ANNALES DE L'I. H. P., SECTION B

A. F. GUALTIEROTTI

Mixtures of gaussian cylinder set measures and abstract Wiener spaces as models for detection

Annales de l'I. H. P., section B, tome 13, nº 4 (1977), p. 333-356

<http://www.numdam.org/item?id=AIHPB_1977__13_4_333_0>

Ann. Inst. Henri Pincaré,
Vol. XIII, n° 4, 1977, p. 333-356.

Section B :
Calcul des Probabilités et Statistique.

Mixtures of Gaussian cylinder set measures and abstract Wiener spaces as models for detection

by

A. F. GUALTIEROTTI
Dept. Mathématiques, École Polytechnique Fédérale, 1007 Lausanne, Suisse

0. INTRODUCTION AND ABSTRACT

Let H be a real and separable Hilbert space. Consider on H a measurable family of weak covariance operators $\{ R_\theta, \theta > 0 \}$ and, on $\mathbb{R}_+ :=]0, \infty[$, a probability distribution F. With each R_θ is associated a weak Gaussian distribution μ^θ. The weak distribution μ is then defined by

$$\mu = \int_{\mathbb{R}_+} \mu^\theta dF(\theta).$$

μ can be studied, as we show below, within the theory of abstract Wiener spaces. The motivation for considering such measures comes from statistical communication theory. There, signals of finite energy are often modelled as random variables, normally distributed, with values in a L_2-space. The theory is then developed as if the parameters, *i. e.* the mean vector and the covariance operator, were known quantities. Actually, these are usually estimated from finite sample data and certain hypotheses, for example: R, the covariance operator, is trace-class, are at best difficult to verify. To protect oneself against possible departures from the basic model, one can ask whether some other model can be built, for which such basic hypotheses play a less fundamental role. It turns out that abstract Wiener spaces are useful to reduce the importance of the trace-class hypothesis [*9 b*].

The parameter θ allows for some further uncertainty in the model, like some form of inertia intrinsically attached to the measuring devices. Some information on the influence of θ in a particular model can be found in [*9 c*].

1. PRELIMINARIES

H = real and separable Hilbert space.
$\langle .,. \rangle$ = inner product of H.
$\| . \|$ = norm of H obtained from $\langle .,. \rangle$.
[H] = bounded linear operators on H.
$\mathscr{F}_H$ = finite dimensional subspaces of H.
$\mathscr{P}_H$ = projections of H with range in $\mathscr{F}_H$.
Π_K = projection of H with range K.
$\mathscr{B}[K]$ = Borel sets of K.
$\mathbb{R}^n$ = Euclidean space.
$\underline{x}$ = $(x_1, \ldots, x_n) \in \mathbb{R}^n$.
$\mathscr{C}[K]$ = cylinder sets with base in K.
$\mathscr{C}[H]$ = cylinder sets of H.
$\mathscr{E}_K$ = orthonormal basis for K.
$\mathbb{R}_+$ = $]0, \infty[$.
N = positive integers.
γ = standard Gaussian weak distribution on H.
$\mathscr{R}[T]$ = range of the map T.

a) Cylinder sets.

Let K be in $\mathscr{F}_H$. $\mathscr{B}[K]$ is generated by the open balls and can be obtained as follows:

if $\mathscr{E}_K = \{ e_1, \ldots, e_n \}$ and $J_{\mathscr{E}_K}(k) = (\langle k, e_1 \rangle, \ldots, \langle k, e_n \rangle) \in \mathbb{R}^n$,

then

$$\mathscr{B}[K] = J_{\mathscr{E}_K}^{-1} \{ \mathscr{B}[\mathbb{R}^n] \}.$$

One also has:

$$J_{\mathscr{E}_K}^{-1} = J_{\mathscr{E}_K}^* \quad \text{and} \quad J_{\mathscr{E}_K}^*(\underline{x}) = \sum_{i=1}^{n} x_i e_i.$$

Finally $\mathscr{C}[K] = \Pi_K^{-1} \{ \mathscr{B}[K] \}$ and $\mathscr{C}[H] = \bigcup_{K \in \mathscr{F}_H} \mathscr{C}[K]$.

b) Gaussian cylinder set measures.

A weak covariance operator R is a positive, self-adjoint and bounded injection of H.

For $\mathscr{H} = \{ h_1, \ldots, h_n \} \subseteq \mathrm{H}$, $\Sigma[\mathrm{R}; \mathscr{H}]$ denotes the matrix with entries $\langle \mathrm{R}h_i, h_j \rangle$, h_i, h_j in $\mathscr{H}$. Then $n_{\Sigma[\mathrm{R};\mathscr{H}]}$ represents the density of a Gaussian vector with mean zero and covariance matrix $\Sigma[\mathrm{R}; \mathscr{H}]$. The corresponding measure on $\mathscr{B}[\mathbb{R}^n]$ is written $\Phi_{\Sigma[\mathrm{R};\mathscr{H}]}$.

Let now K be in $\mathscr{F}_{\mathrm{H}}$. The measure on $\mathscr{B}[\mathrm{K}]$ induced by R is written μ_{K}. If $\mathrm{B} \in \mathscr{B}[\mathrm{K}]$ and $\mathrm{B}' \in \mathscr{B}[\mathbb{R}^n]$ are such that $\mathrm{B} = \mathrm{J}_{\mathscr{E}_{\mathrm{K}}}^{-1} \{ \mathrm{B}' \}$, one sets

$$\mu_{\mathrm{K}}^{\mathrm{R}}[\mathrm{B}] = \Phi_{\Sigma[\mathrm{R};\mathscr{E}_{\mathrm{K}}]}[\mathrm{B}'].$$

The Gaussian cylinder set measure μ^{R} associated with R is obtained as follows:

if $$\mathrm{C} = \mathrm{B} + \mathrm{K}^{\perp},\ \mathrm{B} \in \mathscr{B}[\mathrm{K}],\ \mathrm{K} \in \mathscr{F}_{\mathrm{H}},$$
then $$\mu^{\mathrm{R}}[\mathrm{C}] = \mu_{\mathrm{K}}^{\mathrm{R}}[\mathrm{B}].$$

c) Mixtures.

We consider a map $\rho : \mathbb{R}_+ \to [\mathrm{H}]$, where

i) $\rho(\theta) := \mathrm{R}_\theta$ is a weak covariance operator, F-a. e. θ;

ii) ρ is strongly measurable;

iii) $\rho_h(\theta) := \mathrm{R}_\theta h$ is Bochner integrable with respect to F, for every h in H.

In the present context ρ is strongly measurable if and only if $\rho_{h,k}(\theta) := \langle \mathrm{R}_\theta h, k \rangle$ is measurable, for every (h, k) in $\mathrm{H} \times \mathrm{H}$ [*11*, p. 77].

Let $\mathrm{R} \in [\mathrm{H}]$ be defined by $\mathrm{R}h := \int_{\mathbb{R}_+} \rho_h(\theta) d\mathrm{F}(\theta)$. R is a weak covariance operator [*11*, p. 85].

As a function of $(\underline{x}, \theta)$, $n_{\Sigma[\mathrm{R}_\theta;\mathscr{E}_{\mathrm{K}}]}(\underline{x})$ is measurable, for θ appears only in expressions containing $\langle \mathrm{R}_\theta e_i, e_j \rangle$. Consequently

$$\delta_{\mathscr{E}_{\mathrm{K}}}(\underline{x}) := \int_{\mathbb{R}_+} n_{\Sigma[\mathrm{R}_\theta;\mathscr{E}_{\mathrm{K}}]}(\underline{x}) d\mathrm{F}(\theta)$$

is a density on $\mathbb{R}^n$. Let μ^θ denote the Gaussian cylinder set measure associated with R_θ. The mixture μ is then given by

$$\mathrm{C} = \mathrm{B} + \mathrm{K}^{\perp},\ \mathrm{B} \in \mathscr{B}[\mathrm{K}],\ \mathrm{K} \in \mathscr{F}_{\mathrm{H}}, \qquad \mathrm{B} = \mathrm{J}_{\mathscr{E}_{\mathrm{K}}}^{-1} \{ \mathrm{B}' \},$$

$$\mu[\mathrm{C}] = \mu_{\mathrm{K}}[\mathrm{C}] = \int_{\mathrm{B}'} \delta_{\mathscr{E}_{\mathrm{K}}}(\underline{x}) d\underline{x} = \int_{\mathbb{R}_+} \mu^\theta[\mathrm{C}] d\mathrm{F}(\theta).$$

The covariance structure of μ is given by R and μ is a probability measure if and only if R_θ is trace-class F-a. e. θ.

d) Hypotheses.

(I) $\int_{\mathbb{R}_+} \| \mathrm{R}_\theta \| d\mathrm{F}(\theta) < \infty$

(II) There exists $B \in \mathscr{B}[\mathbb{R}_+]$ such that

1) $F(B) > 0$,

2) R_θ has bounded inverse for $\theta \in B$,

3) for some $\tau > 0$, $\| R_\theta^{-1} \| \quad \| R_\theta \|^{\frac{1}{2}} \leqslant \tau^{-1}$ on B.

REMARKS. — α) (I) and (II) are used to put the stated problem in the abstract Wiener space frame.

β) That the function $f(\theta) : = \| R_\theta \|$ is measurable was remarked for example in [*4*, p. 1, Lemma 1.3].

2. EXTENSION OF μ WHEN IT IS NOT A MEASURE

The theory of abstract Wiener spaces depends on the underlying cylinder set measure only through a few estimations. We will thus consider mostly the results in the theory where these estimations occur. For a pleasant development of the theory one can consult [*3*] and [*13*].

PROPOSITION 1. — If (I) holds, with $[H, \mathscr{C}[H], \mu]$ one can associate a probability space $(\Omega, \mathscr{A}, P)$ and a continuous linear map $\mathscr{L}: H \to L_2[\Omega, \mathscr{A}, P]$ such that

1) $\mathscr{L}h$ has density $\delta_h(x) : = \int_{\mathbb{R}_+} n_{\Sigma[R_\theta;\{h\}]}(x) dF(\theta)$,

2) $E[\mathscr{L}h] = 0$, $V[\mathscr{L}h] = \langle Rh, h \rangle$.

Proof. — Let $\mathscr{E}_H = \{ e_n, n \in N \}$ and define μ_n on $\mathscr{B}[\mathbb{R}^n]$ by

$$\mu_n[B] = \mu \{ h \in H : (\langle h, e_1 \rangle, \ldots, \langle h, e_n \rangle) \in B \}.$$

The family $\{ \mu_n, n \in N \}$ is consistent and thus there exist $(\Omega, \mathscr{A}, P)$ and $\{ X_n, n \in N \}$ such that

$$\mu_n[B] = P \{ (X_1, \ldots, X_n) \in B \}.$$

$\mathscr{L}$ is defined by setting $\mathscr{L}\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} x_i X_i$. Its continuity follows from the inequality:

$$E\left[\mathscr{L}^2\left(\sum_{i=1}^{n} x_i e_i\right)\right] \leqslant \left\{ \int_{\mathbb{R}_+} \| R_\theta \| \, dF(\theta) \right\} \left\| \sum_{i=1}^{n} x_i e_i \right\|^2 .$$

REMARK. — One of the aims of the abstract Wiener space theory is to define $\mathscr{L}$ for as many functions on H as possible ($H^* = H$!) by associating with a function on H a random variable on $(\Omega, \mathscr{A}, P)$. This is easily done for tame functions, *i. e.* functions depending only on a finite dimensional subspace. An « arbitrary » function is then taken to be a limit of tame functions and the random variable associated with it should be the limit in some sense of the random variables associated with the tame sequence. Such a procedure fails in general and motivates the introduction of the notion of measurable norm. We give below an example involving mixtures, which is a modification of the example in [*13*, p. 58].

FACT 1. — Suppose R_θ has bounded inverse F-a. e. θ. Let $\mathscr{E}_H = \{ e_n, n \in N \}$ and Π_n be the projection determined by the first n elements of $\mathscr{E}_H$. Let also Δ be the Euclidean norm on $\mathbb{R}^n$ and define (supposing (I)),

$$f_n(h) = \| \Pi_n h \|^2 = \Delta^2(\langle h, e_1 \rangle, \ldots, \langle h, e_n \rangle),$$
$$X(f_n) = \Delta^2(\mathscr{L}e_1, \ldots, \mathscr{L}e_n).$$

Then $X(f_n)$ does not converge in probability.

Proof. — R_θ being self-adjoint, one can consider $R_\theta^{-1} \in [H]$. Also $\| R_\theta^{-1} \|$ is measurable. Furthermore

$$\| x \|^2 \leqslant \| R_\theta^{-1} \|^2 \| R_\theta x \|^2 \leqslant \| R_\theta^{-1} \|^2 \| R_\theta \| \langle R_\theta x, x \rangle.$$

Let $g(\theta) := \| R_\theta^{-1} \|^{-2} \| R_\theta \|^{-1}$. Then

$$0 \leqslant \langle R_\theta x, x \rangle - \langle g(\theta) id_H x, x \rangle := \langle S_\theta x, x \rangle.$$

Thus the matrix $\Sigma[S_\theta; \mathscr{H}]$ is positive definite. From [*1*, p. 173], one gets, for C convex and symmetric in $\mathbb{R}^n$,

$$\Phi_{\Sigma[g(\theta) id_H; \mathscr{H}]}[C] \geqslant \Phi_{\Sigma[R_\theta; \mathscr{H}]}[C].$$

Now, for $p > n$, and $C_{\sqrt{\varepsilon}} :=$ the cube with sides of length $2\sqrt{\varepsilon}$,

$$\begin{aligned} P \{ | X(f_p) - X(f_n) | > \varepsilon \} & \\ &= 1 - \mu \left\{ \sum_{i=n+1}^{n+p} \langle x, e_i \rangle^2 \leqslant \varepsilon \right\} \\ &\geqslant 1 - \int_{\mathbb{R}_+} dF(\theta) \mu^\theta \{ | \langle x, e_i \rangle | \leqslant \sqrt{\varepsilon}, i = n+1, \ldots, n+p \} \\ &\geqslant 1 - \int_{\mathbb{R}_+} dF(\theta) \Phi_{\Sigma[g(\theta) id_H; \{e_{n+1}, \ldots, e_{n+p}\}]}[C_{\sqrt{\varepsilon}}] \\ &= 1 - \int_{\mathbb{R}_+} dF(\theta) \left\{ \int_{-\sqrt{\varepsilon}}^{+\sqrt{\varepsilon}} [2\Pi g(\theta)]^{-\frac{1}{2}} \exp \left\{ - \frac{x^2}{2g(\theta)} \right\} dx \right\}^{p-n}, \end{aligned}$$

a quantity which tends to one as $p - n$ tends to ∞, by dominated convergence.

DEFINITION [8, p. 374]. — A semi-norm q on H is μ-measurable if and only if $\forall \varepsilon > 0$, $\exists \Pi_\varepsilon \in \mathscr{P}_H$ such that

$$\Pi \in \mathscr{P}_H \quad \text{and} \quad \Pi \perp \Pi_\varepsilon \quad \text{imply} \quad \mu[x \in H : q(\Pi x) > \varepsilon] < \varepsilon.$$

REMARK. — When $\mu = \gamma$, the usual example of a measurable norm is $q(x) = \| Sx \|$, with S Hilbert-Schmidt. Given the importance of such operators for the theory of integration on Hilbert spaces, one expects that the above norm is also μ-measurable for mixtures μ defined earlier. That it is indeed the case is proved similarly. Instead of having the equality

$$\int_H \| S\Pi x \|^2 d\gamma(x) = \text{trace}\ (S^*S\Pi),$$

one has the inequality

$$\int_H \| S\Pi x \|^2 d\mu(x) \leqslant \left\{ \int_{\mathbb{R}_+} \| R_\theta \| \, dF(\theta) \right\} \text{trace}\ (S^*S\Pi).$$

FACT 2. — Let (II) hold and q be a μ-measurable semi-norm. Then

$$q(x) \leqslant \alpha \| x \|, \qquad \alpha > 0, \qquad x \in H.$$

Proof. — Let

$$\sigma_\theta(h) := \langle R_\theta h, h \rangle \qquad \text{and} \qquad n_\sigma(x) := \left(\sigma\sqrt{2\Pi}\right)^{-1} \exp\left\{ -\frac{x^2}{2\sigma^2} \right\}.$$

For $\| h \| = 1$, let

$$\varphi(t\,; h) := 2\int_t^\infty dx \int_{\mathbb{R}_+} dF(\theta) n_{\sigma_\theta(h)}(x) = 2\int_{\mathbb{R}_+} dF(\theta) \int_{t/\sigma_\theta(h)}^\infty n_1(x) dx.$$

We have $\| R_\theta h \|^2 \leqslant \| R_\theta \| \langle R_\theta h, h \rangle$, and,

for $\theta \in B$, $\qquad \sigma_\theta(h) \geqslant \{ \| R_\theta \|^{\frac{1}{2}} \| R_\theta^{-1} \| \}^{-1} \| h \| \geqslant \tau > 0.$

Consequently

$$\varphi(t\,; h) \geqslant 2\int_B dF(\theta) \int_{t/\tau}^\infty n_1(x) dx = 2F(B) \int_t^\infty n_\tau(x) dx.$$

Choose t_0 such that $\displaystyle\int_{t_0}^\infty n_\tau(x) dx = \frac{\varepsilon}{2F(B)}$. Then $\varphi(t_0\,; h) \geqslant \varepsilon$.

The end of proof is then as in [13, p. 61, Lemma 4.2].

REMARK. — Fact 2 is at the core of the theory of abstract Wiener spaces. If a hypothesis of type (II) is not made, the methods used break down. For instance, if R_θ is compact with infinite dimensional range F-a. e. θ and if $\{ e_n, n \in N \}$ is a complete orthonormal set in H, then $\lim_n \langle R_\theta e_n, e_n \rangle = 0$ F-a. e. θ [6, p. 263, Corollary to Thm. 2.5]. Thus $\lim_n \varphi(t; e_n) = 0$, $\forall t$ fixed, and no « universal » t_0 can be chosen.

PROPOSITION 2. — Let (I) and (II) obtain and q be a μ-measurable semi-norm. Denote by E_q the completion of H with respect to q and by i_q the associated inclusion.

Since i_q is continuous (Fact 2), the following makes sense:

i) Let $J_H : H^* \to H$ be the usual identification.
Define $L : E_q^* \to H$ by $L(\eta) := J_H \circ \eta \circ i_q$, $\eta \in E_q^*$.

ii) Let η_i, $i = 1, \ldots, n$, be in E_q^* and B in $\mathscr{B}[\mathbb{R}^n]$.
Define $\mu_q \{ x \in E_q : (\eta_1(x), \ldots, \eta_n(x)) \in B \} :=$

$$\mu \{ x \in H : (\langle x, L(\eta_1) \rangle, \ldots, \langle x, L(\eta_n) \rangle) \in B \}.$$

$\mu_q = \mu \circ i_q^{-1}$ is a cylinder set measure on E_q.

Then μ_q extends to a probability measure $\tilde{\mu}_q$ on the Borel sets of E_q.

Proof. — The proof in [*12*, p. 114, Thm. 1] applies here practically without change. It requires first that i_q be continuous (this is Fact 2, which follows from (II)). Then one must have an associated probability space (this is Fact 1, which follows from (I)). Finally, one must be able to identify a weak limit as the required extension: in our case it follows from dominated convergence.

COROLLARY. — Let $\mu_q^\theta := \mu^\theta \circ i_q^{-1}$. Then μ_q^θ has an extension $\tilde{\mu}_q^\theta$ to a Gaussian probability measure on E_q, F-a. e. θ.

PROPOSITION 3. — Suppose that, for F-a. e. θ, R_θ has bounded inverse and $\| R_\theta^{\frac{1}{2}} \| \leqslant \beta < \infty$. Let q be a continuous semi-norm such that $\mu_q = \mu \circ i_q^{-1}$ extends to a probability measure $\tilde{\mu}_q$ on E_q. Then q is μ-measurable.

Proof. — $R_\theta^{\frac{1}{2}}$ has bounded inverse if and only if R_θ has bounded inverse. That $R_\theta^{\frac{1}{2}}$ has bounded inverse can be written

$$\| R_\theta^{-\frac{1}{2}} \|^{-2} \| x \|^2 \leqslant \| R_\theta^{\frac{1}{2}} x \|^2 \leqslant \| R_\theta^{\frac{1}{2}} \|^2 \| x \|^2,$$

so that

$$R_\theta - \| R_\theta^{-\frac{1}{2}} \|^2 id_H \qquad \text{and} \qquad \| R_\theta^{\frac{1}{2}} \|^2 id_H - R_\theta$$

are both positive definite.

Set $\gamma_q = \gamma \circ i_q^{-1}$. γ_q has weak covariance $\tilde{R}_\gamma := i_q \circ J_H \circ i_q^*$. Since, for $\eta \in E_q^*$ and $L(\eta) = \langle \cdot, h \rangle$,

$$\eta(\tilde{R}_{\gamma_q} \eta) = \langle h, h \rangle \qquad \text{and} \qquad \eta(\tilde{R}_\theta \eta) = \langle R_\theta h, h \rangle,$$

we have that

$$\tilde{R}_\theta - \| R_\theta^{-\frac{1}{2}} \|^2 \tilde{R}_{\gamma_q} \qquad \text{and} \qquad \| R_\theta^{\frac{1}{2}} \|^2 \tilde{R}_{\gamma_q} - \tilde{R}_\theta$$

are both positive definite. It then follows from [*3*, p. 903, Corollary 2.7] that γ_q extends to a Gaussian measure $\tilde{\gamma}_q$. Consequently q is γ-measurable [3, p. 909, *iv*)]. But by [3, p. 913, Corollary 2.5] we then have

$$\mu^\theta[x \in \mathrm{H} : q(\Pi x) > t] \leqslant \gamma\left[x \in \mathrm{H} : q(\Pi x) > \frac{t}{\| \mathrm{R}_\theta^{\frac{1}{2}} \|}\right].$$

If $\| \mathrm{R}_\theta^{\frac{1}{2}} \| \leqslant 1$,

$$\gamma\left[x \in \mathrm{H} : q(\Pi x) > \frac{t}{\| \mathrm{R}_\theta^{\frac{1}{2}} \|}\right] \leqslant \gamma[x \in \mathrm{H} : q(\Pi x) > t],$$

and if $\| \mathrm{R}_\theta^{\frac{1}{2}} \| > 1$,

$$\gamma\left[x \in \mathrm{H} : q(\Pi x) > \frac{t}{\| \mathrm{R}_\theta^{\frac{1}{2}} \|}\right] \leqslant \gamma\left[x \in \mathrm{H} : q(\Pi x) > \frac{t}{\beta}\right].$$

Choose then Π_1 and Π_2 finite dimensional, so that, for Π finite dimensional,

$$\Pi \perp \Pi_1 \quad \text{implies} \quad \gamma[x \in \mathrm{H} : q(\Pi x) > t] < t$$

and

$$\Pi \perp \Pi_2 \quad \text{implies} \quad \gamma\left[x \in \mathrm{H} : q(\Pi x) > \frac{t}{\beta}\right] < \frac{t}{\beta} < t.$$

Consequently, for $\Pi \perp \Pi_0 := \Pi_1 \vee \Pi_2$,

$$\mu^\theta[x \in \mathrm{H} : q(\Pi x) > t] < t, \textit{ i. e. } \mu[x \in \mathrm{H} : q(\Pi x) > t] < t.$$

FACT 3. — Suppose Proposition 2 holds. Then H^* can be regarded as the closure in $\mathrm{L}_2[\tilde{\mu}_q]$ (resp. $\mathrm{L}_2[\tilde{\mu}_q^\theta]$) of E_q^*. Also $\langle\, ., h \,\rangle$ is a representative of the equivalence class of a random variable $\mathrm{X}(h)$ with mean zero, variance $\sigma^2(h) := \langle \mathrm{R}h, h \rangle$ (resp. $\sigma_\theta^2(h) := \langle \mathrm{R}_\theta h, h \rangle$) and density

$$\int_{\mathbb{R}_+} d\mathrm{F}(\theta) n_{\sigma_\theta(h)}(x) \text{ (resp. } n_{\sigma_\theta(h)}).$$

Proof. — Let $\eta \in \mathrm{E}_q^*$ and $i_q^*(\eta) = \langle\, ., h \,\rangle$. Then

$$\mu_q[x \in \mathrm{E}_q : \eta(x) \leqslant t] = \mu[x \in \mathrm{H} : \langle x, h \rangle \leqslant t] = \int_{-\infty}^t dx \int_{\mathbb{R}_+} d\mathrm{F}(\theta) n_{\sigma_\theta(h)}(x).$$

Define

$$\Lambda : \mathrm{H} \to \mathrm{L}_2[\tilde{\mu}_q] \quad \text{by} \quad \Lambda(\mathrm{L}(\eta)) = [\eta]_{\mathrm{L}_2[\tilde{\mu}_q]}.$$

Then

$$\begin{aligned} \| \Lambda(\mathrm{L}(\eta)) \|_{\mathrm{L}_2[\tilde{\mu}_q]}^2 &= \int_{\mathrm{E}_q} \eta^2(x) d\tilde{\mu}_q(x) \\ &= \int_{\mathbb{R}_+} d\mathrm{F}(\theta) \langle \mathrm{R}_\theta(\mathrm{L}(\eta)), \mathrm{L}(\eta) \rangle \\ &\leqslant \left\{ \int_{\mathbb{R}_+} \| \mathrm{R}_\theta \| d\mathrm{F}(\theta) \right\} \| \mathrm{L}(\eta) \|^2. \end{aligned}$$

Thus Λ is continuous and densely defined. It can be continuously extended. Furthermore, if $h = \lim_n h_n$, and $h_n = L(\eta_n)$, one has

$$\int_{E_q} \exp\{it\,\Lambda(h)\}\,d\tilde{\mu}_q = \lim_n \int_{E_q} \exp\{it\Lambda(L(\eta_n))\}\,d\tilde{\mu}_q$$
$$= \lim_n \int_{\mathbb{R}_+} dF(\theta) \exp\left\{-\frac{t^2}{2}\langle R_\theta h_n, h_n\rangle\right\}$$
$$= \int_{\mathbb{R}_+} dF(\theta) \exp\left\{-\frac{t^2}{2}\langle R_\theta h, h\rangle\right\}.$$

N. B. — It is a consequence of the hypotheses that the range of i_q^* is dense in H^*.

FACT 4. — Suppose Proposition 2 holds. Then the support of $\tilde{\mu}_q$ is E_q.

Proof. — The covariance of $\tilde{\mu}_q^\theta$ is $i_q \circ R_\theta \circ J_H \circ i_q^*$, which is injective. Thus the support of $\tilde{\mu}_q^\theta$ is E_q [*3*, p. 911, Thm. 2.5]. It follows that the support of $\tilde{\mu}_q$ has to be E_q.

3. AN EXAMPLE

Let $\alpha : [0, 1] \to \mathbb{R}$ be a continuous function such that $0 < a \leqslant \alpha(x) \leqslant b < \infty$ and let $A(t) := \int_0^t \alpha(x)dx$. If $\theta \in [0, 1]$, $A(\theta t) = \theta \int_0^t \alpha(\theta x)dx$. Write $A_\theta(t)$ for $A(\theta t)$ and $\alpha_\theta(t)$ for $\theta\alpha(\theta t)$.

If W_t is a Wiener process, for $t \in [0, 1]$ and $\theta \in [0, 1]$, set

$$X_t^\theta := X_{\theta t} := W_{A_\theta(t)}.$$

X^θ is a Gaussian process with continuous paths. It induces a measure P_θ on $C[0, 1]$.

Define a separable Hilbert space $(H, \langle .,. \rangle)$ and a unitary map $U : L_2[0, 1] \to H$ as follows:

i) H is the set of functions $h(t) = \int_0^t f(x)dx, f \in L_2[0, 1]$;

ii) if $h_i(t) = \int_0^t f_i(x)dx$, $\langle h_1, h_2 \rangle := \int_0^1 f_1(x)f_2(x)dx$;

iii) $Uf(t) = \int_0^t f(x)dx$.

Now the relation $S_\theta f(x) := \alpha_\theta(x)f(x)$ defines, on $L_2[0, 1]$, a family of transformations S_θ. Set $R_\theta := US_\theta U^*$. One has, for $h_i(t) = \int_0^t f_i(x)dx$,

$$\langle R_\theta h_1, h_2 \rangle = \int_0^1 \alpha_\theta(x) f_1(x) f_2(x) dx,$$

so that
$$\theta^2 a^2 \| h \|^2 \leqslant \| R_\theta h \|^2 \leqslant \theta^2 b^2 \| h \|^2.$$

Consequently R_θ is linear, bounded, positive and self-adjoint. It is also invertible with bounded inverse for $\theta > 0$. Finally, since $\langle R_\theta h_1, h_2 \rangle$ is continuous in θ, the map $\theta \to R_\theta$ is measurable.

Let now Θ be a random variable with values in [0, 1] and density p_Θ. Consider the process $Y_t := X_{\Theta,t}$. Y induces on C[0, 1] a measure P such that

$$P[C] = \int_{[0,1]} P_\theta[C] p_\Theta(\theta) d\theta.$$

Set finally

$$R := \int_{[0,1]} R_\theta p_\Theta(\theta) d\theta.$$

Obviously $\int_{[0,1]} \| R_\theta \| p_\Theta(\theta) d\theta < \infty$ and all the hypotheses considered previously hold for this example.

FACT 1. — P is the extension of $\mu \circ i^{-1}$, where i is the inclusion of H into C[0, 1], and μ is defined through the R_θ's.

Proof. — Let ω be in C[0, 1] and set $ev_t(\omega) := \omega(t)$. Fix $0 \leqslant t_1 < t_2 < \ldots < t_n \leqslant 1$. A standard calculation [9 c] shows that

$$\int_{C[0,1]} \exp\left\{ i \sum_{k=1}^{n} c_k ev_{t_k}(\omega) \right\} dP(\omega)$$

$$= E\left[\exp\left\{ i \sum_{k=1}^{n} c_k Y_{t_k} \right\} \right]$$

$$= \int_{[0,1]} \exp\left\{ -\frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} c_k c_l A_\theta(t_k \wedge t_l) \right\} p_\Theta(\theta) d\theta.$$

Furthermore, if $h(t) = \int_0^t f(x)dx, f \in L_2[0, 1]$,

$$ev_t \circ i(h) = \int_0^t f(x)dx = \int_0^1 I_t(x) f(x) dx = \langle h_t, h \rangle.$$

Thus

$$\int_{C[0,1]} \exp\left\{ i \sum_{k=1}^{n} c_k ev_{t_k}(\omega) \right\} d\mu \circ i^{-1}(\omega)$$

$$= \int_H \exp\left\{ i \sum_{k=1}^{n} c_k \langle h_{t_k}, h \rangle \right\} d\mu(h)$$

$$= \int_{[0,1]} \exp\left\{ -\frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} c_k c_l \langle R_\theta h_{t_k}, h_{t_l} \rangle \right\} p_\Theta(\theta) d\theta.$$

But

$$\langle R_\theta h_{t_k}, h_{t_l} \rangle = \int_0^1 \alpha_\theta(x) I_{t_k}(x) I_{t_l}(x) dx$$
$$= \int_0^1 I_{t_k}(x) I_{t_l}(x) dA_\theta(x) = A_\theta(t_k \wedge t_l).$$

Consequently the restriction of P to the cylinder sets is identical with $\mu \circ i^{-1}$.

FACT 2. — i has dense range and is continuous.

Proof. — [*13*, p. 90, Lemma 5.1, p. 89].

FACT 3. — The sup-norm is μ-measurable.

Proof. — Facts 1 and 2 and Proposition 3 of Part 2.

4. E_q AS A HILBERT SPACE

From a computational point of view it is preferable to deal with Hilbert spaces than with Banach spaces: covariance operators are in particular quite « nicer » on a Hilbert space.

FACT 1. — Let Φ_Σ be a Gaussian measure on $\mathscr{B}[\mathbb{R}^n]$ with mean zero and invertible covariance matrix Σ. Let T be a linear, symmetric and invertible operator on $\mathbb{R}^n$ such that

$$\| T^{-1} \| \| \Sigma^{-\frac{1}{2}} \| \| \Sigma^{\frac{1}{2}} \| \leqslant 1.$$

Then, if C is convex, centrally symmetric and closed,

$$\Phi_\Sigma[C] \leqslant \Phi_\Sigma[TC].$$

Proof. — In the integral giving $\Phi_\Sigma[C]$ make the change of variables $\underline{x} = \Sigma^{\frac{1}{2}}\underline{y}$ and in the one giving $\Phi_\Sigma[TC]$ set $\underline{x} = \Sigma^{\frac{1}{2}}U\underline{y}$, U unitary. Then

$$\Phi_\Sigma[C] = \Phi_n[\Sigma^{-\frac{1}{2}}C] \quad \text{and} \quad \Phi_\Sigma[TC] = \Phi_n[U^*\Sigma^{-\frac{1}{2}}TC],$$

where Φ_n is Φ_Σ, when Σ is the unit matrix of dimension n. Since $\Sigma^{-\frac{1}{2}}C$ is also convex, centrally symmetric and closed, one has to prove:

$$\Phi_n[C] \leqslant \Phi_n[SC], \; S = U^*\Sigma^{-\frac{1}{2}}T\Sigma^{\frac{1}{2}}$$

Choose U such that $\Sigma^{-\frac{1}{2}}T\Sigma^{\frac{1}{2}} = U(\Sigma^{\frac{1}{2}}T\Sigma^{-1}T\Sigma^{\frac{1}{2}})^{\frac{1}{2}}$. S is then symmetric and $\| S^{-1} \| \leqslant 1$. But for such an S the result is known to be true [*2*, p. 184, Lemma (VI, 2, 2)].

FACT 2. — Let H be a finite dimensional Hilbert space and μ_R be the Gaussian measure on H with mean zero and covariance operator R. Let

K be a vector subspace of H and C be a convex, centrally symmetric and closed set.

Then, provided R is an injection,

$$\mu_{R}[C] \leqslant \mu_{R}[C \cap K + K^{\perp}].$$

Proof. — It is as given in [2, p. 186, Lemma (VI, 2, 3)] : it suffices to consider the operator T_p only for $p \geqslant p_0$, where $p_0 \geqslant \| \Sigma^{-\frac{1}{2}} \| \, \| \Sigma^{\frac{1}{2}} \|$.

FACT 3. — Suppose (I) and (II) hold and q is a μ-measurable semi-norm. Let $N^0 := N \cup \{ 0 \}$. Then, for any sequence $\{ a_i > 0, \ i \in N^0 \}$, there exists a sequence $\{ \Pi_i, \ i \in N^0 \} \subseteq \mathcal{P}_H$ such that:

a) $\Pi_i \circ \Pi_j = \delta_{i,j}\Pi_j$,

b) $$id_H = \sum_{i=1}^{\infty} \Pi_i \text{ (strongly)},$$

c) $$q_0(x) := \sum_{i=0}^{\infty} a_i q(\Pi_i x) < \infty, \text{ all } x \text{ in H},$$

d) q_0 is a μ-measurable semi-norm.

Proof. — The proof is as in [13, p. 64, Lemma 4.4], except for two slight changes: the auxiliary sequence α_n has to be manufactured using (II) as in Fact 2 of Part 2. Then the inequality [13, p. 66].

$$\mu\left[q(\Pi_n \circ \Pi x) > \frac{1}{a_n 2^n}\right] \leqslant \mu\left[q(\Pi_n x) > \frac{1}{a_n 2^n}\right]$$

has to be deduced from Fact 2 above, since Lemma 4.3 [13, p. 62] is not available here. Indeed, if H_n is the range of Π_n and K_n the range of $\Pi_n \circ \Pi$, then

$$\mu^{\theta}\left[q(\Pi_n \circ \Pi x) > \frac{1}{a_n 2^n}\right] \leqslant \mu^{\theta}\left[q(\Pi_n x) > \frac{1}{a_n 2^n}\right]$$

can be written:

$$C := \left\{ x \in H_n : q(x) \leqslant \frac{1}{a_n 2^n} \right\},$$

$$\mu^{\theta}\left[x \in H : q(\Pi_n x) > \frac{1}{a_n 2^n}\right] = 1 - \mu^{\theta}_{H_n}[C],$$

$$\mu^{\theta}\left[x \in H : q(\Pi_n \circ \Pi x) > \frac{1}{a_n 2^n}\right] = 1 - \mu^{\theta}_{H_n}[C \cap K_n + K_n^{\perp}],$$

$$\mu^{\theta}_{H_n}[C] \leqslant \mu^{\theta}_{H_n}[C \cap K_n + K^{\perp}],$$

Fact 3 has two corallaries depending only on *a*), *b*), *c*) and *d*) and which are stated here because they will be used in the sequel.

COROLLARY 1. — If (H, μ, q) is as previously, one can find a μ-measurable semi-norm q_0 and an increasing sequence $\{ \Pi_n, n \in \mathrm{N} \}$ in $\mathscr{P}_{\mathrm{H}}$ such that

a) $id_{\mathrm{H}} = \lim_n \Pi_n$ (strongly),

b) q_0 is stronger than q (hence $\mathrm{E}_{q_0} \subseteq \mathrm{E}_q$),

c) each Π_n extends by continuity to a projection $\Pi_n^{q_0}$ on E_{q_0},

d) $id_{\mathrm{E}_{q_0}} = \lim_n \Pi_n^{q_0}$ (strongly).

Proof. — [*13*, p. 66, Corollary 4.2].

COROLLARY 2. — If (H, μ, q) is as previously, there exists a compact operator $\mathrm{T} \in [\mathrm{H}]$ such that

$$q(x) \leqslant \| \mathrm{T}x \|.$$

Proof. — [*13*, p. 71, Corollary 4.3].

PROPOSITION. — Suppose (I) and (II) obtain and q is given by a symmetric bilinear b.

Then $q(x) = \| \mathrm{S}^{\frac{1}{2}}x \|$, where $\mathrm{S}^{\frac{1}{2}}$ is a compact operator such that

a) $b(x, y) = \langle \mathrm{S}x, y \rangle$,

b) $\mathrm{R}_\theta^{\frac{1}{2}}\mathrm{S}^{\frac{1}{2}}$ is Hilbert-Schmidt, F-a. e. θ.

Proof. — q being continuous, b is continuous and thus $b(x, y) = \langle \mathrm{S}x, y \rangle$. Let $\mathrm{E}_q := \mathrm{K}$ and denote the inner product on K by $[.,.]$. Define

$$\mathrm{T} : i_q(\mathrm{H}) \to \mathrm{H} \qquad \text{by} \qquad \mathrm{T}(i_q h) = \mathrm{S}^{\frac{1}{2}}h.$$

T can be extended to a unitary operator from K to H [*13*, p. 76]. Then

$$\int_{\mathrm{K}} \exp\{ i[x, h] \} \, d\tilde{\mu}_q(x) = \int_{\mathbb{R}_+} d\mathrm{F}(\theta) \exp\left\{ -\frac{1}{2} [[\mathrm{T}^*\mathrm{R}_\theta^{\frac{1}{2}} i_q^* h]]^2 \right\}.$$

But $i_q^* = \mathrm{S}^{\frac{1}{2}}\mathrm{T}$ and $\mathrm{T}^*\mathrm{R}_\theta^{\frac{1}{2}} i_q^* = \mathrm{T}^*\mathrm{R}_\theta^{\frac{1}{2}}\mathrm{S}^{\frac{1}{2}}\mathrm{T}$ is Hilbert-Schmidt on K. Thus $\mathrm{R}_\theta^{\frac{1}{2}}\mathrm{S}^{\frac{1}{2}}$ is Hilbert-Schmidt on H. That $\mathrm{S}^{\frac{1}{2}}$ is compact follows from Corollary 2 to Fact 3 above.

REMARKS. — *a)* If $\mathrm{T}_\theta := \mathrm{R}_\theta^{\frac{1}{2}}\mathrm{S}^{\frac{1}{2}}$, $\mathrm{S}^{\frac{1}{2}}$ is a bounded extension of $\mathrm{T}_\theta^*\mathrm{R}_\theta^{-\frac{1}{2}}$.
b) The converse of the above proposition is true [*9*, *a*].

5. SINGULAR AND NON-SINGULAR DETECTIONS FOR SURE SIGNALS

We are now in a position to consider some detection problems. Our aim is to show that there are cases when detection is not unduly affected if one works with cylinder set measures instead of probability ones.

Since singular problems correspond to orthogonal measures and non-singular ones to equivalent measures, we state all results in terms of orthogonality and equivalence.

Set $\mathscr{R}_\theta$ for the range of R_θ and T_a for the translation by a.

Let $k = i_q(h)$ and set $A(k) = \{ \theta > 0 : h \in \mathscr{R}_\theta \}$. Choose a measurable set $A_*(k)$ such that

$$A_*(k) \subseteq A(k) \quad \text{and} \quad F[A_*(k)] = F_*[A(k)] \text{ (inner measure)}.$$

Denote $F[A_*(k)]$ by $\delta(k)$ and define:

if $\delta(k) = 0$, $F_k := 0$;

if $\delta(k) = 1$, $F^k := 0$;

if $0 < \delta(k) < 1$, $F_k[B] := \{ \delta(k) \}^{-1} F[B \cap A_*(k)]$,

$$F^k[B] := \{ 1 - \delta(k) \}^{-1} F[B \cap A^c_*(k)].$$

Then $F = \delta(k)F_k + (1 - \delta(k))F^k$.

Write also

$$\tilde{\lambda}^k_q[C] := \int_{\mathbb{R}_+} \tilde{\mu}^\theta_q[C] dF_k(\theta),$$

$$\tilde{\nu}^k_q[C] := \int_{\mathbb{R}_+} \tilde{\mu}^\theta_q[C] dF^k(\theta),$$

so that $\tilde{\mu}_q = \delta(k)\tilde{\lambda}^k_q + (1 - \delta(k))\tilde{\nu}^k_q$.

PROPOSITION 1. — Suppose (I) and (II) hold. Then there exists a μ-measurable semi-norm q such that

a) $\delta(k) = 1$ yields $\tilde{\mu}_q \circ T_k^{-1} \equiv \tilde{\mu}_q$,

b) $\tilde{\mu}_q \circ T_k^{-1} \perp \tilde{\mu}_q$ forces $\delta(k) = 0$.

Proof. — μ-measurable semi-norms always exist, thus one can choose q as in Corollary 1, Fact 3, Part 4. Then $\tilde{\mu}_q$ and $\tilde{\mu}^\theta_q$ are probability measures and the latter has covariance

$$C_\theta(\eta_1, \eta_2) = \langle R_\theta L(\eta_1), L(\eta_2) \rangle.$$

Denote by $H_\theta(\eta)$ the closure of E^*_q in $L_2[\tilde{\mu}^\theta]$[Fact 3, Part 2]. $H(C_\theta)$, the reproducing kernel Hilbert space of C_θ, is isomorphic to a subset K_θ of E_q which can be regarded as a Hilbert space as follows:

$$K_\theta := \left\{ \int_{E_q} f(x) x d\tilde{\mu}^\theta_q(x), f \in H_\theta(\eta) \right\}$$

and

$$\left[\int_{E_q} f(x) x d\tilde{\mu}^\theta_q(x), \int_{E_q} g(x) x d\tilde{\mu}^\theta_q(x) \right] := \int_{E_q} f(x) g(x) d\tilde{\mu}^\theta_q(x).$$

Fix now $\eta \in E_q^*$. The notations are those of Corollary 1, Fact 3, Part 2. Then

$$q(\eta(\Pi_n^q x)\Pi_n^q x - \eta(x)x) \to 0, \qquad \text{as} \qquad n \to \infty,$$

and

$$q(\eta(\Pi_n^q x)\Pi_n^q x) \leqslant \| \eta \| q^2(x).$$

But, because $\tilde{\mu}_q^\theta$ is Gaussian, $q \in L_2[\tilde{\mu}_q^\theta]$ and dominated convergence yields

$$\int_{E_q} \eta(x)xd\tilde{\mu}_q^\theta(x) = \lim_n \int_{E_q} \eta(\Pi_n^q x)\Pi_n^q x d\tilde{\mu}_q^\theta(x).$$

Now

$$\int_{E_q} \eta(\Pi_n^q x)\Pi_n^q x d\tilde{\mu}_q^\theta(x) = \int_H \eta(\Pi_n^q \circ i_q(x))\Pi_u^q \circ i_q(x)d\mu^\theta(x)$$
$$= i_q \left\{ \int_H \eta(i_q \circ \Pi_n(x))\Pi_n x d\mu^\theta(x). \right\}$$

Writing $i_q^* \eta(\Pi_n x) = \langle \Pi_n x, h \rangle$ and choosing an orthonormal basis for $\Pi_n H$, one sees that

$$i_q \left\{ \int_H \eta(i_q \circ \Pi_n(x))\Pi_n x d\mu^\theta(x) \right\} = \Pi_n^q \circ i_q \circ R_\theta \circ \Pi_n(h).$$

Consequently

$$i_q \circ R_\theta(h) = \int_{E_q} \eta(x)xd\tilde{\mu}_q^\theta(x) = \int_{E_q} \Lambda h(x)xd\tilde{\mu}_q^\theta(x).$$

If now $f \in H_\theta(\eta)$ and $f = \langle \,.\,, h \rangle$, $\tilde{\mu}_q^\theta$-a. s., $h \in H$, choose $h_n = L(\eta_n)$, $\eta_n \in E_q^*$, such that $h = \lim_n h_n$. Then:

$$q(i_q \circ R_\theta(h) - i_q \circ R_\theta(h_n)) \leqslant \alpha \| R_\theta(h - h_n) \| \to 0,$$

$$q\left(\int_{E_q} (\Lambda h(x)xd\tilde{\mu}_q^\theta(x) - \int_{E_q} (\Lambda h_n)(x)xd\tilde{\mu}_q^\theta(x)\right)$$
$$\leqslant \alpha' \left[\left| \int_{E_q} (\Lambda h - \Lambda h_n)d\tilde{\mu}_q^\theta \right| \right]$$
$$= \alpha' \left\{ \int_{E_q} (\Lambda h - \Lambda h_n)^2 d\tilde{\mu}_q^\theta \right\}^{\frac{1}{2}} \to 0.$$

Consequently, for $f = \langle \,.\,, h \rangle$ $\tilde{\mu}_q^\theta$-a. s., one has

$$i_q \circ R_\theta(h) = \int_{E_q} f(x)xd\tilde{\mu}_q^\theta(x).$$

But the right-hand side of the last equality describes exactly the admissible translations for $\tilde{\mu}_q^\theta$. We thus get $\tilde{\mu}_q^\theta \circ T_k^{-1} \equiv \tilde{\mu}_q^\theta$ if k is as in the hypothesis. *a*) then follows from [*14*, p. 97, Corollary].

Finally, if $\tilde{\mu}_q \circ T_k^{-1}[C] = \tilde{\mu}_q[C^c] = 0$, $\tilde{\lambda}_q^k \circ T_k^{-1} \perp \tilde{\lambda}_q^k$. But, by construction, we must have equivalence, which is contradictory unless $\delta(k) = 0$.

REMARK. — Suppose $R_\theta = R \circ S_\theta$, where S_θ has bounded inverse. Then the range of R_θ is the range of R, independently of θ. Thus there is a model for which the signals in RH are admissible, whether or not R_θ is trace-class.

In more general cases, the number of admissible signals may depend on the « size » of $\bigcap_{\theta>0} \mathscr{R}_\theta$ significantly. To study that dependence more information on R_θ is needed.

PROPOSITION 2. — Suppose (II) holds. If $\| R_\theta^{\frac{1}{2}} \| \leqslant \beta < \infty$ and $R_\theta^{\frac{1}{2}}$ has a bounded inverse F-a. s. (so that (I) holds and R_θ has a bounded inverse F-a. s.) and if q is a μ-measurable semi-norm, then

$$\int_{E_q} q^2(x) d\tilde{\mu}_q(x) < \infty .$$

Proof. — As in Proposition 3, Part 2, γ_q extends to a probability measure $\tilde{\gamma}_q$ such that

$$\tilde{\mu}_q^\theta[q(x) > t] \leqslant \tilde{\gamma}_q\left[q(x) > \frac{t}{\| R_\theta^{\frac{1}{2}} \|}\right] .$$

Now, the assertion holds if and only if, for some n_0,

$$\sum_{n=n_0}^{\infty} \tilde{\mu}_q[q(x) > \sqrt{n}] < \infty .$$

Choose t so that $\tilde{\gamma}_q[q(x) \leqslant t] \geqslant s > \frac{1}{2}$, in order to obtain [5], $\forall u \geqslant t$,

$$\tilde{\gamma}_q[q(x) > u] \leqslant s \exp\left\{ -\frac{u^2}{24t^2} \log \frac{s}{1-s} \right\} .$$

Thus, provided $\sqrt{n} \geqslant t \| R_\theta^{\frac{1}{2}} \|$,

$$\begin{aligned} &\tilde{\mu}_q[q(x) > \sqrt{n}] \\ &= \int_{\mathbb{R}_+} dF(\theta) \tilde{\mu}_q^\theta[q(x) > \sqrt{n}] \\ &\leqslant \int_{\mathbb{R}_+} dF(\theta) \tilde{\gamma}_q^\theta\left[q(x) > \frac{\sqrt{n}}{\| R_\theta^{\frac{1}{2}} \|}\right] \\ &\leqslant s \int_{\mathbb{R}_+} dF(\theta) \exp\left\{ -\frac{n}{24t^2 \| R_\theta^{\frac{1}{2}} \|^2} \log \frac{s}{1-s} \right\} . \end{aligned}$$

Consequently, if $n_0 \geqslant (t\beta)^2$ and $c(\theta) := \dfrac{\log s - \log(1-s)}{24t^2 \| R_\theta^{\frac{1}{2}} \|^2}$,

$$\sum_{n=n_0}^{\infty} \tilde{\mu}_q[q(x) > \sqrt{n}] \leqslant s \int_{\mathbb{R}_+} dF(\theta) \frac{e^{-n_0 c(\theta)}}{1 - e^{-c(\theta)}} .$$

Notice that $c(\theta) \geqslant c(\beta) > 0$ and that, for $x > 0$, $\dfrac{e^{-nx}}{1 - e^{-x}}$ is decreasing. Thus finally

$$\sum_{n=n_0}^{\infty} \tilde{\mu}_q[q(x) > \sqrt{n}] \leqslant s \frac{e^{-n_0 c(\beta)}}{1 - e^{-c(\beta)}} < \infty.$$

PROPOSITION 3. — Let Proposition 2 hold. Then, for k not in $i_q(\mathrm{H})$,

$$\tilde{\mu}_q \circ \mathrm{T}_k^{-1} \perp \tilde{\mu}_q.$$

Proof. — It is an immediate consequence of Proposition 2 and [7, p. 201, Thm. 1.2].

REMARK. — Proposition 3 shows that no signals outside of H are admissible, which is as it should be !

PROPOSITION 4. — Let R_0 be a weak covariance operator, with bounded inverse, determining a weak Gaussian distribution ν. For all $\theta > 0$, let S_θ be a weak covariance operator, with bounded inverse, and such that $\mathrm{S}_\theta - id_\mathrm{H}$ is Hilbert-Schmidt.

Suppose now $\mathrm{R}_\theta := \mathrm{R}_0^{\frac{1}{2}} \mathrm{S}_\theta \mathrm{R}_0^{\frac{1}{2}}$ and $\delta(k) = 1$. Then

a) $\nu_q = \nu \circ i_q^{-1}$ extends to a probability measure $\tilde{\nu}_q$ in E_q;

b) $\tilde{\mu}_q \circ \mathrm{T}_k^{-1} \equiv \tilde{\mu}_q$ and

$$\frac{d\tilde{\mu}_q \circ \mathrm{T}_k^{-1}}{d\tilde{\mu}_q} = \left\{ \int_{\mathbb{R}+} \frac{d\tilde{\mu}_q^\theta}{d\tilde{\nu}_q} d\mathrm{F}(\theta) \right\}^{-1} \left\{ \int_{\mathbb{R}+} \frac{d\tilde{\mu}_q^\theta \circ \mathrm{T}_k^{-1}}{d\tilde{\mu}_q^\theta} \cdot \frac{d\tilde{\mu}_q^\theta}{d\tilde{\nu}_q} d\mathrm{F}(\theta) \right\}.$$

Proof. — ν_q has covariance $\tilde{\mathrm{R}}_0 = i_q \circ \mathrm{R}_0 \circ \mathrm{J}_\mathrm{H} \circ i_q^*$; thus, if $\mathrm{L}(\eta) = h$,

$$\eta(\tilde{\mathrm{R}}_0 \eta) = \| \mathrm{R}_0^{\frac{1}{2}} h \|^2.$$

Similarly $\tilde{\mu}_q^\theta$ has covariance $\tilde{\mathrm{R}}_\theta = i_q \circ \mathrm{R}_\theta \circ \mathrm{L}$ and

$$\eta(\tilde{\mathrm{R}}_\theta \eta) = \| \mathrm{S}_\theta^{\frac{1}{2}} \mathrm{R}_0^{\frac{1}{2}} h \|.$$

But $\mathrm{S}_\theta^{\frac{1}{2}}$ has bounded inverse and, from

$$\| \mathrm{S}_\theta^{-\frac{1}{2}} \|^2 \| \mathrm{R}_0^{\frac{1}{2}} h \|^2 \leqslant \| \mathrm{S}_\theta^{\frac{1}{2}} \mathrm{R}_0^{\frac{1}{2}} h \|^2,$$

if follows that $\| \mathrm{S}_\theta^{-\frac{1}{2}} \|^{-2} \tilde{\mathrm{R}}_\theta - \mathrm{R}_0$ is positive definite. Since $\tilde{\mathrm{R}}_\theta$ is a covariance of a measure, so is $\tilde{\mathrm{R}}_0$ [3, p. 914, Corollary 2.7]. ν_q extends thus to a measure $\tilde{\nu}_q$.

Let us now show that $\tilde{\mu}_q^\theta \equiv \tilde{\nu}_q$. To that effect, write $S_\theta = id_H + D_\theta$, $D_\theta = \sum_{i=1}^{\infty} \lambda_i^\theta e_i^\theta \otimes e_i^\theta$, e_i^θ orthonormal and complete. Now both:

$D_\theta = R_0^{\frac{1}{2}}(R_0^{-1}R_\theta R_0^{-\frac{1}{2}} - R_0^{-\frac{1}{2}})$ and $R_0^{-1}R_\theta R_0^{-\frac{1}{2}} - R_0^{-\frac{1}{2}}$ bounded, yield

$$\mathcal{R}(D_\theta) \subseteq \mathcal{R}(R_0^{\frac{1}{2}}).$$

Choose thus h_i^θ such that $e_i^\theta = R_0^{\frac{1}{2}} h_i^\theta$. Since H^* « is » the closure of E_q^* in $L_2[\tilde{\nu}_q]$, $\langle \,.\,, h_i^\theta \rangle = X(h_i^\theta)\tilde{\nu}_q$-a. s. Form

$$g_\theta(\omega_1, \omega_2) = \sum_{i=1}^{\infty} \lambda_i^\theta X(h_i^\theta)(\omega_1)X(h_i^\theta)(\omega_2).$$

Then $g_\theta \in L_2[\tilde{\nu}_q \otimes \tilde{\nu}_q]$, since

$$\int_{E_q \times E_q} g_\theta^2 d(\tilde{\nu}_q \otimes \tilde{\nu}_q) = \sum_{i=1}^{\infty} [\lambda_i^\theta]^2 < \infty.$$

Furthermore, if $L(\eta) = k$ and $L(\xi) = h$,

$$\begin{aligned} &\eta(\tilde{R}_0\xi) - \eta(\tilde{R}_\theta\xi) \\ &= \langle R_0 h, k \rangle - \langle R_\theta h, k \rangle = - \langle R_0^{\frac{1}{2}} D_\theta R_0^{\frac{1}{2}} h, k \rangle \\ &= \sum_{i=1}^{\infty} \lambda_i^\theta \langle R_0 h_i^\theta, h \rangle \langle R_0 h_i^\theta, k \rangle \\ &= \int_{E_q \times E_q} g_\theta(\omega_1, \omega_2)\eta(\omega_1)\xi(\omega_2) d(\tilde{\nu}_q \otimes \tilde{\nu}_q)(\omega_1, \omega_2), \end{aligned}$$

which is a necessary and sufficient condition for $\tilde{\mu}_q^\theta$ and $\tilde{\nu}_q$ to be equivalent.

The result then follows from Proposition 1 and [*14*, p. 97, Lemma 1].

REMARKS. — *a*) If $R = \int_{\mathbb{R}^+} R_\theta dF(\theta)$ as before and $D = \int_{\mathbb{R}^+} D_\theta dF(\theta)$, we have $R = R_0^{\frac{1}{2}}(id_H + D)R_0^{\frac{1}{2}}$, which is formally the expression giving, when all operators considered are properly compact, the equivalence of μ and ν.

b) The expressions $\dfrac{d\tilde{\mu}_q^\theta \circ T_k^{-1}}{d\tilde{\mu}_q^\theta}$ are well known. For example, if $k = i_q(h) = i_q \circ L(\eta)$, it is $\exp\{\eta - \frac{1}{2}\langle R_\theta h, h \rangle\}$. So, to compute $\dfrac{d\tilde{\mu}_q \circ T_k^{-1}}{d\tilde{\mu}_q}$ one needs a formula for $\dfrac{d\tilde{\mu}_q^\theta}{d\tilde{\nu}_q}$. In the next section, we shall consider a particular case for which such a formula is available.

6. NON-SINGULAR DETECTIONS FOR RANDOM SIGNALS

PROPOSITION 1. — Suppose (I) and (II) hold. Let R_θ be F-a. s. invertible and q be a μ-measurable semi-norm.

Let T_θ be a strongly measurable function of θ such that, F-a. s.

a) T_θ is self-adjoint and Hilbert-Schmidt,

b) $id_H + T_\theta$ is a weak covariance with bounded inverse.

Let ν^θ be the weak Gaussian distribution associated with

$$S_\theta = R_\theta^{\frac{1}{2}}(id_H + T_\theta)R_\theta^{\frac{1}{2}} \qquad \text{and set} \qquad \nu := \int_{\mathbb{R}_+} dF(\theta)\nu^\theta.$$

Then ν_q extends to a probability measure $\tilde{\nu}_q$ on E_q which is equivalent to $\tilde{\mu}_q$.

Proof. — As in the proof of Proposition 3, Part 5.

REMARK. — Proposition 1 improves on [9, *b*, p. 21, last §].

FACT 1. — Let q denote a μ-measurable semi-norm and $\Pi \to id_H$ mean that Π tends to id_H strongly as Π stays in $\mathscr{P}_H$. Suppose Proposition 2, Part 5, holds.

$f : H \to \mathbb{C}$ is a map with the following property: there exists a μ-measurable semi-norm $\hat{q}$ such that f restricted to $\{ x \in H : \hat{q}(x) \leqslant t \}$ is q-uniformly continuous, $\forall t > 0$.

With the tame function $f \circ \Pi$, $\Pi \in \mathscr{P}_H$, is associated a random variable $X(f \circ \Pi)$, and with the tame function $\hat{q} \circ \Pi$ another variable $X(\hat{q} \circ \Pi)$, both on E_q (Fact 3, Part 2).

Then, if $\tilde{\mu}_q$-lim stands for limit in $\tilde{\mu}_q$-probability,

a) $\tilde{\mu}_q\text{-}\lim\limits_{\Pi \to id_H} X(\hat{q} \circ \Pi)$ exists and is denoted $X(\hat{q})$;

b) $\tilde{\mu}_q\text{-}\lim\limits_{\Pi \to id_H} X(f \circ \Pi)$ exists and is denoted $X(f)$;

c) if g is a continuous function on E_q, then

$$X(g \circ i_q) = g, \ \tilde{\mu}_q\text{-a. s.}$$

Proof. — « Mutatis mutandis » as in [*13*, p. 59, Lemma 4.1, p. 94, Thm. 6.3]. The stronger hypotheses (stronger than (I) and (II) that is) of Proposition 2, Part 5, are needed in the proof of *b*), where one wants $X(\hat{q})$ integrable.

PROPOSITION 2. — Let $R := \sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i, 0 < \alpha \leqslant \lambda_i \leqslant \beta < \infty$, and $S := R^{\frac{1}{2}}(id_H + T)R^{\frac{1}{2}}$, with $id_H + T$ a weak covariance with bounded inverse and T trace-class. μ is the weak Gaussian distribution associated with R.

Set $(id_H + T)^{\frac{1}{2}} = id_H + A$, $B = R^{\frac{1}{2}}(id_H + A)R^{-\frac{1}{2}}$ and $\nu = \mu \circ B^{-1}$.

Then: *a*) ν is a weak Gaussian distribution with weak covariance S.

Set $B^{-1} := id_H + R^{\frac{1}{2}}CR^{-\frac{1}{2}}$, $R^{\frac{1}{2}}CR^{-\frac{1}{2}} = D$, $q(x) = \| Dx \|$.

Then: *b*) ν_q and μ_q extend to equivalent probability measures and

$$\frac{d\tilde{\nu}_q}{d\tilde{\mu}_q} = \det \{ id_H + C \} \exp \left\{ -\frac{1}{2}[2\langle CR^{-\frac{1}{2}}x, R^{-\frac{1}{2}}x\rangle + \| CR^{-\frac{1}{2}}x \|^2 \right\},$$

where the right hand side is to be interpreted as a random variable on E_q.

Proof. — Write

$$T := \sum_{j=1}^{\infty} \mu_j f_j \otimes f_j$$

and let

$$A := \sum_{j=1}^{\infty} \{ (1 + \mu_j)^{\frac{1}{2}} - 1 \} f_j \otimes f_j.$$

A is trace-class, $id_H + A$ has bounded inverse and $(id_H + T)^{\frac{1}{2}} = id_H + A$. Furthermore, if

$$C := \sum_{j=1}^{\infty} \left\{ \frac{-\mu_j}{(1 + \mu_j)^{\frac{1}{2}}(1 + (1 + \mu_j)^{\frac{1}{2}})} \right\} f_j \otimes f_j,$$

C is trace-class and $(id_H + A)^{-1} = id_H + C$. Thus B has bounded inverse and $B^{-1} = id_H + D$, with D trace-class. Since $q((id_H + D)h) \leqslant cq(h)$, $id_H + D$ is the restriction to H of a bounded linear operator $id_{E_q} + \tilde{D}$ on E_q. Notice also that the characteristic function of ν is

$$\exp\left\{ -\frac{1}{2}\langle Sh, h\rangle \right\}.$$

Since q is a μ-measurable semi-norm, μ_q extends to $\tilde{\mu}_q$, a Gaussian measure. By Proposition 1, ν_q extends to $\tilde{\nu}_q$, a Gaussian measure equivalent to $\tilde{\mu}_q$. Thus we only need to determine the form of the Radon-Nikodym derivative. But $\tilde{\nu}_q = \tilde{\mu}_q(id_{E_q} + \tilde{D})$, so that we can proceed as in [*13*, p. 141, Thm. 5.4].

Let H_n denote the subspace spanned by $e_1, \ldots, e_n$ and Π_n be the associated projection. Denote by K_n the range of $id_H + C$ restricted to H_n and let $\widehat{\Pi}_n$ be the associated projection. Choose, on E_q, a bounded, continuous function φ. φ is $\tilde{\mu}_q$-a. s. equal to the random variable $X(\varphi \circ i_q)$, which is the limit in $\tilde{\mu}_q$-probability of the sequence $X(\varphi \circ i_q \circ \widehat{\Pi}_n)$. The latter, being uniformly bounded, converges in L_1 and thus

$$\int_{E_q} \varphi(x) d\tilde{\mu}_q(x) = \lim_n \int_{K_n} \varphi \circ i_q(x) d\mu_{K_n}(x).$$

Let $\mathscr{G}_n := \{ g_1, \ldots, g_n \}$ be an orthonormal basis for K_n. Then

$$\int_{K_n} \varphi \circ i_q(x) d\mu_{K_n}(x) = \int_{\mathbb{R}_n} \varphi \circ i_q \circ J^*_{\mathscr{G}_n}(\underline{x}) d\mu_{K_n} \circ J^{-1}_{\mathscr{G}_n}(\underline{x}).$$

Adopting momentarily the following notations:

$$\mathscr{E}_n := \{ e_1, \ldots, e_n \} \quad , \qquad C_n := J_{\mathscr{G}_n} \circ (id_H + C) \circ J^*_{\mathscr{E}_n},$$
$$\Sigma_1 := \Sigma[R ; \mathscr{E}_n], \qquad \Sigma_2 := \Sigma[R ; \mathscr{G}_n],$$
$$M := \Sigma_2^{\frac{1}{2}} C_n \Sigma_1^{-\frac{1}{2}}, \qquad c := \frac{|\det C_n|}{(2\Pi)^{n/2} \det \Sigma_1^{\frac{1}{2}}},$$
$$\psi(\underline{x}) := \varphi \circ i_q \circ J^*_{\mathscr{G}_n} \circ M(\underline{x}), \qquad \chi(x) := \exp\left\{ -\frac{1}{2} \langle \Sigma_2^{-1} M\underline{x}, M\underline{x} \rangle \right\},$$

and making the change of variables $\underline{x} = M\underline{y}$, we get

$$\int_{\mathbb{R}_n} \varphi \circ i_q \circ J^*_{\mathscr{G}_n}(\underline{x}) d\mu_{K_n} \circ J^{-1}_{\mathscr{G}_n}(\underline{x}) = c \int_{\mathbb{R}_n} \psi(\underline{x}) \chi(\underline{x}) d\underline{x}.$$

Now

$$\langle \Sigma_2^{-1} M\underline{x}, M\underline{x} \rangle = \langle \Sigma_1^{-1} \underline{x}, \underline{x} \rangle + 2 \langle J^*_{\mathscr{E}_n} \Sigma_1^{-\frac{1}{2}} \underline{x}, C J^*_{\mathscr{E}_n} \Sigma_1^{-\frac{1}{2}} \underline{x} \rangle + \| C J^*_{\mathscr{E}_n} \Sigma_1^{-\frac{1}{2}} \underline{x} \|^2,$$

and $J^*_{\mathscr{E}_n} \Sigma_1^{-\frac{1}{2}} = R^{-\frac{1}{2}} J^*_{\mathscr{E}_n}$ by the choice of $\mathscr{E}_n$.

Furthermore

$$\psi(J_{\mathscr{E}_n} h) = J^*_{\mathscr{G}_n} \Sigma_2^{\frac{1}{2}} J_{\mathscr{G}_n} (id_H + C) R^{-\frac{1}{2}} h, \; h \in H_n.$$

Notice that $J^*_{\mathscr{G}_n} \Sigma_2^{\frac{1}{2}} J_{\mathscr{G}_n}$ is the square root of $J^*_{\mathscr{G}_n} \Sigma_2 J_{\mathscr{G}_n} = \widehat{\Pi}_n R \widehat{\Pi}_n$.

Then finally, if

$$\Delta_n(h) := \exp\left\{ -\frac{1}{2} [2 \langle R^{-\frac{1}{2}} h, C R^{-\frac{1}{2}} h \rangle + \| C R^{-\frac{1}{2}} h \|^2] \right\},$$

$$\int_{K_n} \varphi \circ i_q(x) d\mu_{K_n}(x)$$
$$= |\det C_n| \int_{H_n} \varphi(i_q [\widehat{\Pi}_n R \widehat{\Pi}_n]^{\frac{1}{2}} (id_H + C) R^{-\frac{1}{2}} h) \Delta_n(h) d\mu_{H_n}(h)$$
$$= |\det C_n| \int_{E_q} \varphi(i_q [\widehat{\Pi}_n R \widehat{\Pi}_n]^{\frac{1}{2}} (id_H + C) R^{-\frac{1}{2}} \Pi_n h) \Delta_n(\Pi_n h) d\mu$$

where $\Pi_n h$ is interpreted as a random variable on E_q. The end of the proof consists then is showing that the right hand side of the last equality tends to the expression

$$\det(id_{\mathrm{H}} + \mathrm{C}) \int_{\mathrm{E}_q} \{ \varphi \circ (id_{\mathrm{E}_q} + \tilde{\mathrm{D}}) \} (x) \Delta(x) d\tilde{\mu}_q(x),$$

where Δ is formally the same expression than Δ_n, without restriction on h. The argument goes as follows: one proves that $\mathrm{E}\Delta_n \to \mathrm{E}\Delta$, which implies, since Δ_n goes to Δ in $\tilde{\mu}_q$-probability, that $\Delta_n \to \Delta$ in $\mathrm{L}_1[\tilde{\mu}_q]$ and then that the remaining term converges in $\tilde{\mu}_q$-probability, which is good enough since φ is bounded.

Let us first evaluate $\mathrm{E}\Delta$. To that end write

$$\mathrm{C} := \sum_{i=1}^{\infty} \gamma_i c_i \otimes c_i,$$

$$2\langle \mathrm{R}^{-\frac{1}{2}}h, \mathrm{C}\mathrm{R}^{-\frac{1}{2}}h \rangle + \| \mathrm{C}\mathrm{R}^{-\frac{1}{2}}h \|^2 = \sum_{i=1}^{\infty} (2\gamma_i + \gamma_i^2) \langle h, \mathrm{R}^{-\frac{1}{2}}c_i \rangle^2 .$$

The random variables $\langle h, \mathrm{R}^{-\frac{1}{2}}c_i \rangle$ are, with respect to $\tilde{\mu}_q$, independent, identically distributed and Gaussian with mean zero and variance one. Set

$$\mathrm{Y} := \sum_{i=1}^{n} (2\gamma_i^2 + \gamma_i^2) \langle h, \mathrm{R}^{-\frac{1}{2}}c_i \rangle^2, \qquad \mathrm{X}_n := \exp\left\{ -\frac{1}{2} \mathrm{Y}_n \right\}.$$

Since C is trace-class, Y_n converges $\tilde{\mu}_q$-a. s. and thus $\mathrm{X}_n \to \Delta$, $\tilde{\mu}_q$-a. s. Also

$$\sup_n \int_{\mathrm{E}_q} \mathrm{X}_n^2 d\tilde{\mu}_q = \prod_{i=1}^{\infty} (1 + 4\gamma_i + 2\gamma_i^2)^{-1} < \infty .$$

Consequently

$$\mathrm{E}\Delta = \lim_n \mathrm{E}\mathrm{X}_n = \prod_{i=1}^{\infty} (1 + \gamma_i)^{-1} = \{ \det(id_{\mathrm{H}} + \mathrm{C}) \}^{-1} .$$

Let us now evaluate $\mathrm{E}\Delta_n \circ \Pi_n$. Write

$$\underline{\mathrm{X}} := (\langle \mathrm{R}^{-\frac{1}{2}}e_1, h \rangle, \ldots, \langle \mathrm{R}^{-\frac{1}{2}}e_n, h \rangle)$$

$\Sigma :=$ matrix with entries $\langle (id_{\mathrm{H}} + \mathrm{C})e_i, (id_{\mathrm{H}} + \mathrm{C})e_j \rangle$.

With respect to $\tilde{\mu}_q$, $\underline{\mathrm{X}}$ is a vector of independent Gaussian random variables with mean zero and variance one. Thus

$$\mathrm{E}\Delta_n \circ \Pi_n = \frac{1}{(2\Pi)^{n/2}} \int_{\mathbb{R}_n} \exp\left\{ -\frac{1}{2} \langle \Sigma \underline{x}, \underline{x} \rangle \right\} d\underline{x} = (\det \Sigma)^{-\frac{1}{2}}$$

which also has $\{ \det(id_{\mathrm{H}} + \mathrm{C}) \}^{-1}$ as limit.

For the last step, we have to check that the limit in $\tilde{\mu}_q$-probability of $M_n \circ \Pi_n h$, defined by

$$M_n : = [\widehat{\Pi}_n R \widehat{\Pi}_n]^{\frac{1}{2}}(id_H + C)R^{-\frac{1}{2}},$$

is as desired. Let M_q be the operator $id_{E_q} + \tilde{D}$ and M be the operator $id_H + D$. Since the limit in $\tilde{\mu}_q$-probability of $M_q \Pi_n h$ is $M_q h$, it is sufficient to show that the limit in $\tilde{\mu}_q$-probability, as $p \to \infty$, of $M_p \circ \Pi_n h$ is $M_q \circ \Pi_n h$. But

$$\tilde{\mu}_q[q(M_p \circ \Pi_n h - M_q \circ \Pi_n h) > \varepsilon] \leqslant \frac{1}{\varepsilon^2} \int_{H_n} \| R^{\frac{1}{2}} C R^{-\frac{1}{2}}(M_p \circ \Pi_n(h) - M \circ \Pi_n(h)) \|^2 d\mu_{H_n}(h).$$

Now

$$M_p \circ \Pi_n(h) - M \circ \Pi_n(h) = \sum_{i=1}^{n} \langle h, e_i \rangle (M_p - M)(e_i),$$

so that

$$\int_{H_n} \| R^{\frac{1}{2}} C R^{-\frac{1}{2}}(M_p - M)(\Pi_n h) \|^2 d\mu_{H_n}(h) = \sum_{i=1}^{n} \lambda_i \| R^{\frac{1}{2}} C R^{-\frac{1}{2}}(M_p - M) e_i \|^2 \to 0, \text{ as } p \to \infty.$$

PROPOSITION 3. — Choose R_θ, T_θ, S_θ as in Proposition 2 and R_θ as in Proposition 4, Part 5. Then the conclusion of Proposition 4, Part 5, holds « mutatis mutandis », *i. e.* one can also « compute » a likelihood for mixtures which are « componentwise » equivalent.

REFERENCES

[*1*] T. W. ANDERSON, The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities. *Proc. Amer. Math. Soc.*, vol. **6**, 1955, p. 170-176.

[*2*] A. BADRIKIAN and S. CHEVET, Mesures cylindriques, espaces de Wiener et fonctions aléatoires gaussiennes. *Lecture Notes in Math. 379*, Springer Verlag, Berlin, Heidelberg, New York, 1974.

[*3*] P. BAXENDALE, Gaussian measures on function spaces. *Amer. J. Math.*, vol. **98**, 1976, p. 891-952.

[*4*] J. A. DE ARAYA, A Radon-Nikodym theorem for vector and operator valued measures. *Pacific J. Math.*, vol. **29**, 1969, p. 1-10.

[*5*] X. FERNIQUE, Intégrabilité des vecteurs gaussiens. *C. R. Acad. Sci.* (Paris), vol. **270**, 1970, p. A1698-169.

[6] P. A. FILLMORE and J. P. WILLIAMS, On operator ranges. *Adv. in Math.*, vol. **7**, 1971, p. 254-281.

[7] A. GLEIT and J. ZINN, Admissible and singular translates of measures on vector spaces. *Trans. Amer. Math. Soc.*, vol. **221**, 1976, p. 199-211.

[8] L. GROSS, Measurable functions on Hilbert space. *Trans. Amer. Math. Soc.*, vol. **105**, 1962, p. 372-390.

[9 a] A. F. GUALTIEROTTI, Extension of Gaussian cylinder set measures. *J. Math. Anal. Applic.*, vol. **57**, 1977, p. 12-19.

[9 b] On the robustness of Gaussian detection. *J. Math. Anal. Applic.*, vol. **57**, 1977, p. 20-26.

[9 c] A time-perturbation of Gaussian stochastic processes and some applications to the theory of signal detection. *Submitted for publication.*

[10] O. HANS, Inverse and adjoint transforms of linear bounded random transforms. *Trans. First Prague Conference on Information Theory*, Prague, 1957.

[11] E. HILLE and R. S. PHILLIPS, Functional analysis and semi-groups. *Amer. Math. Soc.*, Providence, R. I., 1957.

[12] G. KALLIANPUR, Abstract Wiener processes and their reproducing kernel Hilbert spaces. *Wahrscheinlichkeitstheorie verw. Geb.*, vol. **17**, 1971, p. 113-123.

[13] H. H. KUO, Gaussian measures in Banach spaces. *Lecture Notes in Math. 463*, Springer Verlag, Berlin, Heidelberg, New York, 1975.

[14] A. V. SKOROHOD, *Integration in Hilbert space*, Springer Verlag, Berlin, Heidelberg, New York, 1974.

(Manuscrit reçu le 13 mai 1977).