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Section B :
Calcul des Probabilités et Statistique.

An operator-valued stochastic integral, II (*)

by

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1. INTRODUCTION AND PRELIMINARIES

In [*7*], Kannan and Bharucha-Reid have defined an operator-valued stochastic integral using the notion of tensor-product of elements of a Hilbert space. In this paper we continue the same with a small difference in the definition of a Brownian motion. In [*7*], the covariance function is defined through the inner-product $\langle \cdot , \cdot \rangle$ and here we define it through the tensor-product $g \otimes h$. Since our integral is operator valued, we obtain four more integrals associated with this integral. The purpose of this paper is to define these integrals and give Itô's formula corresponding to these integrals. In this introductory section we give the basic definitions and notions we use throughout this paper. In section 2, we define the Brownian motion and give a sample-path property. An operator-valued stochastic integral and the stochastic integrals associated with this operator-valued integral are defined in section 3. Section 4 gives the Itô's formula for these integrals. For the standard technique and the results in the scalar case we refer to the book A. V. Skorokhod [*13*], (see also Doob [*2*], Itô [*6*] and Kunita [*9*]). We use the standard technique to prove Itô's formula for the operator-valued stochastic integral and the same method goes for other integrals.

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Let $(\Omega, \mathscr{A}, \mu)$ be a complete probability space and all the sub-σ-algebras of $\mathscr{A}$, that we consider in this paper be complete relative to the probability measure μ. Let H be a real separable Hilbert space with inner-product $\langle \cdot , \cdot \rangle$ and norm $\| \cdot \|$. By **B**(H) we denote the Banach algebra of endomorphisms of H, with $\| \cdot \|_B$ standing for the operator norm.

A mapping $x : \Omega \to H$ is called *a random element in* H, if for each $h \in H$, the scalar function $\langle x(\omega), h \rangle$ is a real random variable. A mapping $T : \Omega \to \mathbf{B}(H)$ is called *a random operator* on H, if for each $h \in H$, $T(\omega)[h]$ is a random element in H.

For a random element x in H, clearly $\| x \|$ is a real random variable (H is separable). The random element $x(\omega)$ is said to be *integrable* if $\| x(\omega) \|$ is integrable. The integrability of a random operator is similarly defined. The *expectation* of a random element $x(\omega)$ in H is an element $Ex \in H$ such that $\langle Ex, h \rangle = E \langle x, h \rangle$, for $h \in H$. For a given sub-σ-algebra $\mathscr{B} \subset \mathscr{A}$, the *conditional expectation* $E\{ x \mid \mathscr{B} \}$ of x relative to $\mathscr{B}$ is defined as follows: $E\{ x \mid \mathscr{B} \}$ is a random element in H such that $\langle h, E\{ x \mid \mathscr{B} \} \rangle$ is $\mathscr{B}$-measurable and $\langle h, E\{ x \mid \mathscr{B} \} \rangle = E\{ \langle h, x \rangle \mid \mathscr{B} \}$ holds for each $h \in H$. A random element $x(\omega)$ in H (respectively a random operator $T(\omega)$ in **B**(H)) is said to be of *second order* if $E\{ \| x(\omega) \|_H^2 \} < \infty$ (respectively $E\{ \| T(\omega) \|^2 \} < \infty$). The collection of equivalence class of second order random elements is the Hilbert space $L_2(\Omega, H)$.

Two random elements x and y in H are said to be *independent* if, for $g, h \in H$, the real random variables $\langle x(\omega), g \rangle$ and $\langle y(\omega), h \rangle$ are independent. If the random elements $T_1 h_1$ and $T_2 h_2$, $h_1, h_2 \in H$, are independent, then we say that the random operators T_1 and T_2 are independent.

For the following notion of tensor product of elements of a Hilbert space we refer to Schatten [*11, 12*]. Let $x, y \in H$. The symbol $x \otimes y$ defines an endomorphism of H through

$$(x \otimes y)h = \langle h, y \rangle_H x \tag{1.1}$$

for all $h \in H$. If x and y are two random elements in H, then, $(x \otimes y)$ is a random operator on H. For, if $g, h \in H$,

$$\langle (x(\omega) \otimes y(\omega))g, h \rangle_H = \langle x(\omega), h \rangle_H \langle g, y(\omega) \rangle_H$$

An operator $T \in \mathbf{B}(H)$ is said to be a *Hilbert-Schmidt-class operator* if, for a complete orthonormal basis $\{ e_i \}$, $i \geq 1$, for H,

$$\sum_{i=1}^{\infty} \| Te_i \|_H^2 < \infty.$$

The collection $[\sigma c]$ of Schmidt-class operators is a Hilbert space with inner-product

$$\langle \mathrm{T}, \mathrm{U} \rangle_\sigma = \sum_{i=1}^{\infty} \langle \mathrm{T}e_i, \mathrm{U}e_i \rangle_{\mathrm{H}} \tag{1.2}$$

$\mathrm{T} \in \mathbf{B}(\mathrm{H})$ is said to be a *trace-class operator* if $\sum_{i=1}^{\infty} | \langle \mathrm{T}e_i, e_i \rangle_{\mathrm{H}} | < \infty$.

For the sake of reference we list the following result.

LEMMA 1.1 (R. Schatten [*11, 12*]). — *Let* $g, h \in \mathrm{H}$. *Then, the operator* $g \otimes h \in \mathbf{B}(\mathrm{H})$ *satisfies the following*

(*a*) $g \otimes h$ is linear in g and also in h;
(*b*) $(g_1 \otimes h_1)(g_2 \otimes h_2) = \langle g_2, h_1 \rangle_{\mathrm{H}}\; g_1 \otimes h_2$;
(*c*) $(g \otimes h)^* = (h \otimes g)$;
(*d*) for any $\mathrm{T} \in \mathbf{B}(\mathrm{H})$, $(\mathrm{T}g \otimes h) = \mathrm{T}(g \otimes h)$ and $(g \otimes \mathrm{T}h) = (g \otimes h)\mathrm{T}^*$;
(*e*) $\|\mathrm{T}\| \leq \|\mathrm{T}\|_\sigma$;
(*f*) for $g_i, h_i, \phi_j, \psi_j \in \mathrm{H}$, $1 \leq i \leq n$, $1 \leq j \leq m$,

$$\left\langle \sum_{i=1}^{n} g_i \otimes h_i, \sum_{j=1}^{m} \phi_j \otimes \psi_j \right\rangle_\sigma = \sum_{i=1}^{n} \sum_{j=1}^{m} \langle g_i, \phi_j \rangle_{\mathrm{H}} \langle \psi_j, h_i \rangle_{\mathrm{H}};$$

(*g*) $g \otimes h \in [\sigma c]$, $g \otimes h \in [\tau c]$, the trace class;
(*h*) $\|g \otimes h\|_\sigma = \|g\|_{\mathrm{H}} \|h\|_{\mathrm{H}}$, $\|g \otimes h\|_\tau = \|g\|_{\mathrm{H}} \|h\|_{\mathrm{H}}$ and

$$\mathrm{tr}\,(g \otimes h) = \langle g, h \rangle_{\mathrm{H}}.$$

The following lemmas are the extensions of the corresponding results in the scalar variables case.

LEMMA 1.2. — *Let* $x(\omega)$ *and* $y(\omega)$ *be two integrable independent random elements in* H. *Then*

$$\mathrm{E}(x \otimes y) = \mathrm{E}x \otimes \mathrm{E}y$$

Proof. — Let $g, h \in \mathrm{H}$. Then

$$\begin{aligned} &\langle \mathrm{E}(x \otimes y)g, h \rangle = \mathrm{E}\langle (x \otimes y)g, h \rangle \\ &= \mathrm{E}\{\langle g, y \rangle \langle x, h \rangle\} = \mathrm{E}\langle g, y \rangle \mathrm{E}\langle x, h \rangle \\ &= \langle g, \mathrm{E}y \rangle \langle \mathrm{E}x, h \rangle = \langle \langle g, \mathrm{E}y \rangle \mathrm{E}x, h \rangle = \langle (\mathrm{E}x \otimes \mathrm{E}y)g, h \rangle. \end{aligned}$$

LEMMA 1.3. — *Let* $\mathscr{B}$ *be a sub-σ-algebra of* $\mathscr{A}$. *Let* x *and* y *be two integrable random elements in* H. *Then, if* x *is* $\mathscr{B}$*-measurable, we have*

$$\mathrm{E}\{(x \otimes y) \mid \mathscr{B}\} = x \otimes \mathrm{E}\{y \mid \mathscr{B}\}.$$

Proof. — As in Lemma 1.2 it is enough if we show that

$$\langle \mathrm{E}\{(x \otimes y) \mid \mathscr{B}\} g, h \rangle = \langle (x \otimes \mathrm{E}\{y \mid \mathscr{B}\}) g, h \rangle,$$

for $g, h \in \mathrm{H}$. Now,

$$\begin{aligned} &\langle \mathrm{E}\{(x \otimes y) \mid \mathscr{B}\} g, h \rangle = \mathrm{E}\{\langle (x \otimes y) g, h \rangle \mid \mathscr{B}\} \\ &= \mathrm{E}\{(\langle g, y \rangle \langle x, h \rangle) \mid \mathscr{B}\} = \langle x, h \rangle \mathrm{E}\{\langle g, y \rangle \mid \mathscr{B}\} \\ &= \langle x, h \rangle \langle g, \mathrm{E}\{y \mid \mathscr{B}\} \rangle = \langle \langle g, \mathrm{E}\{y \mid \mathscr{B}\} \rangle x, h \rangle \\ &= \langle (x \otimes \mathrm{E}\{y \mid \mathscr{B}\}) g, h \rangle. \end{aligned}$$

Hence the lemma.

2. A BROWNIAN MOTION PROCESS IN HILBERT SPACE

In this section we define a Hilbert space valued Brownian motion. A simple observation gives the analogue of the classical result that almost every sample path of a Brownian motion has infinite variation on every finite interval.

Let I be an interval, say $[a, b]$, on the real line and $\{\xi_t(\omega), t \in \mathrm{I}\}$ be an H-valued process. By Ξ we denote the Hilbert space of all equivalence classes of second order H-valued stochastic processes; and the norm in Ξ is given by

$$\|\xi\|_{\Xi} = \left[\int_a^b \mathrm{E}\|\xi_t\|_{\mathrm{H}}^2 dt\right]^{\frac{1}{2}} \tag{2.1}$$

Let $\xi \in \Xi$ and define the operator covariance by

$$\Gamma(s, t) = \mathrm{E}\{\xi(s) \otimes \xi(t)\} \tag{2.2}$$

(Without loss of generality we assume that all the random functions are centered). If $\xi \in \Xi$,

$$\begin{aligned} \mathrm{E}\|\xi(s) \otimes \xi(t)\|_{\mathrm{B}} &\leq \mathrm{E}\|\xi(s) \otimes \xi(t)\|_{\sigma} \\ &= \mathrm{E}\{\|\xi(s)\|_{\mathrm{H}}\|\xi(t)\|_{\mathrm{H}}\} \leq [\mathrm{E}\|\xi(s)\|_{\mathrm{H}}^2 \mathrm{E}\|\xi(t)\|_{\mathrm{H}}^2]^{\frac{1}{2}} < \infty \end{aligned}$$

Thus the covariance function exists for second order processes.

DÉFINITION 2.1. — A process $\beta(t, \omega) \in \Xi$ is called *a Brownian motion in* H, if

(1) $\mathrm{E}\beta(t) = 0$, for all $t \in \mathrm{I}$;
(2) as a function of t, $\beta(t)$ is continuous, almost surely;

(3) $\beta(t)$ has independent increments, that is, if $t_1 < t_2 \leq t_3 < t_4$, the increments $\beta(t_4) - \beta(t_3)$ and $\beta(t_2) - \beta(t_1)$ are independent; and

(4) $$\mathrm{E}\{(\beta(t) - \beta(s)) \otimes (\beta(t) - \beta(s))\} = |t - s|\,\mathrm{S}, \tag{2.3}$$

where S is a positive definite operator in $[\tau c]$, the trace class. (For details of Schmidt-class and trace-class operators we refer to Dunford and Schwartz [3], and Schatten [11, 12].) In [7], the above condition (4) of Definition 2.1 was given by $\mathrm{E}\{\langle \beta(t) - \beta(s), \beta(t) - \beta(s) \rangle\} = |t - s|$. But, using Lemma 1.1 (*h*), condition (2.3) gives

$$\mathrm{E}\{\langle \beta(t) - \beta(s), \beta(t) - \beta(s) \rangle\} = \mathrm{tr}\,(\mathrm{S})\,|t - s|. \tag{2.4}$$

PROPOSITION 2.1. — *Let* H *and* K *be two separable real Hilbert spaces and let* $\mathrm{T} \in \mathbf{B}(\mathrm{H}, \mathrm{K})$, *the Banach space of bounded linear operators from* H *to* K. *If* $\beta(t)$ *is a Brownian motion in* H, *then* $\mathrm{B}(t) = (\mathrm{T}\beta)(t)$ *is a Brownian motion in* K.

The proof is a simple verification of definition 2.1. Conditions (1) and (2) are clear. That $\mathrm{B}(t)$ has independent increments follows from the definition of independent random elements upon noting

$$\langle k, \mathrm{B}(t) - \mathrm{B}(s) \rangle_{\mathrm{K}} = \langle \mathrm{T}^* k, \beta(t) - \beta(s) \rangle_{\mathrm{H}}, \ k \in \mathrm{K}.$$

Condition (4) follows from

$$\begin{aligned} &\mathrm{E}\{(\mathrm{B}(t) - \mathrm{B}(s)) \otimes (\mathrm{B}(t) - \mathrm{B}(s))\} \\ &\qquad = \mathrm{TE}\{(\beta(t) - \beta(s)) \otimes (\beta(t) - \beta(s))\}\,\mathrm{T}^* = |t - s|\,\mathrm{TST}^*, \end{aligned}$$

and that TST^* is a positive definite operator in $[\tau c]$.

From the definition of measurability, (also from proposition 2.1), it follows that $\langle h, \beta(t) \rangle_{\mathrm{H}}$ is a scalar Brownian motion for each $h \in \mathrm{H}$. Hence, from the classical result about the sample path behavior of scalar Brownian motion, it follows that every Brownian motion in H has the property that almost every sample path of $\beta(t)$ has infinite variation in weak sense. A function $f : [a, b] \to \mathrm{H}$ is of *bounded variation* if

$$\sup \left\| \sum_i \{ f(b_i) - f(a_i) \} \right\|_{\mathrm{H}} < \infty,$$

for every choice of a finite number of non-overlapping intervals (a_i, b_i) in $[a, b]$. (For notions and properties of functions of weak bounded variation, bounded variation and strong bounded variation we refer to Hille and Phillips [5]). It is known that $f : [a, b] \to \mathrm{H}$ is of bounded variation if and only if it is of weak bounded variation. Thus we have the following observation.

PROPOSITION 2.2. — *If $\beta(t)$ is a Brownian motion in H, then almost every sample path of $\beta(t)$ has infinite variation on every finite interval.*

3. STOCHASTIC INTEGRALS

In this section we first define an operator valued stochastic integral and obtain few more stochastic integrals with suitable operations on the operator integral. The operator integral is defined the same way as in [7]. The difference here is only in the condition (2.3) in the definition of Brownian motion in H. Here we also assume, without loss of generality, that H has a complete orthonormal basis $\{ e_i \}$ consisting of the eigenvectors of S that appears in condition (2.3). Let $\{ \lambda_i \}$ be the corresponding sequence of eigenvalues. (We remark here that the inner product $\langle \cdot , \cdot \rangle_\sigma$ in $[\sigma c]$ is independent of the choice of basis for H). Let $\{ \mathscr{A}_t, t \in \mathrm{I} \}$ be an increasing family of sub-σ-algebras of such that (a) for every t, $\xi(t)$, $\beta(t) \in \Xi$ are $\mathscr{A}_t$-measurable and (b) for any $t \in \mathrm{I}$, the random elements $\beta(s_1) - \beta(t), \ldots, \beta(s_k) - \beta(t)$ are independent of $\mathscr{A}_t$, for $s_1, \ldots, s_k \in [t, b]$; such a family clearly exists.

A process $\xi(t) \in \Xi$ is said to be *non-anticipatory of the Brownian motion* $\beta(t)$ if, for $r, s, t \in \mathrm{T}$, $r \leq s \leq t$, $\xi(r)$ and $\beta(t) - \beta(s)$ are independent. By Ξ_β we denote the closure of the linear manifold of all processes $\xi(t) \in \Xi$ that are non-anticipatory of β. Let Ξ_β^0 denote all the simple processes non-anticipatory of β. Ξ_β^0 is dense in Ξ_β.

3-A. An operator-valued stochastic integral.

As is customary we first define the integral for $\xi \in \Xi_\beta^0$ and then extend it to all $\xi \in \Xi_\beta$. Let $a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$, and

$$\xi(t) = \begin{cases} \xi(t_i) & \text{for} \quad t_i \leq t \leq t_{i+1} \\ 0 & \text{otherwise.} \end{cases}$$

Define $\mathrm{I}[\xi ; \omega] = \int_a^b \xi(t, \omega) d\beta(t, \omega)$ by

$$\int_a^b \xi(t) d\beta(t) = \sum_{i=0}^{n-1} \xi(t_i) \otimes [\beta(t_{i+1}) - \beta(t_i)]. \tag{3.1}$$

Clearly $\mathrm{I}[\xi ; \omega]$ is a random operator in $[\sigma c]$. Thus our integral is a Schmidt-class random operator. When $\mathrm{H} = \mathrm{R}_1$, the tensor product of elements is just the usual product in R_1 and so (3.1) is a natural extension to Hilbert space case. Basic properties of (3.1) are given in [7].

From Lemma 1.1 (*a*), I is linear on Ξ_β^0. Clearly $E\{ I[\xi; \omega] \} = 0$. An important relation is

$$E\{ \| I[\xi; \omega] \|_\sigma^2 \} = \operatorname{tr}(S) \| \xi \|_\Xi^2 \tag{3.2}$$

For,

$$E\{ \| I[\xi] \|_\sigma^2 \} = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} E\{ \langle \xi(t_i) \otimes \beta(\Delta_i), \xi(t_i) \otimes \beta(\Delta_i) \rangle_\sigma \}$$

where $\beta(\Delta_i) = \beta(t_{i+1}) - \beta(t_i)$, $i = 0, \ldots, n-1$. For $i \lneq j$, we have

$$\begin{aligned} & E\{ \langle \xi(t_i) \otimes \beta(\Delta_i), \xi(t_j) \otimes \beta(\Delta_j) \rangle_\sigma \} \\ & = E\{ E\{ \langle \xi(t_i) \otimes \beta(\Delta_i), \xi(t_j) \otimes \beta(\Delta_j) \rangle_\sigma \mid \mathscr{A}_{t_j} \} \} \\ & = E \langle \xi(t_i) \otimes \beta(\Delta_i), \xi(t_j) \otimes E\{ \beta(\Delta_j) \mid \mathscr{A}_{t_j} \} \rangle_\sigma \\ & = 0, \end{aligned}$$

using $\mathscr{A}_{t_j}$-measurability of $\xi(t_i)$, $\beta(\Delta_i)$ and $\xi(t_j)$ and Lemma 1.3. Thus, from Lemma 1.1, independence and (2.4),

$$\begin{aligned} E\{ \| I[\xi] \|_\sigma^2 \} &= \sum_{i=0}^{n-1} E\{ \langle \xi(t_i) \otimes \beta(\Delta_i), \xi(t_i) \otimes \beta(\Delta_i) \rangle_\sigma \} \\ &= \sum_{i=0}^{n-1} E\{ \langle \xi(t_i), \xi(t_i) \rangle_H \langle \beta(\Delta_i), \beta(\Delta_i) \rangle_H \} = \sum_{i=0}^{n-1} E \| \xi(t_i) \|_H^2 E \| \beta(\Delta_i) \|_H^2 \\ &= \sum_{i=1}^{n-1} E \| \xi(t_i) \|_H^2 \operatorname{tr}(S)(t_{i+1} - t_i) = \operatorname{tr}(S) \int_a^b E \| \xi(t) \|_H^2 dt \\ &= \operatorname{tr}(S) \| \xi \|_\Xi^2 . \end{aligned}$$

Hence the relation (3.2). Next, if T, U $\in$ **B**(H), then

$$\int_a^b (T\xi)(t) d(U\beta)(t) = T \left[\int_a^b \xi(t) d\beta(t) \right] U^* .$$

LEMMA 3-A.1. — *Let* $\{ \xi_n(t) \}$ *be a Cauchy sequence of simple processes in* Ξ_β. *Then the corresponding sequence* $\{ I_n = I[\xi_n] \}$ *form a Cauchy sequence in* $L_2(\Omega, [\sigma c])$.

Proof. — From (3.2),

$$E\{ \| I_n - I_m \|_\sigma^2 \} = \operatorname{tr}(S) \| \xi_n - \xi_m \|_\Xi^2 \to 0,$$

as $n, m, \to \infty$.

Now, for an arbitrary $\xi \in \Xi_\beta$, there exists a sequence $\{ \xi_n \}$ of simple processes in Ξ_β^0 converging to ξ in Ξ. By Lemma 3-A.1, the sequence $\{ I_n \}$ is a Cauchy and hence a convergent sequence in $L_2(\Omega, [\sigma c])$. Let I be the L_2 – limit of I_n. Then define

$$\int_a^b \xi(t)d\beta(t) = L_2 - \lim_n \int_a^b \xi_n(t)d\beta(t).$$

Since a constant multiple of a $[\tau c]$ operator is again a $[\tau c]$ operator, we shall assume that $\mathrm{tr}(S) = 1$. Now (3.2) defines an isometry from Ξ_β^0 into $L_2(\Omega, [\sigma c])$. Since Ξ_β^0 is dense in Ξ_β, the mapping $\xi \xrightarrow{I} \int_a^b \xi(t)d\beta(t)$ extends by continuity to an isomentry from Ξ_β into $L_2(\Omega, [\sigma c])$. Thus we have the following

THEOREM 3-A.1. — *There is a unique isometric operator from Ξ_β into $L_2(\Omega, [\sigma c])$, denoted by*

$$\xi \to \int_a^b \xi(t)d\beta(t)$$

such that, for $t \in [a, b]$

$$I[\xi ; t, \omega] = \int_a^b \chi_{[a,t]}\xi(\tau)d\beta(\tau) = \int_a^t \xi(\tau)d\beta(\tau).$$

For martingale properties of $I[\xi ; t]$ and the covariance operators of $I[\xi]$ see [7]. Hereafter we do *not* assume that $\mathrm{tr}(S) = 1$.

3-B. Trace stochastic integral.

From (3.1) and Lemma 1.1 (*g*) we note that the integral (3.1) is not just a Schmidt-class random operator; it is also a trace-class random operator. Thus from Lemma 1.1 (*h*) we have

$$\mathrm{tr}\left\{ \int_a^b \xi(t)d\beta(t) \right\} = \sum_{i=0}^{n-1} \langle \xi(t_i), \beta(t_{i+1}) - \beta(t_i) \rangle_H. \quad (3.3)$$

Also

$$E\left\{ \mathrm{tr} \int_a^b \xi(t)d\beta(t) \right\} = 0 \quad (3.4)$$

For $\xi \in \Xi_\beta^0$, we define the *trace stochastic integral*

$$\int_a^b \langle \xi(t), d\beta(t) \rangle_H$$

as the Bochner integral

$$\int_a^b \langle \xi(t), d\beta(t) \rangle_H = \sum_{i=0}^{n-1} \langle \xi(t_i), \beta(t_{i+1}) - \beta(t_i) \rangle_H \tag{3.5}$$

Clearly

$$E \int_a^b \langle \xi(t), d\beta(t) \rangle_H = 0.$$

Now

$$E\left\{\left| \int_a^b \langle \xi(t), d\beta(t) \rangle_H \right|^2\right\} = E\left\{\left| \sum_{i=0}^{n-1} \langle \xi(t_i), \beta(t_{i+1}) - \beta(t_i) \rangle_H \right|^2\right\}$$

$$\leq E\left\{ \sum_{i=0}^{n-1} \| \xi(t_i) \|_H^2 \| \beta(\Delta_i) \|_H^2 \right\} = \operatorname{tr}(S) \int_a^b E \| \xi(t) \|_H^2 dt$$

Hence we obtain the relation

$$E\left[\int_a^b \langle \xi(t), d\beta(t) \rangle_H \right]^2 \leq \operatorname{tr}(S) \| \xi \|_\Xi^2 \tag{3.7}$$

If we denote (3.5) by $\langle I \rangle$, (3.7) reads as

$$\| \langle I \rangle \|_{L_2(\Omega)}^2 \leq \operatorname{tr}(S) \| \xi \|_\Xi^2 \tag{3.8}$$

For arbitrary $\xi \in \Xi_\beta$, let $\{ \xi_n \} \in \Xi_\beta^0$ such that $\xi_n \to \xi$ in Ξ_β. Then from (3.8)

$$\| \langle I_n - I_m \rangle \|_{L_2(\Omega)}^2 \leq \operatorname{tr}(S) \| \xi_n - \xi_m \|_\Xi^2 \to 0$$

as $n, m \to \infty$. Thus for $\xi \in \Xi_\beta$, we define

$$\int_a^b \langle \xi(t), d\beta(t) \rangle_H = L_2 - \lim_n \int_a^b \langle \xi_n(t), d\beta(t) \rangle_H \tag{3.9}$$

We call the integral (3.9), *the trace stochastic integral associated with the operator stochastic integral* or simply *the trace integral.*

3-C. **Evaluation stochastic integral.**

The operator integral is a mapping from Ξ_β to $L_2(\Omega, [\sigma c])$;

$$I : \Xi_\beta \to L_2(\Omega, [\sigma c]).$$

Let $h \in H$. Then $\xi \to I[\xi](h)$ defines an operator I_h, by

$$I_h[\xi] = I[\xi]h. \tag{3.10}$$

Now we shall look at the form of the evaluation operator I_h. (I_h goes from Ξ_β into the collection of random elements in H). Let h be a fixed vector in H and $\xi \in \Xi_\beta^0$. From 3-A,

$$I[\xi] = \int_a^b \xi(t)d\beta(t) = \sum_{i=0}^{n-1} \xi(t_i) \otimes \beta(\Delta_i)$$

Operating $I[\xi]$ on h, we get

$$I[\xi](h) = \sum_{i=0}^{n-1} [\xi(t_i) \otimes \beta(\Delta_i)](h) = \sum_{i=0}^{n-1} \langle h, \beta(\Delta_i) \rangle_H \xi(t_i). \tag{3.11}$$

For $\xi \in \Xi_\beta^0$, we define *the evaluation stochastic integral,*

$$\int_a^b \xi(t)d\langle h, \beta(t) \rangle_H,$$

by

$$\int_a^b \xi(t)d\langle h, \beta(t) \rangle_H = \sum_{i=0}^{n-1} \langle h, \beta(\Delta_i) \rangle_H \xi(t_i). \tag{3.12}$$

Clearly,

$$E \int_a^b \xi(t)d\langle h, \beta(t) \rangle_H = 0$$

Let us consider $E \left\| \int_a^b \xi(t)d\langle h, \beta(t) \rangle_H \right\|_H^2$:

$$\begin{aligned} E\left\{ \left\| \int_a^b \xi(t)d\langle h, \beta(t) \rangle_H \right\|_H^2 \right\} &= E\left\{ \left\| \sum_{i=0}^{n-1} \langle h, \beta(\Delta_i) \rangle_H \xi(t_i) \right\|_H^2 \right\} \\ &= E\left\{ \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} [\langle h, \beta(\Delta_i) \rangle_H \langle h, \beta(\Delta_j) \rangle_H \langle \xi(t_i), \xi(t_j) \rangle_H] \right\} \\ &= \sum_{i=0}^{n-1} E\{\langle \xi(t_i), \xi(t_i) \rangle_H \langle h, \beta(\Delta_i) \rangle_H^2\}, \end{aligned} \tag{3.13}$$

where the double sum $\sum_{i \neq j}$ vanishes for the same reasons as given for the operator integral. From $\mathscr{A}_{t_i}$-measurability of $\| \xi(t_i) \|_H^2$ and the independence of $\mathscr{A}_{t_i}$ and $\langle h, \beta(\Delta_i) \rangle_H$, we get

$$E\left\| \int_a^b \xi(t) d\langle h, \beta(t) \rangle_H \right\|_H^2 = \sum_{i=0}^{n-1} E \| \xi(t_i) \|_H^2 E \langle h, \beta(\Delta_i) \rangle_H^2. \quad (3.14)$$

Now

$$\begin{aligned} E\{ \langle h, \beta(\Delta) \rangle_H^2 \} &= E\{ \langle h, \langle h, \beta(\Delta) \rangle_H \beta(\Delta) \rangle_H \} \\ &= E\{ \langle h, [\beta(\Delta) \otimes \beta(\Delta)] h \rangle_H \} \\ &= \langle h, E\{ \beta(\Delta) \otimes \beta(\Delta) \} h \rangle_H \\ &= \mathrm{tr}(S) \| h \|_H^2 | \Delta |. \end{aligned}$$

Using this in (3.14)

$$E\left\| \int_a^b \xi(t) d\langle h, \beta(t) \rangle_H \right\|_H^2 = \mathrm{tr}(S) \| h \|^2 \int_a^b E \| \xi(t) \|_H^2 dt. \quad (3.15)$$

For an arbitrary $\xi \in \Xi$, there exists a sequence $\{ \xi_n \} \in \Xi_\beta^0$ such that $\int_a^b E \| \xi_n - \xi_m \|_H^2 dt \to 0$ as $n, m \to \infty$. By (3.15), the corresponding sequence $\left\{ \int_a^b \xi_n(t) d\langle h, \beta(t) \rangle_H \right\}$, $n \geq 1$, is a Cauchy sequence in $L_2(\Omega, H)$ and hence is a convergent sequence. Thus for an arbitrary $\xi \in \Xi_\beta$, we define the *evaluation stochastic integral associated with the operator integral* by

$$\int_a^b \xi(t) d\langle h, \beta(t) \rangle_H = L_2(\Omega, H) - \lim_n \int_a^b \xi_n(t) d\langle h, \beta(t) \rangle_H.$$

3-D. **Adjoint evaluation integral.**

The operator $I[\xi]$ defined by (3.1) is a random operator on the Hilbert space H. In this section (3-D) we consider the adjoint of I. From Lemma 1.1,

$$(I[\xi])^* = \sum_{i=0}^{n-1} \beta(\Delta_i) \otimes \xi(t_i) \quad (3.16)$$

We want to look at the evaluation operator I_h^* of the adjoint I^*. Fix an h in H. Operating $I^*[\xi]$ on h, where $\xi \in \Xi_\beta^0$, we get

$$I^* h = \sum_{i=0}^{n-1} \langle h, \xi(t_i) \rangle_H [\beta(t_{i+1}) - \beta(t_i)]$$

For $\xi \in \Xi_\beta^0$ and $h \in \mathrm{H}$, we define the *adjoint evaluation stochastic integral* $\int_a^b \langle h, \xi(t) \rangle_{\mathrm{H}} d\beta(t)$, by

$$\int_a^b \langle h, \xi(t) \rangle_{\mathrm{H}} d\beta(t) = \sum_{i=0}^{n-1} \langle h, \xi(t_i) \rangle_{\mathrm{H}} \beta(\Delta_i) \tag{3.17}$$

As in the case of evaluation integral, the adjoint evaluation integral is a Bochner integral taking values in the collection of random elements in H. By non-anticipatory condition $\mathrm{E} \int_a^b \langle h, \xi(t) \rangle_{\mathrm{H}} d\beta(t) = 0$.

$$\begin{aligned}
\mathrm{E} \left\| \int_a^b \langle h, \xi(t) \rangle_{\mathrm{H}} d\beta(t) \right\|_{\mathrm{H}}^2 &= \mathrm{E} \left\| \sum_{i=0}^{n-1} \langle h, \xi(t_i) \rangle_{\mathrm{H}} \beta(\Delta_i) \right\|_{\mathrm{H}}^2 \\
&= \mathrm{E} \left\{ \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \langle h, \xi(t_i) \rangle_{\mathrm{H}} \langle h, \xi(t_j) \rangle_{\mathrm{H}} \langle \beta(\Delta_i), \beta(\Delta_j) \rangle_{\mathrm{H}} \right\} \\
&= \sum_{i=0}^{n-1} \mathrm{E} \langle h, \xi(t_i) \rangle_{\mathrm{H}}^2 \mathrm{E} \langle \beta(\Delta_i), \beta(\Delta_i) \rangle_{\mathrm{H}} \\
&\le \sum_{i=0}^{n-1} \| h \|_{\mathrm{H}}^2 \operatorname{tr}(\mathrm{S}) \mathrm{E} \| \xi(t_i) \|_{\mathrm{H}}^2 (t_{i+1} - t_i) \\
&= \operatorname{tr}(\mathrm{S}) \| h \|_{\mathrm{H}}^2 \int_a^b \mathrm{E} \| \xi(t) \|_{\mathrm{H}}^2 dt.
\end{aligned}$$

Thus we have $\int_a^b \langle h, \xi(t) \rangle_{\mathrm{H}} d\beta(t) \in \mathrm{L}_2(\Omega, \mathrm{H})$ and

$$\mathrm{E} \left\| \int_a^b \langle h, \xi(t) \rangle_{\mathrm{H}} d\beta(t) \right\|_{\mathrm{H}}^2 \le \operatorname{tr}(\mathrm{S}) \| h \|_{\mathrm{H}}^2 \| \xi \|_{\Xi}^2 \tag{3.18}$$

Using (3.18) one extends the definition to an arbitrary $\xi \in \Xi_\beta$ as in the previous cases. We call the integral $\int_a^b \langle h, \xi(t) \rangle_{\mathrm{H}} d\beta(t)$, $\xi \in \Xi_\beta$, $h \in \mathrm{H}$, *the adjoint evaluation stochastic integral* associated with the operator integral, or, simply, the *adjoint evaluation integral.*

3-E. **Inner-product stochastic integral.**

Let g and h be any two fixed elements in H. As already seen, the evaluation integral is a second order random element in H. The purpose of this section is to consider the resulting form of the integral when we take the inner-product of the evaluation integral with $g \in \mathrm{H}$. So, let g, $h \in \mathrm{H}$ and $\xi \in \Xi_\beta^0$. Then,

$$\left\langle \int_a^b \xi(t) d \langle h, \beta(t) \rangle, g \right\rangle_{\mathrm{H}} = \sum_{i=0}^{n-1} \langle \langle h, \beta(\Delta_i) \rangle_{\mathrm{H}} \xi(t_i), g \rangle_{\mathrm{H}}$$

$$= \sum_{i=0}^{n-1} \langle \xi(t_i), g \rangle_{\mathrm{H}} \langle h, \beta(\Delta_i) \rangle_{\mathrm{H}}$$

Thus, for $\xi \in \Xi_\beta^0$ and $g, h \in \mathrm{H}$, we define *the inner-product stochastic integral*, $\int_a^b \langle g, \xi(t) \rangle_{\mathrm{H}} d \langle h, \beta(t) \rangle_{\mathrm{H}}$, by

$$\int_a^b \langle g, \xi(t) \rangle_{\mathrm{H}} d \langle h, \beta(t) \rangle_{\mathrm{H}} = \sum_{i=0}^{n-1} \langle g, \xi(t_i) \rangle_{\mathrm{H}} \langle h, \beta(\Delta_i) \rangle_{\mathrm{H}}. \quad (3.19)$$

Clearly $\mathrm{E} \int_a^b \langle g, \xi(t) \rangle_{\mathrm{H}} d \langle h, \beta(t) \rangle_{\mathrm{H}} = 0$. Moreover,

$$\mathrm{E}\left\{ \int_a^b \langle g, \xi(t) \rangle_{\mathrm{H}} d \langle h, \beta(t) \rangle_{\mathrm{H}} \right\}^2 = \mathrm{E}\left[\sum_{i=0}^{n-1} \langle g, \xi(t_i) \rangle_{\mathrm{H}} \langle h, \beta(\Delta_i) \rangle_{\mathrm{H}} \right]^2$$

$$\leq \mathrm{E}\left\{ \sum_{i=0}^{n-1} \| g \|_{\mathrm{H}}^2 \| h \|_{\mathrm{H}}^2 \| \xi(t_i) \|_{\mathrm{H}}^2 \| \beta(\Delta_i) \|_{\mathrm{H}}^2 \right\}$$

$$= \sum_{i=0}^{n-1} \| g \|_{\mathrm{H}}^2 \| h \|_{\mathrm{H}}^2 \operatorname{tr}(\mathrm{S}) \mathrm{E} \| \xi(t_i) \|_{\mathrm{H}}^2 (t_{i+1} - t_i)$$

$$= \| g \|_{\mathrm{H}}^2 \| h \|_{\mathrm{H}}^2 \operatorname{tr}(\mathrm{S}) \int_a^b \mathrm{E} \| \xi(t) \|_{\mathrm{H}}^2 dt.$$

Hence $\int_a^b \langle g, \xi(t) \rangle_{\mathrm{H}} d \langle h, \beta(t) \rangle_{\mathrm{H}} \in \mathrm{L}_2(\Omega)$ and

$$\mathrm{E}\left| \int_a^b \langle g, \xi(t) \rangle_{\mathrm{H}} d \langle h, \beta(t) \rangle_{\mathrm{H}} \right|^2 \leq \| g \|^2 \| h \|^2 \operatorname{tr}(\mathrm{S}) \| \xi \|_{\Xi}^2 \quad (3.20)$$

Using (3.20) one extends the definition of the integral to an arbitrary $\xi \in \Xi_\beta$ as the $L_2(\Omega)$ limit. We call the integral

$$\int_a^b \langle g, \xi(t) \rangle_H d \langle h, \beta(t) \rangle_H, \ \xi \in \Xi_\beta, \ g, h \in H,$$

the inner-product stochastic integral associated with the operator integral. We note that, if

$$\langle I\ h, g \rangle_H = \int_a^b \langle g, \xi(t) \rangle_H d \langle h, \beta(t) \rangle_H.$$

then

$$\langle I^* h, g \rangle_H = \int_a^b \langle h, \xi(t) \rangle_H d \langle g, \beta(t) \rangle_H.$$

We need the following proposition in the next section.

PROPOSITION 3.1. — *Let $\xi_n(t) \in \Xi_\beta$ and for some $\xi \in \Xi_\beta$, $\xi_n(t) \to \xi(t)$ a. s. for almost all $t \in [a, b]$ (relative to Lebesgue measure). If there is an $\eta(t) \in L_2([a, b])$ such that $\| \xi_n(t) \|_H \leq | \eta(t) |$ a. s., then,*

(A) $\int_a^b \xi_n(t) d\beta(t) \to \int_a^b \xi(t) d\beta(t)$ in $L_2(\Omega, [\sigma c])$;

(B) $\int_a^b \langle \xi_n(t), d\beta(t) \rangle \to \int_a^b \langle \xi(t), d\beta(t) \rangle$ in $L_2(\Omega)$;

(C) $\int_a^b \xi_n(t) d \langle h, \beta(t) \rangle \to \int_a^b \xi(t) d \langle h, \beta(t) \rangle$ in $L_2(\Omega, H)$;

(D) $\int_a^b \langle h, \xi_n(t) \rangle d\beta(t) \to \int_a^b \langle h, \xi(t) \rangle d\beta(t)$ in $L_2(\Omega, H)$;

(E) $\int_a^b \langle g, \xi_n(t) \rangle d \langle h, \beta(t) \rangle \to \int_a^b \langle g, \xi(t) \rangle d \langle h, \beta(t) \rangle$ in $L_2(\Omega)$.

Proof. — (A), (B), (C), (D) and (E) follow respectively from (3.2), (3.8), (3.15), (3.18) and (3.20) upon applying dominated convergence theorem.

4. ITÔ'S FORMULA

Let $\eta(t)$ be a second order Schmidt-class random operator on H, non-anticiparoty of $\beta(t)$ and $\xi(t) \in \Xi_\beta$. A process $\zeta(t) : [a, b] \to L_2(\Omega, [\sigma c])$ is said to have a *stochastic differential*

$$d\zeta(t) = \xi(t) d\beta(t) + \eta(t) dt, \tag{4.1}$$

if for all $t \in [a, b]$, $\zeta(t)$ satisfies the relation

$$\zeta(t) - \zeta(a) = \int_a^t \xi(\tau) d\beta(\tau) + \int_a^t \eta(\tau) d\tau \tag{4.2}$$

almost surely, where the first integral is the operator integral and the second integral is Bochner integral.

A mapping $\Phi : [a, b] \times \mathfrak{H} \to R_1$ is said to be *twice differentiable* if there exist partial derivatives $\Phi_t' : [a, b] \times \mathfrak{H} \to R_1$, $\Phi_x' : [a, b] \times \mathfrak{H} \to \mathfrak{H}$ and $\Phi_{xx}'' : [a, b] \times \mathfrak{H} \to L(\mathfrak{H})$ such that

$$\begin{aligned} \Phi(t + \Delta t, x + \Delta x) - \Phi(t, x) &= \Phi_t'(t, x)\Delta t \\ &+ \langle \Phi_x'(t, x), \Delta x \rangle_{\mathfrak{h}} + \frac{1}{2} \langle \Phi_{xx}''(t, x)\Delta x, \Delta x \rangle_{\mathfrak{h}} \\ &+ o_1(\Delta t) + o_2(\Delta x), \end{aligned} \tag{4.3}$$

where $o_1(\Delta t)$ means that $o(\Delta t)/|\Delta t| \to o$, as $|\Delta t| \to o$ and $o_2(\Delta x)$ means that $o_2(\Delta x)/||\Delta x||^2 \to o$ as $||\Delta x|| \to o$; $\mathfrak{H}$ is a real Hilbert space.

In this section we extend Itô's formula corresponding to the stochastic integrals defined in the last section. Proof is essentially the same as that of one dimensional case. For the sake of completeness we give the proof for operator integral case; we omit many details. The Hilbert space $\mathfrak{H}$ appearing in (4.3) can be any of $[\sigma c]$, H and R_1; for example, $\mathfrak{H} = [\sigma c]$ for the operator integral case.

THEOREM 4-A. — *Let* H *be a real separable Hilbert space,* $[\sigma c]$ *be the (Hilbert-)Schmidt-class operators on* H *and* $\beta(t)$ *be a Brownian motion in* H. *Let* $\Phi(t, x)$ *be a twice differentiable map of* $[a, b] \times [\sigma c]$ *into* R_1 *such that* $\Phi(t, x)$, $\Phi_t'(t, x)$, $\Phi_x'(t, x)$ *and* $\Phi_{xx}''(t, x)$ *are continuous on* $[a, b] \times [\sigma x]$. *Also let* $\zeta(t)$ *be a* $[\sigma c]$*-valued process with stochastic differential* (4.1) *where we further assume that* $||\xi(t)||^2$ *is a second order process. Suppose that* Φ_x' *is symmetric as an operator on* H *and* Φ_{xx}'' *is symmetric as an operator on the Hilbert space* $[\sigma c]$. *Then the process* $z(t) = \Phi(t, \zeta(t))$ *satisfies the relation*

$$\begin{aligned} dz(t) = \Phi_t'(t, \zeta(t))dt &+ \langle \Phi_x'(t, \zeta(t)), \eta(t) \rangle_\sigma dt \\ &+ \frac{1}{2} \sum_{K=1}^{\infty} \langle \Phi_{xx}''(t, \zeta(t))[\xi(t) \otimes \lambda_k e_k], [\xi(t) \otimes e_k] \rangle_\sigma dt \\ &+ \langle \Phi_x'(t, \zeta(t))\xi(t), d\beta(t) \rangle_H \end{aligned} \tag{4.4}$$

where $\{e_k\}$, $1 \leq k \leq \infty$, *is the orthonormal sequence of eigenvectors corresponding to the eigenvalues* $\{\lambda_k\}$ *of the trace operator* S *of definition* 2.1.

Proof. — It is enough if we show (4.4) for the case of constant $\xi(t)$ and $\eta(t)$. Then, by additivity, (4.4) holds for simple processes. Finally the theorem follows, by standard limiting argument, from the definition of the integral. So, let us assume that $\xi(t)$ and $\eta(t)$ are constants, that is, independent of t.

Let π be a partition of the interval $[a, t] \subset [a, b]$, that is, let $t_0, t_1, \ldots, t_n \in [a, t]$ such that $a = t_0 < t_1 < \ldots < t_{n-1} < t_n = t \leq b$. By $[\pi]$ we denote the *mesh* of the partition. Now,

$$\begin{aligned} z(t) - z(a) &= \sum_{i=0}^{n-1} \{ \Phi(t_{i+1}, \zeta(t_{i+1})) - \Phi(t_i, \zeta(t_i)) \} \\ &= \sum_{i=0}^{n-1} \{ \Phi(t_{i+1}, \zeta(t_{i+1})) - \Phi(t_i, \zeta(t_{i+1})) \} \\ &+ \sum_{i=0}^{n-1} \{ \Phi(t_i, \zeta(t_{i+1})) - \Phi(t_i, \zeta(t_i)) \} \end{aligned}$$

From the differentiability of $\Phi(t, x)$, we obtain

$$\begin{aligned} z(t) - z(a) &= \sum_{i=0}^{n-1} \Phi'_t(t_i, \zeta(t_{i+1}))(t_{i+1} - t_i) \\ &+ \sum_{i=0}^{n-1} \langle \Phi'_x(t_i, \zeta(t_i)), \zeta(t_{i+1}) - \zeta(t_i) \rangle_\sigma \\ &+ \frac{1}{2} \sum_{i=0}^{n-1} \langle \Phi''_{xx}(t_i, \zeta(t_i))[\zeta(t_{i+1}) - \zeta(t_i)], \zeta(t_{i+1}) - \zeta(t_i) \rangle_\sigma \\ &+ \sum_{i=0}^{n-1} (\delta_i + \varepsilon_i), \end{aligned} \tag{4.5}$$

where

$$|\delta_i| \leq [\pi] \sup_{0<\theta<1} \| \Phi'_t(t_i + \theta(t_{i+1} - t_i), \zeta(t_{i+1})) - \Phi'_t(t_i, \zeta(t_{i+1})) \|_\sigma,$$

and

$$|\varepsilon_i| \leq \|\zeta(t_{i+1}) - \zeta(t_i)\|_\sigma^2 \sup_{0<\theta<1} \|\Phi''_{xx}(t_i, \zeta(t_i) + \theta(\zeta(t_{i+1}) - \zeta(t_i))) - \Phi''_{xx}(t_i, \zeta(t_i))\|.$$

It is easily seen that $\sum_{i=0}^{n-1}(|\delta_i| + |\varepsilon_i|) \to 0$, a. s., as $[\pi] \to 0$. Using the integral relation (4.2) in (4.5), we obtain $z(t) - z(a)$ as the sum of the following sums plus a negligible error;

$$\Sigma_1 = \sum_{i=0}^{n-1} \Phi'_t(t_i, \zeta(t_i))(t_{i+1} - t_i)$$

$$\Sigma_2 = \sum_{i=0}^{n-1} \langle \Phi'_x(t_i, \zeta(t_i)), \eta \rangle_\sigma (t_{i+1} - t_i)$$

$$\Sigma_3 = \sum_{i=0}^{n-1} \langle \Phi'_x(t_i, \zeta(t_i)), \xi \otimes [\beta(t_{i+1}) - \beta(t_i)] \rangle_\sigma$$

$$\Sigma_4 = \sum_{i=0}^{n-1} \frac{1}{2} \langle \Phi''_{xx}(t_i, \zeta(t_i))\eta, \eta \rangle_\sigma (t_{i+1} - t_i)^2$$

$$\Sigma_5 = \sum_{i=0}^{n-1} \langle \Phi''_{xx}(t_i, \zeta(t_i))\eta, \xi \otimes [\beta(t_{i+1}) - \beta(t_i)] \rangle_\sigma (t_{i+1} - t_i)$$

$$\Sigma_6 = \frac{1}{2} \sum_{i=0}^{n-1} \langle \Phi''_{xx}(t_i, \zeta(t_i))[\xi \otimes \beta(\Delta_i)], \xi \otimes \beta(\Delta_i) \rangle_\sigma$$

where $\beta(\Delta_i) = \beta(t_{i+1}) - \beta(t_i)$.
From the continuity assuptions on $\Phi'_x(t, x)$ and $\zeta(t)$ it follows that,

$$\Sigma_1 \to \int_a^t \Phi'_\tau(\tau, \zeta(\tau))d\tau, \tag{4.6}$$

as $[\pi] \to 0$. Similarly, as $[\pi] \to 0$,

$$\Sigma_2 \to \int_a^t \langle \Phi'_x(\tau, \zeta(\tau)), \eta \rangle_\sigma dt. \tag{4.7}$$

First we rewrite Σ_3 suitably:

$$\begin{aligned}\Sigma_3 &= \sum_{i=0}^{n-1} \langle \Phi'_x(t_i, \zeta(t_i)), \xi \otimes \beta(\Delta_i) \rangle_\sigma \\ &= \sum_{i=0}^{n-1}\sum_{k=1}^{\infty} \langle \Phi'_x(t_i, \zeta(t_i))e_k, [\xi \otimes \beta(\Delta_i)]e_k \rangle_{\mathrm{H}} \\ &= \sum_{i=0}^{n-1}\sum_{k=1}^{\infty} \langle [\beta(\Delta_i) \otimes \xi]\Phi'_x(t_i, \zeta(t_i))e_k, e_k \rangle_{\mathrm{H}} \\ &= \sum_{i=0}^{n-1}\sum_{k=1}^{\infty} \langle \Phi'_x(t_i, \zeta(t_i))e_k, \xi \rangle_{\mathrm{H}} \langle \beta(\Delta_i), e_k \rangle_{\mathrm{H}} \\ &= \sum_{i=0}^{n-1}\sum_{k=1}^{\infty} \langle \beta(\Delta_i), e_k \rangle_{\mathrm{H}} \langle e_k, \Phi'_x(t_i, \zeta(t_i))\xi \rangle_{\mathrm{H}} \\ &= \sum_{i=0}^{n-1} \langle \Phi'_x(t_i, \zeta(t_i))\xi, \beta(\Delta_i) \rangle_{\mathrm{H}} \qquad (4.8)\end{aligned}$$

Define

$$g_{\mathrm{N}}(x) = \begin{cases} 1 & \text{if } \|x\| \leq \mathrm{N} \\ 0 & \text{if } \|x\| > \mathrm{N} \end{cases}$$

Then

$$\int_a^b \mathrm{E}\, \| \Phi'_x[(t, \zeta(t)]g_{\mathrm{N}}(\zeta(t))\xi \|_{\mathrm{H}}^2 dt \leq \sup_{\|\zeta(t)\|_\sigma \leq \mathrm{N}} \| \Phi'_x(t, \zeta(t) \|_\sigma^2 \int_a^b \mathrm{E}\, \| \xi \|_{\mathrm{H}}^2 dt < \infty\,;$$

the finiteness follows from the continuity assuptions. $\zeta(t)$ is bounded a. s. Thus $\Phi'_x(t, \zeta(t))\xi$ is $\mathscr{A}_t$-measurable and is a second order process in H. Hence $\int_a^b \langle \Phi'_x(t, \zeta(t))\xi, d\beta(t) \rangle_{\mathrm{H}}$ is defined. Let $\{ \pi_n, n \geq 1 \}$ be a sequence of partitions such that π_{n+1} is a refinement of π_n and the mesh of the partitions $[\pi_n] \to 0$, as $n \to \infty$. Define

$$f_n(t) = \Phi'_x(t_i, \zeta(t_i))\xi, \qquad \text{for} \qquad t \in [t_i, t_{i+1}], \qquad 0 \leq i \leq n-1$$

Clearly $f_n(t) \to \Phi'_x(t, \zeta(t))\xi$, a. s., as $n \to \infty$.

Moreover, for some constant N, $\| f_n(t) \| \leq \mathrm{N}$ from the continuity of $\Phi'_x(t, x)$. Now Proposition 3.1 implies that

$$\int_a^t \langle f_n(\tau), d\beta(\tau) \rangle \to \int_a^t \langle \Phi'_x(\tau, \zeta(\tau))\xi, d\beta(\tau) \rangle.$$

Thus

$$\Sigma_3 \to \int_a^t \langle \Phi'_x(\tau, \zeta(\tau))\xi, d\beta(\tau) \rangle \quad \text{in probability.} \tag{4.9}$$

From boundedness of $\zeta(t)$ and $\Phi''_{xx}(t, x)$, it easily follows that

$$\Sigma_4 \to 0 \quad \text{and} \quad \Sigma_5 \to 0, \text{ a. s.} \tag{4.10}$$

Let $\beta(t) = \sum_{k=1}^{\infty} \beta_k(t)e_k$ be the Fourier series of $\beta(t)$ and define

$$\beta^{(n)}(t) = \sum_{k=1}^{n} \beta_k(t)e_k.$$

Then

$$\begin{aligned} \mathrm{E}\{\beta_j(\Delta)\beta_k(\Delta)\} = \mathrm{E}\{\langle e_j, \beta(\Delta)\rangle\langle\beta(\Delta), e_k\rangle\} \\ = \langle[\mathrm{E}\{\beta(\Delta)\otimes\beta(\Delta)\}]e_j, e_k\rangle = |\Delta|\langle\lambda_j e_j, e_k\rangle \\ = |\Delta|\lambda_j, \quad \text{if} \quad j = k \end{aligned} \tag{4.11}$$

and zero otherwise; here $\Delta = t - s$, $\beta(\Delta) = \beta(t) - \beta(s)$. Now

$$\begin{aligned} &\mathrm{E}\langle \Phi''_{xx}(s, \zeta(s))[\xi \otimes \beta^{(n)}(\Delta)], [\xi \otimes \beta^{(n)}(\Delta)]\rangle_\sigma \\ &= \sum_{k=1}^{n}\sum_{j=1}^{n} \mathrm{E}\{[\beta_k(\Delta)\beta_j(\Delta)]\langle\Phi''_{xx}(s, \zeta(s))[\xi\otimes e_k], [\xi\otimes e_j]\rangle_\sigma\} \\ &= \sum_{k=1}^{n}\sum_{j=1}^{n} \mathrm{E}\{\langle\Phi''_{xx}(s, \zeta(s))[\xi\otimes e_k], [\xi\otimes e_j]\rangle_\sigma \mathrm{E}\{[\beta_k(\Delta)\beta_j(\Delta)]\} \\ &= \mathrm{E}\sum_{k=1}^{n} \langle\Phi''_{xx}(s, \zeta(s))[\xi\otimes e_k], \xi\otimes e_k\rangle_\sigma \lambda_k(t - s). \end{aligned}$$

This yields that,

$$\begin{aligned} &\mathrm{E}\{\langle\Phi''_{xx}(s, \zeta(s))[\xi\otimes\beta(\Delta)], [\xi\otimes\beta(\Delta)]\rangle_\sigma\} \\ &= \mathrm{E}\left\{\sum_{k=1}^{\infty}\langle\Phi''_{xx}(s, \zeta(s))[\xi\otimes\lambda_k e_k], \xi\otimes e_k\rangle_\sigma(t - s)\right\}. \end{aligned}$$

Thus

$$\Sigma_6 = \frac{1}{2}\sum_{i=0}^{n-1}\sum_{k=1}^{\infty}\langle\Phi''_{xx}(t_i, \zeta(t_i))[\xi\otimes\lambda_k e_k], \xi\otimes e_k\rangle_\sigma(t_{i+1} - t_i) \tag{4.12}$$

Define

$$\chi(t) = \begin{cases} 1 & \text{if } \sup_{s\in[a,t]} \|\zeta(s)\|_\sigma \le M \\ 0 & \text{otherwise} \end{cases}$$

Then

$$E\{\chi(t_i)\|\Phi''_{xx}(t_i, \zeta(t_i))\|^2\} \le \sup_{\substack{s\in[a,t] \\ \|\zeta(s)\|_\sigma \le M}} \|\Phi''_{xx}(s, \zeta(s))\|^2$$
$$\le K, \text{ say.} \tag{4.13}$$

Next, since $\|\cdot\|_\sigma$ is a cross-norm, we have

$$\begin{aligned} E\{\langle \Phi''_{xx}(s, \zeta(s))[\xi \otimes \beta(\Delta)], [\xi \times \beta(\Delta)]\rangle_\sigma^2\} & \\ &\le E\{\|\Phi''_{xx}(s, \zeta(s))[\xi \otimes \beta(\Delta)]\|_\sigma^2 \|\xi \otimes \beta(\Delta)\|_\sigma^2\} \\ &\le E\{\|\Phi''_{xx}(s, \zeta(s))\|^2 \|\xi\|_H^4 \|\beta(\Delta)\|_H^4\} \\ &\le 3K^2 E\|\xi\|_H^4 [\text{tr }(S)]^2 (t-s)^2, \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} E\left\{\sum_{k=1}^{\infty} \langle \Phi''_{xx}(s, \zeta(s))[\xi \otimes \lambda_k e_k], \xi \otimes e_k\rangle_\sigma\right\}^2 (t-s)^2 & \\ &\le K^2 E\left\{\sum_{k=1}^{\infty} \langle \xi \otimes \lambda_k e_k, \xi \otimes e_k\rangle_\sigma\right\}^2 (t-s)^2 \\ &= K^2 E\left\{\sum_{k=1}^{\infty} \langle \xi, \xi\rangle \langle \lambda_k e_k, e_k\rangle\right\}^2 (t-s)^2 \\ &= K^2 E\|\xi\|^4 [\text{tr }(S)]^2 (t-s)^2 \end{aligned} \tag{4.15}$$

Using (4.14) and (4.15),

$$\begin{aligned} E\Bigg\{\sum_{i=0}^{n-1} \chi(t_i)[\langle \Phi''_{xx}(t_i, \zeta(t_i))[\xi \otimes \beta(\Delta_i)], [\xi \otimes \beta(\Delta_i)]\rangle_\sigma & \\ &- \sum_{k=1}^{\infty} \langle \Phi''_{xx}(t_i, \zeta(t_i))[\xi \otimes \lambda_k e_k], \xi \otimes e_k\rangle_\sigma (t_{i+1} - t_i)]\Bigg\}^2 \\ &= \sum_{i=0}^{n-1} E\{\chi(t_i)[\langle \Phi''_{xx}(t_i, \zeta(t_i))[\xi \otimes \beta(\Delta_i)], [\xi \otimes \beta(\Delta_i)]\rangle_\sigma \\ &- \sum_{k=1}^{\infty} \langle \Phi''_{xx}(t_i, \zeta(t_i))[\xi \otimes \lambda_k e_k], \xi \otimes e_k\rangle_\sigma (t_{i+1} - t_i)]\}^2, \end{aligned}$$

(the double sum $\sum_{i<j}$ will vanish),

$$\leq 2\sum_{i=0}^{n-1} \mathrm{E}\{\chi(t_i)\langle \Phi''_{xx}(t_i, \zeta(t_i))[\xi \otimes \beta(\Delta_i)], [\xi \otimes \beta(\Delta_i)]\rangle_\sigma^2\}$$
$$+ 2\sum_{i=0}^{n-1} \mathrm{E}\left\{\sum_{k=1}^{\infty} \chi(t_i)\langle \Phi''_{xx}(t_i, \zeta(t_i))[\xi \otimes \lambda_k e_k], [\xi \otimes e_k]\rangle_\sigma (t_{i+1} - t_i)\right\}^2$$
$$\leq 8\,\mathrm{K}^2\,[\mathrm{tr}\,(\mathrm{S})]^2 \mathrm{E}\,\|\xi\|^4 \Sigma(t_{i+1} - t_i)^2 \tag{4.16}$$

From (4.16), by choosing M sufficiently large, it follows that

$$\Sigma_6 \to \frac{1}{2}\int_a^t \sum_{k=1}^{\infty} \langle \Phi''_{xx}(\tau, \zeta(\tau))[\xi \otimes \lambda_k e_k], [\xi \otimes e_k]\rangle_\sigma d\tau$$

in probability. Hence the theorem.

THEOREM 4-B (Formula for trace-integral). — *Let $\xi \in \Xi_\beta$, η be a second order scalar process non-anticipatory of β and* $\int_a^b \mathrm{E}\,\|\xi(t)\|^4 dt < \infty$. *Let $\zeta(t)$ be a scalar process with*

$$d\zeta(t) = \eta(t)dt + \langle \xi(t), d\beta(t)\rangle .$$

If $\Phi(t, x)$ is defined and continuous and has continuous derivatives $\Phi'_t(t, x)$, $\Phi'_t(t, x)$ and $\Phi''_{xx}(t, x)$ on $[a, b] \times \mathrm{R}_1$, then the process $z(t) = \Phi(t, \zeta(t))$ has the stochastic differential

$$dz(t) = [\Phi'_t(t, \zeta(t)) + \Phi'_x(t, \zeta(t))\eta(t) + \frac{1}{2}\mathrm{tr}\,(\mathrm{S})\Phi''_{xx}(t, \zeta(t))\,\|\xi\|^2]dt$$
$$+ \langle \Phi'_x(t, \zeta(t))\xi(t), d\beta(t)\rangle_{\mathrm{H}}.$$

THEOREM 4-C (Evaluation integral). — *Let $\xi(t), \eta(t) \in \Xi_\beta$ and*

$$\int_a^b \mathrm{E}\,\|\xi(t)\|^4 dt < \infty;$$

$\zeta(t)$ be an H-valued process with stochastic differential

$$d\zeta(t) = \eta(t)dt + \xi(t)d\langle h, \beta(t)\rangle .$$

Let $\Phi(t, x)$ be a twice differentiable map of $[a, b] \times \mathrm{H} \to \mathrm{R}_1$ suchthat $\Phi'_t(t, x)$, $\Phi'_x(t, x)$, $\Phi''_{xx}(t, x)$, $\Phi(t, x)$ are continuous on $[a, b] \times \mathrm{H}$ and $\Phi''_{xx}(t, x)$

is symmetric on H. *Then the process* $z(t) = \Phi(t, \zeta(t))$ *satisfies the relation*

$$z(t) - z(a) = \int_a^t \Phi_t'(t, \zeta(t))dt + \int_a^t \langle \Phi_x'(t, \zeta(t)), \eta(t) \rangle dt + \frac{1}{2} \operatorname{tr}(S) \| h \|^2 \int_a^t \langle \Phi_{xx}''(t, \zeta(t))\xi(t), \xi(t) \rangle dt + \int_a^t \langle \Phi_x'(t, \zeta(t)), \xi(t) \rangle d \langle h, \beta(t) \rangle .$$

We simply remark that the analogues of Σ_3 and Σ_6 can be written as

$$\Sigma_3 = \sum_{i=0}^{n-1} \left\langle \Phi_x'(t_i, \zeta(t_i)), \int_{t_i}^{t_{i+1}} \xi(t) d \langle h, \beta(t) \rangle \right\rangle = \sum_{i=0}^{n-1} \langle \Phi_x'(t_i, \zeta(t_i)), \xi \rangle \langle h, \beta(\Delta_i) \rangle$$

and

$$\Sigma_6 = \frac{1}{2} \sum_{i=0}^{n-1} \left\langle \Phi_{xx}''(t_i, \zeta(t_i)) \int_{t_i}^{t_{i+1}} \xi d \langle h, \beta(t) \rangle, \int_{t_i}^{t_{i+1}} \xi d \langle h, \beta(t) \rangle \right\rangle = \frac{1}{2} \sum_{i=0}^{n-1} \langle \Phi_{xx}''(t_i, \zeta(t_i))[\xi \otimes \beta(\Delta_i)]h, [\xi \otimes \beta(\Delta_i)]h \rangle = \frac{1}{2} \sum_{i=0}^{n-1} \langle \Phi_{xx}''(t_i, \zeta(t_i))\xi, \xi \rangle \langle h, \beta(\Delta_i) \rangle^2 .$$

THEOREM 4-D (Adjoint evaluation integral). — *Let the conditions of Theorem 4-C, for* ξ, η *and* $\Phi(t, x)$ *hold, (*Φ_{xx}'' *need not be symmetric in this case). Let* $\zeta(t)$ *be an H-valued process with stochastic differential*

$$d\zeta(t) = \eta(t)dt + \langle h, \xi(t) \rangle d\beta(t).$$

Then the process $z(t) = \Phi(t, \zeta(t))$ *satisfies the relation*

$$z(t) - z(a) = \int_a^t \Phi_t'(t, \zeta(t))dt + \int_a^t \langle \Phi_x'(t, \zeta(t)), \eta(t) \rangle dt + \frac{1}{2} \int_a^t \sum_{k=1}^{\infty} \langle \Phi_{xx}''(t, \zeta(t)\lambda_k e_k, e_k \rangle \langle h, \xi(t) \rangle^2 dt + \int_a^t \langle \langle h, \xi(t) \rangle \Phi_x'(t, \zeta(t)), d\beta(t) \rangle .$$

First we note that Σ_6 can be written as

$$\Sigma_6 = \frac{1}{2}\sum_{i=0}^{n-1}\left\langle \Phi''_{xx}(t_i, \zeta(t_i)) \int_{t_i}^{t_{i+1}} \langle h, \xi \rangle \, d\beta(t), \int_{t_i}^{t_{i+1}} \langle h, \xi \rangle \, d\beta(t) \right.$$

$$= \frac{1}{2}\sum_{i=0}^{n-1} \langle h, \xi \rangle^2 \langle \Phi''_{xx}(t_i, \zeta(t_i))\beta(\Delta_i), \beta(\Delta_i) \rangle$$

As in the case of Theorem 4-A, we can show that

$$\Sigma_6 \to \frac{1}{2}\int_a^t \sum_{k=1}^{\infty} \langle \Phi''_{xx}(t, \zeta(t)) \langle h, \xi \rangle \lambda_k e_k, \langle h, \xi \rangle e_k \rangle \, dt.$$

THEOREM 4-E (Inner-product integral). — *Let $\eta(t)$ be a second order scalar process non-anticipatory of β and $\xi \in \Xi_\beta$ with*

$$\int_a^b \mathrm{E} \, \| \xi(t) \|^4 dt < \infty.$$

Let $\zeta(t)$ be a numerical process satisfying the relation

$$d\zeta(t) = \eta(t)dt + \langle g, \xi(t) \rangle \, d \langle h, \beta(t) \rangle .$$

If $\Phi(t, x)$ is defined and continuous and has continuous derivatives $\Phi'_t(t, x)$, $\Phi'_x(t, x)$ and $\Phi''_{xx}(t, x)$ on $[a, b] \times \mathrm{R}_1$, then the process $z(t) = \Phi(t, \zeta(t))$ satisfies the relation

$$dz(t) = \Phi'_t(t, \zeta(t))dt + \Phi'_x(t, \zeta(t))\eta(t)dt + \frac{1}{2}\operatorname{tr}(\mathrm{S}) \| h \|^2 \Phi''_{xx}(t, \zeta(t)) \langle g, \xi(t) \rangle^2 dt$$
$$+ \langle \Phi'_x(t, \zeta(t))\xi(t), g \rangle \, d \langle h, \beta(t) \rangle .$$

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