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DAVID GRIFFEATH

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Section B :
Calcul des Probabilités et Statistique.

An ergodic theorem for a class of spin systems

by

David GRIFFEATH
Department of Mathematics, University of Wisconsin,
Madison, Wisconsin, 53706 U. S. A.

RÉSUMÉ. — Dans cet article, nous donnons des conditions nécessaires et suffisantes d'ergodicité de certains systèmes de particules en interaction, en temps discret et temps continu, sur $\mathbb{Z}^d$, les fonctions de transition ayant une certaine forme.

1. INTRODUCTION

We consider certain configuration valued Markov processes $(\xi_t)_{t\in T}$ called *spin systems*. The state space is $\Xi = \{ -1, 1 \}^{\mathbb{Z}^d}$, where $\mathbb{Z}^d$ is the d-dimensional integer lattice; an element $\xi = (\xi(x))_{x\in\mathbb{Z}^d} \in \Xi$ is to be thought of as a configuration of « spins », either + or −, on the « sites » of $\mathbb{Z}^d$. In discrete time ($T = \mathbb{N} = \{ 0, 1, \dots \}$), (ξ_t) is defined by means of a one-step transition kernel of the form

$$p(\xi, \cdot) = \prod_{x\in\mathbb{Z}^d} p_x(\xi, \cdot) \qquad \text{(product measure)}. \tag{1}$$

Here $p(\xi, E)$ is a measure on Ξ for fixed ξ, giving the probability of a transition from ξ into the Borel set $E \subset \Xi$ in one time unit. According to (1), the measure $p(\xi, \cdot)$ is a product of « local » measures $p_x(\xi, \cdot)$ on the two-point space $\{ -1, 1 \}$. Intuitively, each spin observes its environment at time t and then, conditional on this, chooses its value at time $t + 1$ independently of the choices of the other sites. When time is continuous

($T = \mathbb{R}^+ = [0, \infty)$), (ξ_t) is described by non-negative « flip rates » $c_x(\xi)$. Roughly speaking, if $\mathbb{P}_\xi$ governs (ξ_t) starting from ξ, then

$$\mathbb{P}_\xi(\xi_{dt}(x) = -\xi(x)) \sim c_x(\xi)dt .$$

A more precise formulation will be given later.

A spin system (with $T = \mathbb{N}$ or $\mathbb{R}^+$) is called *ergodic* if it tends to an invariant measure (« equilibrium ») as time goes on, i. e. if $\lim_{t\to\infty} \mathbb{P}_\xi(\xi_t \in E) = \mu(E)$ for some measure μ which is independent of ξ. In general it is quite difficult to determine whether or not a given spin system is ergodic. Some specific examples where this question is still open will be mentioned toward the end of the discussion. There is an extensive literature on spin systems, and their ergodic behavior in particular. Numerous relevant papers are listed in the References. Our present purpose is to prove necessary and sufficient conditions for ergodicity of homogeneous spin systems with probabilities/rates of a relatively simple form. Namely, for $T = \mathbb{N}$, assume that the local kernels p_x of (1) are given by

$$p_x(\xi, 1) = \frac{1}{2}\left(1 + a + \sum_{y\in\mathbb{Z}^d} r_y\xi(x + y)\right) \tag{2}$$

for some constants a and r_y. (Here $p_x(\xi, 1) = p_x(\xi, \{1\})$; we omit parentheses from one-point sets whenever it is convenient.) Let $\mathbb{G} = \mathrm{gr}\{y : r_y \neq 0\}$ denote the smallest group containing $\{y \in \mathbb{Z}^d : r_y \neq 0\}$. We assume that (ξ_t) is *irreducible*, i. e. $\mathbb{G} = \mathbb{Z}^d$, for otherwise (ξ_t) breaks up into independent isomorphic subsystems which live on the cosets of $\mathbb{G}$ in $\mathbb{Z}^d$. In this case, each subsystem can be identified with a process on $\mathbb{Z}^\nu$ for some $\nu \leq d$, and (ξ_t) will be ergodic if and only if the subsystems are. Introduce $S^+ = \{y : r_y > 0\}$, $S^- = \{y : r_y < 0\}$. We will prove the following result.

THEOREM 1 *a*. — An irreducible discrete time spin system (ξ_t) with local transition kernels (2) is ergodic if

i) $|a| + \sum_y |r_y| < 1$,

or

ii) $a \neq 0, |a| + \sum_y |r_y| = 1$,

or

iii) $a = 0, \sum_y |r_y| = 1$, and there exist integers m_y, $y \in \mathbb{Z}^d$, only finitely many non-zero, such that

$$\Sigma m_y = \Sigma m_y y = 0 \qquad \text{and} \qquad \sum_{y \in S^-} m_y \qquad \text{is odd}.$$

In all other cases (ξ_t) is not ergodic.

When $T = \mathbb{R}^+$, consider systems with flip rates which can be written in the form

$$c_x(\xi) = \frac{\kappa}{2}\left(1 - \xi(x)\left[a + \sum_{y \in \mathbb{Z}^d} r_y \xi(x+y)\right]\right) \tag{3}$$

for some $\kappa > 0$. By choosing κ appropriately we can (and will) assume that $r_0 = 0$. Again, (ξ_t) is *irreducible* if $\mathbb{G} = \mathrm{gr}\{ y : r_y \neq 0 \} = \mathbb{Z}^d$, and the same remarks apply. With $S^\pm$ defined as before, the continuous time ergodicity criterion is as follows.

THEOREM 1 *b*. — An irreducible continuous time spin system (ξ_t) with flip rates (3) is ergodic if

i) $|a| + \sum_y |r_y| < 1$,

or

ii) $a \neq 0$, $|a| + \sum_y |r_y| = 1$,

or

iii) $a = 0$, $\sum_y |r_y| = 1$, and there exist integers m_y, $y \in \mathbb{Z}^d$, only finitely many non-zero, such that

$$\Sigma m_y y = 0 \qquad \text{and} \qquad \sum_{y \in S^-} m_y \qquad \text{is odd}.$$

In all other cases (ξ_t) is not ergodic.

Examples. — In all cases $d = 1$, $a = 0$.

a) $T = \mathbb{N}$.

i) $r_{-1} = r_0 = \frac{1}{3}$, $r_1 = -\frac{1}{3}$. (ξ_t) is ergodic. (Take $m_{-1} = m_1 = 1$, $m_0 = -2$, $m_y = 0$ otherwise, in Theorem 1 *a*.)

ii) $r_{-1} = -\frac{1}{2}$, $r_1 = \frac{1}{2}$. (ξ_t) is not ergodic. (The configurations with two successive + spins alternating with two successive − spins form a « cycle » for (ξ_t).)

b) $T = \mathbb{R}^+, \kappa = 1$.

i) $r_1 = -1$. (ξ_t) is not ergodic. (The two configurations with alternating + spins and − spins are both traps.)

ii) $r_{-1} = -\frac{1}{2}, r_1 = \frac{1}{2}$. (ξ_t) is ergodic. (Take $m_{-1} = -1, m_1 = 1, m_y = 0$ otherwise, in Theorem 1 *b*.)

The proofs of Theorems 1 *a* and 1 *b* are quite similar, so we will present the discrete case in detail and merely outline the continuous case. The method used is a *duality theory* developed over the past several years by Vasershtein and Leontovich [*12*], Holley and Liggett [*3*], Harris [*2*], and Holley and Stroock [*6*]. Our proof is based on the treatment of the « anti-voter model » in [*6*]. In the next section we give a general formulation of duality for Markov processes, with an outline of its application to spin systems. Section 3 contains the proof of Theorem 1 *a*, then Section 4 sketches the modifications which are necessary to get Theorem 1 *b*. Finally, in Section 5 we discuss briefly the apparent limitations of the duality method, and mention some important unsolved problems.

2. A DUALITY THEOREM FOR MARKOV PROCESSES

The basic idea in [*12*] [*2*] [*3*] and [*6*] is to control the evolution of a Markov process (ξ_t) with uncountable state space Ξ by means of a countable collection $\{ g_i \}$ of functions on Ξ. Under suitable hypotheses, the *index set* of the g_i can be viewed as the state space for a denumerable Markov chain $(\hat{\xi}_t)$ naturally associated with (ξ_t). The ergodic properties of (ξ_t) are then read off from related behavior of the simpler dual process $(\hat{\xi}_t)$.

Let Ξ be a compact polish space, $\mathscr{B}_\Xi$ the Borel σ-algebra on Ξ. $\mathscr{C}$ comprises the continuous functions on Ξ, topologized by the supremum norm. We are given a countable collection $\{ f_i ; i = 1, 2, \dots \}$ of functions in $\mathscr{C}$, with $f_1 = 1$ and $\| f_i \| \leq 1$ for all i. By convention, throw in $f_0 = 0$. Let $\mathscr{F}$ be the finite linear span of $\{ f_i \}$, and assume that $\mathscr{F}$ is dense in $\mathscr{C}$. Write

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{else} \end{cases} \quad i, j \in \mathbb{Z} ; \qquad \operatorname{sgn} r = \begin{cases} 1 & r > 0 \\ 0 & r = 0 \\ -1 & r < 0 \end{cases} \quad r \in \mathbb{R}.$$

a) The discrete time case

$p(\xi, d\eta)$ is a prescribed transition kernel on Ξ, and $(\xi_t)_{t \in \mathbb{N}}$ is the canonical discrete time Markov process on $(\Xi^{\mathbb{N}}, \mathscr{B}, \{ \mathbb{P}_\xi \}_{\xi \in \Xi})$ with one-step kernel p.

Here $\mathscr{B}$ is the usual σ-algebra generated by cylinder events, and $\mathbb{P}_\xi$ is the measure governing (ξ_t) when $\xi_0 = \xi$ a. s. We introduce the operators $P^t : \mathscr{C} \to \mathscr{C}$, $t \in \mathbb{N}$, defined by $(P^t f)(\xi) = \mathbb{E}_\xi[f(\xi_t)]$ ($\mathbb{E}_\xi$ is the expectation operator corresponding to $\mathbb{P}_\xi$.) Abbreviate $P^1 = P$. Assume that P maps $\mathscr{C}$ to $\mathscr{C}$ to ensure that P^t does. In this context, we now formalize the duality method.

THEOREM 2 *a*. — Assume that for each $i \geq 1$,

$$P f_i = \sum_{j=1}^{\infty} r_{ij} f_j \qquad r_{ij} \in \mathbb{R}, \tag{4}$$

with $r_{1j} = \delta_{1j}$, and set $\rho_i = \sum_j | r_{ij} |$.

i) If $\rho_i \leq 1$ for each $i \geq 1$, then there is a denumerable Markov chain $(\hat{\xi}_t)$ on $(\mathbb{Z}^{\mathbb{N}}, \hat{\mathscr{B}}, \{ \hat{\mathbb{P}}_i \}_{i \in \mathbb{Z}})$ such that

$$\mathbb{E}_\xi[f_i(\xi_t)] = \hat{\mathbb{E}}_i\, [(\operatorname{sgn} \hat{\xi}_t) f_{|\hat{\xi}_t|}(\xi)] \qquad \text{for each } \xi \in \Xi,\ i \in \mathbb{Z},\ t \in \mathbb{N}. \tag{5}$$

($\hat{\mathscr{B}}$, $\hat{\mathbb{P}}_i$ and $\hat{\mathbb{E}}_i$ are the obvious analogues of $\mathscr{B}$, $\mathbb{P}_\xi$ and $\mathbb{E}_\xi$.)

ii) If $\rho = \sup_{i \geq 2} \rho_i < 1$, then (ξ_t) is geometrically ergodic. That is to say, there is a measure μ on $(\Xi, \mathscr{B}_\Xi)$ such that $P^t f \to \int f d\mu$ as $t \to \infty$ for all $f \in \mathscr{C}$, and such that for any $f = \sum_{k=1}^{\infty} b_k f_k$ we have

$$\left\| P^t f - \int f d\mu \right\| \leq \left(2 \sum_{k=1}^{\infty} | b_k | \right) \rho^t \qquad t \in \mathbb{N}.$$

Proof. — *i*) For $i \in \mathbb{Z}$, set $g_i = (\operatorname{sgn} i) f_{|i|}$. Define transition probabilities for the chain $(\hat{\xi}_t)$ by

$$\hat{p}_{ij} = \begin{cases} \delta_{ij} & \text{if } \ | i | \leq 1 , \\ 1 - \rho_{|i|} & \text{if } \ | i | \geq 2 , \ \ j = 0 , \\ | r_{|i|\,|j|} | & \text{if } \ | i | \geq 2 , \ \ | j | \geq 1 , \\ 0 & \text{otherwise} . \end{cases}$$

Let $\hat{P}$ be the matrix operator on bounded sequences in $\mathbb{R}^{\mathbb{Z}}$ given by $(\hat{p}_{ij})$. Then (4) and the construction imply $P g_i(\xi) = \hat{P} g_i(\xi)$ for all i and ξ. By

Fubini we also have $\mathrm{P}\hat{\mathrm{P}}g_i(\xi) = \hat{\mathrm{P}}\,\mathrm{P}g_i(\xi)$. Since P and $\hat{\mathrm{P}}$ « commute », an easy induction yields

$$\mathrm{P}^t g_i(\xi) = \hat{\mathrm{P}}^t g_i(\xi) \qquad \text{for all } t, i, \xi, \tag{6}$$

which is precisely (5) when $i \geq 1$.

ii) Let $\tau = \min \{ t \in \mathbb{N} : |\hat{\xi}_t| \leq 1 \}$ ($\tau = \infty$ if no such t exists). When $\rho < 1$, $(\hat{\xi}_t)$ goes from any i such that $|i| \geq 2$ to state 0 with probability at least $1 - \rho > 0$ at each step. Thus $\hat{\mathbb{P}}_i(\tau > t) \leq \rho^t$, and $\tau < \infty$ a. s. Since $\xi_t = \xi_\tau$ for $t \geq \tau$, (6) can be rewritten as

$$\mathbb{E}_\xi[g_i(\xi_t)] = \hat{\mathbb{E}}_i[g_{\hat{\xi}_\tau}(\xi)] + \hat{\mathbb{E}}_i[g_{\hat{\xi}_t}(\xi) - g_{\hat{\xi}_\tau}(\xi)\,;\, \tau > t]\,.$$

Thus

$$|\,\mathbb{E}_\xi[g_i(\xi_t)] - \hat{\mathbb{E}}_i[g_{\hat{\xi}_\tau}(\xi)]\,| \leq 2\hat{\mathbb{P}}_i(\tau > t) \leq 2\rho^t, \tag{7}$$

which tends to 0 as $t \to \infty$. Since $g_{\hat{\xi}_\tau}$ is a constant function, $\mathrm{L}g_i = \lim_{t\to\infty} \mathrm{P}^t g_i(\xi)$ gives rise to a well-defined linear functional on $\mathscr{F}$, independent of the choice of ξ. Extend to $\mathscr{C}$ by approximation, and use Riesz representation to write $\mathrm{L}f = \int f d\mu$, $f \in \mathscr{C}$. The rest is routine, because of (7).

b) The continuous time case

Let $\mathrm{G} : \mathscr{F} \to \mathscr{C}$ be a « pregenerator » with some extension to a generator for a Markov process $(\mathbb{D}, \mathscr{B}, \{ \mathbb{P}_\xi \}_{\xi\in\Xi}, (\xi_t)_{t\in\mathbb{R}^+})$. Here $\mathbb{D}(\mathbb{R}^+, \Xi)$ is the path space of right continuous functions with left limits from $\mathbb{R}^+$ to Ξ, and $\mathscr{B}$ is the usual σ-algebra for this space. (ξ_t) is the coordinate process, and $\mathbb{P}_\xi$ governs it starting from ξ.

THEOREM 2 *b*. — Assume that for each $i \geq 1$,

$$\mathrm{G}f_i = \sum_{j=1}^{\infty} r_{ij} f_j \qquad r_{ij} \in \mathbb{R}, \tag{8}$$

with $r_{1j} \equiv 0$, and let $\lambda_i = -r_{ii} - \sum_{j\neq i} |r_{ij}|$. Suppose also that $\lambda_i \geq 0$ for all i, and define a continuous time Markov chain Q-matrix $(\hat{q}_{ij})_{i,j\in\mathbb{Z}}$ by

$$\hat{q}_{ij} = \begin{cases} 0 & \text{if } |i| \leq 1, \\ \lambda_i & \text{if } |i| \geq 2, \quad j = 0, \\ |r_{|i|\,|j|}| & \text{if } |i| \geq 2, \ |j| \geq 1, \quad \operatorname{sgn} ij = \operatorname{sgn} r_{|i|\,|j|}, \\ r_{|i|\,|i|} & \text{if } |i| \geq 2, \quad j = i, \\ 0 & \text{otherwise}. \end{cases}$$

Assume that the minimal semigroup with Q-matrix $\hat{G} = (\hat{q}_{ij})$ is conservative, let $\hat{\mathbb{P}}_i$ be the law for the minimal process starting at $i \in \mathbb{Z}$, and let $(\hat{\xi}_t)$ be the resulting chain. Then

i) For each $\xi \in \Xi$, $i \geq 1$ and $t \in \mathbb{R}^+$,

$$\mathbb{E}_\xi[f_i(\xi_t)] = \hat{\mathbb{E}}_i\,[(\text{sgn}\;\hat{\xi}_t)f_{|\hat{\xi}_t|}(\xi)]\,. \tag{9}$$

ii) If $\lambda = \inf_{i \geq 2} \lambda_i > 0$, then (ξ_t) is exponentially ergodic. In other words, the conclusions of Theorem 2 *a ii)* hold with ρ^t replaced by $e^{-\lambda t}$.

Proof. — *i)* With g_i as in Theorem 2 *a*, (8) becomes $Gg_i(\xi) = \hat{G}g_i(\xi)$, $i \in \mathbb{Z}$, $\xi \in \Xi$. Write $P^t g_i(\xi) = \mathbb{E}_\xi[g_i(\xi_t)]$, $\hat{P}^t g_i(\xi) = \hat{\mathbb{E}}_i[g_{\hat{\xi}_t}(\xi)]$. Then

$$\frac{dP^t g_i(\xi)}{dt} = P^t G g_i(\xi) = P^t \hat{G} g_i(\xi) = \hat{G} P^t g_i(\xi)\,.$$

So $P^t g_i(\xi)$ solves the same « backward equation » as $\hat{P}^t g_i(\xi)$, with the same initial condition $P^0 g_i(\xi) = \hat{P}^0 g_i(\xi) = g_i(\xi)$. This implies (9) because the solution is unique when the minimal semigroup for G is conservative. For more details in a representative case, see [*6*].

ii) Let $\tau = \min \{ t \in \mathbb{R}^+ : |\hat{\xi}_t| \leq 1 \}$. From any i such that $|i| \geq 2$ we go to 0 with rate at least λ, so $\hat{\mathbb{P}}_i(\tau > t) \leq e^{-\lambda t}$. $\tau < \infty$ a. s. since $\lambda > 0$, and the proof is completed just as in the discrete case.

Remarks. — Geometric/exponential ergodicity clearly obtains if there is a positive minimal probability/rate that the dual process goes to $\{ -1, 0, 1 \}$, but for the application we have in mind one needs the stronger assumptions of the Theorem. In this paper duality will only be applied to spin systems. The same general method, however, can be used to study certain diffusions on the d-dimensional torus, for example. See [*6*].

Turning to spin systems, where $\Xi = \{ -1, 1 \}^{\mathbb{Z}^d}$, we now sketch the duality theory of [*5*] and [*12*]. Let $\mathscr{F}$ consist of all functions depending on only finitely many sites in $\mathbb{Z}^d$. We want to choose a countable « basis » $\{ f_i \}$ such that $\overline{\text{span}}\,\{ f_i \} = \overline{\mathscr{F}}$. Since $\mathscr{F}$ is dense in $\mathscr{C}$, the duality theory will then apply. The most useful choices are the « multiplicative bases » (cf. [*5*] [*12*]), consisting of the functions

$$f_x(\xi) = \frac{\xi(x) + \alpha_x}{1 + |\alpha_x|} \qquad x \in \mathbb{Z}^d\,, \tag{10}$$

and all finite products of these over distinct sites x, for some fixed sequence $\alpha = (\alpha_x)_{x \in \mathbb{Z}^d} \subset [-1, 1]$. The empty product is $f_1 = 1$, and the remaining products are indexed by $\{ i \geq 2 \}$ with the aid of a one-to-one correspondence. We will be concerned almost exclusively with the simplest case, where $\alpha_x \equiv 0$. When $T = \mathbb{N}$, and $p(\xi, d\eta)$ is given by (1) and (2), one easily checks that P takes $\mathscr{C}$ into $\mathscr{C}$. If x_k are distinct sites of $\mathbb{Z}^d$ and f_{x_k} are of the form (10), then

$$\text{(11)} \qquad P\left(\prod_k f_{x_k}\right) = \prod_k P f_{x_k}.$$

(Equation (11) simply expresses the conditional independence of the one-step local transitions.) When $T = \mathbb{R}^+$ we have yet to define precisely the spin system (ξ_t) with flip rates c. This is done by identifying its pre-generator G:

$$\text{(12)} \qquad Gf(\xi) = \sum_{x \in \mathbb{Z}^d} c_x(\xi)[f({}_x\xi) - f(\xi)],$$

where ${}_x\xi$ is « ξ flipped at x » $\left(\text{i. e. } {}_x\xi(y) = \begin{cases} \xi(y) & y \neq x \\ -\xi(y) & y = x \end{cases}\right)$. In order that G takes $\mathscr{F}$ into $\mathscr{C}$, one assumes that $c_x(\xi) \in \mathscr{C}$ for each fixed x. A sufficient condition for uniqueness of the Markov generator extending G is

$$\text{(13)} \qquad \sup_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} \sup_{\xi \in \Xi} | c_x({}_y\xi) - c_x(\xi) | < \infty$$

(cf. [7] [4] [1]). Condition (13) implies that each $c_x(\cdot) \in \mathscr{C}$, and is easy to check whenever the flip rates have the form (3). In fact $\sum_y | r_y | \leq 1$ because $c_x \geq 0$, while the quantity $| c_x({}_y\xi) - c_x(\xi) |$ is evidently dominated by $\frac{\kappa}{2} | r_{y-x} |$ when $y \neq x$ and κ when $y = x$. Thus, the spin system (ξ_t) is the unique strong Feller process whose generator extends G as given by (3) and (12). The continuous time analogue of (11) is

$$\text{(14)} \qquad G\left(\prod_k f_{x_k}\right) = \sum_k \left(\prod_l f_{x_l}\right) G f_{x_k},$$

where the second product is over all $l \neq k$. Now, it follows easily from (11) and (14) that the conditions of Theorems 2 *a ii*) and 2 *b ii*) need only be checked for the one site functions f_x — they are automatically inherited by products. This is the key property of the multiplicative bases for spin systems. In continuous time, the requirement that $\hat{G}$ uniquely determine $\{ \hat{\mathbb{P}}_i \}$ imposes an extra regularity condition on general spin systems (see (2.5) in [5]). But again, when the flip rates have the simple form (3) this conditions is always satisfied.

Next, for $T = \mathbb{N}$, note that the inequalities $0 \leq p_x(\xi, 1) \leq 1$ force $|a| + \sum_y |r_y| \leq 1$ if we choose ξ appropriately in (2). Taking $\alpha_x \equiv 0$ we have

$$(15) \qquad \mathrm{P}f_x(\xi) = 2p_x(\xi, 1) - 1 = a + \sum_{y \in \mathbb{Z}^d} r_y f_{x+y}(\xi).$$

Together with the discussion of the previous paragraph, (15) shows that any of the discrete time processes (ξ_t) in the class we are considering satisfies condition *i)* of Theorem 2 *a* with respect to the $\alpha = 0$ multiplicative basis. Similarly, if $T = \mathbb{R}^+$, and the flip rates $c_x(\xi) \geq 0$ satisfy (3), then $|a| + \Sigma |r_y| \leq 1$. Hence, again with $\alpha = 0$,

$$(16) \quad \mathrm{G}f_x(\xi) = -2c_x(\xi)\xi(x) = -\kappa f_x(\xi) + \kappa\left[a + \sum_{y \in \mathbb{Z} - 0} r_y f_{x+y}(\xi)\right].$$

so condition *i)* of Theorem 2 *b* is automatic. In other words, all of the spin systems we are considering, with $T = \mathbb{N}$ or $\mathbb{R}^+$, have $\alpha = 0$ dual processes. Until further notice, the only dual processes considered will be $\alpha = 0$ duals.

We are already in a position to dispense with case *i)* of Theorems 1 *a* and 1 *b*. From (15) and (16) it is evident that these are precisely the situations in which Theorems 2 *a ii)* and 2 *b ii)* apply. Note that for these processes convergence to equilibrium is geometric/exponential. From now on, interest will center around the « critical cases »: $|a| + \Sigma |r_y| = 1$. In these cases we have to look more carefully at the dual process, but it turns out that the following condition for ergodicity is actually necessary and sufficient over the classes (2) and (3). We formulate it in the general setting of Theorem 2 (*a* or *b*).

LEMMA (Holley-Stroock). — If condition *i)* of Theorem 2 holds, and

$$(17) \quad \lim_{t \to \infty} \sum_{j \geq 2} | \hat{\mathbb{P}}_i(\hat{\xi}_t = j) - \hat{\mathbb{P}}_i(\hat{\xi}_t = -j) | = 0 \qquad \text{for all } i \geq 2,$$

then (ξ_t) is ergodic.

Proof. — By (5) and (9),

$$P^t f_i(\xi) = \hat{\mathbb{P}}_i(\hat{\xi}_t = 1) - \hat{\mathbb{P}}_i(\hat{\xi}_t = -1) + \sum_{j \geq 2} [\hat{\mathbb{P}}_i(\hat{\xi}_t = j) - \hat{\mathbb{P}}_i(\hat{\xi}_t = -j)] f_j(\xi) .$$

Under (17), with τ as in Theorem 2,

$$\lim_{t \to \infty} P^t f_i(\xi) = \hat{\mathbb{P}}_i(\tau < \infty, \hat{\xi}_\tau = 1) - \hat{\mathbb{P}}_i(\tau < \infty, \hat{\xi}_\tau = -1),$$

a constant independent of ξ. This implies ergodicity of (ξ_t) as before.

We will use the Lemma in the next two sections to handle the critical cases of Theorem 1.

Remarks. — Theorem 2 *a ii*) was proved by Vasershtein and Leontovich [*12*] for spin systems with the $\alpha = 1$ multiplicative basis, and stated in greater generality. Dual processes are not explicit in their arguments; they simply manipulate the relevant equations. Continuous spin system duality with $\alpha = 1$ and $r_{ij} \geq 0$ was studied by Holley and Liggett [*3*], and Harris [*2*]. The idea of introducing $g_{-i} = - f_i$ to handle $r_{ij} < 0$ was devised by Holley and Stroock [*5*], who identified the general α-multiplicative bases for continuous time systems, and gave numerous applications. Theorem 2 *b* for spin systems is due to them. The Lemma above is based on Corollary 3.13 in [*5*]; they note later in their paper that (17) is *not* necessary for ergodicity of arbitrary spin systems. For a related class of systems, Schwartz [*9*] used the state « 0 » (= Δ) to handle strict contractions.

3. PROOF OF THEOREM 1*a*

Throughout this section $T = \mathbb{N}$ and (ξ_t) is a spin system with local transition probabilities (3). As already explained, we need only consider the cases where $|a| + \Sigma |r_y| = 1$, and $\mathbb{G} = \text{gr}\{ y : r_y \neq 0 \} = \mathbb{Z}^d$. These critical case systems may be interpreted as « voter models » — The spin value at each site represents the position of that site's occupant on some issue ($+1$ = for, -1 = against). At each transition, the person at site x chooses a site $x + y$ with probability $\pi_y = |r_y|$. If $y \in S^+$ is chosen, then the position of $x + y$ at time t is adopted by x at time $t + 1$; if $y \in S^-$, then the *opposite* of the position of $x + y$ is chosen. When $a \neq 0$, the position at x becomes sgn a with the remaining probability $|a|$.

To begin, consider the ($\alpha = 0$) dual $(\hat{\xi}_t)$. In the last section we thought of this process as living on $\mathbb{Z}$, by means of an arbitrary bijection of the non-empty products of f_{x_k}'s with $\{ i \geq 2 \}$. For an in-depth analysis, a much more natural identification is $\prod_{k=1} f_{x_k} \sim \{ x_1, \ldots, x_n \}$. Under this scheme, the state space for $(\hat{\xi}_t)$ consists of the finite subsets A of $\mathbb{Z}^d$ and « negatives » of these, which we denote by $\bar{A}$. The points 1 and -1 in the proof of Theorem 2 correspond to $\emptyset$ and $\bar{\emptyset}$ respectively, and if $i \geq 2$ corresponds to A, then $-i$ corresponds to $\bar{A}$. There is no need to retain a point corresponding to 0, since $\rho_i \equiv 1$. Let $\hat{\Xi}^+ = \{ \text{finite } A \subset \mathbb{Z}^d \}$, $\hat{\Xi}^- = \{ \bar{A} : A \in \hat{\Xi}^+ \}$, $\hat{\Xi} = \hat{\Xi}^+ \cup \hat{\Xi}^-$. From now one we take $\hat{\Xi}$ as the state space for $(\hat{\xi}_t)$. For $A \in \hat{\Xi}^+$, let $|A| = A$; if $B = \bar{A} \in \hat{\Xi}^-$, put $|B| = A$. Also, set $\operatorname{sgn} B = \begin{cases} 1 & B \in \hat{\Xi}^+ \\ -1 & B \in \hat{\Xi}^- \end{cases}$. Finally, denote $\# B =$ the cardinality of $|B|$, $B \in \hat{\Xi}$. In order to identify the transition mechanism of the dual, it suffices to combine equations (11) and (15). If $a = 0$, one easily verifies that starting from $B \in \hat{\Xi}$, $(|\hat{\xi}_t|)$ is $\# B$ particles performing independent random walks on $\mathbb{Z}^d$ with common displacement measure π, one walk starting from each site of $|B|$, but with the following « collision rule »: whenever an even number of particles attempt to occupy the same site at time t they all disappear; when an odd number try to occupy a site then one remains and the rest are removed. (This follows from the fact that $[\xi(y)]^{2n} = 1$, while $[\xi(y)]^{2n+1} = \xi(y)$.) If $a \neq 0$, each particle simply disappears with the remaining probability $1 - |a|$. The sgn $\hat{\xi}_t$ are given by

$$\operatorname{sgn} \hat{\xi}_{t+1} = (-1)^{N_{t+1}} \operatorname{sgn} \hat{\xi}_t, \tag{18}$$

where N_{t+1} is the number of particles in the dual which attempt displacements in S^- from time t to time $t+1$ (*or* simply disappear when $a < 0$). Together, $(|\hat{\xi}_t|)$ and $(\operatorname{sgn} \hat{\xi}_t)$ totally determine $(\hat{\xi}_t)$.

The proof of ergodicity under *ii*) or *iii*) of Theorem 1 *a* is based on comparison of $(\hat{\xi}_t)$ with processes $(\hat{\xi}_t^s)$, $s \in \mathbb{N}$, which ignore the collision rule after time s. Thus $\hat{\mathbb{P}}_A^s$ denotes the (nonhomogeneous) Markov measure which behaves exactly like $\hat{\mathbb{P}}_A$ *through time s*, and thereafter prescribes independent transitions for $|\hat{\xi}_t^s|$. In other words, when two or more particles of the new process attempt to occupy the same site after time s, they are allowed to do so. Define sgn $\hat{\xi}_t^s$ as before for all t, i. e. by (18). Naturally we must enlarge the state space and path space to allow for multiple (but finite) occupancy of sites. Assume that this has been done, and let $\check{\Xi}$, $\check{\Xi}^+$ be the respective extensions of $\hat{\Xi}$, $\hat{\Xi}^+$. Abbreviate $\hat{p}_{AB}(t) = \hat{\mathbb{P}}_A(\hat{\xi}_t = B)$, $\hat{p}_{AB}^s(t) = \hat{\mathbb{P}}_A^s(\hat{\xi}_t^s = B)$. According to (17) we want to show

$$\lim_{t \to \infty} \sum_{\substack{B \in \hat{\Xi}^+ \\ B \neq \bar{\emptyset}}} |\hat{p}_{AB}(t) - \hat{p}_{A\bar{B}}(t)| = 0 \qquad A \in \hat{\Xi}^+,\ A \neq \emptyset. \tag{19}$$

The sum may be extended to $B \in \check{\Xi}^+$, and majorized by

$$\sum_{\substack{B\in\check{\Xi} \\ B\neq\emptyset,\bar{\emptyset}}} |\, \hat{p}_{AB}(t) - \hat{p}^s_{AB}(t)\,| + \sum_{\substack{B\in\check{\Xi}^+ \\ B\neq\emptyset}} |\, \hat{p}^s_{AB}(t) - \hat{p}^s_{AB}(t)\,|$$
$$= \qquad \Sigma^s_1(t) \qquad + \qquad \Sigma^s_2(t).$$

To estimate $\Sigma^s_1(t)$ we use the « fundamental coupling inequality »: if X_1 and X_2 are (general) random variables governed by a common probability measure Pr, and if μ_1 and μ_2 are their respective laws, then

$$\|\, \mu_1 - \mu_2 \,\| \leq \Pr(X_1 \neq X_2).$$

Paste $(\hat{\xi}_t)$ and $(\hat{\xi}^s_t)$ together until time s, then let them run independently. The conclusion is that for $t > s$, $\Sigma^s_1(t) \leq 2\,\|\, \hat{\mathbb{P}}_A(\hat{\xi}_t \in \cdot) - \hat{\mathbb{P}}^s_A(\hat{\xi}_t \in \cdot)\,\| \leq 2\hat{\mathbb{P}}_A$ (the dual has a collision between times s and t) $\leq 2\hat{\mathbb{P}}_A$ (the dual has a collision after s). So $\lim_{s\to\infty} \sup_{t\geq s} \Sigma^s_1(t) = 0$, because $\hat{\xi}_0$ has finitely many particles at least one of which disappears at each collision. Next, apply the Markov property at time s to $\Sigma^s_2(t)$. It follows that (19) holds if

$$\text{(20)} \qquad \lim_{t\to\infty} \sum_{\substack{B\in\check{\Xi}^+ \\ B\neq\emptyset}} |\, \hat{p}^0_{AB}(t) - \hat{p}^0_{A\bar{B}}(t)\,| = 0 \qquad A \in \hat{\Xi}^+,\ A \neq \emptyset,$$

i. e. if the analogue of (19) for the totally independent process is valid. To treat case *ii*), simply note that $(\#\ \hat{\xi}^0_t)$ is a Galton-Watson process with mean offspring per particle equal to $1 - |\,a\,| < 1$. Hence $(\hat{\xi}^0_t)$ is absorbed at $\emptyset$ or $\bar{\emptyset}$, at a geometric rate. Thus (20) holds, and so $(\hat{\xi}_t)$ is geometrically ergodic. In case *iii*), $\#\ \hat{\xi}^0_t = \#\ \hat{\xi}^0_0$ for all t. From (18),

$$\hat{\mathbb{P}}^0_A(\hat{\xi}^0_t = B) = \hat{\mathbb{P}}^0_A(\{\,|\,\hat{\xi}^0_t\,| = |\,B\,|\,\} \cap E_e),$$

where E_e is the event that the total number of displacements in S^- through time t by the A particles is even. Similarly,

$$\hat{\mathbb{P}}^0_A(\hat{\xi}^0_t = \bar{B}) = \hat{\mathbb{P}}^0_A(\{\,|\,\hat{\xi}_t\,| = |\,B\,|\,\} \cap E_o),$$

E_o denoting an odd number of displacements in S^-. We conclude that if $A = \{\, x_1, \ldots, x_l \,\} \in \hat{\Xi}$, $\sigma = (\sigma_1, \ldots, \sigma_l)$ is a generic permutation of $\{\, 1, \ldots, l\,\}$, and $y_1, \ldots, y_l$ run over $\mathbb{Z}^d$, then the sum in (20) is majorized by

$$\sum_{y_1,\ldots,y_l} |\, \Sigma[\hat{p}_{x_1 y_{\sigma_1}}(t) + \hat{p}_{x_1 \bar{y}_{\sigma_1}}(t)] \cdot \ldots \cdot [\hat{p}_{x_{l-1} y_{\sigma_{l-1}}}(t) + \hat{p}_{x_{l-1} \bar{y}_{\sigma_{l-1}}}(t)]$$
$$\cdot [\hat{p}_{x_l y_{\sigma_l}}(t) - \hat{p}_{x_l \bar{y}_{\sigma_l}}(t)]\,| \leq l! \sum_{y\in\mathbb{Z}^d} |\, \hat{p}_{0y}(t) - \hat{p}_{0\bar{y}}(t)\,|\,.$$

But $\hat{p}_{0\bar{y}}(t) = \hat{p}_{\bar{0}y}(t)$ for all y. Hence the proof of part *iii)* of Theorem 1 *a* will be complete once we show that

$$\lim_{t\to\infty} \sum_{y\in\mathbb{Z}^d} | \hat{p}_{0y}(t) - \hat{p}_{\bar{0}y}(t) | = 0 . \tag{21}$$

Consider $\mathbb{A} = \mathbb{Z}^d \times \{ 0, 1 \}$ as an additive group with addition mod 2 in the second component. Let $S = \{ (y, 0), y \in S^+ \} \cup \{ (y, 1), y \in S^- \}$, and put $\mathbb{B} = \text{gr}(S - S)$, the group generated by differences of elements of S. The condition in *iii)* of Theorem 1*a* says simply that $(0, 1) \in \mathbb{B}$. It is well known that

$$|| \mathbb{P}_x(\xi_t \in \cdot) - \mathbb{P}_y(\xi_t \in \cdot) || \to 0 \quad \text{as} \quad t \to \infty \qquad \text{(variation norm)}$$

for any x, y in the same cyclic subclass of a discrete time random walk on a countable abelian group. But $(0, 1) \in \mathbb{B}$ says that 0 and $\bar{0}$ belong to the same class in $\mathbb{A}$, so (21) holds.

To finish the proof of Theorem 1*a*, it remains only to show nonergodicity in the remaining cases. Suppose now that $(0, 1) \notin \mathbb{B}$ so that $\mathbb{B}$ and $(0, 1) + \mathbb{B}$ are distinct cosets of $\mathbb{A}/\mathbb{B}$. Choose $y_i \in \mathbb{Z}^d$, $1 \leq i \leq \iota - 2$, so that $(y_i, \varepsilon_i) + \mathbb{B}$, $\varepsilon_i = 0$ or 1, are the remaining cosets ($2 \leq \iota = \text{card } \mathbb{A}/\mathbb{B} < \infty$). To any $y \in \mathbb{Z}^d$ there corresponds a unique $\varepsilon_y \in \{ 0, 1 \}$ such that $(y, \varepsilon_y) = (y_i, 0) + \beta$ for some y_i and $\beta \in \mathbb{B}$. The uniqueness follows from the fact that $(0, 1) \notin \mathbb{B}$. Define a configuration ξ^+ by

$$\xi^+(y) = \begin{cases} +1 & \text{if} \quad \varepsilon_y = 0 \\ -1 & \text{if} \quad \varepsilon_y = 1 , \end{cases} \tag{22}$$

and let $\xi^- = -\xi^+$. If $| a | = 0$, $| r_y | = 1$, and $(0, 1) \notin \mathbb{B}$, then when the process (ξ_t) starts in ξ^+ it will move deterministically through a cycle of states (possibly of length 1). Starting from ξ^-, at each time t the system will be in the « negative » of the state reached from ξ^+. This shows that (ξ_t) is not ergodic. The details are left to the reader.

4. PROOF OF THEOREM 1*b* (AN OUTLINE)

Apart from technicalities the continuous time version of Theorem 1 is simpler that the discrete case, due to the absence of time periodicity. Assume now that $T = \mathbb{R}^+$, and that (ξ_t) has flip rates (3). Since ergodicity is unaffected by constant change of time scale, it suffices to set $\kappa = 1$. We consider the cases where $| a | + \Sigma | r_y | = 1$ (and, as always, $\mathbb{G} = \mathbb{Z}^d$). These may be thought of as continuous time voter models: the rate at which an individual at x changes opinion is a linear combination of people's positions, those in $x + S^+$ « accepted », those in $x + S^-$ « rejected ». When $a = 0$, the cases $S^- = \emptyset$, and $S^+ = \emptyset$, which have been studied in great detail ([*3*] [*5*] [*8*]), are sometimes called the « voter model » and « anti-voter model » respectively.

The continuous time ($\alpha = 0$) dual $(\hat{\xi}_t)$ is described as follows (cf. [5]). If $a = 0$, then starting from $B \in \hat{\Xi}$, $(|\hat{\xi}_t|)$ is $\# B$ particles performing continuous time random walks on $\mathbb{Z}^d$ with common displacement rates $\pi_y = |r_y|$. Only one particle attempts a jump at any time t. If the new site is unoccupied the jump takes place, but if there is another particle there then both disappear. If $a \neq 0$, each particle simply disappears at rate $|a|$. (sgn $\hat{\xi}_t$) changes sign every time a displacement in S^- is attempted, and also when a particle simply disappears if $a < 0$.

The discrete time proof of case *ii*), and case *iii*) up to (21), are easily translated to $T = \mathbb{R}^+$. Now consider the group $\mathbb{A}$ and collection $S \subset \mathbb{A}$ introduced at the end of the last section. Let $\mathbb{C} = \text{gr}(S)$. The condition in *iii*) of Theorem 1 *b* states that $(0, 1) \in \mathbb{C}$. This implies (21) for the continuous time random walk on $\mathbb{A}$ performed by the one particle dual.

Finally, suppose that $(0, 1) \notin \mathbb{C}$. Then evidently any $\gamma \in \mathbb{C}$ has a unique representation as $\gamma = (y, \varepsilon_y)$, $\varepsilon_y \in \{0, 1\}$. Define ξ^+ in terms of these ε_y, as in (22). Write $\xi^- = -\xi^+$. When $\Sigma |r_y| = 1$ and $(0, 1) \notin \mathbb{C}$, it is easy to check that ξ^+ and ξ^- are traps for (ξ_t). Thus (ξ_t) is not ergodic.

Remarks. — Suppose $\Sigma |r_y| = 1$. If $S^- = \emptyset$ (« voter model ») the configurations « all + 1's » and « all − 1's » are traps, so (ξ_t) is never ergodic. If $S^+ = \emptyset$ (« anti-voter model »), then the condition *iii*) of Theorem 1 *b* is equivalent to even period for π, i. e. card $\mathbb{Z}^d/\text{gr}(S^- - S^-) =$ an even positive integer. These two models have been analysed in the nonhomogeneous case, where the one particle dual is a Markov chain. One can obtain a detailed description of the class of invariant measures for (ξ_t) when the system is not ergodic. See [*3*] and [*8*] for details. Conceivably a similar analysis could be carried out for the general class of processes considered here. A recent paper by Schwartz [*10*] contains several additional applications of duality to spin systems, e. g. to certain « biased » voter models which are not of the form (3). Obvious generalizations of Theorem 1*a* and 1*b* hold when $\mathbb{Z}^d$ is replaced by any countable abelian group. Our proof goes through virtually without change.

5. SOME OPEN PROBLEMS

We conclude the discussion by mentioning a few open problems on spin systems which illustrate the seeming limitations of the duality method.

PROBLEM 1. — « Majority voting » (cf. [*13*]). T = $\mathbb{N}$. For some $\theta \in \left[0, \frac{1}{2}\right]$,

(23) $$p_x(\xi, 1) = \frac{1}{2}(1 + \theta[\xi(x-1) + \xi(x) + \xi(x+1) - \xi(x-1)\xi(x)\xi(x+1)]) .$$

Theorem 2 *a ii*) applies with the $\alpha = 0$ multiplicative basis if $\theta < \frac{1}{4}$. It is conjectured that the process given by (23) is ergodic whenever $\theta > 0$. Indeed, one of the leading unsolved questions about spin systems, in discrete *or* continuous time, is the much more general « positive probabilities/rates problem »: Is every homogeneous spin system on $\mathbb{Z}$ with finite range interactions and strictly positive local transition probabilities/rates ergodic? It has been widely conjectured that the answer is yes. Even the simplest cases of the general positive probabilities/rates problem seem beyond the scope of duality. Sticking to discrete time for the moment, suppose that for some reals *a*, *b*, *c*, *d*,

$$0 < p_x(\xi, 1) = \frac{1}{2}[1 + a + b\xi(x) + c\xi(x+1) + d\xi(x)\xi(x+1)] < 1 \qquad x \in \mathbb{Z},\ \xi \in \Xi,$$

i. e. (ξ_t) is a homogeneous linear system with strictly positive one-step transition kernels, p_x depending only on the spins at x and $x + 1$. Computer simulations of these systems ([*14*]) support the ergodicity claim. If one chooses the best α-multiplicative basis, by far the greater portion of the 4-dimensional region parameterized by (a, b, c, d) can be shown to correspond to ergodic systems. But only 8 of the 16 possible cases of sign for *a*, *b*, *c* and *d* can be checked completely. One elusive example is

PROBLEM 2. — T = $\mathbb{N}$. For some $\theta \in \left[0, \frac{1}{2}\right]$,

(24) $$p_x(\xi, 1) = \frac{1}{2}(1 - \theta[1 + \xi(x) + \xi(x+1) - \xi(x)\xi(x+1)]) .$$

Again, $\alpha = 0$ duality applies if $\theta < \frac{1}{4}$. However, no other θ gives more information. Is the system (24) ergodic whenever $\theta < \frac{1}{2}$?

The continuous time « right neighbor » homogeneous flip rates are of the form

$$c_x(\xi) = 1 + a\xi(x) + b\xi(x+1) + c\xi(x)\xi(x+1) > 0 \qquad x \in \mathbb{Z},\ \xi \in \Xi$$

(up to a positive constant κ). These rates are approximated by discrete time systems $(\xi^{\varepsilon}_{t\varepsilon})$, where

$$p^{\varepsilon}_x(\xi, 1) = \frac{1}{2}(1 - 2a\varepsilon + [1 - 2\varepsilon]\xi(x) - 2c\varepsilon\xi(x + 1) - 2b\varepsilon\xi(x)\xi(x + 1))$$

and ε is small. Thus the continuous time problem resides in a 3-dimensional « corner » of the 4-dimensional discrete time problem. The fact that there is one less dimension allows Holley and Stroock [*5*] to handle 6 of the 8 cases of sign for a, b and c. In all these instances the positive rates conjecture is confirmed. But the remaining two cases cannot be handled completely; for instance,

PROBLEM 3. — $T = \mathbb{R}^+$. For some $\theta \in [0, 1]$,

$$c_x(\xi) = 1 + \theta[\xi(x) + \xi(x + 1) + \xi(x)\xi(x + 1)]. \quad (25)$$

As noted in [*5*], multiplicative bases only yield ergodicity for the spin system with rates (25) if $\theta < 3/7$. Is this system ergodic whenever $\theta < 1$? (*).

One might ask whether better results can be obtained by replacing the α-multiplicative basis with a basis which contains functions f_i depending on more than one site. In a few specific cases this is manageable, but by and large, any more complicated bases rapidly become unwieldy.

In conclusion, it should be noted that the positive probabilities/rates conjecture is false when $\mathbb{Z}$ is replaced by $\mathbb{Z}^d$, $d > 1$. Nonergodicity examples with $d = 2$ are provided by the work of Toom [*11*] when $T = \mathbb{N}$, and by the stochastic Ising model (cf. [*4*]) when $T = \mathbb{R}^+$.

(*) *Added in proof :* HOLLEY and LIGGETT have now answered this question in the affirmative (private communication). For general positive a, b and c the problem remains open.

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