ANNALES DE L'I. H. P., SECTION B

DANG-NGOC-NGHIEM

On the classification of dynamical systems

Annales de l'I. H. P., section B, tome 9, nº 4 (1973), p. 397-425

<http://www.numdam.org/item?id=AIHPB_1973__9_4_397_0>

Ann. Inst. Henri Poincaré, Section B :
Vol. IX, n° 4, 1973, p. 397-425. *Calcul des Probabilités et Statistique.*

On the classification of dynamical systems

by

DANG - NGOC - NGHIEM (*)
Université Paris VI. Laboratoire de Probabilités.
Tour 56. 9, quai Saint-Bernard, Paris V^e (France).

SUMMARY. — We study the classification of dynamical systems (under the weak equivalence) introduced by H. A. Dye and developped by W. Krieger by a method similar to that of the classification of von Neumann algebras. We study the properties of finite, semi-finite, properly infinite, purely infinite, discrete, continuous, type $I_1, \ldots, I_{\mathcal{N}}$, II_1, II_∞, III systems. We also introduce the notion of induced systems generalizing a construction of Kakutani for conservative automorphisms and prove that the classification of measurable subsets (under Hopf's equivalence relation) is closely related to the classification of induced systems. Finally we give a complete description of the structure of « separable » discrete systems and some properties of the products of dynamical systems.

I. INTRODUCTION AND NOTATIONS

The aim of this paper is to study the classification of general dynamical systems (under the weak equivalence) initiated by H. A. Dye (cf. [*8*], [9]) and developped by W. Krieger (cf. [*24*], [*25*]). Dye studied particularly dynamical systems admitting a finite invariant measure, W. Krieger

(*) Équipe de Recherche n° 1 « Processus stochastiques et applications » dépendant de la Section n° 1 « Mathématiques, Informatique » associée au C. N. R. S. (France).

succeeded in the search of an infinite number of non isomorphie « hyper-finite » ergodic systems that do not admit any invariant measure (finite of σ-finite). Furthermore, Krieger introduced an important construction which establishes a very close connection between the classification of ergodic systems and the classification of factors in the W*-category; Krieger's construction generalizes Von Neumann's (cf. [*26*]).

We use a method very similar to that of the classification of Von Neumann algebras and provide some definitions that are invariant under the weak equivalence: discrete, continuous, type I_n, type II_1, type II_∞, type-III-systems; in a subsequent paper we shall show that these definitions are closely related to that of Von Neumann algebras *via* Krieger's construction.

In the part IV, we study systematically some important properties of induced systems, particularly the relations between the types of induced systems and that of the global system.

In the part V, we study the comparison of measurable subsets under the action of the group of automorphisms (See also [*21*] [*34*] [*35*]).

In the part VI, we investigate the classification of measurable subsets by using the classification of corresponding induced systems, we also reproduce the Hopf's theorem on the characterization of finite systems and give some of its important consequences.

In the part VII, we give a complete description of discrete systems: decomposition into homogeneous systems.

In the part VIII, we study the products of dynamical systems.

Certain results obtained in this paper will be generalized to « non-commutative dynamical systems » (i. e. groups acting on Von Neumann algebras by $*$-automorphisms) but we prefer to keep the paper independent of the von Neumann algebra context.

Throughout this note we shall use the following notations: let $(M, \mathscr{B}, m)$ be a σ-finite measure space i. e. M is a set, $\mathscr{B}$ a σ-algebra of subsets of M, m a σ-finite measure on M, $\mathscr{B}$ is supposed to be complete with respect to m; by automorphism of $(M, \mathscr{B}, m)$ we shall mean an invertible bi-measurable transformation T of $(M, \mathscr{B}, m)$ that is non-singular (i. e. $m(B) = 0$ implies $m(TB) = m(T^{-1}B) = 0$). We shall denote by Aut $(M, \mathscr{B}, m)$ (or simply Aut (m)) the group of all automorphisms of $(M, \mathscr{B}, m)$.

Let G be a subgroup of Aut $(M, \mathscr{B}, m)$, let [G] be the *full group* (cf. [*8*]) of G i. e. the set of automorphisms h such that there exist a *sequence* $g_n \in G$ and a partition (M_{g_n}) of M such that

$$h|_{M_{g_n}} = g_n|_{M_{g_n}}$$

where $h|_{M_g}$ denotes the restriction of h on M_{g_n}. It is easy to see that [G] is a group and [[G]] = [G].

Let μ be a measure absolutely continuous with respect to $m(\mu \ll m)$, $T \in \text{Aut}(m)$; we define

$$(T\mu)(B) = \mu(T^{-1}B) \qquad \forall B \in \mathscr{B}.$$

The measure μ is said to be G-invariant if $T\mu = \mu$ for all $T \in G$.

If μ is G-invariant then μ is [G]-invariant (by definition of [G]).

DEFINITION 1. — We call dynamical system (or simply system) a couple $(M, \mathscr{B}, m; G)$ where $(M, \mathscr{B}, m)$ is a σ-finite measure space and G is a group of automorphisms of $(M, \mathscr{B}, m)$.

Let $\mathscr{F} = (M, \mathscr{B}, m; G)$ and $\mathscr{F}' = (M', \mathscr{B}', m'; G')$ be two systems by *weak morphism* of $\mathscr{F}$ into $\mathscr{F}'$ we mean a measurable mapping ϕ of $(M, \mathscr{B}, m)$ into $(M', \mathscr{B}', m')$ such that

α) $\phi m \ll m'$,

β) $\forall h' \in [G']$ there exists a $h \in [G]$ such that $\phi \circ h = h' \circ \phi$,

γ) $\forall h \in [G]$, we have $h\phi^{-1}(\mathscr{B}') = \phi^{-1}(\mathscr{B}')$ and, there exists $h' \in [G']$ such that $\phi \circ h = h' \circ \phi$.

If ϕ is an isomorphism of $(M, \mathscr{B}, m)$ on $(M', \mathscr{B}', m')$, the conditions α and β mean that

$$\phi^{-1}[G']\phi \subset [G]$$

If furthemore we suppose that $\phi^{-1}[G']\phi = [G]$, ϕ is said a *weak isomorphism* of $\mathscr{F}$ into $\mathscr{F}'$ and we say that the systems $\mathscr{F}$ and $\mathscr{F}'$ are *weakly isomorphic* (or *weakly equivalent*) (cf. [*8*] [*9*] [*23*] [*24*]).

We have also a category called the $\mathscr{S}$*-category* whose objects are dynamical systems and whose morphisms are weak morphisms of dynamical systems.

A system $\mathscr{F} = (M, \mathscr{B}, m; G)$ is said *separable* if $(M, \mathscr{B}, m)$ is a Lebesgue-space (cf. [*24*] [*25*]) and G is countable, we define also a full subcategory (called the $\mathscr{S}_s$*-category*) whose objects are separable systems.

If $\mathscr{F} = (M, \mathscr{B}, m; G)$ and $\mathscr{F}' = (M', \mathscr{B}', m', \mathscr{S}')$ are two separable systems then (cf. [*24*])

α) $[G] = \{ h \in \text{Aut}(m)/hx \in \text{Orb}_G\, x \text{ for almost all } x \in M \}$ where $\text{Orb}_G\, x = \{ gx/g \in G \}$.

β) Let ϕ be a isomorphism of $(M, \mathscr{B}, m)$ or $(M', \mathscr{B}', m')$; ϕ is a weak isomorphism of $\mathscr{F}$ or $\mathscr{F}'$ if and only if

$$\phi(\text{Orb}_G\, x) = \text{Orb}_{G'}\, \phi x$$

(cf. [*24*]).

We denote by $\mathcal{I}(\mathcal{F})$ or simply $\mathcal{I}$ the σ-algebra of G-invariant elements of $\mathcal{B}$. Let $\mathcal{F} = (M, \mathcal{B}, m; G)$ be a dynamical system, let E be a non null subset of M *invariant* under the action of G, E is provided with the induced σ-algebra $\mathcal{B}_E$, the induced measure m_E and the induced action G_E, the canonical system $(E, \mathcal{B}_E, m_E, G_E)$ will be called a *subsystem* of $\mathcal{F}$ and noted $\mathcal{F}_E$ of $(M, \mathcal{B}, m; G)_E$.

Let $\mathcal{B}'$ be a sub-σ-algebra of $\mathcal{B}$ and let

$$G' = \{ g \in G / g\mathcal{B}' = \mathcal{B}' \}$$

Let m' be the restriction of m to $\mathcal{B}'$ (if the restriction of m to $\mathcal{B}'$ is not σ-finite we replace m by a finite equivalent measure), the dynamical system $(M, \mathcal{B}', m'; G')$ will be called a *quotient* of $\mathcal{F}$.

Let $(B_i)_{i \in I}$ be a partition of M, $B_i \in \mathcal{B}$, $m(B_i) > 0$ (I is necessarily countable by the σ-finiteness of m), and we suppose that each B_i is G-invariant, in this case, we say that $\mathcal{F}$ is the *direct sum* of the $\mathcal{F}_{B_i}$ and we write

$$\mathcal{F} = \sum_{i \in I} \mathcal{F}_{B_i}$$

We can verify that all these definitions agree (up to a weak isomorphism) with the usual definitions in a category (namely the $\mathcal{S}$-category).

II. FIRST CLASSIFICATION OF DYNAMICAL SYSTEMS

DEFINITION 1. — (See also [*21*] [*24*] [*35*]).

Let $\mathcal{F} = (M, \mathcal{B}, m; G)$ be a dynamical system

α) $\mathcal{F}$ is said to be *finite* if for every $E \in \mathcal{B}$ with $m(E) > 0$, there exists a finite G-invariant measure $\mu(\ll m)$ such that $\mu(E) > 0$.

β) $\mathcal{F}$ is said to be *semi-finite* if for every $E \in \mathcal{B}$ with $m(E) > 0$ there exists a σ-finite G-invariant measure $\mu(\ll m)$ such that $\mu(E) > 0$.

γ) $\mathcal{F}$ is said to be *properly infinite* (resp. purely infinite) if there exists no finite (resp. σ-finite) G-invariant measure absolutely continuous with respect to m other than the null-measure.

δ) $\mathcal{F}$ is said to be *infinite* if it is not finite.

Let μ be a measure on $(M, \mathcal{B})$, $\mu \ll m$, let supp μ be the support of μ; then μ and m are equivalent on supp μ and supp μ is G-invariant if μ is G-invariant.

PROPOSITION 1 (Classical). — The system $\mathcal{F}$ is finite (resp. semi-finite) if and only if there exists a finite (resp. σ-finite) G-invariant measure μ equivalent to m.

THEOREM 1 (Classical). — Any dynamical system $\mathcal{F} = (M, \mathcal{B}, m; G)$ splits uniquely as the direct sum of three systems:

$$\mathcal{F} = \mathcal{F}_{M_1} + \mathcal{F}_{M_2} + \mathcal{F}_{M_3}$$

where $M_1, M_2, M_3 \in \mathcal{B}$, $\Sigma M_i = M$, each M_i is G-invariant and:

— $\mathcal{F}_{M_1}$ is finite,
— $\mathcal{F}_{M_2}$ is semi-finite properly infinite,
— $\mathcal{F}_{M_3}$ is purely infinite.

THEOREM 2. — Let $\mathcal{F} = (M, \mathcal{B}, m; G)$ be an *ergodic* system and let H be a group of *conservative* automorphisms (e. g. $(M, \mathcal{B}, m; H)$ finite) such that $hGh^{-1} \subset [G]$ for all $h \in H$. If $\mu(\sim m)$ is a σ-finite G-invariant measure then μ is also H-invariant. It follows that:

α) If the system $(M, \mathcal{B}, m; H)$ is purely infinite, so is the system $\mathcal{F}$.
β) If $(M, \mathcal{B}, m; H)$ is ergodic and finite (resp. semi-finite infinite) then the system $\mathcal{F}$ is either finite (resp. semi-finite infinite) or purely infinite.

— *Proof* : Let $\mu(\sim m)$ be a σ-finite G-invariant measure, μ is [G]-invariant; let $h \in H$, $g \in G$, there exists $g' \in [G]$ such that $gh = hg'$, hence

$$\begin{aligned} g(h\mu) &= h(g'\mu) \\ &= h\mu \end{aligned}$$

Therefore $h\mu$ is G-invariant, as G is ergodic

$$h\mu = \lambda(h)\mu \qquad \text{with} \qquad \lambda(h) \in R^+$$

By the « Jacobian theorem » (cf. [*15*], p. 89) $\lambda(h) = 1$, hence μ is H-invariant. q. e. d. ■

Remark : This theorem provides a very short proof that the systems constructed in [*32*] [*2*] [*1*] [*24*] are purely infinite (consider H the group generated by the shift on the product space).

III. SECOND CLASSIFICATION OF DYNAMICAL SYSTEMS

We shall continue in this paragraph the study of the classification of dynamical systems by defining invariants similar to that used in the classification of W*-algebras.

DEFINITION 1. — α) Let $\mathscr{F} = (M, \mathscr{B}, m; G)$ be a dynamical system, $\mathscr{I}(\mathscr{F})$ (or $\mathscr{I}$) the sub-σ-algebra of invariant elements of $\mathscr{B}$; let $B \in \mathscr{B}$, we shall call the G-*support* of B, that we note $\mathrm{supp}_G B$, the least element of $\mathscr{I}$ containing B. We verify immediately

$$\mathrm{supp}_G B = \operatorname*{ess\,inf}_{\substack{E \in \mathscr{I} \\ E \supset B}} E = \operatorname*{ess\,sup}_{g \in G} gB$$

β) An element $B \in \mathscr{B}$ is said to be a G-*atom* (or $\mathscr{I}$-atom (cf. Neveu-Hanen [*31*])) or a $\mathscr{I}$-abelian subset fo M (cf. [*8*] [*10*]) if $\mathscr{B}_B = \mathscr{I}_B$, where $\mathscr{B}_B$ and $\mathscr{I}_B$ denote the traces of $\mathscr{B}$ and $\mathscr{I}$ on B.

Let B be a G-atom and let $E \in \mathscr{B}$, $E \subset B$, we can write $E = B \cap F$ with $F \in \mathscr{I}$, we can take $F = \mathrm{supp}_G E$, then:

$$E = B \cap \mathrm{Supp}_G E \quad \text{if} \quad E \subset B.$$

Then B is minimal among the elements of $\mathscr{B}$ which have the same G-support as E; the converse is true (cf. [*10*]).

It is clear that an atom is a G-atom, and if $\mathscr{F}$ is ergodic (i. e. $\mathscr{I} = \{ \phi, M \}$), these two notions are equivalent.

If B is a G-atom, then gB is a G-atom for all $g \in [G]$: let $E \in \mathscr{B}$, $gB \cap E = g(B \cap g^{-1}E) = g(B \cap F)$ with a $F \in \mathscr{I}$ then $gB \cap E = gB \cap F$.

Every subset of a G-atom is a G-atom.

Remark. — For separable systems, the notion of G-atom coincides with that of « orbit-section » (cf. Prop. VII-2).

DEFINITION 2. — A system $\mathscr{F} = (M, \mathscr{B}, m, G)$ is said to be *discrete* if every non null subset of $\mathscr{B}$ contains a non null G-atom.

$\mathscr{F}$ is said to be *continuous* if M does not contain any non null G-atom.

THEOREM 1 (See also [*31*]). — Let $\mathscr{F} = (M, \mathscr{B}, m; G)$ be a dynamical system, the following conditions are equivalent:

(*i*) $\mathscr{F}$ is discrete.

(*ii*) There exists a G-atom B whose G-support is the whole space M.

(*iii*) The space M is a disjoint union of a sequence of G-atoms.

Proof. — (*i*) $\Rightarrow$ (*ii*) Let $(B_i)_{i \in I}$ be a maximal family of G-atoms such that $\mathrm{supp}_G B_i$ are disjoint; by the maximality of the family we have:

$$M = \operatorname*{ess\,sup}_{i \in I} (\mathrm{supp}_G B_i)$$

As the measure m is σ-finite, we can suppose $I = N$; let $F = \bigcup_{n \in N} F_n$; B is a G-atom (the G-supports of F_n are disjoint) and

$$\mathrm{supp}_G\, B = \bigcup_n \mathrm{supp}_{G\, B_n} = M.$$

(*ii*) $\Rightarrow$ (*i*) is immediate by remarking that gF is a G-atom.

(*i*) $\Rightarrow$ (*iii*) is an immediate consequence of Zorn's lemma and the σ-finiteness of m.

(*iii*) $\Rightarrow$ (*i*) is immediate. q. e. d. ■

THEOREM 2. — Any dynamical system splits uniquely as the direct sum of a discrete system and a continuous system.

Proof. — Let $\mathscr{F} = (M, \mathscr{B}, m; G)$ be a dynamical system, let $M_{c = M \setminus M_d}$ and $M_d = \text{ess sup} \{ B/B \text{ G-atom} \}$. It is clear that M_d and M_c are G-invariant and $\mathscr{F} = \mathscr{F}_{M_d} + \mathscr{F}_{M_c}$ with $\mathscr{F}_{M_d}$ discrete and $\mathscr{F}_{M_c}$ continuous; the uniqueness of the decomposition is clear. q. e. d. ■

IV. INDUCED SYSTEMS

§ 1. Induced automorphisms

We shall give a definition of induced automorphism generalizing that of Kakutani (cf. [*20*]) for conservative automorphisms. For this purpose we shall need some special notions.

Let $(M, \mathscr{B}, m)$ be a σ-finite measure space, let T be an automorphism of $(M, \mathscr{B}, m)$. Let $x \in M$ we denote :

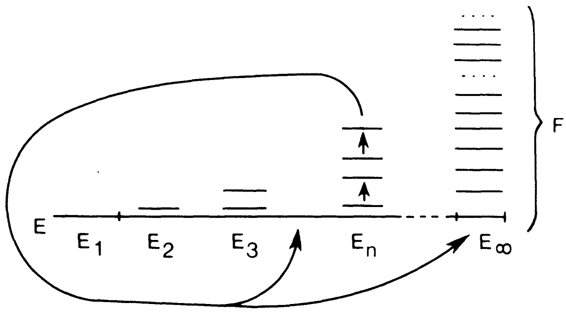
$$\mathrm{Orb}_T\, x = \{ T^n x /_{n \in Z} \}$$
$$\mathrm{Orb}_T^+\, x = \{ T^n x /_{n > 0} \} \qquad (\text{or } x^+)$$
$$\mathrm{Orb}_T^-\, x = \{ T^n x /_{n > 0} \} \qquad (\text{or } x^-)$$

$\mathrm{Orb}_T\, x$ is ordered in the natural way if x is not periodic. Let $E \in \mathscr{B}$, $m(E) > 0$. Let

$$E_1 = E \cap T^{-1} E$$
$$E_n = (E \cap T^{-n} E) \setminus \left(\bigcup_{1 \leq k < n} E_k \right)$$
$$E_\infty = E \setminus \bigcup_{n \geq 1} E_n = \{ x \in E / T^n x \notin E, \ \forall n \geqq 1 \}$$

(1) $$E_\infty = E \cap \left(\bigcap_{n \geqq 1} T^{-n}(E^c) \right) \quad \text{with} \quad E^c = M \setminus E.$$

Let $F = \bigcup_{n \geq 0} T^n E$, the restriction of T to F has the following classical form: Kakutani's skyscraper (here with an infinite column)



We define similarly $E_{-1}, E_{-2}, \ldots, E_{-\infty}$ with the automorphism T^{-1}. We have

$$E_{-\infty} = E \cap \left(\bigcap_{n \geq 1} \right) T^n E^c. \tag{2}$$

Let

$$\begin{aligned} E_{+\mathrm{fin}} &= \{ x \in E / x^+ \cap E \text{ finite and } x \text{ non periodic} \} \\ E_{-\mathrm{fin}} &= \{ x \in E / x^- \cap E \text{ finite and } x \text{ non periodic} \} \\ E_{+\mathrm{fin}} &= \{ x \in E / x^+ \cap E \text{ infinite or } x \text{ periodic} \} \\ E_{-\mathrm{fin}} &= \{ x \in E / x^- \cap E \text{ infinite or } x \text{ periodic} \} \end{aligned}$$

We obtain immediately the following lemma

LEMMA 1.

$$\begin{aligned} E_{+\mathrm{fin}} &= E \cap \left(\bigcup_{n \geq 1} T^{-n} E_{\infty} \right) \\ E_{-\mathrm{fin}} &= E \cap \left(\bigcup_{n \geq 1} T^{n} E_{-\infty} \right) \\ E_{+\mathrm{fin}} &= E \cap \left(\bigcap_{n \geq 1} T^{-n} (E^c_{\infty}) \right) \\ E_{-\mathrm{fin}} &= E \cap \left(\bigcap_{n \geq 1} T^{n} (E^c_{-\infty}) \right) \end{aligned}$$

DEFINITION 1. — We can define now the *induced automorphism* T_E of T on M:

α) On $E_{+\mathrm{fin}} \cap E_{-\mathrm{fin}}$: we define $T_E x = T^{n(x)} x$ where $n(x)$ is the first entry time of x (as in the Kakutani's definition).

β) On $E_{+\text{fin}} \cap E_{-\text{fin}}$: let $x \in E_{+\text{fin}} \cap E_{-\text{fin}}$

$$\text{Orb}_T\, x \cap E = \{ x_{-n}, x_{-n+1}, \ldots, x_0 = x, \ldots, x_{n'} \}$$

We define

$$T_x = \begin{cases} T^{n(x)}x & \text{if} \quad n(x) < \infty \\ x_{-n} & \text{if} \quad n(x) = \infty \end{cases}$$

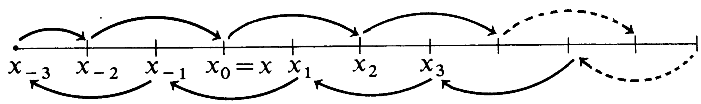
γ) On $E_{+\text{fin}} \cap E_{-\text{fin}}$: let $x \in E_{+\text{fin}} \cap E_{-\text{fin}}$:

$$E \cap \text{Orb}_T\, x = \{ x_{-n} x_{-n+1}, \ldots, x_0 = x, x_1, \ldots \}$$

We define

$$T_E x = \begin{cases} x_1 & \text{if} \quad n = 0 \\ x_{-2} & \text{if} \quad n \quad \text{even}, \quad n \neq 0. \\ x_2 & \text{if} \quad n \quad \text{odd} \end{cases}$$

$E \cap \text{Orb}_T\, x$:

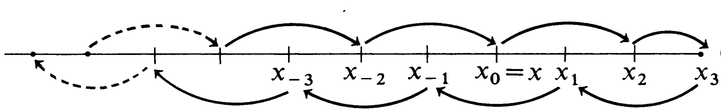


δ) On $E_{+\text{fin}} \cap E_{-\text{fin}}$: let $x \in E_{+\text{fin}} \cap E_{-\infty}$

$$E \cap \text{Orb}_T\, x = \{ \ldots, x_{-1}, x_0 = x, x_1, \ldots, x_n \}$$

We define

$$T_E x = \begin{matrix} x_{-2} & \text{if} & n & \text{even} \\ x_1 & \text{if} & n = 1 & \\ x_2 & \text{if} & n & \text{odd}, \quad n \neq 1 \end{matrix}$$



$\text{Orb}_T\, x \cap E$

ε) On $M \backslash E$, we define $T_E x = x$.

PROPOSITION 1.

1) $(T^{-1})_E = (T_E)^{-1}$
2) $T_E \in [T]$
3) $E \cap \text{Orb}_T\, x = \text{Orb}_{T_E}\, x, \quad \forall x \in E$
4) $T_E x = x, \quad \forall x \in M \backslash E.$

Proof. — Properties 1), 3) and 4), result immediately form the definition of T_E ; it is easy to verify that T_E and T_E^{-1} and T_E^{-1} are measurable, and

$T_E \in \mathrm{Aut}(E, \mathscr{B}_E, m_E)$ (in fact, we can find a countable partition of E on each element of which T_E is simply defined); 2) results from 3) and the lemma (2-1) of Krieger [*24*]. q. e. d. ■

§ 2. Induced systems

Let $\mathscr{F} = (M, \mathscr{B}, m ; G)$ be a dynamical system, $E \in \mathscr{B}$, $m(E) > 0$; let

$$[G]_E = \{ h \in [G]/hE = E, hx = x \quad \forall x \in M \backslash E \}$$

Then $[G]_E$ is a subgroup of $[G]$ and $[G]_E = \{ h_E/h \in [G] \}$ we can also identify $[G]_E$ to a subgroup of $\mathrm{Aut}(E, \mathscr{B}_E, m_E)$ and $[G]_E$ is full i. e.

$$[[G]_E] = [G]_E$$

DEFINITION 2. — The system $\mathscr{F}_E = (E, \mathscr{B}_E, m_E, [G]_E)$ is called the *induced-system* of $\mathscr{F}$ on E (cf. Dye [*8*]). Following proposition 1, if $h \in [G]$ then $h_E \in [G]_E$. Consider G_E the *group generated by* $\{ g_E \}_{g \in G}$ we have:

$$G_E \subset [G]_E$$

Let $x \in E$, $g \in G$, $g' \in G$

$$g_E g'_E x \in g_E \,\mathrm{Orb}_G\, x = \mathrm{Orb}_G\, x$$

It is also true that $g_E g'_E \dots g''_E x \in \mathrm{Orb}_G\, x$, $\forall x \in E$, $g, g' \dots g'' \in G$; the proposition 1 implies

$$\mathrm{Orb}_{G_E}\, x = \mathrm{Orb}_G\, x \cap E \qquad \forall x \in E.$$

Let $h \in [G]$, there exists a partition (M_{g_n}) of M such that

$$g_n \in G$$
$$h \,|_{M_{g_n}} = g_n \,|_{M_{g_n}}.$$

Let G′ be the *countable* group generated by $\{ g_n \}_{n \in N}$. As we have

$$hx \in \mathrm{Orb}_{G'}\, x \qquad \forall x \in M$$

this implies

$$h_E x \in \mathrm{Orb}_{G'_E}\, x = E \cap \mathrm{Orb}_{G'}\, x \qquad \forall x \in M.$$

Hence, by the lemma 2.1 of ([*24*]) (since G'_E is *countable*).

$$h_E \in (G'_E)$$

We have shown that: if $h \in [G]$ then $h_E \in [G_E]$, i. e.

$$[G]_E \subset [G_E)$$

As $[G_E]$ is clearly inclued in $[G]_E$ we obtain

$$[G]_E = [G_E]$$

We can write without any ambiguity $\mathscr{F}_E = (E, \mathscr{B}_E, m_E ; G_E)$; we have proved:

PROPOSITION 2. — Let $\mathscr{F} = (M, \mathscr{B}, m ; G)$ be a dynamical system, $E \in \mathscr{B}$, $m(E) > 0$; the induced system $\mathscr{F}_E = (E, \mathscr{B}_E, m ; G_E)$ possesses following properties:

1) $[G_E] = [G]_E$
2) $\mathrm{Orb}_{G_E}\, x = E \cap \mathrm{Orb}_G\, x$, $\forall x \in E$
3) If G is countable then so is G_E ; if $\mathscr{F}$ is separable so is $\mathscr{F}_E$.

Let $\mathscr{I}(\mathscr{F})$ (resp. $\mathscr{I}(\mathscr{F}_E)$) be the sub-$\sigma$-algebra of invariant elements of $\mathscr{B}$ (resp. $\mathscr{B}_E$).

Then, the above proposition implies immediately:

PROPOSITION 3 (See also [*8*], Lemma 3.5). — The invariant sub-σ-algebra of the induced system $\mathscr{F}_E$ is the trace on E of the invariant sub-σ-algebra of the global system $\mathscr{F}$.

COROLLARY. — Let $E \in \mathscr{B}$, $m(E) > 0$; the induced system $\mathscr{F}_E$ is ergodic if and only $\mathscr{F}_{\mathrm{supp}_G E}$ is ergodic.

Proof. — The condition is clearly sufficient. Now suppose that $\mathscr{F}_E$ is ergodic, we can also suppose without restricting the generality that $M = \mathrm{supp}_G\, E$; let $B \in \mathscr{B}$ be an invariant subset of M, $m(B) > 0$. As $M = \operatorname{ess\,sup}_{g \in G} gE$ if $B \neq M$ we must have $B \cap E \neq 0$, $B \cap E \neq E$ and as $B \cap E$ is G_E-invariant that contradicts the ergodicity of $\mathscr{F}_E$. q. e. d. ■

DEFINITION 3. — Let $\mathscr{F} = (M, \mathscr{B}, m ; G)$ and $\mathscr{F}' = (M', \mathscr{B}', m' ; G')$ be two dynamical systems; we shall call *partial isomorphism* a triple (E, E', h) where $E \in \mathscr{B}$, $E' \in \mathscr{B}'$, $m(E) > 0$, $m'(E') > 0$ and h weak isomorphism of $\mathscr{F}_E$ onto $\mathscr{F}'_{E'}$.

If G and G′ are countable then h is a weak isomorphism of $\mathscr{F}_E$ onto $\mathscr{F}'_{E'}$ of and only if:

— h is a isomorphism of measure space (E, m_E) onto $(E', m'_{E'})$,
— $h(E \cap \mathrm{Orb}_G\, x) = E' \cap \mathrm{Orb}_{G'}\, hx$ for almost all x in M.

Let $E, F \in \mathscr{B} m(E) > 0$, $m(F) > 0$; let h be an isomorphism of (E, m_E) onto (F, m_F) such that:

— there exist a sequence $(g_n)_{n\in N}$, $g_n \in G$, and a partition (E_{g_n}) of E such that

$$h\,|_{E_{g_n}} = g_n|_{E_{g_n}}$$

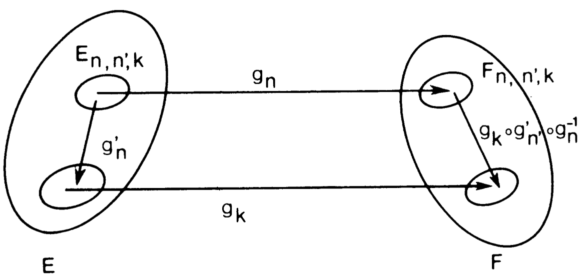
The triple (E, F, h) is called a *partial* G-*isomorphism;* if two elements E, F $\in \mathscr{B}$, $m(E) > 0$, $m(F) > 0$, are such that there exists a mapping $h: E \to F$ giving a partial G-isomorphism (E, F, h), we say that E and F are G-*equivalent* (or simply equivalent) and we note $E \underset{G}{\sim} F$ (or $E \sim F$); we can verify easily that $\underset{G}{\sim}$ is an equivalence.

LEMME 2. — Any partial G-isomorphism is a partial isomorphism.

Proof. — We use the preceding notations: let $g' \in [G_E]$ and let $E'_{g'_n}$ ($g'_n \in G$) be a partition of E such that

$$g'\,|_{E'_{g'_n}} = g'_n\,|_{E'_{g'_n}}$$

and let $E_{n,n',k} = E_{g_n} \cap E'_{g'_{n'}}\, g'^{-1}_{n'}(E_{g_k})$; $(E_{n,n',k})_{(n,n',k)\in N^3}$ form a partition of E, let $F_{n,n',k} = g_n(E_{n,n',k})$; $(F_{n,n',k})_{(n,n',k)\in N^3}$ form a partition of F.



We have

$$h' \circ g' \circ h^{-1}\,|_{F_{n,n'k}} = g_k \circ g'_{n'} \circ g_n^{-1}\,|_{F_{n,n',k}}$$

Therefore

$$h \circ g' \circ h^{-1} \in [G_F] \qquad \forall g' \in [G_E]$$

It follows that

$$h[G_E]h^{-1} = [G_F]$$

and (E, F, h) is a partial isomorphism. q. e. d. ■

LEMME 3. — Let $\mathscr{F} = (M, \mathscr{B}, m; G)$ be a dynamical system, $B \in \mathscr{B}$, $m(B) > 0$, let $E \subset B$, $F \subset B$. The following conditions are equivalent:

(*i*) E and F are G-equivalent.

(*ii*) E and F are G_B-equivalent.

Proof. — (*i*) $\Rightarrow$ (*ii*) Let $h : E \to F$ such that (E, F, h) is a partial G-isomorphism. Let

$$E = \sum_{n \leq 1} E_n \quad \text{such that} \quad h|_{E_n} = g_n|_{E_n}, \qquad g_n \in G$$

Let G′ be the (countable) group generated by $(g_n)_{n\in N}$ and G'_E the induced group generated by $(g'_E)_{g' \in G'}$; by Prop. 2.2

$$hx \in \mathrm{Orb}_{G'_E}\, x \qquad \forall x \in E.$$

Let

$$G'_E = \{ g_1, g_2, \dots \}$$
$$E'_n = \{ x \in E / hx = g'_n x \}$$
$$H_1 = E'_1$$
$$H_n = E'_n \Big\backslash \Big(\bigcup_{k \leq n-1} E'_k \Big)$$

We have

$$E = \Sigma H_n$$

and

$$h|_{H_n} = g'_n|_{H_n} \qquad \text{with} \qquad g'_n \in G'_E \subset G_E$$

Hence

$$E \underset{G_B}{\sim} F.$$

The converse is trivial since $G_E \subset [G]$. q. e. d. ■

§ 3. **Induced systems and invariant measures**

THEOREM 3. — Let $\mathscr{F} = (M, \mathscr{B}, m ; G)$ be a dynamical system, let $E \in \mathscr{B}$, $m(E) > 0$. If $\mu(\ll m)$ is an invariant measure for $\mathscr{F}$ then the restriction μ_E of μ on E is invariant for $\mathscr{F}_E$. Conversely let $\nu(\ll m_E)$ be an invariant measure for $\mathscr{F}_E$, then there exists a unique G-invariant $\mu \ll m$ on M such that $\nu = \mu_E$, and

$$\mathrm{supp}\,(\mu) = \mathrm{supp}_G\,(\mathrm{supp}\,\nu)$$

Proof. — It is clear, by definition of induced system, that if μ is G-invariant then μ_E is G_E-invariant. Conversely let $\nu(\ll, m_E)$ be a G_E-invariant measure on E, as supp ν is G_E-invariant; using Prop. 3 we can shall construct a G-invariant measure $\mu \sim m$ such that $\mu_E = \nu$.

As $M = \operatorname{ess\,sup}_{g \in G} gE$ and m is σ-finite, there exists a sequence $g_n \in G$, $g_0 = id_M$ such that

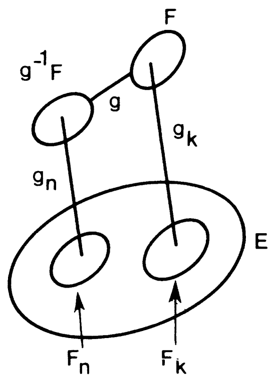
$$M = \bigcup_{n \geq 0} g_n E$$

let $M_0 = E$, $M_n = g_n E \setminus \left(\bigcup_{0 \leq k \leq n-1} g_k E \right)$, we have

$$M = \sum_{n \geqq 0} M_n$$

Let $M'_n = g_n^{-1} M_n$, we define $\mu_{M_n} = g_1 \nu_{M'_n}$, and μ is completely defined on M.

Let $g \in G$, $F \in \mathscr{B}$ a non null subset, we have to show that $\mu(g^{-1}F) = \mu(F)$, as $M = \sum_{n \geqq 0} M_n$ and g is an automorphism, we can suppose $F \subset M_k$ for a certain k. Let $F'_n = M_n \cap g^{-1}F$, $F_n = gF'_n$, we have $g^{-1}F = \sum_{n \geqq 0} F'_n$ and $F = \sum_{n \geqq 0} F_n$, we can finally suppose that $F \subset M_k$, $g^{-1}F \subset M_n$ for certain k, n.



Let $F_k = g_k^{-1}F$, $F_n = g_n^{-1}(g^{-1}F)$; $F_k \subset E$, $F_n \subset E$ and the triple $(F_n, F_k ; g_k^{-1} \circ g \circ g_n|_{F_n})$ is a partial G_E-isomorphism of $\mathscr{F}_E$ (Lemma 3), as ν is G_E-invariant, we have:

$$\nu(F_n) = \nu(F_k)$$

Therefore

$$\mu(g^{-1}F) = \nu(F_n) = \nu(F_k) = \mu(F)$$

We have proved that μ is G-invariant, the uniqueness of μ is immediate. q. e. d. ■

PROPOSITION 4 (See also [*34*]). — A discrete system is semi-finite.

Proof. — Let B be a G-atom such that $\text{supp}_G B = M$ (Th. III.1), we can suppose $0 < m(B) < \infty$; by Proposition 3 and the definition of a G-atom, every measurable subset of B is invariant for the induced system $\mathscr{F}_B$, the measure $\nu = m_B$ is G_B-invariant and the theorem 3 gives a measure $\mu(\sim m)$ G-invariant, so $\mathscr{F}$ is semi-finite (Proposition II.1). q. e. d. ■

§ 4. Types and induced systems

THEOREM 4. — Let $\mathscr{F} = (M, \mathscr{B}, m; G)$ be a dynamical system, $E \in \mathscr{B}$, $m(E) > 0$. If $\mathscr{F}$ is semi-finite (resp. purely infinite, discrete, continuous) then induced system $\mathscr{F}_E$ is of the same type. The converse is true if the G-support of E is the whole space M.

Proof. — We remark first that by Proposition IV.3, for measurable subsets of E the notions of G-atom and G_E-atom concide; the theorem follows from Theorem 3, Proposition II.1, and Theorem III.1. q. e. d. ■

Remark. — It is clear that if $\mathscr{F}$ is finite then so is $\mathscr{F}_E$; but the converse is not true, as one can easily see: if $\mathscr{F}$ is semi-finite there always exists $E \in \mathscr{B}$ such that $\mathscr{F}_E$ is finite and $\text{supp}_G E = M$.

V. COMPARISON OF MEASURABLE SUBSETS

DEFINITION 1 (cf. [*21*] [*34*] [*35*]). — Let $\mathscr{F} = (M, \mathscr{B}, m; G)$ be a dynamical system, $E, F \in \mathscr{B}$ we say that E is *G-bounded* by F and we write $E \underset{G}{\prec} F$ (or simply $E \prec F$) if there exists $F_1 \subset F$ such that $E \underset{G}{\sim} F_1$.

For the convenience of the reader, we summarize some important results on the relation $\prec$.

LEMMA 1 (cf. [*21*] [*34*] [*35*]).

a) $E \prec F$ implies $\text{supp}_G E \subset \text{supp}_G F$.

b) If $E \prec F$ and $B \in \mathscr{I}$ then $E \cap B \prec F \cap B$.

c) If $(E_i)_{i \in I}$ (resp. $(F_i)_{i \in I}$) is a measurable partition of E (resp. F) such that $E_i \prec F_i$ then $E \prec F$.

Proof (cf. [*21*] [*34*] [*35*]).

PROPOSITION 1 (cf. [*34*] [*35*]). — Let $E, F \in \mathscr{B}$. The following conditions are equivalent:

(*i*) $E \sim F$.

(*ii*) $E \prec F$ and $F \prec E$.

Proof (cf. [*34*] [*35*]).

THEOREM 1 (comparability theorem) (cf. [*34*] [*35*]). — Let E and F ne two measurable subsets of M. Then there exists an invariant measurable subset B such that

$$F \cap B \prec E \cap B$$
$$E \cap (M \backslash B) \prec F \cap (M \backslash B)$$

Proof (cf. [*34*] [*35*]).

COROLLARY 1. — Let $\mathscr{F} = (M, \mathscr{B}, m; G)$ be an ergodic system, let $E, F \in \mathscr{B}$. Then $E \underset{G}{\prec} F$ or $F \underset{G}{\prec} E$.

Proof. — Immediate $B = \emptyset$ or M. ■

COROLLARY 2. — Let $\mathscr{F} = (M, \mathscr{B}, m; G)$ be a dynamical system, $(E_i)_{i \in I}$ be a family of disjoint G-equivalent measurable subsets of M. There exist an invariant non null subset $B \in \mathscr{I}$ and a family $(F_k)_{k \in K}$ of disjoint equivalent non null subsets of B such that.

1) $I \subset K$.
2) $\forall k \in I$, $F_k \sim E_k \cap B$.
3) Let $F_0 = B \backslash \underset{k \in K}{\text{ess sup}}\, F_k$.

Then $F_0 \prec F_k$ and $F_0 \neq F_k$. If furthermore we suppose that I is infinite then we can suppose $F_0 = \emptyset$.

Proof. — Let $(E_k)_{k \in K}$ be a maximal family of disjoint G-equivalent subsets of M completing the family $(E_i)_{i \in I}$ $(I \subset K)$ and let $E = M \backslash (\underset{k \in K}{\text{ess sup}}\, E_k)$ and let $B \in \mathscr{I}$ such that

$$E_0 \cap B \prec E_k \cap B$$
$$E_k \cap B \prec E_0 \cap B.$$

(The choice of $k \in K$ has no consequence since E_k's are equivalent). Let

$$F_0 = E_0 \cap B, \qquad F_k = E_k \cap B, \qquad k \in K.$$

The relation $F_k \prec F_0$ would imply $E_k \prec E_0$, a contradiction with the maximality of the family $(E_k)_{k \in K}$. Therefore

$$F_k \neq \emptyset, \qquad k \in K.$$
$$F_0 \prec F, \qquad F_0 \sim F_k$$
$$F_0 = E_0 \cap B = B \backslash \underset{k \in K}{\text{ess sup}}\, E_k \cap B = B \backslash \underset{k \in K}{\text{ess sup}}\, F_k$$

If I is infinite; K is infinite, let $K' = K - \{ k_0 \}$, K′ is equipotent ot K and we have:

$$B = F_0 + \sum_{k \in K} F_k \sim F_0 + \sum_{k \in K'} F_k \prec \sum_{k \in K} F_k \subseteq B$$

Then $G \sim \sum_{k \in K} F_k$; replacing F_k by disjoint equivalent subsets, we can suppose $G = \sum_{k \in K} F_k$. q. e. d. ■

COROLLARY 3 (Characterization of continuous systems). — Let $\mathscr{F} = (M, \mathscr{B}, m; G)$ be a dynamical system. Then $\mathscr{F}$ is continuous of and only if every measurable subset of M is a sum of two disjoint equivalent measurable subsets.

Proof. — The condition is sufficient, for if $E \in \mathscr{B}$ is a non null G-atom of M, let $E = E_1 + E_2$ h a partial G-isomorphism from E_1 on E_2. We define $h' \in \text{Aut}(M, \mathscr{B}, m)$ by

$$h'x = \begin{cases} x & \text{if} \quad x \in M \backslash E \\ hx & \text{if} \quad x \in E_1 \\ h^{-1} & \text{if} \quad x \in E_2 \end{cases}$$

It is easy to see that $h' \in [G_E]$ and $h'E_1 = E_2$; by Proposition IV.3, $h'E_1 = E_1 \neq E_2$; this contradiction proves that $\mathscr{F}$ is continuous.

Suppose now that $\mathscr{F}$ is continuous, let $E \in \mathscr{B}$, $m(E) > 0$; by Zorn's lemma and the lemma 1; it suffices to show that there exist $E_1, E_2 \subset E$, $E_1 \underset{G}{\sim} E_2$ and $m(E_1) > 0$; by the lemma IV.3 and theorem IV.4, we can replace $\mathscr{F}$ by $\mathscr{F}_E$ and suppose $E = M$. Since M is not a G-atom, there exist $F \subset M$, $m(F) > 0$, and $g \in G$ such that $gF \backslash F \neq \emptyset$, let $E_2 = (gF) \backslash F$ and $E_1 = g^{-1}E_2$, we have $E_1 \sim E_2$, $E_1 \cap E_2 \neq \emptyset$. q. e. d. ■

VI. CLASSIFICATION OF MEASURABLE SUBSETS

§ 1. Classification of measurable subsets

DEFINITION 1. — Let $\mathscr{F} = (M, \mathscr{B}, m; G)$ be a dynamical system, $E \in \mathscr{B}$, we say that E is finite (resp. semi-finite, properly infinite, purely infinite, discrete, continuous) if the induced system $\mathscr{F}_E$ possesses the corresponding property. If there is any risk of confusion we write « G-finite », « G-semi-finite », etc. instead of finite, semi-finite, etc.

LEMMA 1. — Let $\mathscr{F} = (M, \mathscr{B}, m; G)$ be a dynamical system, let $(E_i)_{i\in I}$ be a family of measurable subsets of M such that the G-supports of E are disjoint and let $E = \sum_{i\in I} E_i\,(= (\operatorname{ess\,sup}_{i\in I} E_i))$. Then E is finite (resp. semi-finite, properly infinite, purely infinite discrete, continuous if and only if each E_i possesses the corresponding property.

Proof. — It is clear that $\mathscr{F}_E = \sum_{i\in I} \mathscr{F}_{E_i}$, and the lemma follows from the lemma II.1 and theorem III.1. q. e. d. ■

LEMMA 2. — Let $E \in \mathscr{B}$ be a finite (resp. semi-finite, purely infinite, discrete continuous) measurable subset of M and let $F \in \mathscr{B}$, $F \underset{G}{\prec} E$. Then F is finite (resp. semi-finite, purely infinite, discrete continuous).

Proof. — By the lemma IV.3 we can suppose $F \subset E$ and the lemma follows from theorem IV.4. q. e. d. ■

LEMMA 3. — Let $E \in \mathscr{B}$, $F \in \mathscr{B}$ be such that $\operatorname{supp}_G E \subset \operatorname{supp}_G F$. If E is a G-atom, then $E \underset{G}{\prec} F$.

Proof. — We can suppose $\operatorname{supp}_G F = M$; by the comparability theorem, let $B \in \mathscr{I}$ such that

$$E \cap B \prec F \cap B$$
$$F \cap B^c \prec E \cap B^c$$

Let h be a partial G-isomorphism from $F \cap B^c$ onto a subset of $E \cap B^c$, then $\operatorname{supp}_G h(F \cap B^c) = \operatorname{supp}_G (F \cap B^c) = B^c$ (Lemma V.1 *a*) and $\operatorname{supp}_G F = M$.

Since $E \cap B^c$ is a G-atom, and $h(F \cap B^c) \subset E \cap B^c$ we have

$$\begin{aligned} h(F \cap B^c) &= (\operatorname{supp}_G h(F \cap B^c)) \cap (E \cap B^c) \qquad \text{(Def. III.1)} \\ &= E \cap B^c \end{aligned}$$

Therefore $F \cap B^c \underset{G}{\sim} E \cap B^c$, hence $E \underset{G}{\prec} F$. q. e. d. ■

PROPOSITION 1 (Characterization of continuous semi-finite systems). — Let $\mathscr{F} = (M, \mathscr{B}, m; G)$ be a semi-finite system. Then $\mathscr{F}$ is continuous if and only if there exists a decreasing sequence $(E_n)_{n\in N}$ of G-finite measurable subsets of M, such that $\mathrm{supp}_G E_n = M$ and $(E_n \backslash E_{n+1}) \underset{G}{\sim} E_{n+1}$, $\forall n \in N$.

Proof. — The condition is necessary following the corollary 3 of theorem V.1. Now suppose that there exists such a sequence $(E_n)_{n\in N}$; replacing $\mathscr{F}$ by the induced system $\mathscr{F}_{E_1}$, and using the lemma IV.3, we can suppose that $E_1 = M$ and m is a *finite invariant* measure; let $F \in \mathscr{B}$ be a G-atom, by the lemma 3 we have

$$F \prec E_n \quad \text{for all} \quad n \in N.$$

Therefore

$$m(F) \leqq m(E_n) \leqq \frac{1}{2^n} m(M) \to 0, \qquad n \to \infty.$$

Hence $\mu(F) = 0$ and the system $\mathscr{F}$ is continuous. q. e. d. ■

§ 2. Hopf's theorem and its consequences

In [17] E. Hopf gave a characterization of the finiteness of a system generated by one automorphism, but the proof is valid for a general system if we replace the sentence « E is an image by division of F » by « E is G-equivalent to F » for E, $F \in \mathscr{B}$.

THEOREM 1 (Hopf's Theorem). — Let $\mathscr{F} = (M, \mathscr{B}, m; G)$ be a dynamical system. Then $\mathscr{F}$ is finite if and only if every subset of M that is G-equivalent to M coincides with M.

Let $E \in \mathscr{B}$; applying the theorem to the induced system $\mathscr{F}_E$ and using the lemma IV.3 we obtain:

COROLLARY 1. — Let $\mathscr{F} = (M, \mathscr{B}, m; G)$ be a dynamical system, $E \in \mathscr{B}$. Then E is finite if and only if every subset of E that is G-equivalent to E coincides with E.

COROLLARY 2. — Let $\mathscr{F} = (M, \mathscr{B}, m; G)$ be a dynamical system, $E \in \mathscr{B}$. Then E is properly infinite if and only if E is the sum of a sequence $\{E_n\}_{n\in N}$ of disjoint measurable subsets G-equivalent to E.

Proof. — Using the induced system $\mathscr{F}_E$ and the lemma IV.3 we can suppose E = M.

The condition is sufficient: suppose $M = \sum_{n \geq 0} E_n$, to show that $\mathscr{F}$ is properly infinite it is sufficient (and necessary) to prove that for all $B \in \mathcal{I}$, $\mathcal{F}_B$ is infinite (consequence of theorem II.1), since

$$B = M \cap B \sim E_n \cap B \qquad \text{(Lemma V.1)}$$

We have (Lemma V.1)

$$B = \sum_{n \geq 0} (E_n \cap B) \sim \sum_{n \geq 1} (E_n \cap B)$$

and the corollary 1 implies that B is infinite.

Now suppose that $\mathcal{F}$ is properly infinite, it is sufficient to show that there exist two disjoint measurable subsets F_1, F_2 of M such that $M = F_1 \cup F_2$ and $M \sim F_1 \sim F_2$ (for we can take $E_1 = F_1$ and we apply the result to F_2 that is properly infinite, etc.); using Zorn's lemma, it suffices to prove that there exist $B \in \mathcal{I}$, $F_1 \in \mathcal{B}$, $F_2 \in \mathcal{B}$, $m(B) > 0$, $F_1 \cap F_2 = \emptyset$, $B = F_1 \cup F_2$, and $B \sim F_1 \sim F_2$. Since $\mathcal{F}$ is infinite, let $M_2 \underset{\neq}{\subset} M = M_1$, $M_2 \sim M_1$ and h be a partial G-isomorphism form M_1 onto M_2, let $H_1 = M_1 \backslash M_2$, $H_2 = hH_1$, $H_n = hH_{n-1}$, we have $\emptyset \neq H_1 \sim H_2 \sim H_3 \sim \ldots$, by the corollary 2 of theorem V.1, there exist $B \in \mathcal{I}$, $m(B) > 0$ and $\{K_n\}_{n \in N}$, $\phi \neq K_n \in \mathcal{B}$, $K_n \cap K_{n'} = \emptyset$ of $n \neq n'$, such that

$$K_n \sim H_n \cap B, \quad n \in N, \qquad \text{and hence} \qquad K_n \sim K_{n'}, \quad \forall n, n' \in N.$$

$$B = \sum_{n \geq 0} K_n$$

Let

$$F_1 = K_0 + K_2 + K_4 + \ldots$$
$$F_2 = K_1 + K_3 + K_5 + \ldots$$

It is clear, by lemma V.1, that $B \sim F_1 \sim F_2$. q. e. d. ■

COROLLARY 3. — Let $\mathcal{F} = (M, \mathcal{B}, m; G)$ be a dynamical system, let E, $F \in \mathcal{B}$ such that $\text{supp}_G E \subset \text{supp}_G F$. If F is properly infinite then $E \prec F$. In particular if $\mathcal{F}$ is ergodic, two infinite measurable subsets are equivalent.

Proof. — Let $(E_i)_{i \in I}$ be a maximal family of non null disjoint measurable subsets of E such that $E_i \prec F$; as $E \supset \operatorname{ess\,sup}_{i \in I} E_i$ is in the G-support of F, the maximality of the family $(E_i)_{i \in I}$ implies that $E = \operatorname{ess\,sup}_{i \in I} E_i$, by the σ-finiteness of m, I must be countable, we can suppose $I \subset N$, by the corollary 2, $F = \Sigma F_n$, $F \sim F_1 \sim F_2 \sim \ldots$, hence

$$E = \sum_{i \in I} E_i \prec \sum_{n \in N} F_n = F. \quad \text{q. e. d.} \quad ■$$

VII. STRUCTURE OF DISCRETE SYSTEMS

§ 1. Homogeneous systems

DEFINITION 1. — Let $\mathcal{F} = (M, \mathcal{B}, m; G)$ be a dynamical system, we say that $\mathcal{F}$ is homogeneous if M is the sum of a family of non null disjoint equivalent G-atoms (cf. [5] [8]).

LEMMA 1. — Let $\mathcal{F} = (M, \mathcal{B}, m; G)$ be a homogeneous system, let $(E_i)_{i\in I}$ (resp. $(F_j)_{j\in J}$) be a partition of M into a family of equivalent G-atoms. Then card I = card J.

Proof. — If I is finite, the system $\mathcal{F}$ is finite and consequently J is finite. Therefore I and J are finite or infinite at the same time. If I and J are infinite, they are infinite countable, hence card I = card J.

Now suppose that I and J are finite. As the G-supports of E_i and E_j are M, by the lemma VI.3 we have $E_i \sim F_j$, $i \in I$, $j \in J$; we can suppose that card I = card J′ with $J' \subseteq J$.

$$\sum_{i\in I} E_i \sim \sum_{j\in J'} F_j$$

As $\mathcal{F}$ is finite, the Hopf's theorem implies $\sum_{j\in J'} F_j = M$ and $J = J'$ or card I = card J. q. e. d. ■

DEFINITION 2. — The above lemma proves that n = card I is an algebraic invariance of the homogeneous system $\mathcal{F}$. We say that $\mathcal{F}$ is a *type*-I_n-*system*, $n = 1, 2, \ldots, \mathcal{N}$ or a *n-homogeneous system.*

A type-I_n-system is finite if and only if n is finite, a type $I_{\mathcal{N}}$-system is semi-finite properly infinite (proposition IV.4 and corollary 2 of theorem VI.1).

§ 2. Structure of discrete systems

PROPOSITION 1. — A discrete system $\mathcal{F} = (M, \mathcal{B}, m; G)$ splits canonically as the direct sum of type-I_n-systems, $n = 1, 2, \ldots, \mathcal{N}$. $\mathcal{F}$ is finite if and only if its type-$I_{\mathcal{N}}$-subsystem is null.

Proof. — We can apply the corollary 2 of theorem V.1 to the family $(E_i)_{i\in I}$ reduced to one non null G-atom; using the same notations, $(F_j)_{j\in J}$

are non null disjoint G-equivalent G-atoms, F_0 is also a G-atoms (since $F_0 \prec F_j$). By the lemma 1 $\mathrm{supp}_G\, F_0 \neq \mathrm{supp}_G\, F_j$, there exists $C \in \mathscr{I}$, $C \subset B$ such that

$$C \cap F_j \neq \emptyset \qquad \forall j \in J$$
$$B = F_0 + \sum_{j \in J} F_j$$
$$C = \sum_{j \in J} C \cap F_j$$

As $C \cap F_j \sim C \quad F_j$ $(j, j' \in I)$ (lemma V.1). The induced system $\mathscr{F}_E$ is homogeneous and non-null and we apply the infinite recurrence and the proposition follows. q. e. d. ■

Product of dynamical systems

Definition 3. — Let $\mathscr{F}_i = (M_i, \mathscr{B}_i, m_i\,; G_i)_{i \in I}$ be a family of dynamical systems, let $M = \prod_I M_i$, $\mathscr{B} = \bigotimes_I \mathscr{B}_i$, $m = \bigotimes_I m_i$ let $g \in G_j$, $j \in J$ we define $\tilde{g}$ on M by:

$$M \ni (x_i)_{i \in I} \to (x'_i) = \tilde{g}((x_i)) \quad \text{with} \quad \begin{cases} x'_i = x_i & \text{if} \quad i \neq j \\ x'_i = g x_i & \text{if} \quad i = j. \end{cases}$$

It is clear that $\tilde{g} \in \mathrm{Aut}\,(M, \mathscr{B}, m)$; let G be the group generated by the set of $\tilde{g}$, $g \in G_i$, $i \in I$; we say that the system $\mathscr{F} = (M, \mathscr{B}, m\,; G)$ is the *product* of the systems $\mathscr{F}_i$ and we write $\mathscr{F} = \prod_I \mathscr{F}_i$.

Permutation systems

Let $n = 1, 2, \ldots, \mathscr{N}$, $M_n = [1, n + 1[$, $\mathscr{A}_n = \mathscr{P}(M_n)$ (the set of all subsets of M_n), m_n the discrete measure with unit mass at each point of M_n, G_n the permutation group at finite distance of M_n, the canonical dynamical system $\mathscr{P}_n = (M_n, \mathscr{A}_n, m_n\,; G_n)$ is called the *n-permutation system.* It is clear that $\mathscr{P}_n$ is an ergodic semi-finite discrete system, and $\mathscr{P}_n$ is finite if and only if n is finite; conversely it is easy to see that any ergodic discrete system is weakly isomorphic to a n-permutation system (since for ergodic system a G-atom is an atom).

CHARACTERIZATION OF THE G-ATOMS OF A SEPARABLE SYSTEM

PROPOSITION 2. — Let $\mathscr{F} = (M, \mathscr{B}, m ; G)$ be a separable system, let $E \in \mathscr{B}$. The following conditions are equivalent:

i) E is a G-atom.

ii) $G_E = \{ id_E \}$.

iii) For almost all x of M, the intersection of $\mathrm{Orb}_G\, x$ and E is reduced to an one-point set or an empty set.

Proof. — It is an immediate consequence of the proposition IV.1, 3), proposition IV.2, 2), proposition IV.3 the definition of a G-atom and the properties of Lebesgue spaces. q. e. d. ■

DEFINITION 4. — A system $\mathscr{T} = (M, \mathscr{B}, m ; G)$ with $G = \{ id_M \}$ is called a *motionless system.*

THEOREM 1. — A separable n-homogeneous system $\mathscr{F}$ is weakly isomorphic to the product of the n-permutation system $\mathscr{P}_n$ by a separable motionless system.

Proof. — Let $\mathscr{F} = (M, \mathscr{B}, m ; G)$, $M = \sum_{1 \leq k \leq n} E_k$, $E_1 \sim E_k$, $1 \leq k \leq n$, E_1 a G-atom. Let $\mathscr{T}_n = (E_1, \mathscr{B}_1, m_1 ; G_1)$ be the induced system, $\mathscr{T}_n$ is a motionless system (cf. Prop. 2).

Let h_k be a G-partial isomorphism from E_1 onto E_k, $h_1 = id_{E_1}$. Consider the mapping $\phi : (M_n, E_1) \in (k, x) \to h_k x$; it is easy to see that ϕ is a weak isomorphism from $\mathscr{P}_n \times \mathscr{T}_n$ onto $\mathscr{F}$. q. e. d. ■

This theorem and the proposition 1 imply:

COROLLARY 1. — A separable discrete system $\mathscr{F}$ is weakly isomorphic to a system of the form

$$\sum_{1 \leq n \leq \mathscr{N}} \mathscr{P}_n \times \mathscr{T}_n$$

where $\mathscr{P}_n$ is the n-permutation system and $\mathscr{T}_n$ is a motionless system.

COROLLARY 2. — Let $\mathscr{F} = (M, \mathscr{B}, m ; G)$ be a separable discrete system, let $\mathscr{I}$ be the invariant sub-σ-algebra of $\mathscr{B}$. Any automorphism of $(M, \mathscr{B}, m)$ leaving fixed the elements of $\mathscr{I}$ belongs to the full group of G.

Proof. — It is easy to see that the corollary is true for a system of the form $\mathscr{P}_n \times \mathscr{T}_n$, where $\mathscr{T}_n$ is a separable motionless system (properties of Lebesgue spaces), the general case follows from the corollary 2. ■

We summarize some important notions of type in the following definition.

DEFINITION 5 :

— A type-I_n-system is a n-homogeneous system $n = 1, 2, \ldots, \mathscr{N}$.

— A type-II_1-system is a finite continuous system.

— A type-II_∞-system is a semi-finite properly infinite continuous system.

— A type-III-system is a purely infinite system (it is automatically continuous).

A general dynamical system splits uniquely as the direct sum of these different systems.

Remarks. — Consider a dynamical system $\mathscr{F} = (M, \mathscr{B}, m; G)$, we suppose that G is a separable topological group and there exists a continuous unitary representation U of G into $L^2(M, \mathscr{B}, m)$ such that

$$({}_g f)h = U_g f U_g^* h \quad \forall f \in \mathscr{L}^\infty(M, \mathscr{B}, m), \quad \forall h \in L^2(M, \mathscr{B}, m), \quad \forall g \in G.$$

Let K be a countable dense subgroup of G,

$$\mathscr{I}(K) = \{ B \in \mathscr{B} \mid kB = B, \ \forall k \in K \},$$

then $\mathscr{I}(K) = \mathscr{I}(G)$. Applying the above results to $(M, \mathscr{B}, m; K)$ and using the density of K in G, we can see that *the theorem* 1 *and its corollaries remain valid for separable group* G.

VIII. PRODUCTS OF DYNAMICAL SYSTEMS

We summarize in this paragraph some important properties concerning the products of dynamical systems.

Let $\mathscr{F}_i = (M_i, \mathscr{B}_i, m_i; G_i)$, $i = 1, 2$ be two dynamical systems and let, $\mathscr{F} = \mathscr{F}_1 \times \mathscr{F}_2$, $\mathscr{F} = (M, \mathscr{B}, m; G)$ let $\mathscr{I}(\mathscr{F})$ (resp. $\mathscr{I}(\mathscr{F}_1)$, $\mathscr{I}(\mathscr{F}_2)$) be the sub-$\sigma$-algebra of invariant elements of $\mathscr{B}$ (resp. $\mathscr{B}_1$, $\mathscr{B}_2$).

§ 1. Products and invariant sub-σ-algebras

PROPOSITION 1. — We have $\mathscr{I}(\mathscr{F}_1 \times \mathscr{F}_2) = \mathscr{I}(\mathscr{F}_1) \otimes \mathscr{I}(\mathscr{F}_2)$; in particular, $\mathscr{F}$ is ergodic if and only if $\mathscr{F}_1$ and $\mathscr{F}_2$ are both ergodic.

Proof. — As $(M, \mathscr{B}, m)$ is the product of $(M_1, \mathscr{B}_1, m_1)$ and $(M_2, \mathscr{B}_2, m_2)$ we can identify canonically

$$L^2(M, \mathscr{B}, m) = L^2(M_1, \mathscr{B}_1, m_1) \otimes L^2(M_2, \mathscr{B}_2, m_2).$$

Let U^i the canonical representation of G_i into $L^2(M_i, \mathscr{B}_i, m_i)$, $i = 1, 2$, defined by (cf. [5], p. 134),

$$U^i_g f = \left(\frac{dg^{-1}m_i}{dm_i}\right)^{1/2} \cdot {}_g f \quad \forall g \in G_i, \quad f \in L^2(M_i, \mathscr{B}_i, m_i)$$

where $({}_g f)(x) = f(g^{-1}x)$, $\forall x \in M_i$.

We define the representation U of G into $L^2(M, \mathscr{B}, m)$ by the same way.

We identify canonically :

$$\begin{aligned} L^\infty(\mathscr{B}_1) &\simeq L^\infty(\mathscr{B}_1) \otimes 1_{L^2(\mathscr{B}_2,m_2)} \\ L^\infty(\mathscr{B}_2) &\simeq 1_{L^2(\mathscr{B}_1,m_1)} \otimes L^\infty(\mathscr{B}_2) \\ U^1_g &\simeq U^1_g \otimes 1_{L^2(\mathscr{B}_2,m_2)} \qquad \forall g \in G_1 \\ U^2_{g'} &\simeq 1_{L^2(\mathscr{B}_1,m_1)} \otimes U^2_{g'} \qquad \forall g' \in G_2. \\ L^\infty(\mathscr{I}(\mathscr{F}_1)) &\simeq L^\infty(\mathscr{I}(\mathscr{F}_1)) \otimes 1_{L^2(\mathscr{B}_2,m_2)} \\ L^\infty(\mathscr{I}(\mathscr{F}_2)) &\simeq 1_{L^2(\mathscr{B}_1,m_1)} \otimes L^\infty(\mathscr{I}(\mathscr{F}_2)) \end{aligned}$$

Let

$$\begin{aligned} \mathscr{A}_1 &= (L^\infty(\mathscr{B}_1) \cup U^1(G_1))'' \\ \mathscr{A}_2 &= (L^\infty(\mathscr{B}_2) \cup U^2(G_2))'' \\ \mathscr{A} &= (L^\infty(\mathscr{B}) \cup U(G))'' \end{aligned}$$

By the proposition 6, p. 26 of [5], we have

$$\mathscr{A} = \mathscr{A}_1 \otimes \mathscr{A}_2$$

It follows that (cf. [37] [38])

$$\mathscr{A}' = \mathscr{A}'_1 \otimes \mathscr{A}'_2$$

As $\mathscr{A}' = L^\infty(\mathscr{I}(\mathscr{F}))$, $\mathscr{A}'_i = L^\infty(\mathscr{I}(\mathscr{F}_i))$, $i = 1, 2$; we have :

$$L^\infty(\mathscr{I}(\mathscr{F}_1 \times \mathscr{F}_2)) = L^\infty(\mathscr{I}(\mathscr{F}_1)) \otimes L^\infty(\mathscr{I}(\mathscr{F}_2))$$

Hence

$$\mathscr{I}(\mathscr{F}_1 \times \mathscr{F}_2) = \mathscr{I}(\mathscr{F}_1) \otimes \mathscr{I}(\mathscr{F}_2).$$

q. e. d. ■

§ 2. **Products and induced systems**

PROPOSITION 2. — Let $\mathscr{F} = \mathscr{F}_1 \times \mathscr{F}_2$, let $E_1 \in \mathscr{B}_1$, $E_2 \in \mathscr{B}_2$. Then

$$(\mathscr{F}_1 \times \mathscr{F}_2)_{E_1 \times E_2} = \mathscr{F}_{1\,E_1} \times \mathscr{F}_{2\,E_2}.$$

Proof. — We have

$$(\mathcal{F}_1 \times \mathcal{F}_2)_{E_1 \times E_2} = (E_1 \times E_2, (\mathcal{B}_1 \otimes \mathcal{B}_2)_{E_1 \times E_2}(m_1 \otimes m_2)_{E_1 \times E_2} ; G_{E_1 \times E_2})$$
$$(\mathcal{F}_1 \times \mathcal{F}_2)_{E_1 \times E_2} = (E_1 \times E_2, \mathcal{B}_{1\,E_1} \otimes \mathcal{B}_{2\,E_2}, m_{1\,E_1} \otimes m_{2\,E_2} ; G_{1\,E_1} \vee G_{2\,E_2})$$

as one can easily verify from the definition VII.3 and the definition of induced systems. Hence

$$(\mathcal{F}_1 \times \mathcal{F}_2)_{E_1 \times E_2} = \mathcal{F}_{1\,E_1} \times \mathcal{F}_{2\,E_2}.$$

q. e. d. ■

§ 3. Products and discrete systems

PROPOSITION 3. — If $\mathcal{F}_1$ and $\mathcal{F}_2$ are discrete, then so is $\mathcal{F}_1 \times \mathcal{F}_2$. If $\mathcal{F}_i$ is of type I_{n_i}, $i = 1, 2$; then $\mathcal{F}_1 \times \mathcal{F}_2$ is of type $I_{n_1 \times n_2}$.

Proof. — Let E_i is a G_i-atom of $\mathcal{F}_i$ such that $\text{supp}_{G_i} E_i = M_i$ (cf. theorem III.1), $i = 1, 2$. We have:

$$\begin{aligned} \mathcal{I}(\mathcal{F}_1 \times \mathcal{F}_2)_{E_1 \times E_2} &= \mathcal{I}(\mathcal{F}_1)_{E_1} \otimes \mathcal{I}(\mathcal{F}_2)_{E_2} \\ &= \mathcal{B}_{1\,E_1} \otimes \mathcal{B}_{2\,E_2} \\ &= (\mathcal{B}_1 \otimes \mathcal{B}_2)_{E_1 \times E_2} \end{aligned}$$

Therefore, $E_1 \times E_2$ is a G-atom of $\mathcal{F}_1 \times \mathcal{F}_2$, and one can verify immediately that $\text{supp}_G (E_1 \times E_2) = M_1 \times M_2$; the theorem III.1 implies that $\mathcal{F}_1 \times \mathcal{F}_2$ is discrete. The second part of the proposition is immediate.

q. e. d. ■

§ 4. Products and semi-finite systems

PROPOSITION 4. — The product system $\mathcal{F}_1 \times \mathcal{F}_2$ is finite if and only if both of $\mathcal{F}_1$ and $\mathcal{F}_2$ are finite. $\mathcal{F}_1 \times \mathcal{F}_2$ is semi-finite if both of $\mathcal{F}_1$ and $\mathcal{F}_2$ are semi-finite.

Proof. — The proposition results immediately from the definitions and the proposition II.1. q. e. d. ■

§ 5. Products and type-II-systems

PROPOSITION 5. — The product $\mathcal{F}_1 \times \mathcal{F}_2$ is of type II if both of $\mathcal{F}_1$ and $\mathcal{F}_2$ are semi-finite and one of them is of type II.

Proof. — It follows from proposition 4 that $\mathcal{F}_1 \times \mathcal{F}_2$ is semi-finite, we have to show that it is continuous; suppose that $\mathcal{F}_1$ is of type II, it follows from proposition VI.1 that there exists a decreasing sequence

$\{ E_n^1 \}_{n\in N}$ of G_1-finite measurable subsets of M_1 such that $\text{supp}_{G_1} E_n^1 = M_1$ and $(E_n^1 \backslash E_{n+1}^1) \underset{G_1}{\sim} E_{n+1}^1$; let $E_n = E_n^1 \times M_2$, it is clear that the sequence $(E_n)_{n\in N}$ satisfies the conditions of proposition VI.1, that implies that $\mathscr{F}_1 \times \mathscr{F}_2$ is continuous. q. e. d. ■

§ 6. Products and type-III-systems

PROPOSITION 6. — If the product $\mathscr{F}_1 \times \mathscr{F}_2$ is of type III, then one of $\mathscr{F}_1$ and $\mathscr{F}_2$ is of type III.

Proof. — The proposition results from the definition a type-III-systems (cf. Def. VII.5), the theorem II.1 and the proposition 4. q. e. d. ■

§ 7. Products and continuous systems

PROPOSITION 7. — If the product $\mathscr{F}_1 \times \mathscr{F}_2$ is continuous, then one of $\mathscr{F}_1$ and $\mathscr{F}_2$ is continuous.

Proof. — The proposition follows from theorem III.2 and proposition 2.
q. e. d. ■

Remark. — In a subsequent paper, we shall show that the converse of proposition 6 an proposition 7 is true if G is a separable topological group acting continuously on L^∞, for the topology $\sigma(L^\infty, L^1)$.

§ 8. Factorization of a type-II_∞-system

PROPOSITION 8. — Let $\mathscr{F} = (M, \mathscr{B}, m; G)$ be a properly infinite system; then there exists a non null measurable subset E of M such that $\mathscr{F}$ is weakly isomorphic to $\mathscr{F}_E$ and $\mathscr{F}_E \times \mathscr{P}_{\mathscr{N}}$; in particular, $\mathscr{F}$ is weakly isomorphic to $\mathscr{F} \times \mathscr{P}_{\mathscr{N}}$.

Proof. — By the corollary 2 of theorem VI.1, M is the disjoint union of a sequence $(E_n)_{n\in N}$ of measurable subsets such that $E_n \underset{G}{\sim} M$, $\forall n \in N$; let $E = E_0$, by the lemma IV.3, $\mathscr{F}$ is weakly isomorphic to $\mathscr{F}_E$. By a method similar to that of the proof of theorem VII.1, we can see that $\mathscr{F}$ is weakly isomorphic to $\mathscr{F}_E \times \mathscr{P}_{\mathscr{N}}$. q. e. d. ■

PROPOSITION 9. — Let $\mathscr{F} = (M, \mathscr{B}, m; G)$ be a type II_∞-system; then there exists a non null measurable subset E of M such that:

— $\mathcal{F}_E$ is a type-II_1-system.
— $\mathcal{F}$ is weakly isomorphic to $\mathcal{F}_E \times \mathcal{P}_{\mathcal{N}}$.

Proof. — By the corollary 2 of theorem VI.1, M is a disjoint union of a sequence $(E_n)_{n \in N}$ of measurable subsets such that $E_n \underset{G}{\sim} E_{n'}$, $n, n' \in N$; as each E_n is semi-finite, there exists a sequence $(F_n)_{n \in N_n}$ of non-null G-*finite* measurable subsets such that:

$$F_n \underset{G}{\sim} F_{n'}, \qquad \forall n, n' \in N.$$

By the corollary 2 of theorem V.1, the corollary 2 of theorem VI.1 and an exhaustive induction, we can suppose that M is the disjoint union of the F_n's; let $E = F_0$, $\mathcal{F}_E$ is of type II_1; a proof similar to that of theorem VII.1 shows that $\mathcal{F}$ is weakly isomorphic to $\mathcal{F}_E \times \mathcal{P}_{\mathcal{N}}$.

q. e. d. ■

BIBLIOGRAPHY

[1] L. K. ARNOLD, On σ-finite invariant measures. *Z. Wahrscheinlichkeitstheorie verw. Geb.*, t. **9**, 1968, p. 85-97.
[2] A. BRUNEL, Sur les mesures invariantes. *Z. Wahrscheinlichkeitstheorie verw. Geb.*, t. **5**, 1966, p. 330-333.
[3] A. CALDERON, Sur les mesures invariantes. *C. R. Acad. Sci.* (Paris), t. **240**, 1955, p. 1960-1962.
[4] R. V. CHACON, A class of linear transformations. *Proc. Amer. Math. Soc.*, t. **15**, 1964, p. 560-564.
[5] J. DIXMIER, *Les algèbres d'opérateurs dans l'espace hilbertien.* Gauthier-Villars, 1969.
[6] J. DIXMIER, *Les C*-algèbres et leurs représentations.* Gauthier-Villars, 1969.
[7] Y. N. DOWKER, On measurable transformations in finite measure spaces. *Ann. of Math.*, t. **62** (2), 1955, p. 504-516.
[8] H. A. DYE, On groups of measure preserving transformations I. *Amer. J. Math.*, t. **81**, 1959, p. 119-159.
[9] H. A. DYE, On groups of measure preserving transformations II. *Amer. J. Math.*, t. **85**, 1963, p. 551-576.
[10] A. GUICHARDET, Une caractérisation des algèbres de Von Neumann discrètes. *Bull. Soc. Math. France*, t. **89**, 1961, p. 77-101.
[11] A. B. HAJIAN, On ergodic measure preserving transformations defined on an infinite measure space. *Proc. Amer. Math. Soc.*, t. **16**, 1965, p. 45-48.
[12] A. B. HAJIAN, Y. ITO and S. KAKUTANI, Invariant measures and orbits of dissipative transformations. *Advances in Math.*, t. **8**, 1972, p. 52-65.
[13] A. B. HAJIAN and S. KAKUTANI, Weakly wandering sets and invariant measures. *Trans. Amer. Math. Soc.*, t. **110**, 1964, p. 136-151.
[14] A. B. HAJIAN and S. KAKUTANI, Transformations of σ-Type. *To appear.*
[15] P. HALMOS, *Lectures on ergodic theory.* Math. Soc. of Japan, N. Y., Chelsea Publishing Co., 1956.
[16] P. HALMOS, Invariant measures. *Ann. of Math., Ser. II*, t. **48**, 1947, p. 735-754.
[17] E. HOPF, Theory of measure and invariant integrals. *Trans. Amer. Math. Soc.*, 1932, p. 373-393.

[18] K. JACOBS, Invariant and non invariant measures. *Lecture Notes in Math.* (Springer-Verlag), n° 31, 1967, p. 118-135.

[19] L. K. JONES and U. KRENGEL, On transformations without finite invariant measure. *To appear.*

[20] S. KAKUTANI, Induced measure perserving transformations. *Proc. Imp. Acad. Tokyo*, t. **19**, 1943, p. 635-641.

[21] Y. KAWADA, Uber die exitenz der invarianten Integrale. *Jap. J. Math.*, t. **19**, 1944, p. 81-95.

[22] U. KRENGEL, Entropy of conservative transformations. *Z. Wahrscheinlichkeitstheorie verw. Geb.*, t. **7**, 1967, p. 161-181.

[23] W. KRIEGER, On the isomorphy problem for ergodic equivalence relations. *Math. Zeitschr.*, t. **103**, 1968, p. 78-84.

[24] W. KRIEGER, On Non-singular Transformations of a measure space I. *Z. Wahrscheinlichkeitstheorie verw. Geb.*, t. **11**, 1969, p. 83-97.

[25] W. KRIEGER, On Non-singular Transformations of a measure space II. *Z. Wahrscheinlichkeistheorie Verw. Geb.*, t. **11**, 1969, p. 98-119.

[26] W. KRIEGER, On constructing non-*isomorphic Hyperfinite factors of Type III. *J. Funct. Analysis*, t. **6**, 1970, p. 97-109.

[28] J. NEVEU, Sur l'existence des mesures invariantes en théorie ergodique. *C. R. Acad. Sci.* (Paris), t. **260**, 1965, p. 393-396.

[29] J. NEVEU, Existence of bounded inv. measures in ergodic theory. *Proc. fifth Berkeley Sym. Math. Stat. Prob.*, University of California Press, Vol. II, Part 2, 1967, p. 461-472.

[30] J. NEVEU, *Bases Mathématiques du Calcul des Probabilités.* Masson, 1970.

[31] J. NEVEU, Atomes conditionnels d'espaces de Probabilité et Théorie de l'Information. Symposium on Probability Methods in Analysis. *Lecture Notes in Math.*, Springer-Verlag, t. **31**, 1967, p. 256-271.

[32] D. ORNSTEIN, On invariant measures. *Publ. Amer. Math. Soc.*, t. **66**, 1960.

[33] S. SAKAI, *C*-Algebras and W*-algebras.* Springer-Verlag, 1971.

[34] E. STØRMER, Automorphisms and equivalence in Von Neumann Algebras. *To appear.*

[35] E. STØRMER, Invariant measures and Von Neumann Algebras. *To appear.*

[36] L. SUCHESTON, On existence of finite invariant measures. *Math. Z.*, t. **86**, 1964, p. 327-336.

[37] M. TAKESAKI, Tomita's Theory of Modular Hilbert Algebras. *Lecture Notes*, n° 128, Springer-Verlag, Berlin, 1970.

[38] M. TAKESAKI, The Theory of Operator Algebras. Univ. of California, Los Angeles, C. A., 1969-1970.

(Manuscrit reçu le 5 juillet 1973)