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V. MANDREKAR

H. SALEHI

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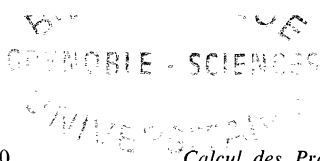
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Subordination of infinite-dimensional stationary stochastic processes (*)

by

V. MANDREKAR and H. SALEHI

Department of Statistics and Probability

RÉSUMÉ. — The notion of subordination for infinite-dimensional stationary processes is introduced and its analytic characterization in terms of its spectral and cross-spectral distributions is studied. These results constitute a natural extension to the infinite-dimensional case of Kolmogorov's work on subordination for the univariate processes.

INTRODUCTION AND PRELIMINARY RESULTS

1. The study of subordination of weakly stationary stochastic processes was initiated by A. N. Kolmogorov in his fundamental paper [6]. Based on the isomorphism between the time and spectral domain of the univariate stationary processes ([6], 2.7), he was able to obtain analytic characterizations of the notion of subordination, which enabled him to establish various interesting results concerning the prediction of the stationary stochastic processes.

The importance of the concept of subordination for the multivariate case was emphasized and initiated by P. Masani [9] basing his results on

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the work of M. Rosenberg [16] and, was furthered in [15], [17] ⁽¹⁾. In our recent work [7] we obtained an analogue of Kolmogorov isomorphism theorem for operator-valued measures. In this paper we utilize this theorem to investigate subordination of the infinite-dimensional stationary processes studied by Payen [14] (see also [2] and [5]).

The notion of subordination in finite-dimensional case has lent a new insight into the problems of analysis ([10], [17]). It is hoped that a similar insight may lead to the solution of hitherto untractable problems in the infinite-dimensional case (§ 5).

In section 3 we present an extended version of Kolmogorov's results on subordination after the preliminary results in section 2. In section 4 the results are applied to the weakly stationary processes studied by Payen [14].

2. For any two separable (complex) Hilbert-spaces \mathcal{H} , \mathcal{K} with inner products $(\cdot, \cdot)_{\mathcal{H}}$, $(\cdot, \cdot)_{\mathcal{K}}$ and norms $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{\mathcal{K}}$ we denote by

- a) $O(\mathcal{H}, \mathcal{K})$, the class of all linear operators from \mathcal{H} into \mathcal{K} ;
- b) $L(\mathcal{H}, \mathcal{K})$, the class of all operators in $O(\mathcal{H}, \mathcal{K})$ with domain \mathcal{H} ;
- c) $B(\mathcal{H}, \mathcal{K})$, the class of all bounded operators in $L(\mathcal{H}, \mathcal{K})$;
- d) $C(\mathcal{H}, \mathcal{K})$, the class of all compact operators in $B(\mathcal{H}, \mathcal{K})$;
- e) $HS(\mathcal{H}, \mathcal{K})$, the class of all Hilbert-Schmidt operators [1], in $B(\mathcal{H}, \mathcal{K})$;
- f) $T(\mathcal{H}, \mathcal{K})$, the class of all non-negative definite operators in $C(\mathcal{H}, \mathcal{K})$ of finite-trace.

2.1. DÉFINITION. — Let \mathfrak{B} be a σ -algebra of subsets of a space Ω . A $B(\mathcal{H}, \mathcal{K})$ -valued set function M is said to be

- (i) countably additive (c. a.) if for each $x \in \mathcal{H}$, A_i 's disjoint in \mathfrak{B} ,

$$M(\cup_i A_i)x = \sum_{i=1}^{\infty} M(A_i)x$$

where the series converges in \mathcal{K} ;

- (ii) μ -bounded if there exists a non-negative finite c. a. measure μ on \mathfrak{B} such that $\|M(A)x\|_{\mathcal{K}} \leq \mu(A)\|x\|_{\mathcal{H}}$.

It can be easily checked that if M is μ -bounded then M is countably additive in the uniform norm on $B(\mathcal{H}, \mathcal{K})$ ([7], § 2).

⁽¹⁾ We thank Professor M. Rosenberg for making available a copy of the galley proof of his paper [17].

2.2. DEFINITION. — Let $\mathcal{H}, \mathcal{K}, \mathcal{O}$ be separable Hilbert-spaces, Φ be a measurable $\mathcal{O}(\mathcal{H}, \mathcal{O})$ -valued function ([7], 2.1) and M be a μ -bounded $\mathcal{B}(\mathcal{H}, \mathcal{K})$ -valued measure. Then Φ is said to be integrable M if $\Phi M'_\mu$ ⁽¹⁾ is bounded and $\Phi M'_\mu$ is Bochner integrable. We write

$$\int \Phi dM = \int \Phi M'_\mu d\mu.$$

Remark. — (i) The integral $\int \Phi dM$ is independent of the bounding measure ([7], 4.5).

(ii) If M is non-negative definite $\mathcal{B}(\mathcal{H}, \mathcal{H})$ -valued and $\mathcal{H} = \mathcal{O} = \mathcal{K}$, then $\int \Phi dM$ exists if $(\Phi, 1)$ is M -integrable ([7], 4.8) and in this case

$$\int \Phi dM = (\Phi, 1)_M.$$

The following is our definition of the concept of absolute continuity.

2.3. DEFINITION. — Let M and N be $\mathcal{B}(\mathcal{H}, \mathcal{K}), \mathcal{B}(\mathcal{H}, \mathcal{O})$ -valued c. a. measures respectively. We say that N is absolutely continuous (a. c.) with respect to (w. r. t.) M and write $N \ll M$ if there exists a finite-valued non-negative measure μ such that M and N are μ -bounded and the null space \mathfrak{N}_M of M is included in \mathfrak{N}_N .

2.4. RADON-NIKODYM THEOREM. — Let M be a $\mathcal{C}(\mathcal{H}, \mathcal{K})$ -valued c. a. measure and let N be a $\mathcal{B}(\mathcal{H}, \mathcal{O})$ -valued c. a. measure. Then $N \ll M$ if there exists an $\mathcal{O}(\mathcal{H}, \mathcal{O})$ -valued function Φ integrable M such that

$$(2.5) \quad N(B) = \int_B \Phi dM, \quad \text{for all } B \in \mathfrak{B}.$$

Proof. — Let $N \ll M$. Define $\Phi = N'M'^{-}$ ⁽²⁾. Φ is a measurable $\mathcal{O}(\mathcal{H}, \mathcal{K})$ -valued function ⁽³⁾. $N \ll M$ implies $\overline{R(N'^*)} \subseteq \overline{R(M'^*)}$, where $\overline{R(\cdot)}$ denotes the closure of the range. Hence, $N'^* = P_{\overline{R(M'^*)}} N'^*$, giving $N' = N' P_{\mathfrak{R}_M}^\perp$. Put $\Phi M' = N'M'^{-} M' = N' P_{\mathfrak{R}_M}^\perp$. Therefore $N' = \Phi M'$

⁽¹⁾ M'_μ denotes the Radon-Nykodym derivative of M with respect to μ (see [7], 3.9).

⁽²⁾ A^- denotes the generalized inverse of an operator A ([7], 2.11).

⁽³⁾ $A^- = (A^*A)^- A^*$. A measurable $\Rightarrow A^*$ measurable $\Rightarrow A^*A$ measurable $\Rightarrow (A^*A)^-$ measurable $\Rightarrow A^-$ is measurable.

i. e. $N(B) = \int_B \Phi dM$ for all $B \in \mathfrak{B}$. Let $N'(B) = \int_B \Phi dM$. This implies that $N' = \Phi M'$ and therefore $\mathfrak{N}_{M'} \subseteq \mathfrak{N}_{N'}$, i. e. $N \ll M$.

2.6. *Remark.* — We observe that if Φ, Ψ are such that

$$N(B) = \int_B \Phi dM = \int_B \Psi dM$$

for all $B \in \mathfrak{B}$, then we only deduce that $\Phi M' = \Psi M'$ a. e. $[\mu]$, that is $\Phi \equiv \Psi \pmod{M}$ in the terminology of ([15], p. 360). We shall mean by dN/dM any function Φ satisfying (2.5) and call it the Radon-Nikodym derivative of N w. r. t. M .

In most of our work we shall be dealing with measures N , absolutely continuous with respect to a non-negative $C(\mathcal{H}, \mathcal{H})$ -valued measure F such that a version of dN/dF is in $L_{2,F}$ ([7], 4.10), i. e., N is Hellinger square-integrable w. r. t. F in the sense of [18].

This necessitates the introduction of the notion of the Hellinger square-integrability for operator-valued measures. The Hellinger integrability for matrix-valued measures arose in connection with the multivariate stochastic processes in the work of H. Salehi [19]. His work will be basic in our extension of the notion to the operator-valued measures. Since definition 1 of [18] does not seem to extend to the infinite-dimensional case directly we use the characterization given in 2(b) of [18] as our definition.

2.7. **DEFINITION.** — a) Let \mathcal{H} be a separable Hilbert space and F be a $T(\mathcal{H}, \mathcal{H})$ -valued c. a. measure on the σ -algebra \mathfrak{B} of subsets of a space Ω . We say that a $B(\mathcal{H}, \mathcal{H})$ -valued c. a. measure N is Hellinger square integrable F if $N \ll F$ and a version of dN/dF is in $(\mathcal{H}, \mathcal{H})L_{2,F}$ ⁽¹⁾.

b) The class of all $B(\mathcal{H}, \mathcal{H})$ -valued c. a. measures N , Hellinger square integrable F is denoted by $H_{2,F}$.

On $H_{2,F}$ we define the Gramian

$$(2.8) \quad (M, N)_F = \int (dM/dF)dF(dN/dF)^*.$$

It can be easily shown that $H_{2,F}$ is a Hilbert space over the ring $B(\mathcal{H}, \mathcal{H})$ with the inner product

$$(2.9) \quad ((M, N))_F = \text{trace } (M, N)_F.$$

⁽¹⁾ $(\mathcal{H}, \mathcal{H})L_{2,F}$ denotes the space $L_{2,F}$ of functions with values in $O(\mathcal{H}, \mathcal{H})$. This notation will be handy when several $L_{2,F}$'s are involved.

In fact, $H_{2,F}$ is isomorphically isometric to $L_{2,F}$ under the map T given by

$$(2.10) \quad T\Phi = N, \quad \Phi \in L_{2,F}.$$

Let $M \in H_{2,F}$. For any $C \in \mathfrak{B}$, M^C denotes the $B(\mathcal{H}, \mathcal{H})$ -valued c. a. measure given by

$$M^C(B) = M(C \cap B), \quad B \in \mathfrak{B}.$$

Clearly $M^C \in H_{2,F}$. For $M, N \in H_{2,F}$ we also have

$$(2.11) \quad (M^C, M^D)_F = \int_{C \cap D} (dM/dF)dF(dN/dF)^*.$$

Using the orthogonality (in the sense of Gramian) in $H_{2,F}$ we give the following definition of F singularity for elements of $H_{2,F}$.

2.12. DEFINITION. — Let M, N be $B(\mathcal{H}, \mathcal{H})$ -valued c. a. measures. Then we say that M and N are F -singular if $M, N \in H_{2,F}$ and $(M^B, N^B)_F = 0$ for all $B \in \mathfrak{B}$ and we write $M \perp_F N$.

2.13. Remark. — In view of lemma 6.7 of [15], the notion of singularity in finite-dimensional case is independent of the measure F . Hence we can choose F such that M, N lie in the space $H_{2,F}$. Thus the above definition reduces to the definition of singularity given in [15].

SPECTRAL SUBORDINATION

3. Let $\mathcal{H}, \mathcal{K}, \mathcal{O}$ be separable Hilbert spaces. Let E be a spectral measure on a σ -algebra \mathfrak{B} of subsets of a space Ω for the Hilbert space \mathcal{O} . We denote by $HS(\mathcal{H}, \mathcal{O})$ and $HS(\mathcal{K}, \mathcal{O})$ the Hilbert spaces of all Hilbert-Schmidt operators on \mathcal{H} into \mathcal{O} and \mathcal{K} into \mathcal{O} respectively. For an $X \in HS(\mathcal{H}, \mathcal{O})$ we define by

a) \mathcal{M}_X , the closed subspace of \mathcal{O} generated by the images of the operators $\{E(B)X, B \in \mathfrak{B}\}$;

b) $\mathcal{S}_X(\mathcal{H}, \mathcal{H})$, the closed subspace of $HS(\mathcal{H}, \mathcal{O})$ generated by the family $\{E(B)XA; B \in \mathfrak{B} \text{ and } A \in B(\mathcal{H}, \mathcal{H})\}$ over the ring $B(\mathcal{H}, \mathcal{H})$;

c) $\mathcal{S}_X(\mathcal{K}, \mathcal{H})$, the closed subspace of $HS(\mathcal{K}, \mathcal{O})$ generated by the family $\{E(B)XA; B \in \mathfrak{B} \text{ and } A \in B(\mathcal{K}, \mathcal{H})\}$ over the ring $B(\mathcal{K}, \mathcal{H})$.

The spaces $\mathcal{M}_Y, \mathcal{S}_Y(\mathcal{H}, \mathcal{H})$ and $\mathcal{S}_Y(\mathcal{K}, \mathcal{H})$ are similarly defined.

Following M. Rosenberg ([17], § 1) we define the notion of spectral subordination.

3.1. DEFINITION. — Let $X \in \text{HS}(\mathcal{H}, \mathcal{O})$ and $Y \in \text{HS}(\mathcal{H}, \mathcal{O})$. We say that Y is spectral subordinate to $X(Y \overset{E}{\ll} X)$ if $\mathcal{M}_Y \subseteq \mathcal{M}_X$.

The following theorem gives various characterizations of the spectral subordination.

3.2. THEOREM. — The following conditions are equivalent :

- (i) $Y \overset{E}{\ll} X$; (ii) $\mathcal{S}_Y(\mathcal{H}, \mathcal{H}) \subseteq \mathcal{S}_X(\mathcal{H}, \mathcal{H})$; (iii) $\mathcal{S}_Y(\mathcal{H}, \mathcal{H}) \subseteq \mathcal{S}_X(\mathcal{H}, \mathcal{H})$.
- The proof of the theorem depends on the following lemma.

3.3. LEMMA. — Let X be in $\text{HS}(\mathcal{H}, \mathcal{O})$. Denote by σ_X the closed subspace of \mathcal{O} generated by the image of X and by Σ_X the closure of the set $\{XA, A \in \mathcal{B}(\mathcal{H}, \mathcal{H})\}$ in $\text{HS}(\mathcal{H}, \mathcal{O})$. Then $\text{HS}(\mathcal{H}, \sigma_X) = \Sigma_X$.

Proof. — Obviously $\Sigma_X \subseteq \text{HS}(\mathcal{H}, \sigma_X)$. Let $Z \in \text{HS}(\mathcal{H}, \sigma_X)$ and $\{e_i\}$ be a complete orthonormal system (CONS) in \mathcal{H} . Then $\sum_{i=1}^{\infty} \|Ze_i\|^2$ is finite. Hence for any $\varepsilon > 0$, there exists an integer N , such that

$$(3.4) \quad \sum_{N+1}^{\infty} \|Ze_i\|_{\mathcal{O}}^2 < \varepsilon/2.$$

Let $Z_N = ZP_N$ where P_N is the projection onto the subspace of \mathcal{H} generated by $e_1 \dots e_N$. Clearly $Z_N \in \text{HS}(\mathcal{H}, \sigma_X)$. By ([14], p. 335) there exists $A \in \mathcal{B}(\mathcal{H}, \mathcal{H})$ such that

$$(3.5) \quad \sum_{i=1}^N \|XAe_i - Z_Ne_i\|_{\mathcal{O}} < \varepsilon/2.$$

Let $B = AP_N$ then from (3.5) we get

$$(3.6) \quad \sum_{i=1}^{\infty} \|XB e_i - Z_N e_i\|_{\mathcal{O}}^2 < \varepsilon/2.$$

Thus by (3.4) and (3.6) we get

$$(3.7) \quad \|XB - Z\|_E^2 < \varepsilon \quad (1).$$

Hence $\text{HS}(\mathcal{H}, \sigma_X) \subseteq \Sigma_X$.

(1) $\|\cdot\|_E$ denote the Hilbert-Schmidt norm.

Proof of theorem 3.2. — Let us denote by \mathbf{I} , the cartesian product of \mathfrak{B} and $\mathbf{B}(\mathcal{H}, \mathcal{H})$ and for each $\alpha \in \mathbf{I}$, $Z_\alpha = \mathbf{E}(\mathbf{B})\mathbf{X}\mathbf{A}$. Then it can be easily checked that

$$(3.8) \quad \mathcal{S}_{\mathbf{X}}(\mathcal{H}, \mathcal{H}) = \text{the subspace of } \mathbf{HS}(\mathcal{H}, \mathcal{O}) \text{ generated by } \bigcup_{\alpha} \Sigma_{Z_\alpha}$$

Clearly, $\mathcal{M}_{\mathbf{X}}$ is equal to the subspace of \mathcal{O} generated by $\bigcup_{\alpha} \sigma_{Z_\alpha}$. Since $\mathbf{HS}(\mathcal{H}, \mathcal{M}_{\mathbf{X}})$ is the tensor-product of \mathcal{H} with $\mathcal{M}_{\mathbf{X}}$ ([13], p. 130 and [1], p. 22-23), it follows from ([12], p. 132) that

$$(3.9) \quad \mathbf{HS}(\mathcal{H}, \mathcal{M}_{\mathbf{X}}) = \text{closed subspace generated by } \bigcup_{\alpha} \mathbf{HS}(\mathcal{H}, \sigma_{Z_\alpha})$$

From (3.8), (3.9) and lemma 3.3 we get $\mathcal{S}_{\mathbf{X}}(\mathcal{H}, \mathcal{H}) = \mathbf{HS}(\mathcal{H}, \mathcal{M}_{\mathbf{X}})$. By a similar argument we have $\mathcal{S}_{\mathbf{Y}}(\mathcal{H}, \mathcal{H}) = \mathbf{HS}(\mathcal{H}, \mathcal{M}_{\mathbf{Y}})$. Hence (i) and (ii) are equivalent. It can also be shown that $\mathcal{S}_{\mathbf{X}}(\mathcal{H}, \mathcal{H}) = \mathbf{HS}(\mathcal{H}, \mathcal{M}_{\mathbf{X}})$ and $\mathcal{S}_{\mathbf{Y}}(\mathcal{H}, \mathcal{H}) = \mathbf{HS}(\mathcal{H}, \mathcal{M}_{\mathbf{Y}})$, hence (i) is equivalent to (iii).

Let $\mathbf{X} \in \mathbf{HS}(\mathcal{H}, \mathcal{O})$, $\mathbf{Y} \in \mathbf{HS}(\mathcal{H}, \mathcal{O})$ and \mathbf{E} be a spectral measure on \mathfrak{B} for \mathcal{O} . We denote by $\mathbf{M}_{\mathbf{X}\mathbf{X}}$, $\mathbf{M}_{\mathbf{X}\mathbf{Y}}$, $\mathbf{M}_{\mathbf{Y}\mathbf{X}}$, $\mathbf{M}_{\mathbf{Y}\mathbf{Y}}$ the c. a. measures $\mathbf{X}^*\mathbf{E}(\cdot)\mathbf{X}$, $\mathbf{X}^*\mathbf{E}(\cdot)\mathbf{Y}$, $\mathbf{Y}^*\mathbf{E}(\cdot)\mathbf{X}$, $\mathbf{Y}^*\mathbf{E}(\cdot)\mathbf{Y}$ on \mathfrak{B} .

For $\mathbf{X} \in \mathbf{HS}(\mathcal{H}, \mathcal{O})$ it can be easily observed that the $\mathbf{HS}(\mathcal{H}, \mathcal{O})$ -valued function ξ on \mathfrak{B} given by $\xi(\mathbf{B}) = \mathbf{E}(\mathbf{B})\mathbf{X}$, $\mathbf{B} \in \mathfrak{B}$, is an $\mathbf{HS}(\mathcal{H}, \mathcal{O})$ -valued countably additive orthogonally scattered (c. a. o. s.) measure in the sense of § 6 of our previous work [7], with the associated c. a. measure $\mathbf{M}_{\mathbf{X}\mathbf{X}}$ on \mathfrak{B} given by $\mathbf{M}_{\mathbf{X}\mathbf{X}}(\mathbf{B}) = \mathbf{X}^*\mathbf{E}(\mathbf{B})\mathbf{X}$. Clearly $\mathbf{M}_{\mathbf{X}\mathbf{X}}$ is a $\mathbf{T}(\mathcal{H}, \mathcal{H})$ -valued measure. For $\mathbf{T}(\mathcal{H}, \mathcal{H})$ -valued measures the integral $\int \Phi d\mathbf{M}_{\mathbf{X}\mathbf{X}}\Psi^*$ was defined by us for measurable $\mathbf{O}(\mathcal{H}, \mathcal{H})$ -valued functions Φ, Ψ , and it was shown that the space

$$(3.10) \quad (\mathcal{H}, \mathcal{H})\mathbf{L}_{2,\mathbf{M}} = \left\{ \Phi : \int \Phi d\mathbf{M}_{\mathbf{X}\mathbf{X}}\Phi^* \in \mathbf{T}(\mathcal{H}, \mathcal{H}) \right\}$$

is a complete Hilbert space over $\mathbf{B}(\mathcal{H}, \mathcal{H})$ under the norm

$$\|\Phi\|_{\mathbf{M}} = \text{trace} \int \Phi d\mathbf{M}\Phi^*,$$

and simple functions are dense in it ([7], 4.19, 4.22). Using this result we defined the stochastic integral $\int \Phi d\xi$ ([7], 6.8). Let $\int \Phi d\mathbf{E}\mathbf{X} = \int \Phi d\xi$,

for $\Phi \in L_{2, M_{XX}}$, where the integral on the right is the stochastic integral in the sense of ([7], 6.8). Now we observe that the closed subspace \mathcal{S}_ξ of $HS(\mathcal{H}, \mathcal{O})$ generated by the set $\{\xi(B)A^*, B \in \mathfrak{B}, A \in B(\mathcal{H}, \mathcal{H})\}$ over $B(\mathcal{H}, \mathcal{H})$ ([7], § 6) is precisely $\mathcal{S}_X(\mathcal{H}, \mathcal{H})$. Hence by ([7], 6.9) we get that

$$(3.11) \quad \mathcal{S}_X(\mathcal{H}, \mathcal{H}) = \left\{ \int \Phi dEX; \Phi \in L_{2, M_{XX}} \right\}.$$

In addition the map $\Phi \rightarrow \int \Phi dEX$ is an isometry in the sense that the Gramian $\left(\int \Phi dEX, \int \Psi dEX \right)$ given by $\left(\int \Phi dEX \right)^* \left(\int \Psi dEX \right)$ satisfies

$$(3.12) \quad \left(\int \Phi dEX, \int \Psi dEX \right) = \int \Phi dM_{XX} \Psi^*,$$

and hence, in particular,

$$(3.13) \quad \left\| \int \Phi dEX \right\|_E^2 = \|\Phi\|_M^2.$$

From the relation $\mathcal{S}_X(\mathcal{H}, \mathcal{H}) = HS(\mathcal{H}, M_X)$ and Payen's definition of the orthogonal projection onto the subspace $HS(\mathcal{H}, M_X)$ of an element Y in $HS(\mathcal{H}, \mathcal{O})$, we get that the orthogonal projection of Y into $\mathcal{S}_X(\mathcal{H}, \mathcal{H})$ is the element $\hat{Y} \in HS(\mathcal{H}, M_X)$ given by $\hat{Y}a =$ projection of Ya onto M_X .

3.14. THEOREM. — Let $X \in HS(\mathcal{H}, \mathcal{O})$, $Y \in HS(\mathcal{H}, \mathcal{O})$ and \hat{Y} be the orthogonal projection of Y onto $\mathcal{S}_X(\mathcal{H}, \mathcal{H})$. Then

$$(i) \quad \hat{Y} = \int \Phi_{\hat{Y}} dEX \quad \text{where} \quad \Phi_{\hat{Y}} \in (\mathcal{H}, \mathcal{H})L_{2, M_{XX}},$$

$$(ii) \quad M_{YX}(B) = M_{\hat{Y}X}(B) = \int_B \Phi_{\hat{Y}} dM_{XX}, \quad B \in \mathfrak{B},$$

$$(iii) \quad M_{\hat{Y}\hat{Y}}(B) = \int_B \Phi_{\hat{Y}} dM_{XX} \Phi_{\hat{Y}}^* = \int_B \Phi_{\hat{Y}} dM_{X\hat{Y}}, \quad B \in \mathfrak{B}.$$

Proof. — (i) follows from (3.11). For (ii) observe that

$$(E(B)Xh, Yk)_\mathcal{O} = (E(B)Xh, \hat{Y}k)_\mathcal{O}$$

for all $h \in \mathcal{H}, k \in \mathcal{K}$. Hence $(Y - \hat{Y})^*E(B)X = 0$ giving $M_{YX}(B) = M_{\hat{Y}X}(B)$. Also, $M_{\hat{Y}X}(B) = \hat{Y}^*E(B)X = (E(B)\hat{Y})^*X = \left(\int_B \Phi_{\hat{Y}}dEX\right)^* X$ by (i). Clearly, $\left(\int \Psi dEX\right)^* X = \int \Psi dM_{XX}$ for Ψ a simple function in $L_{2,M_{XX}}$. Since simple functions are dense in $L_{2,M_{XX}}$ ([7], 4.22), we have

$$\left(\int_B \Phi_{\hat{Y}}dEX\right) X = \int_B \Phi_{\hat{Y}}dM_{XX}.$$

(iii) follows from (i), (ii) and (3.12).

From theorem 3.14 (i) and (ii) we get that

$$(3.15) \quad \hat{Y} = \int \Phi_{\hat{Y}}dEX \quad \text{and} \quad M_{YX} \in H_{2,M_{XX}} \quad \text{with} \quad T\Phi_{\hat{Y}} = M_{YX}$$

where T is an in (2.10).

The following is an extended version of Kolmogorov's theorems ([6], theorems 8, 9).

3.16. THEOREM (spectral subordination). — Let $X \in HS(\mathcal{H}, \mathcal{O})$ and $Y \in HS(\mathcal{K}, \mathcal{O})$. Then the following conditions are equivalent.

(i) $Y \stackrel{E}{\ll} X$.

(ii) There exists a $\Phi \in (\mathcal{K}, \mathcal{H})L_{2,M_{XX}}$ such that $Y = \int \Phi dEX$.

(iii) $M_{YX} \in H_{2,M_{XX}}$, $M_{YX} = T\Phi$, and $M_{YY}(B) = (M_{YX}^B, M_{YX}^B)_{M_{XX}}$ for all $B \in \mathfrak{B}$.

Proof. — (i) \Rightarrow (ii) \Rightarrow (iii) follow from theorem 3.14. For (iii) \Rightarrow (i) observe that for \hat{Y} as in theorem 3.14

$$(3.17) \quad (Y - \hat{Y})^*(Y - \hat{Y}) = Y^*Y - \hat{Y}^*\hat{Y}.$$

From $M_{YX}(B) = \int_B \Phi dM_{XX}$, $\Phi \in (\mathcal{K}, \mathcal{H})L_{2,M_{XX}}$ and (3.15), we get $T\Phi = M_{YX}$ is in $(\mathcal{K}, \mathcal{H})L_{2,M_{XX}}$. Hence,

$$(3.18) \quad M_{YY}(B) = M_{\hat{Y}\hat{Y}}(B).$$

Thus the result follows from (3.17) and (3.18).

Remark. — In view of (3.15) and theorem 3.16 the following conditions are equivalent

$$\begin{aligned}
 (3.19) \quad & (i) \quad Y \stackrel{E}{\ll} X, \\
 & (ii) \quad Y = \int \Phi dEX, \\
 & (iii) \quad M_{YY}(B) = \int_B \Phi dM_{XX} \Phi^* \quad B \in \mathfrak{B}.
 \end{aligned}$$

The next theorem is our extension of Kolmogorov’s theorem on mutual subordination ([6], theorem 10).

3.20. THEOREM (mutual spectral subordination). — Let $X \in HS(\mathcal{H}, \mathcal{O})$, $Y \in HS(\mathcal{K}, \mathcal{O})$ and $Y \stackrel{E}{\ll} X$. Then $X \stackrel{E}{\ll} Y$ if f .

$$M_{XX} \ll M_{YY} \quad \text{and} \quad dM_{XX}/dM_{YY} \in (\mathcal{H}, \mathcal{K})L_{2, M_{YY}}.$$

Proof. — Let M'_{YX} , M'_{XX} , M'_{XY} and M'_{YY} be the derivatives of M_{YX} , M_{XX} , M_{XY} and M_{YY} respectively w.r.t. the finite-valued non-negative c.a. measure trace $M_{XX} + \text{trace } M_{YY}$ on \mathfrak{B} . $X \stackrel{E}{\ll} Y$ implies

$$(3.21) \quad M'_{XY} = \Psi M'_{YY} \quad \text{and} \quad M'_{XX} = (\Psi \sqrt{M'_{YY}})(\Psi \sqrt{M'_{YY}})^*$$

where $\Psi = T^{-1}M_{XY}$. Since $\Psi \in (\mathcal{H}, \mathcal{K})L_{2, M}$ we get that

$$(3.22) \quad M'_{YX} = M'^*_{XY} = (\Psi M'_{YY})^* = \sqrt{M'_{YY}}(\Psi \sqrt{M'_{YY}})^*$$

therefore from (3.21) and (3.22)

$$M'_{XX} = \Psi \sqrt{M'_{YY}}(\Psi \sqrt{M'_{YY}})^* = \Psi M'_{YX}.$$

Hence $M_{XX} \ll M_{YX}$ and $dM_{XX}/dM_{YX} \in (\mathcal{H}, \mathcal{K})L_{2, M_{YX}}$.

Conversely, from polar decomposition for a bounded operator and theorem 3.16 (iii) we get $\Phi \sqrt{M'_{XX}} = \sqrt{M'_{YY}}V$, where V is a partial isometry. Hence from the fact that $\Psi = dM_{XX}/dM_{XY} \in (\mathcal{H}, \mathcal{K})L_{2, M_{YY}}$ it follows that $\Psi \Phi \sqrt{M'_{XX}}$ is a bounded operator on \mathcal{H} . Now $M_{XX} \ll M_{YX}$ and $Y \stackrel{E}{\ll} X$ imply

$$(3.23) \quad M'_{XX} = \Psi \Phi M'_{XX}.$$

By (3.23) and the fact that $\Psi \Phi \sqrt{M'_{XX}}$ is a bounded operator on \mathcal{H} we conclude

$$(3.24) \quad \Psi \Phi \sqrt{M'_{XX}} = \sqrt{M'_{XX}} \quad \text{on } \mathcal{H}.$$

From theorem 3.16 (iii),

$$\begin{aligned}
 M'_{YY} &= (\Phi\sqrt{M'_{XX}})(\Phi\sqrt{M'_{XX}})^* \\
 \Rightarrow \Psi M'_{YY} &= \Psi\Phi\sqrt{M'_{XX}}(\Phi\sqrt{M'_{XX}})^* \\
 \Rightarrow \Psi M'_{YY} &= \sqrt{M'_{XX}}(\Phi\sqrt{M'_{XX}})^*, \text{ by (3.24).} \\
 \Rightarrow (\Psi\sqrt{M'_{YY}}\sqrt{M'_{YY}})^* &= (\sqrt{M'_{XX}}(\Phi\sqrt{M'_{XX}})^*)^* \\
 \Rightarrow \sqrt{M'_{YY}}(\Psi\sqrt{M'_{YY}})^* &= (\Phi\sqrt{M'_{XX}})\sqrt{M'_{XX}} \\
 \Rightarrow (\Psi\sqrt{M'_{YY}})(\Psi\sqrt{M'_{YY}})^* &= M'_{XX} \text{ by (3.24).}
 \end{aligned}$$

Thus

$$(3.25) \quad M_{XX}(B) = \int_B \Psi dM_{YY} \Psi^*$$

Also $M'_{XY} = M'_{YX}^* = ((\Phi\sqrt{M'})\sqrt{M'})^* = \sqrt{M'}(\Phi\sqrt{M'})^*$. Hence by (3.24)

$$M'_{XY} = \Psi(\Phi\sqrt{M'})^*(\Phi\sqrt{M'})^* = \Psi M'_{YY}$$

Hence

$$(3.26) \quad M_{XY}(B) = \int \Psi dM_{YY}$$

From (3.25), (3.26) and theorem 3.16 we get $X \stackrel{E}{\ll} Y$.

3.27. *Remark.* — a) The finite-dimensional extension of Kolmogorov theorem given by Rosenberg ([17], 1.12) is a consequence of our above theorem. The conditions on rank, which are valid only in the finite-dimensional case are easy consequences of the measure theoretic results (see e. g. [15], p. 362).

b) If we take the definition of rank given in Halmos [4] we get from (3.19) (iii) and (4.12) of [7] that the rank $(M'_{YY}) \leq \text{rank}(\sqrt{M'_{XX}})$ and hence $\text{rank } M'_{YY} \leq \text{rank } M'_{XX}$. The condition on rank in ([17], 1.12) can be shown to be necessary in the infinite dimensional case.

We now introduce the concept of E-orthogonality.

3.28. **DEFINITION.** — Let X and Y be in $\text{HS}(\mathcal{H}, \mathcal{O})$. We say that X is orthogonal Y if $\mathcal{M}_X \perp \mathcal{M}_Y$ in \mathcal{O} and write $X \perp_E Y$.

The following theorem is extended version of the result due to Kolmogorov ([6], theorem 12) on subordination and orthogonality.

3.29. **THEOREM.** — Let $X, Y, Z \in \text{HS}(\mathcal{H}, \mathcal{O})$ such that $X = Y + Z$. If $Y \perp_E Z$, then Y and Z are subordinate to X if $f M_{YY} \perp_{M_{XX}} M_{ZZ}$.

Proof. — Assume $Y, Z \stackrel{E}{\ll} X$. Then $M_{YX} \in H_{2, M_{XX}}, M_{ZX} \in H_{2, M_{XX}}$ with

$$M_{YX} = \int \Phi dM_{XX}, \quad M_{ZX} = \int \Psi dM_{XX}$$

where

$$Y = \int \Phi dEX \quad \text{and} \quad Z = \int \Psi dEX.$$

Therefore $Y \perp_E Z$ if $f M_{YX} \perp_{M_{XX}} M_{ZX}$ by (2.11) and (3.12). But

$$M_{YX}(B) = M_{YY}(B) \quad \text{and} \quad M_{ZX}(B) = M_{ZZ}(B),$$

since $Y \perp_E Z$. Hence $M_{YY} \perp_{M_{XX}} M_{ZZ}$.

Assume $M_{YY} \perp_{M_{XX}} M_{ZZ}$. Then $M_{YX} \perp_{M_{XX}} M_{ZX}$. But

$$M_{YX}(B) = \int_B \Phi_{\hat{Y}} dM_{XX} \quad \text{and} \quad M_{ZX}(B) = \int_B \Phi_{\hat{Z}} dM_{XX}.$$

Since $X = Y + Z$ we get $\Phi_{\hat{Y}} + \Phi_{\hat{Z}} = I$ in $L_{2, M}$. Also since $Y \perp_E Z$,

$$\begin{aligned} M_{YY}(B) = M_{YX}(B) &= \int_B \Phi_{\hat{Y}} dM_{XX} = \int_B \Phi_{\hat{Y}} dM_{XX} (\Phi_{\hat{Y}} + \Phi_{\hat{Z}})^* \\ &= \int_B \Phi_{\hat{Y}} dM_{XX} \Phi_{\hat{Y}} + \int_B \Phi_{\hat{Y}} dM_{XX} \Phi_{\hat{Z}}^*. \end{aligned}$$

But $\int_B \Phi_{\hat{Y}} dM_{XX} \Phi_{\hat{Z}}^* = (M_{YY}^B, M_{ZZ}^B)_{M_{XX}} = 0$, because $M_{YY} \perp_{M_{XX}} M_{ZZ}$, so that

$M_{YY}(B) = \int \Phi_{\hat{Y}} dM_{XX} \Phi_{\hat{Y}}^*$. Therefore by theorem 3.16 we get $Y \stackrel{E}{\ll} X$.

Hence $Z \stackrel{E}{\ll} X$.

SUBORDINATION OF WEAKLY STATIONARY PROCESSES

4. Let $\mathcal{H}, \mathcal{K}, \mathcal{O}$ be separable Hilbert-spaces and $(G, +)$ be a locally compact abelian group. Following Payen [14] we say that $\{X_t, t \in G\}$ is an $HS(\mathcal{H}, \mathcal{O})$ -valued weakly stationary process if $X_t^* X_s$ is a function $s-t$. If $\{X_t, t \in G\}$ and $\{Y_t, t \in G\}$ are $HS(\mathcal{H}, \mathcal{O})$ and $HS(\mathcal{K}, \mathcal{O})$ -valued stationary processes respectively then we say that X_t and Y_t are *stationarily correlated* if $X_t^* Y_s$ is a function of $s-t$.

Let \mathfrak{M}_X and \mathfrak{M}_Y be the closed subspaces of \mathcal{O} generated by the ranges of $\{X_t, t \in G\}$ and $\{Y_t, t \in G\}$ respectively. By a slight extension of

an argument of Payen ([14], p. 356) there exists a group of unitary operator $\{ U_t, t \in G \}$ ⁽¹⁾ on \mathcal{O} to \mathcal{O} such that

$$U_t X_s = X_{t+s} \quad \text{and} \quad U_t Y_s = Y_{s+t}.$$

Under the condition that $t \rightarrow X_t$ and $t \rightarrow Y_t$ is a continuous map we get ([14], p. 363)

$$(4.1) \quad X_t = \int_{\hat{G}} \overline{\langle t, \lambda \rangle} E(d\lambda) X_0 \quad \text{and} \quad Y_t = \int_{\hat{G}} \overline{\langle t, \lambda \rangle} E(d\lambda) Y_0$$

where \hat{G} is the character group of G and E is a spectral measure on the Borel algebra \mathfrak{B} of sets generated by the open subsets of \hat{G} for \mathcal{O} .

We call the measures $M_{X_0 X_0}$ and $M_{Y_0 Y_0}$ the spectral measures of X_t and Y_t respectively following our terminology in [7]. In the terminology of Kolmogorov [6] we call the measures $M_{X_0 Y_0}$ and $M_{Y_0 X_0}$ the cross-spectral measures of X_t and Y_t .

We observe that

$$(4.2) \quad \mathfrak{M}_X = \mathcal{M}_{X_0}$$

We now give the definition of subordination of stochastic process as a direct extension of the classical one in [6].

4.3. DEFINITION. — Let X_t and Y_t be $HS(\mathcal{H}, \mathcal{O})$ and $HS(\mathcal{H}, \mathcal{O})$ -valued stationary stochastic processes which are stationarily correlated. Then Y_t is said to subordinate to X_t if $\mathfrak{M}_Y \subseteq \mathfrak{M}_X$.

4.4. Remark. — In view of (4.2) we get that Y_t is subordinate to X_t if $\int Y_0 \stackrel{E}{\ll} X_0$.

From the above remark the following theorems are the immediate consequences of theorems 3.16, 3.20.

4.5. THEOREM. — Let X_t and Y_t be $HS(\mathcal{H}, \mathcal{O})$ and $HS(\mathcal{H}, \mathcal{O})$ -valued stationary processes which are stationarily correlated. Then

a) the following conditions are equivalent:

- (i) Y_t is subordinate to X_t ,
- (ii) There exists a function $\Phi \in (\mathcal{H}, \mathcal{H})L_{2, M_{X_0 X_0}}$ such that $Y_0 = \int \Phi dEX_0$.
- (iii) The cross-spectral measure $M_{Y_0 X_0} \in H_{2, M_{X_0 X_0}}$ with $T\Phi = M_{Y_0 X_0}$ and $M_{Y_0 Y_0}(B) = (M_{Y_0 X_0}^B, M_{Y_0 X_0}^B)_{M_{X_0 X_0}}$ for all $B \in \mathfrak{B}$.

⁽¹⁾ For each t , the operator U_t is uniquely defined on the closure of $(\mathfrak{M}_X + \mathfrak{M}_Y)$ onto itself.

b) if Y_t is subordinate to X_t , then X_t is subordinate to Y_t if $f M_{X_0 X_0} \ll M_{Y_0 X_0}$ and $dM_{X_0 X_0} / dM_{Y_0 X_0} \in (\mathcal{H}, \mathcal{H})L_{2, M_{Y_0 Y_0}}$.

4.6. DEFINITION. — Let X_t and Y_t be $HS(\mathcal{H}, \mathcal{O})$ and $HS(\mathcal{H}, \mathcal{O})$ -valued stationary processes which are stationarily correlated. Then X_t, Y_t are orthogonal if $X_t^* Y_s = 0$ for all t, s .

The following theorem is an extension of Kolmogorov's result ([6], theorem 12) to the processes studied by Payen [14]. The proof is immediate from theorem 3.29 and (4.2) and is omitted.

4.7. THEOREM. — Let X_t, Y_t and Z_t be $HS(\mathcal{H}, \mathcal{O})$ -valued stationary processes which are stationarily correlated such that $X_t = Y_t + Z_t$, where Y_t, Z_t are orthogonal. Then Y_t, Z_t are subordinate to X_t if $f M_{Y_0 Y_0}$ and $M_{Z_0 Z_0}$ are $M_{X_0 X_0}$ -singular.

5. In the prediction theory of univariate processes Szegő theorem has played a very important role [3]. The generalization of Szegő theorem with applications to multivariate prediction theory depends on the following fact:

$$(5.1) \quad \Phi \in L_2^{0+} \Rightarrow \text{rank } \Phi = \text{const. a. e.},$$

where L_2^{0+} denotes the class of all matrix-valued functions square-integrable with vanishing negative Fourier coefficients. The proof of (5.1) was given by Masani [8] and Matveev [11] applying theorems on Hardy class functions. Recently, Rosenberg [17] presented a new proof of (5.1) based on the subordination of stationary stochastic processes. We extend the result (5.1) to the infinite-dimensional case without recourse to the theory of $HS(\mathcal{H}, \mathcal{O})$ -valued Hardy class functions about which the results are fragmentary. Let \mathcal{H}, \mathcal{O} be two separable Hilbert spaces and L_2 be the Hilbert space of $HS(\mathcal{H}, \mathcal{O})$ -valued functions Φ on the interval $[0, 2\pi]$ such that $\int_0^{2\pi} |\Phi(e^{i\theta})|_{\mathbb{E}}^2 d\theta$ is finite. The space L_2^{0+} will denote the subspace of L_2 , consisting of functions with negative Fourier coefficients zero.

5.2. THEOREM. — Let $\Phi \in L_2^{0+}$ then $\text{rank } \Phi(e^{i\theta}) = \text{const. a. e.}$

Proof. — We know by the very definition that

$$(5.3) \quad \Phi(e^{i\theta}) = \sum_{k=0}^{\infty} A_k e^{ik\theta} \quad \text{with} \quad \sum_{k=0}^{\infty} |A_k|_{\mathbb{E}}^2 < \infty.$$

Choose a $B(\mathcal{O}, L_2(\Omega))$ -valued stationary process ξ_n with $\xi_n^* \xi_m = \delta_{nm} I$. Define the $HS(\mathcal{H}, L_2(\Omega))$ -valued stationary process

$$(5.4) \quad X_n = \sum_{k=0}^{\infty} \xi_{n-k} A_k.$$

From (5.4) and Payen ([14], p. 364 and p. 375), we get that

$$(5.5) \quad M_{X_0 X_0}(B) = \frac{1}{2\pi} \int_B \Phi^*(e^{i\theta}) \Phi(e^{i\theta}) d\theta$$

where B is a Borel subset of $[0, 2\pi]$. Clearly X_n is purely non-deterministic ([14], p. 359). Let Z_n be the innovation process of X_n ([2], p. 897). Then X_n, Z_n are mutually subordinate ([2], p. 899, see also [5]). Also $Z_n^* Z_m = \delta_{nm} G$ where $G \in T(\mathcal{H}, \mathcal{H})$ (see e. g. [2], p. 897). Hence

$$(5.6) \quad M_{Z_0 Z_0} = \frac{1}{2\pi} G \cdot Leb.$$

From remark 3.27 we get $\text{rank } \Phi^*(\theta)\Phi(\theta) = \text{rank } G$ a. e.

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Michigan State University,
East Lansing, Michigan 48823.
