ANNALES DE L'I. H. P., SECTION B

E. B. DAVIES

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Annales de l'I. H. P., section B, tome 11, nº 3 (1975), p. 265-273

<http://www.numdam.org/item?id=AIHPB_1975__11_3_265_0>

Ann. Inst. Henri Poincaré, Section B :
Vol. XI, n° 3, 1975, p. 265-273. *Calcul des Probabilités et Statistique.*

Markovian master equations. III

by

E. B. DAVIES

St. John's College,
Oxford, England

ABSTRACT. — We discuss the relationship between two recent series of papers concerning the form in the weak coupling limit of the solutions of certain evolution equations in a Banach space.

RÉSUMÉ. — Nous discutons la relation entre deux séries récentes d'articles concernant la forme dans une limite faible des solutions de certaines équations d'évolution dans une espace de Banach.

1. INTRODUCTION

In the last three years two series of papers have appeared studying Banach space evolution equations of the form

$$\frac{df}{dt} = \{ Z + \lambda A(t) \} f(t) \tag{1.1}$$

in the weak coupling limit, $\lambda \to 0$. While the technical hypotheses of these papers are very different, the conclusions are so similar that an attempt to find a common core is clearly required.

The first series [*1*]-[*5*] [*10*], which we shall call the « abstract » approach, takes such an evolution equation on a Banach space $\mathscr{B}$ with a projection P and studies the form of $Pf(t)$ in the limit $\lambda \to 0$, where $f(t)$ is a solution of the evolution equation and $Pf(0) = f(0)$. The second series [6]-[9], which we shall call the « stochastic » approach, studies the evolution in a Banach space $\mathscr{B}_0$ under the assumption that $A(t)$ is a random time-dependent operator, and studies the form of the expected value of $f(t)$ in the limit $\lambda \to 0$. We list some of the significant ways in which one approach goes beyond the other.

(1) The abstract approach has only considered the case where $A(t)$ is independent of t, while the stochastic approach has only considered the case where the free evolution is trivial, that is $Z = 0$.

(2) The stochastic approach has allowed unbounded operators $A(t)$ while the abstract approach has only been carried out for bounded $A(t)$.

(3) The stochastic approach uses probabilistic notions and so is more particular than the abstract approach.

(4) The stochastic approach assumes that the mean value of the operator $A(t)$ vanishes, while this condition is not necessary to the abstract approach.

In Section 2 of this paper we shall outline an extension of the abstract approach which includes the essential properties of the stochastic approach. We do not, however, allow $A(t)$ to be unbounded since this necessitates further complications, such as the introduction of a scale of Banach spaces [9]. In Section 3 we make more precise how the theorems of the stochastic approach may be extracted, concluding with a comparison of our estimates with those in [9]. Section 4 is devoted to the study of an alternative abstract version of the stochastic approach [7].

2. THE GENERAL THEORY

We let P be a projection of norm one on a Banach space $\mathscr{B}$ and put $\mathscr{B}_0 = P\mathscr{B}$ and $\mathscr{B}_1 = (1 - P)\mathscr{B}$ so that $\mathscr{B} = \mathscr{B}_0 \oplus \mathscr{B}_1$.

We let $A(t)$ be a strongly continuous bounded operator valued function on $\mathscr{B}$ and wish to consider the evolution equation

$$f'(t) = \lambda A(t) f(t) \tag{2.1}$$

which is equivalent to the integral equation

$$f(t) = f(0) + \lambda \int_{s=0}^{t} A(s) f(s) ds . \tag{2.2}$$

We suppose that

$$PA(t)P = 0 \tag{2.3}$$

for all $t \geqslant 0$ and put

$$B(t) = (1 - P)A(t)(1 - P). \tag{2.4}$$

Then we consider the operator equation in $\mathscr{B}$

$$\frac{\partial}{\partial t} V_\lambda(t, s)\phi = \lambda B(t) V_\lambda(t, s)\phi \tag{2.5}$$

where $\phi \in \mathscr{B}$ and $0 \leqslant s \leqslant t < \infty$, or the equivalent integral equation

$$V_\lambda(t, s) = 1 + \lambda \int_{u=s}^{t} B(u)V_\lambda(u, s)ds. \tag{2.6}$$

It is easy to show that

$$PV_\lambda(t, s) = V_\lambda(t, s)P = P. \tag{2.7}$$

The equation (2.2) may now be replaced by the equivalent

$$\begin{aligned} f(t) &= V_\lambda(t, 0)f(0) \\ &+ \lambda \int_{s=0}^{t} V_\lambda(t, s)\{ PA(s)(1 - P) + (1 - P)A(s)P \} f(s)ds. \end{aligned} \tag{2.8}$$

Then if $Pf(0) = f(0) = a \in \mathscr{B}_0$

$$Pf(t) = a + \lambda \int_{s=0}^{t} PA(s)(1 - P)f(s)ds \tag{2.9}$$

and

$$(1 - P)f(s) = \lambda \int_{u=0}^{s} V_\lambda(s, u)(1 - P)A(u)Pf(u)du. \tag{2.10}$$

Putting

$$x_\lambda(\lambda^2 t) = Pf(t) \tag{2.11}$$

and using (2.3) we obtain the integral equation in $\mathscr{B}_0$,

$$x_\lambda(\lambda^2 t) = a + \lambda^2 \int_{s=0}^{t} \int_{u=0}^{s} PA(s)V_\lambda(s, u)A(u)Px_\lambda(\lambda^2 u)duds \tag{2.12}$$

upon which all our subsequent calculations are based. Putting $\lambda^2 t = \tau$ and $\lambda^2 u = \sigma$ this may be rewritten as

$$x_\lambda(\tau) = a + \int_{\sigma=0}^{\tau} K_\lambda(\tau, \sigma)x_\lambda(\sigma)d\sigma \tag{2.13}$$

where

$$K_\lambda(\tau, \alpha) = \int_{s=\lambda^{-2}\sigma}^{\lambda^{-2}\tau} PA(s)V_\lambda(s, u)A(u)Pds. \tag{2.14}$$

This is an operator valued Volterra integral equation which we rewrite as

$$x_\lambda = a + \tilde{K}_\lambda x_\lambda \tag{2.15}$$

on the Banach space

$$X = C\{[0, \tau_0], \mathscr{B}_0\} \tag{2.16}$$

of continuous $\mathscr{B}_0$-valued functions on $[0, \tau_0]$, where τ_0 is an arbitrary finite number.

THEOREM 2.1. — *Suppose that for all* $|\lambda| \leqslant 1$ *and all* $0 \leqslant \sigma \leqslant \tau \leqslant \tau_0$

$$\| K_\lambda(\tau, \sigma) \| \leqslant C. \tag{2.17}$$

Suppose also that the operators $\tilde{K}_\lambda$ *on* X *converge strongly to the Volterra operator* $\tilde{K}_0$ *on* X *with constant kernel* K_0. *Then*

$$\lim_{\lambda \to 0} x_\lambda(\tau) = e^{K_0 \tau} a \tag{2.18}$$

uniformly for $0 \leqslant \tau \leqslant \tau_0$.

Proof. — This is a minor variation of corresponding results in [3] [4].

In order to investigate further the convergence of the operators $\tilde{K}_\lambda$ we expand

$$V_\lambda(s, u) = 1 + \lambda \int_{x_1 = u}^{s} B_{x_1} dx_1 + \lambda^2 \int_{x_2 = u}^{s} \int_{x_1 = u}^{x_2} B_{x_2} B_{x_1} dx_1 dx_2 + \dots \tag{2.19}$$

to write $K_\lambda(\tau, \sigma)$ as a sum of an infinite series

$$K_\lambda(\tau, \sigma) = \sum_{n=0}^{\infty} \lambda^n K_\lambda^{(n)}(\tau, \sigma) \tag{2.20}$$

where

$$K_\lambda^{(0)}(\tau, \sigma) = \int_{s = \sigma\lambda^{-2}}^{\tau\lambda^{-2}} PA_s A_{\sigma\lambda^{-2}} P ds \tag{2.21}$$

$$K_\lambda^{(1)}(\tau, \sigma) = \int_{s = \sigma\lambda^{-2}}^{\tau\lambda^{-2}} \int_{x = \sigma\lambda^{-2}}^{s} PA_s B_x A_{\sigma\lambda^{-2}} P dx ds \tag{2.22}$$

with similar expressions for larger n. Each of these integral kernels is estimated by a procedure adapted to the application [3] [4]. The estimates which are relevant to the stochastic application are the following.

THEOREM 2.2. — *Suppose that for all integers n and all*

$$0 \leqslant u \leqslant x_1 \leqslant \dots \leqslant x_n \leqslant s < \infty$$

$$\| PA_s B_{x_n} B_{x_{n-1}} \dots B_{x_1} A_u P \| \leqslant p(s - x_n) p(x_n - x_{n-1}) \dots p(x_1 - u) b^{n+2} \tag{2.23}$$

where

$$\int_0^\infty p(t)dt \equiv c < \infty . \tag{2.24}$$

Suppose also that there is a bounded operator K_0 *on* $\mathscr{B}_0$ *such that for all* $a \in \mathscr{B}_0$ *and all integers* n

$$\lim_{\alpha \to \infty} \frac{n+1}{\alpha^{n+1}} \int_{s=0}^{\alpha} \int_{u=0}^{s} u^n \mathrm{P} \mathrm{A}_s \mathrm{A}_u a du ds = \mathrm{K}_0 a . \tag{2.25}$$

Then the operators $\tilde{\mathrm{K}}_\lambda$ *on* X *converge strongly to* $\tilde{\mathrm{K}}_0$ *as* $\lambda \to 0$.

Proof. — We first show that

$$\lim_{\lambda \to 0} \left\| \sum_{n=1}^{\infty} \lambda^n \tilde{\mathrm{K}}_\lambda^{(n)} \right\| = 0 . \tag{2.26}$$

Since uniform convergence of the kernels implies norm convergence of the operators we examine

$$\left\| \sum_{n=1}^{\infty} \lambda^n \mathrm{K}_\lambda^{(n)}(\tau, \sigma) \right\|$$

$$\leqslant \sum_{n=1}^{\infty} |\lambda|^n \int_{s=\sigma\lambda^{-2}}^{\infty} \int_{x_n=\sigma\lambda^{-2}}^{s} \cdots \int_{x_1=\sigma\lambda^{-2}}^{x_2} \| \mathrm{P}\mathrm{A}_s \mathrm{B}_{x_n} \ldots \mathrm{B}_{x_1} \mathrm{A}_{\sigma\lambda^{-2}} \mathrm{P} \| \, dx_1 \ldots dx_n ds$$

$$\leqslant \sum_{n=1}^{\infty} |\lambda|^n b^{n+2} \int_{s=0}^{\infty} \cdots \int_{x_1=0}^{x_2} p(s - x_n) p(x_n - x_{n-1}) \ldots p(x_1) dx_1 \ldots dx_n ds$$

$$\leqslant \sum_{n=1}^{\infty} |\lambda|^n b^{n+2} c^{n+1} = b^3 c^2 |\lambda| \{ 1 - |\lambda| bc \}^{-1} . \tag{2.27}$$

It now remains to show that the operator $\tilde{\mathrm{K}}_\lambda^{(0)}$ converges strongly to $\tilde{\mathrm{K}}_0$. For this it would suffice that the kernel $\mathrm{K}_\lambda^{(0)}(\tau, \sigma)$ converged uniformly to K_0, but our conditions are actually much weaker. We first note that the kernel $\mathrm{K}_\lambda^{(0)}(\tau, \sigma)$ is uniformly bounded by (2.23) so it suffices to prove strong convergence on any set of functions which generates X. We choose $f(\tau) = \tau^n$ a where $a \in \mathscr{B}_0$.

$$(\tilde{\mathrm{K}}_\lambda^{(0)} f)(\tau) = \int_{\sigma=0}^{\tau} \mathrm{K}_\lambda^{(0)}(\tau, \sigma) \sigma^n a d\sigma = \int_{\sigma=0}^{\tau} \int_{s=\sigma\lambda^{-2}}^{\tau\lambda^{-2}} \sigma^n \mathrm{P}\mathrm{A}_s \mathrm{A}_{\sigma\lambda^{-2}} a ds d\sigma$$

$$= \tau^{n+1} (\tau\lambda^{-2})^{-n-1} \int_{s=0}^{\tau\lambda^{-2}} \int_{u=0}^{s} u^n \mathrm{P}\mathrm{A}_s \mathrm{A}_u a du ds . \tag{2.28}$$

This proves that if $\tau > 0$

$$\lim_{\lambda \to 0} (\tilde{K}_\lambda^{(0)} f)(\tau) = \frac{\tau^{n+1}}{n+1} K_0 a = \int_{\sigma=0}^{\tau} K_0 f(\sigma) d\sigma \tag{2.29}$$

and also if $0 < \tau_1 \leqslant \tau_0 < \infty$, that the convergence is uniform for $\tau_1 \leqslant \tau \leqslant \tau_0$. The fact that the convergence is uniform for $0 \leqslant \tau \leqslant \tau_0$ follows from the bound

$$\| (\tilde{K}_\lambda^{(0)} f)(\tau) \| \leqslant \frac{\tau^{n+1}}{n+1} c \| a \| \tag{2.30}$$

which is valid for all λ.

3. STOCHASTIC DIFFERENTIAL EQUATIONS

Given a Banach space $\mathscr{B}_0$ and a probability space $(\Omega, \mathscr{F}, d\omega)$ we can define the Banach space

$$\mathscr{B} = L^\infty(\Omega, \mathscr{F}, d\omega, \mathscr{B}_0) \tag{3.1}$$

as the space of essentially bounded, strongly $\mathscr{F}$-measurable, $\mathscr{B}_0$-valued functions on Ω. The Banach space $\mathscr{B}_0$ can be identified with the constant functions on Ω and the expectation is then a projection $P : \mathscr{B} \to \mathscr{B}_0$. Now suppose that for $0 \leqslant s \leqslant t < \infty$ there are σ-fields $\mathscr{F}$ such that if $0 \leqslant s \leqslant s' \leqslant t' \leqslant t < \infty$ then

$$\mathscr{F}_{s'}^{t'} \subseteq \mathscr{F}_s^t \tag{3.2}$$

Then the conditional expectation with respect to $\mathscr{F}_s^t$ is a projection P_s^t on $\mathscr{B}$ and its range is the subspace of $\mathscr{B}$ of those functions measurable with respect to $\mathscr{F}_s^t$. If $0 \leqslant s \leqslant t < \infty$ the following equalities may be easily verified.

$$P = PP_0^t = P_0^t P = P_t^\infty P = PP_t^\infty \tag{3.3}$$

$$P_0^s = P_0^s P_0^t = P_0^t P_0^s \tag{3.4}$$

$$P_t^\infty = P_t^\infty P_s^\infty = P_s^\infty P_t^\infty \tag{3.5}$$

$$\| P \| = \| P_0^t \| = \| P_t^\infty \| = 1 . \tag{3.6}$$

We assume a strong mixing hypothesis of the following form. If $Pf = 0$ and $P_0^s f = f$ then for all $t \geqslant 0$

$$\| P_{t+s}^\infty f \| \leqslant p(t) \| P_s^\infty \psi \| \tag{3.7}$$

where

$$\int_0^\infty p(t) dt \equiv c < \infty . \tag{3.8}$$

This holds in particular if for some $a > 0$ and all $t \geqslant 0$

$$P_{t+a}^{\infty} P_0^t = P \tag{3.9}$$

that is, if the σ-fields $\mathscr{F}_0^t$ and $\mathscr{F}_{t+a}^{\infty}$ are independent.

Given a random operator $A(t, \omega)$ on $\mathscr{B}_0$ which for every $t \geqslant 0$ is essentially bounded we define a bounded operator A_t on $\mathscr{B}$ by

$$(A_t f)(\omega) = A(t, \omega) f(\omega) . \tag{3.10}$$

If $A(t, \omega)$ is $\mathscr{F}_t^t$ measurable then for $0 \leqslant a \leqslant t \leqslant b < \infty$

$$A_t P_0^b = P_0^b A_t , \qquad A_t P_a^{\infty} = P_a^{\infty} A_t . \tag{3.11}$$

Finally we suppose that $A(t, \omega)$ has expectation zero for all $t \geqslant 0$, or in operator terms

$$PA_t P = 0 . \tag{3.12}$$

Apart from the fact that they allow unbounded operators, the above is essentially the situation of [9]. We now abstract it by supposing that $\mathscr{B}$ is a general Banach space with projections P, P_0^t, P_t^{∞} defined for all $t \geqslant 0$ and satisfying (3.3-3.8). We also suppose that A_t is a strongly continuous operator valued function satisfying (3.11), (3.12) and also

$$\| A_t \| \leqslant c \tag{3.13}$$

for all $0 \leqslant t < \infty$.

THEOREM 3.1. — *The above conditions imply that* (2.23) *is satisfied.*

Proof. — If $a \in \mathscr{B}_0$ then

$$\begin{aligned} K \equiv \| PA_s B_{x_n} \ldots B_{x_1} A_u P a \| & \\ = \| PA_s (1 - P) A_{x_n} (1 - P) A_{x_{n-1}} \ldots (1 - P) A_u P a \| & \\ = \| PA_s b \| = \| PP_s^{\infty} A_s b \| & \\ = \| PA_s P_s^{\infty} b \| \leqslant c \| P_s^{\infty} b \| & \end{aligned} \tag{3.14}$$

where

$$b = (1 - P) A_{x_n} (1 - P) \ldots (1 - P) A_u P a \tag{3.15}$$

satisfies

$$Pb = 0, \qquad P_0^{x_n} b = b . \tag{3.16}$$

Therefore by (3.7)

$$\begin{aligned} k \leqslant cp(s - x_n) \| P_{x_n}^{\infty} b \| = cp(s - x_n) \| (1 - P) P_{x_n}^{\infty} A_{x_n} (1 - P) \ldots (1 - P(A_n P a \| \\ \leqslant cp(s - x_n) \| P_{x_n}^{\infty} A_{x_n} (1 - P) A_{x_{n-1}} \ldots (1 - P) A_n P a \| \end{aligned} \tag{3.17}$$

after which we may proceed by induction.

We finally comment that the key estimates required for the above theorems are (2.25), (3.7), (3.8) and (3.13). These correspond respectively to (2.14), (2.1), (2.18) and (2.3) of [9].

4. AN ALTERNATIVE APPROACH

An alternative approach to the abstraction of the above results on stochastic differential equations has been followed by T. G. Kurtz [7]. In order to compare his work with [3] we reformulate his Theorem 2.2 in our notation, restricting attention to the case where the perturbation is bounded.

Kurtz supposes that e^{Zt} is a strongly continuous one-parameter contraction semi-group on the Banach space $\mathscr{B}$, to that Z is a dissipative operator. He assumes

$$s - \lim_{\lambda \to 0} \lambda \int_0^\infty e^{-\lambda t} e^{Zt} dt = \mathrm{P} \tag{4.1}$$

exists; P is then a projection. Putting $\mathscr{B}_0 = \mathrm{P}\mathscr{B}$ and $\mathscr{B}_1 = (1 - \mathrm{P})\mathscr{B}$, it may be seen that Z leaves each subspace $\mathscr{B}_r$ invariant, $\mathrm{Z} = 0$ on $\mathscr{B}_0$ and $\mathrm{Z}_1 = \mathrm{P}_1\mathrm{Z}$ is one-one on $\mathscr{B}_1$. If A is a bounded dissipative operator on $\mathscr{B}$ then for all $\lambda > 0$, $\exp(\mathrm{Z} + \lambda\mathrm{A})t$ is a one-parameter contraction semi-group on $\mathscr{B}$.

PROPOSITION 4.1 (Kurtz). — *Suppose* $\mathrm{C} \equiv \mathrm{PAP} = 0$ *and let*

$$\mathrm{D}_0 = \{ f \in \mathscr{B}_0 : \mathrm{A}f \in \text{Range } \mathrm{Z}_1 \} \tag{4.2}$$

Define the operator C_0 *on* D_0 *by*

$$\mathrm{C}_0 f = -\mathrm{PAZ}_1^{-1}\mathrm{A}f \tag{4.3}$$

and suppose

$$\text{Range } (\lambda - \mathrm{C}_0) \supseteq \overline{\mathrm{D}_0} \tag{4.4}$$

for some $\lambda > 0$. *Then the closure of* C_0 *restricted so that* $\mathrm{C}_0 f \in \overline{\mathrm{D}_0}$ *is the infinitesimal generator* $\tilde{\mathrm{C}}_0$ *of a strongly continuous contraction semi-group* T_t *defined on* $\overline{\mathrm{D}_0}$. *Moreover for all* $f \in \overline{\mathrm{D}_0}$ *and* $t \geqslant 0$.

$$\lim_{\lambda \to 0} \exp \{ (\mathrm{Z} + \lambda\mathrm{A})\lambda^{-2}t \} f = \mathrm{T}_t f. \tag{4.5}$$

The operator C_0 is somewhat similar to the operator K defined in [3] [4] by

$$\mathrm{K}f = \lim_{\alpha \to \infty} \int_0^\alpha \mathrm{PA}e^{\mathrm{Z}_1 t}\mathrm{A}f dt \tag{4.6}$$

where $f \in \mathscr{B}_0$. The following considerations show that although Kurtz' theorem is completely satisfactory for describing evolution in a Banach space controlled by a Markov process, it will not encompass the quantummechanical applications which first led to our interest in the problem [*1*] [*2*].

Consider the special case where $\mathscr{B}$ is a Hilbert space and e^{Zt} is a unitary group on $\mathscr{B}$. Then (iZ) is a self-adjoint operator and P is the orthogonal projection onto the null-space of Z. If (iA) is self-adjoint then it is easily seen that (iC_0) and hence $(i\tilde{C}_0)$ are symmetric operators, a very special case in terms of [*3*] [*4*].

This point may be made more explicitly for the example worked out in [*2*], § 2, where $\mathscr{B}$ is a Hilbert space and $\mathscr{B}_0$ is one-dimensional. It may then be shown that if $D_0 \neq \{0\}$ then $\tilde{C}_0 = K$ and T_t is unitary, which is the case only when there is no dissipative effect.

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(Manuscrit reçu le 6 janvier 1975)