .$$

For each k, G_k is a distribution function on $[0, 1]$.

THEOREM 2. — Let us assume that (A 2.2) and (A 2.3) hold and that

$$\sup_k (\mathrm{B}(k+1) - \mathrm{B}(k)) < \infty \tag{2.2}$$

Further let us assume that for some $0 < \alpha \leq 1$ the limit

$$\lim_{k\to\infty} \mathrm{G}_k(\alpha) = \mathrm{G}(\alpha) \qquad \text{exists}. \tag{2.3}$$

Then

$$\mathscr{P}(\mathrm{Z}_d \leq \alpha) \underset{d\to\infty}{\longrightarrow} \mathrm{G}(\alpha).$$

Proof. — We shall apply Th. 1 with the parameter p called d and with $x_{k,d} = \frac{1}{d}\mathrm{B}(k)$, $y_{k,d} = \frac{1}{d}\mathrm{B}(k+\alpha)$. Then $\mathrm{R}_d(z) = \mathrm{G}_k(\alpha)$ if $z \in \mathrm{I}_{k,d}$. The conditions (A 1.1)-(A 1.3) are satisfied trivially, in fact $\sup_k (x_{k+1,d} - x_{k,d}) \underset{d\to\infty}{\longrightarrow} 0$. Further, because of this last relation, for any $z > 0$ and any k_0, $z \in \mathrm{I}_{k,d}$ implies $k \geq k_0$ if d is sufficiently large. Hence $\mathrm{R}_d(z) \underset{d\to\infty}{\longrightarrow} \mathrm{G}(\alpha)$ for any $z > 0$. Then by Th. 1

$$\mathscr{P}(\mathrm{Z}_d \leq \alpha) \underset{d\to\infty}{\longrightarrow} \int_0^\infty \mathrm{G}(\alpha) f(v) dv = \mathrm{G}(\alpha).$$

3. MODELS WITHOUT A BRAKING FORCE

In the whole paper, $\phi(t, v)$ will denote the a. a. d. of the pointer at time t under the initial velocity v. In this paragraph, we shall assume that ϕ is defined on $[0, \infty) \times (0, \infty)$ and that it satisfies the following three conditions:

(A 3.1) for each $v > 0$, $\phi(., v)$ is strictly increasing and continuous on $[0, \infty)$, $\phi(0, v) = 0$, $\phi(\infty, v) = \infty$

(A 3.2) for each $t > 0$, $\phi(t, .)$ is strictly increasing and continuous on $(0, \infty)$, $\phi(t, 0+) = 0$, $\phi(t, \infty) = \infty$

(A 3.3) if, for a given $v > 0$, t_1 is defined by $\phi(t_1, v) = 1$, then

$$\phi(t + t_1, v) = \phi(t, v) + 1.$$

The last condition expresses the fact that each subsequent cycle is a replica of the previous one, i. e. the pointer rotates by inertia without any systematic braking or accelerating force. The condition $\phi(\infty, v) = \infty$ is superfluous; it follows from (A 3.3). To give v its proper interpretation, we should have also assumed that $\phi'(0, v) = v$, however we shall not need this explicitly in the proofs. (In the whole paper, several functions of two variables will occur with v as the second variable; if the function is F, e. g., then F′ and F″ will always denote the first and second derivative with respect to the first variable.) In formulating and proving the theorems of this section it is more convenient to replace ϕ by its inverse with respect to t, i. e. by a function H of two variables β and v defined on $[0, \infty) \times (0, \infty)$ by $\phi(H(\beta, v), v) = \beta$. Clearly $H(\beta, v)$ denotes the time the pointer needs to reach β under v. H satisfies the following three conditions:

(A 3.4) for each $v > 0$, $H(., v)$ is strictly increasing and continuous on $[0, \infty)$, $H(0, v) = 0$, $H(\infty, v) = \infty$

(A 3.5) for each $\beta > 0$, $H(\beta, .)$ is strictly decreasing and continuous on $(0, \infty)$, $H(\beta, 0+) = \infty$, $H(\beta, \infty) = 0$

(A 3.6) for each integer $k \geq 0$, each $0 \leq \alpha \leq 1$ and each $v > 0$

$$H(k + \alpha, v) = kH(1, v) + H(\alpha, v).$$

3.1 The (dV, $d \to \infty$, s) case

In this section, V, f and Z_d have the same meaning as in paragraph 2; s is the (artificial) deterministic stopping time ($s > 0$). The function A of paragraph 2 is defined by $A(v) = \phi(s, v)$. It is strictly increasing and continuous on $(0, \infty)$ with $A(0+) = 0$. Hence B is strictly increasing and continuous on $[0, \infty)$ and it satisfies

$$s = H(\beta, B(\beta)) \qquad \text{for any } \beta > 0. \tag{3.1.1}$$

In addition to (A 3.3)-(A 3.6), we will need another condition on H, namely

(A 3.1.1) $\begin{cases} \text{for each } \beta \geq 0, \text{ the derivative } H_2(\beta, v) = \dfrac{\partial}{\partial v} H(\beta, v) \text{ and a} \\ \text{finite limit} \\ \lim\limits_{v \to \infty} (- H_2(\beta, v))v^2 = L(\beta) \\ \text{exist, with } L(1) > 0. \end{cases}$

THEOREM 3.1. — Under (A 3.3)-(A 3.6) and (A 3.1.1),

$$\mathscr{P}(Z_d \leq \alpha) \underset{d \to \infty}{\longrightarrow} \frac{L(\alpha)}{L(1)} \qquad \text{for any } 0 \leq \alpha \leq 1.$$

Proof. — By (A 3.1.1) and the l'Hospital rule

$$H(\beta, v)v \underset{v \to \infty}{\longrightarrow} L(\beta). \tag{3.1.2}$$

Put $H^*(\beta, w) = H\left(\beta, \frac{1}{w}\right)$, $H^*(\beta, 0) = 0$. Then $H^*(\beta, .)$ is continuous on $[0, \infty)$ and, by (A 3.1.1)

$$(3.1.3) \quad H_2^*(\beta, w) = \frac{\partial}{\partial w} H^*(\beta, w) \qquad \text{exists and } H^*(\beta, 0+) = L(\beta).$$

We are going to apply Th. 2. We must show that (2.2) and (2.3) hold with $G(\alpha) = \frac{L(\alpha)}{L(1)}$ for each $0 \leq \alpha \leq 1$. Both (2.2) and (2.3) will be proved, if we show that

$$(3.1.4) \qquad B(k + \alpha) - B(k) \xrightarrow[k\to\infty]{} \frac{L(\alpha)}{s} \qquad \text{for any } 0 \leq \alpha \leq 1\,.$$

By (3.1.1) and (A 3.6),

$$s = H(k, B(k)) = kH^*\left(1, \frac{1}{B(k)}\right) = k\left[H^*(1, 0) + H_2^*(1, w_k)\frac{1}{B(k)}\right],$$

where $0 < w_k < \frac{1}{B(k)}$. $B(k) \to \infty$ and $H^*(1, 0) = 0$. Hence by (3.1.3)

$$(3.1.5) \qquad \frac{B(k)}{k} \xrightarrow[k\to\infty]{} \frac{L(1)}{s}.$$

For any two sequences v_k, z_k such that $B(k) \leq v_k \leq z_k \leq B(k + 1)$, $1 \leq \frac{z_k}{v_k} \leq \frac{B(k + 1)}{B(k)}$. Hence by (3.1.4)

$$(3.1.6) \qquad \frac{z_k}{v_k} \xrightarrow[k\to\infty]{} 1\,.$$

By (3.1.1) and (A 3.6),

$$\begin{aligned} s = H(k + \alpha, B(k + \alpha)) &= H(k, B(k + \alpha)) + H(\alpha, B(k + \alpha)) \\ &= H(k, B(k)) + H_2(k, v_{k,\alpha})(B(k + \alpha) - B(k)) + H(\alpha, B(k + \alpha)) \end{aligned}$$

where $B(k) < v_{k,\alpha} < B(k + \alpha)$. As $H(k, B(k)) = s$, we have

$$(3.1.7) \qquad B(k + \alpha) - B(k) = -\frac{H(\alpha, B(k + \alpha))}{kH_2(1, v_{k,\alpha})}.$$

Hence

$$B(k+\alpha) - B(k) = \frac{H(\alpha, B(k+\alpha))B(k+\alpha)}{-H_2(1, v_{k,\alpha})v_{k,\alpha}^2} \cdot \frac{B(k)}{k} \cdot \frac{v_{k,\alpha}^2}{B(k)B(k+\alpha)} \xrightarrow[k\to\infty]{} \frac{L(\alpha)}{L(1)} \frac{L(1)}{s}$$

by (A 3.1.1), (3.1.2), (3.1.5) and (3.1.6). This proves (3.1.4).

As a particular case we will now consider the general dynamic model, i. e. a model in which the motion of the pointer is controlled by the differential equation

$$\phi''(t) = q(\phi(t))\,.$$

Put $Q(y) = \int_0^y q(x)dx$. We shall assume that

(A 3.1.2) q is continuous 1-periodic and satisfies the ordinary Lipschitz condition

(A 3.1.3) $$Q(1) = 0$$

(A 3.1.4) $$Q(y) \geq 0 \quad \text{for all } y$$

The function $\phi(t, v)$ is defined as the (only) solution of the differential equation with $\phi(0, v) = 0$, $\phi'(0, v) = v$. It is easy to see that the function H introduced in this paragraph and relating t and $\phi(t, v)$ through $t = H(\phi(t, v), v)$ has the form

(3.1.8) $$H(\beta, v) = \int_0^\beta (v^2 + 2Q(y)^{-1/2}dy.$$

In the classical dynamic model, H is related to the incomplete elliptic integral of the first kind

$$F(\psi, k) = \int_0^\psi (1 - k^2 \sin^2 y)^{-1/2}dy$$

by

$$H(\beta, v) = \frac{k}{\sqrt{2\pi c}}\left[F\left(\frac{\pi}{2}, k\right) - F\left(\pi\left(\frac{1}{2} - \beta\right), k\right)\right],$$

where $k = \sqrt{\dfrac{2c}{\pi v^2 + 2c}}$, however we will not need this result. It is easy to see from (3.1.8) that H satisfies (A 3.4)-(A 3.6). In particular, (A 3.1.3) implies that Q is 1-periodic, hence (A 3.6) holds. From (A 3.1.2) and (A 3.1.4) it also follows that $Q(y) \leq Ky^2$ so that $\int_0^\beta Q(y)^{-1/2}dy = \infty$; hence $H(\beta, 0+) = \infty$. The assumption (A 3.1.1) of Th. 3.1) also holds with $L(\beta) = \beta$. Hence

COROLLARY TO THEOREM 3.1. — In the general dynamic model with q satisfying (A 3.1.2)-(A 3.1.4), Z_d has in the limit (as $d \to \infty$) the uniform distribution on [0, 1].

3.2 The (V, $s \to \infty$) case

In this section, V is a random variable on $[0, \infty)$ representing the initial velocity with a given density function f; s is the (artificial) deterministic stopping time. B is defined as in 3.1, i. e. by (3.1.1), however to express its dependence on s, we shall write B_s instead of B. Similarly, we shall denote the random variable representing the stopping angle under the stopping time s by Z_s. Hence $Z_s = B_s^{-1}(V) \bmod 1$. In addition to (A 3.4)-(A 3.6), we shall assume that

(A 3.2.1) for each $\beta \geq 0$, the derivative $H_2(\beta, v) = \frac{\partial}{\partial v} H(\beta, v)$ exists and is continuous on $(0, \infty)$.

THEOREM 3.2. — Under (A 3.4)-(A 3.6) and (A 3.2.1),

$$\mathscr{P}(Z_s \leq \alpha) \underset{s \to \infty}{\longrightarrow} \int_0^\infty \frac{H(\alpha, v)}{H(1, v)} f(v) dv \qquad \text{for any } 0 \leq \alpha \leq 1 .$$

Proof. — Take a fixed $0 < \alpha < 1$. We shall apply Th. 1 with p replaced by s and with $x_{k,s} = B_s(k)$, $y_{k,s} = B_s(k + \alpha)$. The assumptions (A 1.1) and (A 1.2) are clearly satisfied. To show that (A 1.3) also holds, take an arbitrary $z \geq 0$ and define integers k_s by $B_s(k_s) \leq z < B_s(k_s + 1)$. We must show that $B_s(k_s) \underset{s \to \infty}{\longrightarrow} z$, $B_s(k_s + 1) \underset{s \to \infty}{\longrightarrow} z$. Assume that $B_s(k_s) \not\to z$. Hence there exist $0 < z' < z$ and $s_j \to \infty$ such that for all $k^{(j)} = k_{s_j}$, $B_{s_j}(k^{(j)}) \leq z'$. By (3.1.1), (A 3.5) and (A 3.6), $s_j = k^{(j)} H(1, B_{s_j}(k^{(j)})) \geq k^{(j)} H(1, z')$ and $s_j < (k^{(j)} + 1) H(1, z)$. The second inequality implies $k^{(j)} \to \infty$ and then $H(1, z') \leq \frac{s_j}{k^{(j)}} < H(1, z) \frac{k^{(j)} + 1}{k^{(j)}}$ leads to a contradictory inequality $H(1, z') \leq H(1, z)$. Hence $B_s(k_s) \to z$. The proof of $B_s(k_s + 1) \to z$ would be similar.

The function R_s of Th. 1 is defined by

$$R_s(z) = \frac{B_s(k + \alpha) - B_s(k)}{B_s(k + 1) - B_s(k)} \qquad \text{if } z \in I_{k,\alpha} .$$

We must show that for any z

$$R_s(z) \underset{s \to \infty}{\longrightarrow} R(z) = \frac{H(\alpha, z)}{H(1, z)} . \tag{3.2.1}$$

By (3.1.7)

$$R_s(z) = \frac{H(\alpha, B_s(k + \alpha))}{H(1, B_s(k + 1))} \cdot \frac{H_2(1, v_s^{(1)})}{H_2(1, v_s^{(2)})}$$

where $B_s(k) < v_s^{(j)} < B_s(k + 1)$, $j = 1, 2$. Hence (3.2.1) follows from (A 1.3) and from the continuity of H and H_2.

3.3 The ($v \to \infty$, S) case

In this section, the initial velocity v is deterministic, the stopping time S is a random variable on $[0, \infty)$ with a density function f. The stopping angle under v will be denoted by Z_v, i. e. $Z_v = \phi(S, v) \bmod 1$. By (A 3.1) $\{ s : \phi(s, v) \leq \beta \} = \{ s : s \leq H(\beta, v) \}$. Hence for $0 \leq \alpha \leq 1$,

$$\mathscr{P}(Z_v \leq \alpha) = \sum_{k=0}^{\infty} \int_{H(k,v)}^{H(k+\alpha,v)} f(s)ds .$$

In addition to (A 3.1)-(A 3.3) we will need the following assumption:

(A 3.3.1) for each $\beta \geq 0$ a finite limit $\dfrac{H(\beta, v)}{H(1, v)} \underset{v \to \infty}{\longrightarrow} G(\beta)$ exists.

THEOREM 3.3. — Under (A 3.3)-(A 3.6) and (A 3.3.1),

$$\mathscr{P}(Z_v \leq \alpha) \underset{v \to \infty}{\longrightarrow} G(\alpha) \qquad \text{for any } 0 \leq \alpha \leq 1 .$$

Proof. — Take a fixed $0 < \alpha < 1$. We shall apply Th. 1 with $p = v$ and $x_{k,v} = H(k, v)$, $y_{k,v} = H(k + \alpha, v)$. (A 1.1) and (A 1.2) hold by (A 3.4). By (A 3.5) and (A 3.6), $H(k + \alpha, v) - H(k, v) = H(\alpha, v) \underset{v \to \infty}{\longrightarrow} 0$. Hence $\sup_k (x_{k+1,v} - x_{k,v}) \underset{v \to \infty}{\longrightarrow} 0$ and (A 1.3) is satisfied. The function R_v occurring in Th. 1 is defined by $R_v(z) = \dfrac{H(k + \alpha, v) - H(k, v)}{H(k + 1, v) - H(k, v)}$ if $z \in I_{k,v}$. However, by (A 3.6), $R_v(z) = \dfrac{H(\alpha, v)}{H(1, v)} \underset{v \to \infty}{\longrightarrow} G(\alpha)$.

Remark. — It is easy to see that (A 3.1.1) or its weaker form (3.1.2) imply (A 3.3.1) with $G(\alpha) = \dfrac{L(\alpha)}{L(1)}$. Hence, under the assumptions of Th. 3.1, Z_d of Th. 3.1 and Z_v of Th. 3.3 have in the limit the same distribution. In particular:

COROLLARY TO THEOREM 3.3. — In the general dynamic model (defined in Section 3.1) with q satisfying (A 3.1.2)-(A 3.1.4), Z_v has (as $v \to \infty$) the uniform distribution on [0, 1].

3.4 The (v, dS, $d \to \infty$) case

In this section, the initial velocity v is deterministic (and fixed) and the stopping time is dS where S is a random variable on $[0, \infty)$ with a density function f. The stopping angle under the stopping time dS will be denoted by Z_d, so that $Z_d = \phi(dS, v) \bmod 1$. This time

$$\mathscr{P}(Z_d \leq \alpha) = \sum_{k=0}^{\infty} \int_{\frac{1}{d}H(k,v)}^{\frac{1}{d}H(k+\alpha,v)} f(s)ds .$$

THEOREM 3.4. — Under (A 3.4)-(A 3.6)

$$\mathcal{P}(Z_d \leq \alpha) \underset{d\to\infty}{\longrightarrow} \frac{H(\alpha, v)}{H(1, v)} \qquad \text{for any } 0 \leq \alpha \leq 1 .$$

Proof. — The theorem follows again from Th. 1 with $p = d$, $x_{k,d} = \frac{1}{d} H(k, v)$, $y_{k,d} = \frac{1}{d} H(k + \alpha, v)$. The details would be similar to those of the proof of Th. 3.3.

4. THE (dV, $d \to \infty$) CASE WITH A BRAKING FORCE

As in paragraph 2, V is a random variable on $[0, \infty)$ representing the initial velocity with a density function f, $d > 0$ the dilating parameter. We shall assume that the motion of the pointer is controlled by the differential equation

(4.1) $$\phi''(t) = q_1(\phi'(t)) + q(\phi(t))$$

where q_1 represents the braking force. We shall consider only two cases, namely $q_1(x) \equiv - b (b > 0)$, i. e. a constant braking force and $q_1(x) = - ax$ $(a > 0)$, i. e. a braking force by friction. The a. a. d. $\phi(t, v)$ at time t under v is defined as the solution of (4.1) with the initial conditions $\phi(0, v) = 0$, $\phi'(0, v) = v$. If q satisfies (A 3.1.2), then ϕ is well defined in both particular cases mentioned above. The stopping time $C(v)$ under v is defined by $C(v) = \sup \{ s > 0 : \phi'(t, v) > 0 \text{ for all } 0 \leq t \leq s \}$. $C(v)$ may be infinite. If $C(v)$ is finite, $\phi'(C(v), v) = 0$ by continuity. The stopping a. a. d. $A(v)$ under v is defined by $A(v) = \phi(C(v), v)$ if $C(v) < \infty$ and by $A(v) = \lim_{t\to\infty} \phi(t, v)$ if $C(v)$ is infinite. We will see that in the two cases mentioned above $A(v) < \infty$ even if $C(v) = \infty$. As $\phi(., v)$ is strictly increasing in $[0, A(v))$, $H(\beta, v)$ is again well defined by $\beta = \phi(H(\beta, v), v)$ for any $\beta \in [0, A(v))$.

4.1 The (dV, $d \to \infty$) case with a constant braking force

In this section we shall assume that the motion is controlled by the differential equation

(4.1.1) $$\phi''(t) = - b + q(\phi(t))$$

with $b > 0$. We shall require q to satisfy (A 3.1.2) and (A 3.1.3). Put $Q_0(y) = 2by - 2Q(y)$, where again $Q(y) = \int_0^y q(x)dx$. It is easy to see that

$$H(\beta, v) = \int_0^{\beta} (v^2 - Q_0(y))^{-1/2} dy \qquad \text{for any } 0 \leq \beta < A(v)$$

and that

(4.1.2) $\phi'(t, v) = (v^2 - Q_0(\phi(t, v)))^{1/2}$ for any $0 \le t < C(v)$.

As $Q_0(y) \underset{y \to \infty}{\longrightarrow} \infty$, we see from (4.1.2) that $A(v) < \infty$ even if $C(v) = \infty$, and that

(4.1.3) $A(v) = \sup \{ y \ge 0 : Q_0(x) < v^2 \text{ for all } 0 \le x \le y \}$.

From (4.1.3) we can see easily that A satisfies (A 2.1) and is in fact strictly increasing.

THEOREM 4.1. — Let the function q occurring in (4.1.1) satisfy (A 3.1.2) and (A 3.1.3). Then

$$\mathscr{P}(Z_d \le \alpha) \underset{d \to \infty}{\longrightarrow} \frac{(b + R(\alpha) - R(1))^+}{b}$$

where $R(\alpha) = \max_{0 \le y \le \alpha} \{ by - Q(y) \}$ and $(z)^+ = \max \{ 0, z \}$.

Proof. — We will apply Th. 2. The function B occurring in this theorem was defined in paragraph 2 by $B(\beta) = \max \{ v : A(v) \le \beta \}$. It is easy to see that in our case

$$B(\beta) = \max_{0 \le y \le \beta} \sqrt{(Q_0(y))^+} \qquad \text{for any } \beta \ge 0.$$

Hence $B^2(\beta) = \max_{0 \le y \le \beta} (Q_0(y))^+$. As $Q_0(1) = 2b > 0$ we can write $B^2(\beta) = \max_{0 \le y \le \beta} Q_0(y)$ for $\beta \ge 1$. Hence, for any $k \ge 1$ and any $0 \le \alpha \le 1$

$$\begin{aligned} B^2(k + \alpha) &= \max \{ \max_{0 \le y \le k} Q_0(y), \ \max_{k \le y \le k+\alpha} Q_0(y) \} \\ &= \max \{ B^2(k), \ 2bk + \max_{0 \le y \le \alpha} Q_0(y) \}. \end{aligned} \tag{4.1.4}$$

In the last step we used the fact that Q is 1-periodic by (A 3.1.2) and (A 3.1.3) Substituting $\alpha = 1$ we have $B^2(k + 1) = \max \{ B^2(k), 2bk + B^2(1) \}$ and, using this relation, we can prove by induction that

(4.1.5) $B^2(k) = 2b(k - 1) + B^2(1)$ for all $k \ge 1$.

Substituting (4.1.5) into (4.1.4) we finally get

(4.1.6) $B^2(k + \alpha) = 2b(k - 1) + \max \{ B^2(1), \ 2b + \max_{0 \le y \le \alpha} Q_0(y) \}$.

By (4.1.6)

$$\frac{B(k + \alpha)}{\sqrt{k}} \underset{k \to \infty}{\longrightarrow} 2b \qquad \text{for any } 0 \le \alpha \le 1 \tag{4.1.7}$$

and

$$B(k + 1) - B(k) = \frac{B^2(k + 1) - B^2(k)}{B(k + 1) + B(k)} = \frac{2b}{B(k + 1) + B(k)} \underset{k \to \infty}{\longrightarrow} 0$$

by (4.1.5) and (4.1.7). Hence the condition (2.2) of Th. 2 is satisfied. Finally,

$$\frac{B(k+\alpha)-B(k)}{B(k+1)-B(k)} = \frac{B^2(k+\alpha)-B^2(k)}{B^2(k+1)-B^2(k)} \cdot \frac{B(k+1)+B(k)}{B(k+\alpha)+B(k)}$$

$$\underset{k\to\infty}{\longrightarrow} \frac{\max\{B^2(1),\ 2b+\max_{0\le y\le\alpha} Q_0(y)\} - B^2(1)}{2b} \cdot 1$$

by (4.1.5), (4.1.6) and (4.1.7). Hence, (2.3) of Th. 2 is also satisfied with $G(\alpha) = \frac{(b + R(\alpha) - R(1))^+}{b}$. (A 2.3) is satisfied by (4.1.5).

Remark 1. — If $b \ge \max_{0\le x\le 1} q(x)$, then $Q_0(y)$ is non-decreasing on [0, 1], so that $R(\alpha) = b\alpha - Q(\alpha)$ and the limit distribution function of Z_d is

$$G(\alpha) = \alpha - \frac{Q(\alpha)}{b} = \frac{Q_0(\alpha)}{2b}. \tag{4.1.8}$$

Remark 2. — In the classical dynamic case with a constant braking force $b > 0$, $\frac{Q_0(y)}{2} = by - \frac{c}{2\pi}(1 - \cos 2\pi y)$. If $b \ge c$, the limit distribution function G is strictly increasing and (4.1.8) holds. If $b < c$, there exists $0 < \alpha_0 < \frac{1}{4} < \alpha_1 < 1$ (depending on b, c) such that G is constant on $(\alpha_0\alpha_1)$ (i. e. the pointer cannot stop between α_0 and α_1) and G is strictly increasing and equals $\frac{Q_0(\alpha)}{2b}$ outside (α_0, α_1). The point α_0 is the solution of $b = c \sin 2\pi\alpha$ in $\left(0, \frac{1}{4}\right)$, α_1 is the solution of $Q_0(\alpha) = Q_0(\alpha_0)$ in $\left(\frac{1}{4}, 1\right)$.

4.2 The (dV, $d \to \infty$) case with a friction

In this section we shall assume that the motion is controlled by the differential equation

$$\phi''(t) = -a\phi'(t) + q(\phi(t)) \tag{4.2.1}$$

with $a > 0$ and q and Q satisfying (A 3.1.2) and (A 3.1.3). To be able to apply Th. 2, we will have to establish the existence of the limit (2.3). By (4.2.15) below, $B(k+\alpha) = W^{(k)}(B(\alpha))$, where $W^{(k)}$ is the k-th iterate of a function W. Limits of ratios $\frac{W^{(k)}(z) - W^{(k)}(z_0)}{W^{(k+1)}(z_0) - W^{(k)}(z_0)}$ were studied by Lévy, Szekers and others. A systematic treatment may be found in [3]. The most relevant part of [3] is Chap. VII, § 4 and it is likely that Th. 7.7 of [3] would apply to our situation. However it seems easier to establish the existence of the limit directly, rather than to try to show that the assumptions of Th. 7.7 are satisfied in our case.

The functions ϕ, C, A, H are defined as at the beginning of paragraph 4. For any $v > 0$ and $\beta \in [0, A(v))$, $D(\beta, v)$ will denote the velocity of the pointer at the a. a. d. β under v, i. e.

$$D(\beta, v) = \phi'(H(\beta, v), v) \tag{4.2.2}$$

or

$$D(\beta, v) = \frac{1}{H'(\beta, v)}. \tag{4.2.3}$$

D is strictly positive and differentiable with respect to β on $[0, A(v))$. If $C(v) < \infty$ (in which case $A(v) < \infty$ trivially), $D(., v)$ is continuous on $[0, A(v)]$ and $D(A(v), v) = 0$. Later we shall see that this is true even if $C(v) = \infty$. We will now prove a series of lemmas we will need in the main theorem.

LEMMA 4.2.1. — For any $v > 0$, $D(., v)$ satisfies in $[0, A(v))$ the differential equation

$$y'(\beta) = -a + \frac{q(\beta)}{y(\beta)}. \tag{4.2.4}$$

Proof. — (4.2.4) follows immediately from (4.2.1)-(4.2.3).

Sometimes it is more convenient to use $D_0 = D^2$ instead of D. The next assertion follows immediately from L 4.2.1.

LEMMA 4.2.2. — For any $v > 0$, $D_0(., v)$ satisfies in $[0, A(v))$ the differential equation

$$\frac{1}{2} y'(\beta) = -a\sqrt{y(\beta)} + q(\beta). \tag{4.2.5}$$

Under a solution D of (4.2.4) in an open interval (β_1, β_2) we will always understand a strictly positive function satisfying (4.2.4) on (β_1, β_2). A function D will be called a solution of (4.2.4) on a closed interval $[\beta_1, \beta_2]$, if it is a solution on (β_1, β_2) and continuous on $[\beta_1, \beta_2]$. Let, for any γ and any $z > 0$, $D(., z \mid \gamma)$ denote the solution of (4.2.4) under the initial condition $D(\gamma, z \mid \gamma) = z$. The function $-a + \frac{q(\beta)}{z}$ satisfies the Lipschitz condition in any interval $[z_1, z_2]$ with $z_1 > 0$. Therefore $D(., z \mid \gamma)$ is well defined in some neighbourhood of γ and can be continued until it reaches the boundary 0 or escapes to $+\infty$. When extending $D(., z \mid \gamma)$ to the left of γ we need not be worried about the possible escape to ∞, if we know that the possible extension to $[\gamma_1, \gamma]$ is strictly separated from 0 by an $\varepsilon > 0$, because it is easy to see that $D(\beta, z \mid \gamma) \geq \varepsilon$ for $\beta \leq \gamma$ implies

$$D(\beta, z \mid \gamma) \leq -\left(a + \frac{m}{\varepsilon}\right)(\beta - \gamma) + z .$$

The same conventions and notations will apply to solutions D_0 of (4.2.5); $D_0(., z \mid \gamma)$ will denote the solution of (4.2.5) under $D_0(\gamma, z \mid \gamma) = \gamma$. By uniqueness, $D_0(\beta, z^2 \mid \gamma) = D(\beta, z \mid \gamma)^2$ for any β from the domain of their definition. Integrating (4.2.4) and (4.2.5) we obtain for any γ, any $z > 0$ and any β from the domain of definition of D or D_0

$$D(\beta, z \mid \gamma) = z - a(\beta - \gamma) + \int_\gamma^\beta \frac{q(y)}{D(y, z \mid \gamma)} dy \tag{4.2.6}$$

and

$$\frac{1}{2} D_0(\beta, z^2 \mid \gamma) = \frac{1}{2} z^2 - a \int_\gamma^\beta \sqrt{D_0(y, z^2 \mid \gamma)} dy + Q(\beta) - Q(\gamma) . \tag{4.2.7}$$

It follows from (4.2.7) that any solution D_0 on (β_1, β_2) can be uniquely extended to $[\beta_1, \beta_2]$ by continuity. The same must hold for $D = \sqrt{D_0}$. For any $v > 0$, the function $D(., v)$ defined by (4.2.2) can be identified with $D(., v \mid 0)$ by uniqueness. Similarly $D_0(., v^2) = D_0(., v^2 \mid 0)$.

LEMMA 4.2.3. — Let $0 < z_1 < z_2$, let $D(., z_1 \mid \gamma)$ and $D(., z_2 \mid \gamma)$ exist on an interval I (closed or open) containing γ and let at least one of the two functions be strictly positive on I. Then

$$D(\beta, z_1 \mid \gamma) < D(\beta, z_2 \mid \gamma) \qquad \text{for all } \beta \in I .$$

Proof. — As one of the solutions is strictly positive on I, the other solution cannot cross it in I because of uniqueness.

LEMMA 4.2.4. — Let, for some $v > 0$ and γ

$$D(., v) > 0 \text{ on } [0, \gamma), \quad D(\gamma, v) = 0 . \tag{4.2.8}$$

Then $D(., z \mid \gamma)$ is defined and strictly positive on $[0, \gamma]$ for any $z > 0$. For any $\beta \in [0, \gamma]$, $D(\beta, . \mid \gamma)$ is strictly increasing and continuous on $(0, \infty)$.

Proof. — The existence of $D(., z \mid \gamma)$ on $[0, \gamma]$ follows from the fact that the extension to the left of γ is separated from 0 by $D(., v)$ because of L 4.2.3. The fact that $D(\beta, . \mid \gamma)$ is strictly increasing in z follows from L 4.2.3. The continuity with respect to z follows from well known theorems, but it can be also obtained directly by an uniqueness argument, e. g. if $z_n \to z > 0$ and $D(\beta, z_n \mid \gamma) \to l > D(\beta, z \mid \gamma)$ for some $\beta \in [0, \gamma]$, then there are infinitely many solutions asuming their values between l and $D(\beta, z \mid \gamma)$ at β and reaching z at γ.

LEMMA 4.2.5. — Let (4.2.8) hold with $\gamma = 1$. Then there exists $z_0 > 0$ such that

$$D(\beta, z \mid 1) \geq \frac{1}{2} z \qquad \text{for all } z \geq z_0 \text{ and all } \beta \in [0, 1].$$

Proof. — By L 4.2.4, $D(., 1 \mid 1)$ exists on [0, 1] and is strictly positive. Hence $D(\beta, z \mid 1) \geq D(\beta, 1 \mid 1) \geq \varepsilon$ for some $\varepsilon > 0$, all $\beta \in [0, 1]$ on all $z \geq 1$. Let $m = \max_{0 \leq \alpha \leq 1} \mid q(\alpha) \mid$. Then $\int_\beta^1 \frac{q(y)}{D(y, z \mid 1)} dy > -\frac{m}{\varepsilon}$ for all $\beta \in [0, 1]$ and all $z \geq 1$. Then by (4.2.6), $D(\beta, z \mid 1) \geq z - \frac{m}{\varepsilon}$ for all $z \geq 1$ and all $\beta \in [0, 1]$. If (4.2.8) holds with $\gamma = 1$, $W(z) = D(0, z \mid 1)$ is well defined for all $z > 0$ by L 4.2.4.

LEMMA 4.2.6. — Let (4.2.8) hold with $\gamma = 1$, Then

(4.2.9) W is continuous and strictly increasing on $(0, \infty)$

(4.2.10) $$W(z) > z \qquad \text{for all } z > 0$$

(4.2.11) $$W(z) - z \underset{z \to \infty}{\longrightarrow} a.$$

Proof. — (4.2.9) follows from L 4.2.4. By (4.2.7)

$$\frac{1}{2} W(z)^2 = \frac{1}{2} z^2 + a \int_0^1 D(y, z \mid 1) dy > \frac{1}{2} z^2$$

which implies (4.2.10). By L 4.2.5 $D(y, z \mid 1) \underset{z \to \infty}{\longrightarrow} \infty$ for each $y \in [0, 1]$ and $\frac{1}{D(y, z \mid 1)} \leq \frac{2}{z_0}$ for all $y \in [0, 1]$ and all $z \geq z_0$. Hence

(4.2.12) $$\int_\beta^1 \frac{q(y)}{D(y, z \mid 1)} dy \underset{z \to \infty}{\longrightarrow} 0 \qquad \text{uniformly in } \beta \in [0, 1].$$

By (4.2.6), $W(z) - z = a - \int_0^1 \frac{Q(y)}{D(y, z \mid 1)} dy$ and (4.2.11) follows by (4.2.12).

LEMMA 4.2.7. — Let (4.2.8) hold with $\gamma = 1$. Then there exist $z_0 > 0$ and $d_0 > 0$ such that

$$|(W(z_1) - z_1) - (W(z_2) - z_2)| \leq d_0 \frac{|z_1 - z_2| + 1}{z_1 z_2}$$

for all $z_1, z_2 \geq z_0$.

Proof. — By (4.2.6)

$$(D(\beta, z_1 \mid 1) - z_1) - (D(\beta, z_2 \mid 1) - z_2) = \int_\beta^1 q(y)\left(\frac{1}{D(y, z_2 \mid 1)} - \frac{1}{D(y, z_1 \mid 1)}\right)dy.$$

Hence, by (4.2.12)

$$|D(\beta, z_1 \mid 1) - D(\beta, z_2 \mid 1)| \leq |z_1 - z_2| + 1$$

for all $\beta \in [0, 1]$ and all sufficiently large z_1, z_2. Hence

$$|(W(z_1) - z_1) - (W(z_2) - z_2)|$$
$$\left|\int_0^1 q(y)\frac{D(y, z_1 \mid 1) - D(y, z_2 \mid 1)}{D(y, z_1 \mid 1)D(y, z_2 \mid 1)}dy\right| \leq 4m\frac{|z_1 - z_2| + 1}{z_1 z_2}$$

for all sufficiently large z_1, z_2 by L 4.2.5.

The next few lemmas show that the function A (the stopping a. a. d.) satisfies (A 2.1).

LEMMA 4.2.8. — For any $v > 0$, $0 < A(v) < \infty$.

Proof. — $A(v) > 0$ is trivial as $\phi(., v)$ is strictly increasing on $[0, C(v))$. If $q \equiv 0$ (a non-biased case), (4.2.1) can be solved explicitly and $C(v) = \infty$, $A(v) = \frac{v}{a}$ for any $v > 0$. Hence we can assume $q \not\equiv 0$. Let $A(v) = \infty$ for some v. Then $D_0(., v^2)$ is defined on $[0, \infty)$ and by (4.2.7)

$$\frac{1}{2}D_0(\beta, v^2) = \frac{1}{2}v^2 + Q(\beta) - a\int_0^\beta \sqrt{D_0(y, v^2)}\,dy.$$

Q is bounded and $D_0(\beta, v^2) > 0$ for all $\beta \geq 0$. Hence $\int_0^\infty D(y, v)dy < \infty$. Put $w = \frac{1}{2}v^2 - \int_0^\infty D(y, v)dy$. As Q is 1-periodic,

$$0 \leq \lim_{k\to\infty} \frac{1}{2}D_0(k\alpha, v^2) = w + Q(\alpha)$$

for all $0 \leq \alpha \leq 1$. Since $q \not\equiv 0$, there must be an $\varepsilon > 0$ and an interval $I \subset [0, 1]$ such that $w + Q(\alpha) \geq \varepsilon$ if $\alpha \in I$. Then $\frac{1}{2}D_0(k\alpha, v^2) \geq w + Q(\alpha) \geq \varepsilon$ for all k and all $\alpha \in I$, which implies $\int_0^\infty D(y, v)dy = \infty$ contradicting the fact that the integral is finite.

LEMMA 4.2.9. — A $\gamma > 0$ is a stopping a. a. d. if and only if, for some $v > 0$, $D(., v)$ is defined and strictly positive on $[0, \gamma)$ and $D(\gamma, v) = 0$.

Proof. — Let γ be a stopping a. a. d., i. e. $\gamma = A(v)$ for some $v > 0$. Then $D(., v)$ is defined and strictly positive on $[0, \gamma)$ by definition. If $C(v) < \infty$, $D(\gamma, v) = 0$ by definition. Assume $C(v) = \infty$ and $D(\gamma, v) = \delta > 0$. Then $\phi'(t, v) \underset{t\to\infty}{\longrightarrow} \delta > 0$, which contradicts $A(v) = \lim_{t\to\infty} \phi(t, v) < \infty$.
The converse follows from the definition of C and A.

LEMMA 4.2.10. — A is non-decreasing on $(0, \infty)$.

Proof. — $v_1 < v_2$ and $A(v_2) < A(v_1)$ contradicts L 4.2.9 and L 4.2.3 with $I = [0, A(v_2)]$.

LEMMA 4.2.11. — A is left-continuous.

Proof. — Let $v_n \underset{n\to\infty}{\nearrow} v$. By L 4.2.6, $A(v_n) \underset{n\to\infty}{\nearrow} \gamma \leq A(v)$. Assume $\gamma < A(v)$. By L 4.2.9, $D(\gamma, v) > 0$. Take any $0 < z < D(\gamma, v)$. $D(., z \mid \gamma)$ is well defined and strictly positive in some $[\gamma_1, \gamma]$, $\gamma_1 < \gamma$. There exists an n such that $A(v_n) > \gamma_1$, and $D(\gamma_1, z \mid \gamma) > D(\gamma_1, v_n)$. Hence the extension of $D(., z \mid \gamma)$ to the left of γ_1 is separated from 0 by $D(., v_n)$ and can be extended to $[0, \gamma_1]$. Then by 4.2.3, $D(\beta, v_n) < D(\beta, z \mid \gamma) < D(\beta, v)$ for all n and all $\beta < A(v_n)$. In particular $v_n < D(0, z \mid \gamma) < v$, which contradicts $v_n \to v$.

LEMMA 4.2.12. — $A(v) \underset{v\to\infty}{\longrightarrow} \infty$.

Proof. — Assume $\lim_{v\to\infty} A(v) = \gamma < \infty$. Take a $z > 0$. As in the proof of L 4.2.11, $D(., z \mid \gamma)$ is separated from 0 by any $D(., v)$ and can be extended to $[0, \gamma]$. Then we have a contradiction $v = D(0, v) < D(0, z \mid \gamma) < \infty$ for all $v > 0$.

LEMMA 4.2.13. — There exist at least one stopping a. a. d. $\alpha \in (0, 1]$.

Proof. — Take a $v > 0$ and put $\gamma = A(v)$. There exists an integer $k \geq 0$ and $0 < \alpha \leq 1$ such that $\gamma = k + \alpha$. If $k > 0$, define $\bar{D}(\beta) = D(\beta + k, v)$. $\bar{D}(\beta) > 0$ for $\beta \in [0, \alpha)$, $\bar{D}(\alpha) = 0$ by L 4.2.9. As q is 1-periodic, $\bar{D}$ satisfies (4.2.4) and, therefore $\bar{D}(\beta) = \bar{D}(\beta, v_0)$ with $v_0 = \bar{D}(0)$. Hence, by L 4.2.9, α is a stopping a. a. d. In the rest of this section we will assume, without always mentioning it explicitly, that

(A 4.2.1) $\quad$ 1 is a stopping a. a. d.

This is no restriction of generality, as we can always take α of L 4.2.13 as a new origin or, equivalently, replace $q(\beta)$ by $q_0(\beta) = q(\beta - \alpha)$.

LEMMA 4.2.14. — For any $0 < \alpha \leq 1$, either all $\alpha + k$, $k = 0, 1, 2, \ldots$ are stopping a. a. d. or all $\alpha + k$ are non-stopping a. a. d.

Proof. — Let for some $0 < \alpha \leq 1$ and an integer k, $\alpha + k$ be a stopping time. By the proof of L 4.2.13, α is a stopping a. a. d., i. e. $\alpha = A(v)$ for some $v > 0$. Put $\bar{D}(\beta) = D(\beta - 1, v)$ for $\beta \in [1, \alpha + 1]$. $\bar{D}$ satisfies (4.2.4) on $(1, \alpha + 1)$ and, by L 4.2.9, $\bar{D}(\beta) > 0$ on $[1, \alpha + 1)$ and $\bar{D}(\alpha + 1) = 0$. By (A 4.2.1), there exists v_1 such that $A(v_1) = 1$ and the extension of D to the left of 1 is separated from 0 by $D(., v_1)$ on [0, 1]. Hence D can be extended to $[0, \alpha + 1]$ and identified with some $D(., v_2)$. By L 4.2.9, $\alpha + 1$ is a stopping a. a. d. Hence, by induction, all $\alpha + k$ are stopping a. a. d.'s.

The next three lemmas provide some more information about the nature of stopping a. a. d.'s. As they are not used in the proofs of the main theorems, their proofs will be omitted.

LEMMA 4.2.15. — If $q(\beta) > 0$, then β is not a stopping a. a. d.

LEMMA 4.2.16. — If $q(\beta) < 0$ and if β is a stopping a. a. d., i. e. if $\beta = A(v)$ for some v, then $C(v) < \infty$ and v is unique.

LEMMA 4.2.17. — Let β_1 be a stopping a. a. d. and let $q(\beta) < 0$ in (β_1, β_2) for some $\beta_2 > \beta_1$. Then all $\beta \in [\beta_1, \beta_2]$ are stopping a. a. d.'s.

The lemmas 4.2.8, 10, 11, 12 show that A satisfies (A 2.1). Hence the function B defined by (2.1) satisfies (A 2.2).

LEMMA 4.2.18. — If β is a stopping a. a. d., then

$$B(\beta) = \lim_{z \searrow 0} D(0, z \mid \beta).$$

Proof. — By the definition of $B(\beta) = \max \{ v : A(v) = \beta \}$. By L 4.2.5, $D(., B(\beta)) > 0$ on $[0, \beta)$ and $D(\beta, B(\beta)) = 0$. Hence, $D(0, z \mid \beta)$ is well defined by L 4.2.4 for each $z > 0$, $D(0, z \mid \beta) > B(\beta)$ by L 4.2.3 and $D(0, z \mid B) \underset{z \to 0}{\to} B(\beta)$ by an argument similar to that one at the end of the proof of L 4.2.4.

A function W was defined on $(0, \infty)$ between L 4.2.5 and L 4.2.6. By L 4.2.6, it is strictly increasing so that may define $W(0) = \lim_{z \to 0} W(z)$

LEMMA 4.2.19. — For any $\beta \geq 0$

$$B(\beta + 1) = W(B(\beta)). \tag{4.2.13}$$

Proof. — Assume first that $\beta > 0$ is a stopping a. a. d. By L 4.2.9, L 4.2.4 and L 4.2.14, $D(., z \mid \beta)$ and $D(., z \mid \beta + 1)$ are defined on $[0, \beta]$ and $[0, \beta + 1]$ respectively for any $z > 0$. For $z > 0$ the uniqueness argument applies and $D(0, z \mid \beta + 1) = D(0, D(1, z \mid \beta + 1) \mid 1) = W(D(1, z \mid \beta + 1))$. By an argument similar to that of the proof of L 4.2.14,

$$D(1, z \mid \beta + 1) = D(0, z \mid \beta).$$

Hence $D(0, z \mid \beta + 1) = W(D(0, z \mid \beta))$ and (4.2.13) follows from L 4.2.18. If $\beta > 0$ is not a stopping a. a. d., then β belongs to an interval $[\beta_1, \beta_2)$ over which B is constant, so that $B(\beta) = B(\beta_1)$. At the same time, β_1 is a stopping a. a. d. by definition of D and left-continuity of A, By L 4.2.14, the same holds for $\beta + 1$ and $\beta_1 + 1$, i. e. $B(\beta + 1) = B(\beta_1 + 1)$ and (4.2.13) follows. Finally, $B(0)=0$ by definition and $W(0)=\lim_{z\to 0} W(z)=B(1)$ by definition and L 4.2.18.

LEMMA 4.2.20. — For any integer l and any $\beta \geq 0$

$$B(k + l + \beta) - B(k + \beta) \underset{k\to\infty}{\longrightarrow} la. \tag{4.2.14}$$

Proof. — As B satisfies (A 2.2), $B(\beta) \underset{\beta\to\infty}{\longrightarrow} \infty$. For any integer k, $B(k + 1 + \beta) - B(k + \beta) = W(B(k + \beta)) - B(k + \beta) \underset{k\to\infty}{\longrightarrow} a$ by (4.2.11) and $B(k + \beta) \underset{k\to\infty}{\longrightarrow} \infty$. Hence (4.2.14) holds for $l = 1$. Finally

$$B(k + l + \beta) - B(k + \beta) = \sum_{j=1}^{l} [B(k + j + \beta) - B(k + j - 1 + \beta)].$$

In the following theorems, $W^{(k)}$ will denote the k-th iterate of W, i. e. $W^{(k+1)}(z) = W(W^{(k)}(z))$. It follows from L 4.2.19 that

$$B(k + \beta) = W^{(k)}(B(\beta)). \tag{4.2.15}$$

We shall also write $N(z) = W(z) - z$.

THEOREM 4.2.1. — For any $z_0 > 0$ the limit

$$\lim_{k\to\infty} [W^{(k)}(z) - W^{(k)}(0)] = M(z)$$

exists uniformly in $[0, z_0]$. M is non-decreasing and continuous on $[0, \infty)$.

Proof. — There exists an integer l_0 such that $z_0 \leq B(l_0)$. Then $W^{(k)}(z) \leq W^{(k)}(z_0) \leq B(k + l_0)$, $W^{(k)}(z) - W^k(0) \leq B(k + l_0) - B(j)$ for all k and all $z \leq z_0$. By (4.2.14),

$$W^{(k)}(z) - W^{(k)}(0) \leq d_1 \tag{4.2.16}$$

for some $d_1 > 0$, all k and all $z \leq z_0$. Further

$$\frac{B(k)}{k} = \frac{1}{k}\sum_{j=0}^{k-1}(B(j+1) - B(j)) \to a$$

by (4.2.14). Hence

$$W^{(k)}(z) \geq W^{(k)}(0) = B(k) \geq a_0 k \qquad \text{for some } a_0 > 0,$$

and all k and all $z \geq 0$. Then by L 4.2.7 and (4.2.16)

$$|\,N(W^{(k)}(z)) - N(W^{(k)}(0))\,| \leq \frac{d_0(d_1+1)}{a_0{}^2k^2} \tag{4.2.17}$$

for all sufficiently large k and all $z \leq z_0$. Substituting $W(z)$, $W^{(2)}(z)$, ... in N, and summing, we obtain $W^{(k)}(z) - z = \sum_{j=0}^{k-1} N(W^{(j)}(z))$. Hence $W^{(k)}(z) - W^{(k)}(0) = \sum_{j=0}^{k-1} [N(W^{(j)}(z)) - N(W^{(j)}(0))]$. Finally, by (4.2.17), the series $\sum_{j=0}^{\infty} [N(W^{(j)}(z)) - N(W^{(j)}(0))]$ is convergent uniformly in $[0, z_0]$.

Remark. — It follows easily from Th. 4.2.1 and (4.2.14) that M satisfies the Abel functional equation

$$M(W(z)) = M(z) + a \qquad \text{on } [0, \infty). \tag{4.2.18}$$

There are of course infinitely many functions M satisfying (4.2.18) with the initial condition $M(0) = 0$, however it would be possible to show that M is the only solution within a class $\mathcal{F}$ of functions K on $[0, \infty)$ satisfying the following smoothness condition at ∞: For any $0 < \varepsilon < \delta$ the limit

$$\lim_{\substack{x\to\infty \\ y\to\infty \\ \varepsilon < |y-x| < \delta}} \frac{K(y) - K(x)}{y - x} = l \qquad \text{exists}. \tag{4.2.19}$$

l is allowed to be infinite and depends generally on K; trivially it is independent of ε, δ.

The results of this section can now be summarized in

THEOREM 4.2.2. — Let (A 3.1.2), (A 3.1.3) and (A 4.2.1) hold. Then for any $0 \leq \alpha \leq 1$

$$\mathcal{P}(Z_d \leq \alpha) \underset{d\to\infty}{\longrightarrow} \frac{M(B(\alpha))}{a}$$

where the function M is defined by Th. 4.2.1 or, alternatively, as a unique solution of (4.2.18) in $\mathscr{F}$ with $M(0) = 0$ (see the remark following Th. 4.2.1).

Proof. — We have proved that (A 2.1) holds; hence (A 2.2) holds. Further, (A 2.3) holds by (4.2.10) and (4.2.15). (2.2) of Th. 2 holds by (4.2.14) with $l = 1$. Finally (2.3) of Th. 2 holds with $G(\alpha) = \dfrac{M(B(\alpha))}{a}$ by (4.2.14), (4.2.15) and Th. 4.2.1.

REFERENCES

[1] M. Fréchet, *Recherches théoriques modernes sur le calcul des probabilités.* 2^d éd., Paris, Gauthier-Villars, 1950-1952.

[2] B. Hostinsky, Méthodes générales du calcul des probabilités. *Mémor. Sci. Math.*, n° 51, 1938, p. 1-63.

[3] M. Kuczma, *Functional equations in a single variable.* Warszawa, Polish Sci. Publ., 1968.

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