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J. W. COHEN

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Extreme value distribution for the M/G/1 and the G/M/1 queueing systems

by

J. W. COHEN

Technological University Delft, Holland.

SUMMARY. — For the supremum of the virtual delay time in a busy cycle and for the supremum of the actual waiting times of the customers served in a busy cycle the Laplace-Stieltjes transforms of the distribution functions have been found recently. Also for the supremum of the number of customers simultaneously present in the system during a busy cycle the generating function of the distribution is known. For every one of these variables the limit distribution of the maximum of these variables over a finite number of busy cycles is derived in the present paper. These limit distributions are obtained for the queueing systems M/G/1 and G/M/1 and for traffic intensities equal to one and less than one.

1. SOME RELATIONS FOR THE M/G/1 SYSTEM

For the M/G/1 queueing system denote by $\underline{v}_t$ the virtual waiting time at time t, by $\underline{x}_t$ the number of customers in the system at time t and by $\underline{w}_n$ the actual waiting time of the *nth* arriving customer with $\underline{w}_1 = 0$. Further $\underline{c}$ will denote the duration of a busy cycle and n the number of customers served in a busy cycle. Define

$$\underline{v}_{\max} \stackrel{\text{def}}{=} \sup_{0<t<\underline{c}} \underline{v}_t,$$

$$\underline{w}_{\max} \stackrel{\text{def}}{=} \sup_{1\leq n\leq \underline{n}} \underline{w}_n,$$

$$\underline{x}_{\max} \stackrel{\text{def}}{=} \sup_{0<t<\underline{c}} \underline{x}_t,$$

so that $\underline{v}_{\max}$ is the supremum of the virtual waiting time in a busy cycle, $\underline{w}_{\max}$ is the supremum of all actual waiting times of a busy cycle and $\underline{x}_{\max}$ the maximum number of customers simultaneously present in a busy cycle.

Denoting by $B(t)$ the distribution function of the service times and by α the average interarrival time then with

$$\beta(\rho) = \int_0^\infty e^{-\rho t} dB(t), \; \operatorname{Re} \rho \geqq 0, \; B(0+) = 0, \; \beta = \int_0^\infty t dB(t) < \infty,$$

we have

$$(1.1) \quad \Pr\{\underline{v}_{\max} < v\} = \frac{\dfrac{1}{2\pi i}\displaystyle\int_{C_\eta} e^{\eta v} \frac{\beta(\eta)}{\beta(\eta) + \alpha\eta - 1} d\eta}{\dfrac{1}{2\pi i}\displaystyle\int_{C_\eta} e^{\eta v} \frac{d\eta}{\beta(\eta) + \alpha\eta - 1}}, \; \operatorname{Re} \eta > \delta; \quad v > 0,$$
$$= 0. \qquad v < 0,$$

$$(1.2) \quad \Pr\{\underline{w}_{\max} < w\} = \frac{\dfrac{1}{2\pi i}\displaystyle\int_{C_\eta} e^{\eta w} \frac{d\eta}{\beta(\eta) + \alpha\eta - 1}}{\dfrac{1}{2\pi i}\displaystyle\int_{C_\eta} \frac{e^{\eta w}}{1 - \alpha\eta} \frac{d\eta}{\beta(\eta) + \alpha\eta - 1}}, \; \frac{1}{\alpha} > \operatorname{Re} \eta > \delta; \; w > 0,$$
$$= 0, \qquad w < 0,$$

and for $x = 1, 2, \ldots,$

$$(1.3) \quad \Pr\{\underline{x}_{\max} \leqq x\} = \frac{\dfrac{1}{2\pi i}\displaystyle\int_{D_\omega} \frac{d\omega}{\omega^x} \frac{\beta\left\{\frac{1}{\alpha}(1-\omega)\right\}}{\beta\left\{\frac{1}{\alpha}(1-\omega)\right\} - \omega}}{\dfrac{1}{2\pi i}\displaystyle\int_{D_\omega} \frac{d\omega}{\omega^{x+1}} \frac{\beta\left\{\frac{1}{\alpha}(1-\omega)\right\}}{\beta\left\{\frac{1}{\alpha}(1-\omega)\right\} - \omega}}, \; |\omega| < \mu.$$

Here we used the notation

$$\frac{1}{2\pi i}\int_{C_\eta} \ldots d\eta = \lim_{b\to\infty} \int_{R-ib}^{R+ib} \ldots d\eta, \quad R = \operatorname{Re} \eta,$$

and D_ω is a circle in the complex ω-plane with center at $\omega = 0$ and radius $|\omega|$, the positive direction of integration being counter clockwise. By δ is denoted the larger zero of $\beta(\eta) + \alpha\eta - 1$ with $\operatorname{Re} \eta \geqq 0$, while μ is the smaller zero inside or on the unit circle of $\beta\left\{\frac{1}{\alpha}(1-\omega)\right\}-\omega$. It is well known (cf. Takacs [1]) that if $a \stackrel{\text{def}}{=} \beta/\alpha \leqq 1$ then $\delta=0$, $\mu=1$; the zeros δ and μ have multiplicity one if $a \neq 1$, if $a=1$ they have multiplicity two. The relations (1.1) and (1.3) have been derived by Takacs [2] and by Cohen [3], [4], [5], while the relation (1.2) has been obtained by Cohen [6].

Let $\underline{w}$ and $\underline{x}$ be stochastic variables with distribution functions given by

$$(1.4)\quad \mathrm{E}\{e^{-\rho\underline{w}}\}=\frac{(1-a)\alpha\rho}{\beta(\rho)+\alpha\rho-1},\qquad \mathrm{Re}\,\rho\geqq 0,\qquad a<1,$$

$$(1.5)\quad \mathrm{E}\{\omega^{\underline{x}}\}=(1-a)\frac{(1-\omega)\beta\left\{\frac{1}{\alpha}(1-\omega)\right\}}{\beta\left\{\frac{1}{\alpha}(1-\omega)\right\}-\omega},\qquad |\omega|\leqq 1,\qquad a<1,$$

so that the distribution of $\underline{w}$ is the stationary distribution of the (virtual or actual) waiting time for the M/G/1 queue, and the distribution of $\underline{x}$ is the stationary distribution of the number of customers present in the M/G/1 queueing system.

Further let $\underline{\sigma}$ be a negative exponentially distributed variable with expectation α and $\underline{\tau}$ a variable with distribution function $\mathrm{B}(t)$. Assume that $\underline{w}$ and $\underline{\sigma}$ are independent, and also that $\underline{w}$ and $\underline{\tau}$ are independent. It follows from (1.1), ..., (1.5) that for $a<1$,

$$\begin{aligned}
\Pr\{\underline{v}_{\max}<v\}&=\frac{\Pr\{\underline{w}+\underline{\tau}<v\}}{\Pr\{\underline{w}<v\}}, && v>0,\\
&=0, && v<0;\\
\Pr\{\underline{w}_{\max}<w\}&=\frac{\Pr\{\underline{w}<w\}}{\Pr\{\underline{w}<w+\underline{\sigma}\}}, && w>0,\\
&=0, && w<0,\\
\Pr\{\underline{x}_{\max}\leqq x\}&=1-\frac{\Pr\{\underline{x}=x\}}{\Pr\{\underline{x}\leqq x\}}, && x=0,1,\ldots.
\end{aligned}$$

From (1.1) for $v>0$, $\mathrm{Re}\,\eta>0$, $a\leqq 1$,

$$(1.6)\qquad 1-\Pr\{\underline{v}_{\max}<v\}=\frac{\dfrac{\alpha}{2\pi i}\displaystyle\int_{C_\eta}e^{\eta v}\frac{\eta\,d\eta}{\beta(\eta)+\alpha\eta-1}}{\dfrac{1}{2\pi i}\displaystyle\int_{C_\eta}e^{\eta v}\frac{d\eta}{\beta(\eta)+\alpha\eta-1}};$$

from (1.2) for $w > 0, \frac{1}{\alpha} > \operatorname{Re} \eta > 0, a \leqq 1$,

$$(1.7) \qquad 1 - \Pr\{\underline{w}_{\max} < w\} = \frac{\dfrac{\alpha}{2\pi i}\displaystyle\int_{C_\eta} \frac{e^{\eta w}}{1-\alpha\eta}\frac{\eta d\eta}{\beta(\eta)+\alpha\eta-1}}{\dfrac{1}{2\pi i}\displaystyle\int_{C_\eta} \frac{e^{\eta w}}{1-\alpha\eta}\frac{d\eta}{\beta(\eta)+\alpha\eta-1}};$$

and from (1.3) for $x = 2, 3, \ldots, |\omega| < 1$,

$$(1.8) \qquad 1 - \Pr\{\underline{x}_{\max} \leqq x\} = \frac{\dfrac{1}{2\pi i}\displaystyle\int_{D_\omega} \frac{d\omega}{\omega^x}\frac{1-\omega}{\beta\left\{\frac{1}{\alpha}(1-\omega)\right\}-\omega}}{\dfrac{1}{2\pi i}\displaystyle\int_{D_\omega} \frac{d\omega}{\omega^x}\frac{1}{\beta\left\{\frac{1}{\alpha}(1-\omega)\right\}-\omega}}$$

Define

$$\begin{aligned} H(t) &\stackrel{\text{def}}{=} \frac{1}{\beta}\int_0^t \{1 - B(\tau)\}\, d\tau, & h(t) &\stackrel{\text{def}}{=} \frac{1}{\beta}\{1 - B(t)\}, & t > 0, \\ &= 0, & &= 0, & t < 0, \end{aligned}$$

so that H(t) is a distribution function having a bounded and monotone density function $h(t)$. Define for $a \leqq 1$

$$(1.9) \qquad K(t, a) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} a^n H^{n*}(t),$$

so that

$$(1.10) \qquad \int_{0-}^{\infty} e^{-\eta t} d_t K(t, a) = \frac{\alpha\eta}{\beta(\eta)+\alpha\eta-1}, \qquad \operatorname{Re} \eta > 0.$$

Obviously, K(t, 1) is the renewal function of a renewal process with H(t) as renewal distribution. Since H(t) has a density which is monotone and bounded K(t, a) has for $a \leqq 1$ a bounded derivative $k(t, a)$ (cf. Feller [7], p. 358) and

$$(1.11) \qquad k(t, a) = \frac{d}{dt} K(t, a), \qquad t > 0,$$

$$(1.12) \qquad \int_0^{\infty} e^{-\eta t} k(t, a) dt = \frac{\alpha\eta}{\beta(\eta)+\alpha\eta-1} - 1, \qquad \operatorname{Re} \eta > 0.$$

Since for $a \leqq 1$

$$\int_0^\infty \mathrm{K}(w+\tau, a)e^{-\tau/\alpha}\frac{d\tau}{\alpha} = e^{w/\alpha}\int_{t=w}^\infty e^{-t/\alpha}\mathrm{K}(t, a)\frac{dt}{\alpha},$$

$$= \mathrm{K}(w, a) + e^{w/\alpha}\int_{t=w}^\infty e^{-t/\alpha}k(t, a)dt,$$

we have for $w > 0$, $a \leqq 1$,

$$\text{(1.13)}\qquad \frac{d}{dw}\int_0^\infty \mathrm{K}(w+\tau, a)e^{-\tau/\alpha}\frac{d\tau}{\alpha} = e^{w/\alpha}\int_{t=w}^\infty e^{-t/\alpha}k(t, a)\frac{dt}{\alpha}.$$

It is easily seen that for $a \leqq 1$, $0 < \operatorname{Re} \eta < \dfrac{1}{\alpha}$,

$$\text{(1.14)}\qquad \int_0^\infty e^{-\eta w}d_w\int_0^\infty \mathrm{K}(w+t, a)e^{-t/\alpha}\frac{dt}{\alpha} = \frac{1}{1-\alpha\eta}\,\frac{\alpha\eta}{\beta(\eta)+\alpha\eta-1},$$

$$\int_0^\infty e^{-\eta w}d_w\int_0^\infty k(w+t, a)e^{-t/\alpha}\frac{dt}{\alpha} = \frac{1}{1-\alpha\eta}\,\frac{\alpha\eta^2}{\beta(\eta)+\alpha\eta-1}.$$

Further for $|\omega| < 1$, $x = 0, 1, \ldots,$

$$\text{(1.15)}\qquad \frac{1}{2\pi i}\int_{\mathrm{D}_\omega}\frac{d\omega}{\omega^{x+1}}\,\frac{1-\omega}{\beta\left\{\dfrac{1}{\alpha}(1-\omega)\right\}-\omega} = \int_{0-}^\infty \frac{(t/\alpha)^x}{x!}e^{-t/\alpha}d_t\mathrm{K}(t, a),$$

$$\frac{1}{2\pi i}\int_{\mathrm{D}_\omega}\frac{d\omega}{\omega^{x+1}}\,\frac{1}{\beta\left\{\dfrac{1}{\alpha}(1-\omega)\right\}-\omega} = \int_{0-}^\infty \frac{(t/\alpha)^x}{x!}e^{-t/\alpha}\mathrm{K}(t, a)\frac{dt}{\alpha}.$$

From (1.6), (1.7) and (1.8) it follows easily by using the inversion formula for the Laplace-Stieltjes transform that for $a \leqq 1$,

$$\text{(1.16)}\qquad 1 - \Pr\{\underline{v}_{\max} < v\} = \alpha\frac{k(v, a)}{\mathrm{K}(v, a)} = \alpha\frac{d}{dv}\log \mathrm{K}(v, a), \quad v > 0,$$

$$= \alpha\frac{d}{dv}\log \Pr\{\underline{w} < v\} \quad \text{if} \quad a < 1;$$

$$\text{(1.17)}\qquad 1 - \Pr\{\underline{w}_{\max} < w\} = \frac{\alpha\displaystyle\int_0^\infty k(w+t, a)e^{-t/\alpha}dt}{\displaystyle\int_0^\infty \mathrm{K}(w+t, a)e^{-t/\alpha}dt}$$

$$= \alpha\frac{d}{dw}\log\int_0^\infty \mathrm{K}(w+t, a)e^{-t/\alpha}dt,$$

$$= \alpha\frac{d}{dw}\log \Pr\{\underline{w} < w + \underline{\sigma}\} \text{ if } a < 1, \quad w > 0,$$

$$(1.18) \quad 1 - \Pr\{\underline{x}_{\max} < x\} = \frac{\displaystyle\int_{0-}^{\infty} \frac{(t/\alpha)^{x-1}}{(x-1)!} e^{-t/\alpha} d_t \mathrm{K}(t, a)}{\displaystyle\int_{0-}^{\infty} \frac{(t/\alpha)^{x-1}}{(x-1)!} e^{-t/\alpha} \mathrm{K}(t, a) \frac{dt}{\alpha}}, \quad x = 2, 3, \ldots,$$

From the relations

$$\mathrm{E}\{\underline{v}_{\max}\} = \int_0^{\infty} \{1 - \Pr\{\underline{v}_{\max} < v\}\}\, dv,$$

$$\mathrm{E}\{\underline{v}_{\max}^2\} = 2\int_0^{\infty} v\{1 - \Pr\{\underline{v}_{\max} < v\}\}\, dv,$$

and

$$\Pr\{\underline{w} < 0+\} = 1 - a \quad \text{if} \quad a < 1,$$

it is found that for $a < 1$ (cf. (1.16) and (1.17))

$$\mathrm{E}\{\underline{v}_{\max}\} = \frac{\beta}{a} \log \frac{1}{1-a},$$

$$\mathrm{E}\{\underline{v}_{\max}^2\} = -2\frac{\beta}{a} \int_0^{\infty} \log\{1 - \Pr\{\underline{w} \geqq v\}\}\, dv,$$

$$\mathrm{E}\{\underline{w}_{\max}\} = \frac{\beta}{a} \log \frac{\beta\left(\frac{1}{\alpha}\right)}{1-a},$$

$$\mathrm{E}\{\underline{w}_{\max}^2\} = -2\frac{\beta}{a} \int_0^{\infty} \log \Pr\{\underline{w} < w + \underline{\sigma}\}\, dw.$$

Since

$$\mathrm{E}\{\underline{v}_{\max}^2\} = 2\frac{\beta}{a} \sum_{n=1}^{\infty} \int_0^{\infty} \frac{1}{n} [\Pr\{\underline{w} \geqq w\}]^n dw,$$

and

$$\Pr\{\underline{w} \geqq w\} < a,$$

we have

$$2\frac{\beta}{a} \int_0^{\infty} \Pr\{\underline{w} \geqq w\}\, dw < \mathrm{E}\{\underline{v}_{\max}^2\} < 2\frac{\beta}{a} \sum_{n=1}^{\infty} \frac{a^{n-1}}{n} \int_0^{\infty} \Pr\{\underline{w} \geqq w\}\, dw$$

so that since

$$\int_0^{\infty} \Pr\{\underline{w} \geqq w\}\, dw = \frac{1}{2} \frac{a\beta}{1-a} \frac{\beta_2}{\beta^2},$$

with β_2 the second moment of B(t), we obtain

$$\frac{\beta_2}{1-a} < \mathrm{E}\{\underline{v}_{\max}^2\} < \frac{\beta_2}{1-a}\frac{1}{a}\log\frac{1}{1-a}.$$

It is seen that the second moment of $\underline{v}_{\max}$ is finite if $\beta_2 < \infty$, a similar conclusion holds for $\underline{w}_{\max}$ and $\underline{x}_{\max}$. It is noted that $\mathrm{E}\{\underline{w}_{\max}\}$ is finite if $a < 1$, while $\mathrm{E}\{\underline{w}\}$ is finite if $a < 1$ and $\beta_2 < \infty$.

2. EXTREME VALUE DISTRIBUTIONS FOR M/G/1

Suppose the server is idle at time $t = 0$. Denote by $\underline{v}_{\max}^{(j)}$, $\underline{w}_{\max}^{(j)}$ and $\underline{x}_{\max}^{(j)}$ the supremum of $\underline{v}_t$, of $\underline{w}_n$ and of $\underline{x}_t$ in the jth busy cycle of the queueing system M/G/1, $j = 1, 2, \ldots$. Obviously, $\underline{v}_{\max}^{(j)}$, $j = 1, 2, \ldots$, are independent, identically distributed variables with finite first moment if $a < 1$ and with finite second moment if $\beta_2 < \infty$. If $a < 1$ then the strong law of large numbers applies for the sequence $\underline{v}_{\max}^{(j)}$, $j = 1, 2, \ldots$; whereas if $\beta_2 < \infty$ the central limit theorem applies also for this sequence. Similar statements hold for the other sequences $\underline{w}_{\max}^{(j)}$, $j = 1, 2, \ldots$, and $\underline{x}_{\max}^{(j)}$, $j = 1, 2, \ldots$

Define for $n = 1, 2, \ldots$,

$$\underline{\mathrm{V}}_n \stackrel{\text{def}}{=} \max_{1 \leq j \leq n} \underline{v}_{\max}^{(j)}, \qquad \underline{\mathrm{W}}_n \stackrel{\text{def}}{=} \max_{1 \leq j \leq n} \underline{w}_{\max}^{(j)},$$
$$\underline{\mathrm{X}}_n \stackrel{\text{def}}{=} \max_{1 \leq j \leq n} \underline{x}_{\max}^{(j)},$$

i. e. $\underline{\mathrm{V}}_n$ is the supremum of the virtual waiting time in n busy cycles, $\underline{\mathrm{W}}_n$ that of the actual waiting times in n busy cycles and $\underline{\mathrm{X}}_n$ the supremum of the number of customers present simultaneously in the system during n busy cycles. For these variables we shall derive some limit theorems.

THEOREM 1. — If $a = 1$ and β_2, the second moment of B(t), is finite then the distributions of $\frac{1}{n\beta}\underline{\mathrm{V}}_n$, of $\frac{1}{n\beta}\underline{\mathrm{W}}_n$ and of $\frac{1}{n}\underline{\mathrm{X}}_n$ all converge for $n \to \infty$ to the distribution G(x) with

$$\mathrm{G}(x) = e^{-x^{-1}} \quad \text{for} \quad x > 0, \quad = 0 \quad \text{for} \quad x < 0.$$

Proof. Since $\beta_2/2\beta$ is the first moment of H(t), and since $h(t)$ is monotone we have from renewal theory (cf. Feller [7], p. 358)

$$\lim_{t\to\infty} \frac{\mathrm{K}(t, 1)}{t} = \frac{2\beta}{\beta_2}, \qquad \lim_{t\to\infty} k(t, 1) = \frac{2\beta}{\beta_2}. \tag{2.1}$$

Hence from (1.16) since $a = 1$

$$\text{(2.2)} \qquad \lim_{v\to\infty} v\{1 - \Pr\{\underline{v}^{(j)}_{\max} < v\}\} = \alpha = \beta.$$

From this relation and from

$$\Pr\left\{\frac{1}{n\beta}\mathrm{V}_n < x\right\} = [\Pr\{v^{(j)}_{\max} < n\beta x\}]^n = \left\{1 - \frac{\beta}{n\beta x} + o\left(\frac{1}{n}\right)\right\}^n, \qquad x > 0,$$

for $n \to \infty$ it follows immediately that

$$\lim_{n\to\infty} \Pr\left\{\frac{1}{n\beta}\underline{\mathrm{V}}_n < x\right\} = e^{-x^{-1}}, \qquad x > 0,$$
$$= 0, \qquad x < 0,$$

and the statement for $\underline{\mathrm{V}}_n$ has been proved.

From (1.17) for $a = 1$

$$\text{(2.4)} \quad 1 - \Pr\{\underline{w}_{\max} < w\} = \alpha \frac{\displaystyle\int_0^\infty k(w+\tau, 1)e^{-\tau/\alpha}\frac{d\tau}{\alpha}}{\displaystyle\int_0^\infty \mathrm{K}(w+\tau, 1)e^{-\tau/\alpha}\frac{d\tau}{\alpha}}, \qquad w > 0.$$

For given $\varepsilon > 0$ a finite number $\mathrm{W}(\varepsilon) > 0$ exists such that

$$\left|k(w, 1) - \frac{2\beta}{\beta_2}\right| < \varepsilon \qquad \text{for all} \qquad w > \mathrm{W}(\varepsilon),$$

so that

$$\left|k(w+t, 1) - \frac{2\beta}{\beta_2}\right| < \varepsilon \qquad \text{for all} \qquad w > \mathrm{W}(\varepsilon), \quad t \geqq 0.$$

Consequently, since $k(t, 1)$ is bounded

$$\lim_{w\to\infty} \int_0^\infty k(t+w, 1)e^{-t/\alpha}\frac{dt}{\alpha} = \frac{2\beta}{\beta_2}\int_0^\infty e^{-t/\alpha}\frac{dt}{\alpha} = \frac{2\beta}{\beta_2}.$$

Using (2.1) the same argumentation yields

$$\lim_{w\to\infty} \frac{1}{w}\int_0^\infty \mathrm{K}(w+t, 1)e^{-\tau/\alpha}\frac{d\tau}{\alpha} = \lim_{w\to\infty}\int_0^\infty \frac{\mathrm{K}(w+t, 1)}{w+t}e^{-t/\alpha}\left\{1 + \frac{t}{w}\right\}\frac{dt}{\alpha} = \frac{2\beta}{\beta_2}.$$

Hence from (2.4)

$$\lim_{w\to\infty} w\{1 - \Pr\{\underline{w}^{(j)}_{\max} < w\}\} = \alpha = \beta,$$

so that, as above the statement for $\underline{\mathrm{W}}_n$ follows.

For $x = 1, 2, \ldots,$

$$\int_{W(\varepsilon)}^{\infty} \frac{(t/\alpha)^{x-1}}{(x-1)!} e^{-t/\alpha} \left| k(t, 1) - \frac{2\beta}{\beta_2} \right| \frac{dt}{\alpha} \leqq \varepsilon \int_{W(\varepsilon)}^{\infty} \frac{(t/\alpha)^{x-1}}{(x-1)!} e^{-t/\alpha} \frac{dt}{\alpha} \leqq \varepsilon,$$

and

$$\lim_{x\to\infty} \int_0^{W(\varepsilon)} \frac{(t/\alpha)^{x-1}}{(x-1)!} e^{-t/\alpha} k(t, 1) \frac{dt}{\alpha} \leqq \max_{0 \leqq t \leqq W(\varepsilon)} k(t, 1) . \lim_{x\to\infty} \int_0^{W(\varepsilon)} \frac{(t/\alpha)^{x-1}}{(x-1)!} e^{-t/\alpha} \frac{dt}{\alpha} = 0.$$

It follows

$$\lim_{x\to\infty} \int_0^{\infty} \frac{(t/\alpha)^{x-1}}{(x-1)!} e^{-t/\alpha} \left(k(t, 1) - \frac{2\beta}{\beta_2} \right) \frac{dt}{\alpha} = 0,$$

or

$$\lim_{x\to\infty} \int_0^{\infty} \frac{(t/\alpha)^{x-1}}{(x-1)!} e^{-t/\alpha} k(t, 1) \frac{dt}{\alpha} = \frac{2\beta}{\beta_2}.$$

In the same way it is shown that

$$\lim_{x\to\infty} \int_0^{\infty} \frac{(t/\alpha)^{x}}{x!} e^{-t/\alpha} \frac{K(t, 1)}{t} \frac{dt}{\alpha} = \frac{2\beta}{\beta_2}.$$

Hence from (1.12)

$$\lim_{x\to\infty} x \{ 1 - \Pr \{ \underline{x}^{(j)}_{\max} \leqq x \} \} = 1,$$

the last relation leads as above to the statement for X_n. The theorem is proved.

THEOREM 2. — If $a < 1$, $\rho_0 > 0$ and $-\rho_0$ is the abcissa of convergence of $\beta(\rho)$ and if $\beta(-\rho_0 + 0) = \infty$ then for $-\infty < x < \infty$,

$$\lim_{n\to\infty} \Pr \left\{ \frac{1}{\beta} \underline{V}_n < \frac{x + \log (nb_1)}{-\varepsilon\beta} \right\} = e^{-e^{-x}},$$

$$\lim_{n\to\infty} \Pr \left\{ \frac{1}{\beta} \underline{W}_n < \frac{x + \log (nb_2)}{-\varepsilon\beta} \right\} = e^{-e^{-x}},$$

$$\lim_{n\to\infty} \Pr \left\{ \underline{X}_n < \frac{x + \log (nb_3)}{\log (1 - \alpha\varepsilon)} \right\} = e^{-e^{-x}},$$

with

$$b_1 = \frac{\alpha - \beta}{\alpha + \beta'(\varepsilon)} \alpha\varepsilon, \quad b_2 = \frac{\alpha - \beta}{\alpha + \beta'(\varepsilon)} \frac{\alpha\varepsilon}{1 - \alpha\varepsilon}, \quad b_3 = \frac{\alpha - \beta}{\alpha + \beta'(\varepsilon)} \alpha\varepsilon(1 - \alpha\varepsilon),$$

$$\beta'(\rho) = - \int_0^{\infty} t e^{-\rho t} dB(t), \quad \text{Re } \rho > -\rho_0,$$

and ε is the zero of $\beta(\eta) + \alpha\eta - 1$, Re $\eta < 0$ which is nearest to the imaginary axis Re $\eta = 0$.

Proof. Since $\rho_0 > 0$ and $a < 1$, the function $\beta(\eta) + \alpha\eta - 1$ has for Re $\eta < 0$ a real zero. Denote by ε its real zero nearest to the axis Re $\eta = 0$. Clearly $\varepsilon > -\rho_0$. From

$$|\beta(\eta)| \leqq \beta(\text{Re } \eta) = 1 - \alpha\varepsilon < |1 - \alpha\eta| \quad \text{for} \quad \text{Re } \eta = \varepsilon, \eta \neq \varepsilon,$$

it follows that ε is the only zero with Re $\eta = \varepsilon$. From

$$|\beta(\eta)| \leqq \beta(\text{Re } \eta) < |1 - \alpha\eta| \quad \text{for} \quad \text{Re } \eta > \varepsilon$$

and from Rouche's theorem it is seen that $\beta(\eta) + \alpha\eta - 1$ has only one zero with Re $\eta > \varepsilon$; this zero is $\eta = 0$. Hence ε is the zero with Re $\eta < 0$ nearest to the axis Re $\eta = 0$. Moreover, ε is a single zero, since

$$\beta'(\varepsilon) + \alpha = \beta'(\varepsilon) + \frac{1 - \beta(\varepsilon)}{\varepsilon} = -\sum_{n=1}^{\infty} \int_0^{\infty} (n-1) \frac{(-\varepsilon)^{n-1} t^n}{n!} dB(t) < 0,$$

the series being convergent. If $\beta(\eta) + \alpha\eta - 1$ has a second zero ε_1 with Re $\varepsilon_1 < 0$ then $-\rho_0 <$ Re $\varepsilon_1 < \varepsilon$. Let C_ξ be a line parallel to the imaginary axis with Re $\varepsilon_1 <$ Re $\xi < \varepsilon$ if ε_1 exists, otherwise $-\rho_0 <$ Re $\xi < \varepsilon$. The function $\beta(\eta) + \alpha\eta - 1$ is analytic for Re $\eta >$ Re ξ and has single zeros at $\eta = \varepsilon$ and $\eta = 0$. From Cauchy's theorem it follows for

$$\text{Re } \eta > 0 > \varepsilon > \text{Re } \xi > \text{Re } \varepsilon_1 > -\rho_0$$

$$\frac{\alpha}{2\pi i} \int_{C_\eta} e^{\eta v} \frac{\eta d\eta}{\beta(\eta) + \alpha\eta - 1} = \frac{\alpha\varepsilon}{\alpha + \beta'(\varepsilon)} e^{\varepsilon v} + \frac{\alpha}{2\pi i} \int_{C_\xi} e^{\xi v} \frac{\xi d\xi}{\beta(\xi) + \alpha\xi - 1}, \qquad v > 0,$$

$$\frac{1}{2\pi i} \int_{C_\eta} e^{\eta v} \frac{d\eta}{\beta(\eta) + \alpha\eta - 1} = \frac{1}{\alpha - \beta} + \frac{1}{\alpha + \beta'(\varepsilon)} e^{\varepsilon v} + \frac{1}{2\pi i} \int_{C_\xi} e^{\xi v} \frac{d\xi}{\beta(\xi) + \alpha\xi - 1}, \qquad v > 0.$$

It is easily verified that

$$\lim_{v \to \infty} \frac{e^{\varepsilon v}}{2\pi i} \int_{C_\xi} e^{\xi v} \frac{\xi d\xi}{\beta(\xi) + \alpha\xi - 1} = 0, \quad \lim_{v \to \infty} \frac{e^{\varepsilon v}}{2\pi i} \int_{C_\xi} e^{\xi v} \frac{d\xi}{\beta(\xi) + \alpha\xi - 1} = 0.$$

Hence from (1.6) we obtain

$$\lim_{v \to \infty} e^{-\varepsilon v} \{ 1 - \Pr \{ \underline{v}_{\max}^{(j)} < v \} \} = \frac{\alpha - \beta}{\alpha + \beta'(\varepsilon)} \alpha\varepsilon = b_1 > 0.$$

Therefore

$$\Pr\left\{\frac{1}{\beta}\underline{V}_n < \frac{x + \log(nb_1)}{-\varepsilon\beta}\right\} = \left[\Pr\left\{\underline{v}_{\max}^{(j)} < \frac{x + \log(nb_1)}{-\varepsilon\beta}\right\}\right]^n,$$

so that for $n \to \infty$

$$\Pr\left\{\frac{1}{\beta}\underline{V}_n < \frac{x + \log(nb_1)}{-\varepsilon\beta}\right\} = \left[1 - b_1 e^{-x-\log(nb_1)} + o\left(\frac{1}{n}\right)\right]^n,$$

i. e.

$$\lim_{n\to\infty} \Pr\left\{\frac{1}{\beta}\underline{V}_n < \frac{x + \log(nb_1)}{-\varepsilon\beta}\right\} = e^{-e^{-x}}, \quad -\infty < x < \infty.$$

This proves the statement for $\underline{V}_n$, that for $\underline{W}_n$ is proved in the same way. The statement for $\underline{X}_n$ is also analogous. Start from (1.3) and move the path of integration D_ω to a circle with radius $|\omega| > 1$ and such that the first zero of

$$\beta\left\{\frac{1}{\alpha}(1-\omega)\right\} - \omega$$

outside the circle $|\omega| = 1$ is an interior point of this circle.

COROLLARY to theorem 2. For $a < 1$ the variables $\frac{1}{\beta}\frac{\underline{V}_n}{\log n}$, $\frac{1}{\beta}\frac{\underline{W}_n}{\log n}$ and $\frac{\underline{X}_n}{\log n}$ converge for $n \to \infty$ in probability to $-\frac{1}{\varepsilon\beta}$, $-\frac{1}{\varepsilon\beta}$ and $\frac{1}{\log(1-\alpha\varepsilon)}$, respectively.

Proof. For every fixed $x > 0$ it follows from theorem 2 that for $n \to \infty$

$$\Pr\left\{\left|\frac{1}{\beta}\frac{\underline{V}_n}{\log n} + \frac{1}{\varepsilon\beta} + \frac{\log b_1}{\varepsilon\beta\log n}\right| > \frac{x}{-\varepsilon\beta\log n}\right\} \to \{e^{-e^{x}} + 1 - e^{-e^{-x}}\},$$

so that for every $z > 0$,

$$\Pr\left\{\left|\frac{1}{\beta}\frac{\underline{V}_n}{\log n} + \frac{1}{\varepsilon\beta} + \frac{\log b_1}{\varepsilon\beta\log n}\right| > \frac{z}{-\varepsilon\beta}\right\} \to e^{-n^z} + 1 - e^{-n^{-z}} \to 0 \quad \text{for} \quad n \to \infty,$$

and hence the statement for $\underline{V}_n$ follows; the other statements are proved similarly.

During a busy cycle a realisation of $\underline{v}_t$ may have a number of intersections with level K. There are no intersections at all if during the busy cycle the virtual delay time is always less than K. Denote by $\underline{\Pi}_K^{(j)}$ the number of intersections from above with level K of $\underline{v}_t$ in the jth busy cycle, $j = 1, 2, \ldots$.

Obviously, the variables $\underline{\Pi}_K^{(j)}, j = 1, 2, \ldots$, are independent and identically distributed variables. It has been shown in [8] that if $a \leqq 1$ then

$$\Pr\{\underline{\Pi}_K^{(j)} = m\} = f(0), \qquad m = 0,$$
$$= \{1 - f(0)\}\{1 - h(0)\}^{m-1}h(0), \quad m = 1, 2, \ldots,$$

where

$$f(0) = \Pr\{\underline{v}_{\max} < K\},$$
$$h(0) = \left[\frac{1}{2\pi i}\int_{C_\eta} e^{\eta K}\frac{\alpha d\eta}{\beta(\eta) + \alpha\eta - 1}\right]^{-1}, \quad \operatorname{Re}\eta > 0.$$

Denote by E_K the state with K customers left behind in the system at a departure. Let $\underline{\Lambda}_K^{(j)}$ represent the number of times that state E_K occurs during the *jth* busy cycle. Obviously, $\underline{\Lambda}_K^{(j)}, j = 1, 2, \ldots$, are independent and identically distributed variables. It has been shown in [9] that if $a \leqq 1$ then

$$\Pr\{\underline{\Lambda}_K^{(j)} = m\} = f(1), \qquad m = 0,$$
$$= \{1 - f(1)\}\{1 - h(1)\}^{m-1}h(1), \quad m = 1, 2, \ldots,$$

where

$$f(1) = \Pr\{\underline{x}_{\max} \leqq K\},$$
$$h(1) = \left[\frac{1}{2\pi i}\int_{D_\omega}\frac{d\omega}{\omega^{K+1}}\frac{\beta\left\{\frac{1}{\alpha}(1-\omega)\right\}}{\beta\left\{\frac{1}{\alpha}(1-\omega)\right\} - \omega}\right]^{-1}, \quad |\omega| < 1.$$

Define

$$\underline{P}_{K,n} \stackrel{\text{def}}{=} \max_{1 \leqq j \leqq n} \underline{\Pi}_K^{(j)}, \quad \underline{L}_{K,n} \stackrel{\text{def}}{=} \max_{1 \leqq j \leqq n} \underline{\Lambda}_K^{(j)},$$

then we have :

THEOREM 3. — If $a \leqq 1$ then

$$\lim_{n\to\infty}\Pr\left\{\underline{P}_{K,n} < \frac{x + \log\left\{n\dfrac{1 - f(0)}{1 - h(0)}\right\}}{-\log\{1 - h(0)\}}\right\} = e^{-e^{-x}}, \quad -\infty < x < \infty,$$
$$\lim_{n\to\infty}\Pr\left\{\underline{L}_{K,n} < \frac{x + \log\left\{n\dfrac{1 - f(1)}{1 - h(1)}\right\}}{-\log\{1 - h(1)\}}\right\} = e^{-e^{-x}}, \quad -\infty < x < \infty.$$

Proof. It is easily verified that

$$\Pr\{\underline{\Pi}_K^{(j)} \geqq m\} = \frac{1 - f(0)}{1 - h(0)}\exp\{m\log(1 - h(0))\}$$

from which the statement of the theorem follows as in the preceding theorem. Similarly for $\underline{L}_{K,n}$.

As before we obtain.

COROLLARY to theorem 3. For $a \leq 1$ the variables $\frac{\underline{P}_{K,n}}{\log n}$ and $\frac{\underline{L}_{K,n}}{\log n}$ converge for $n \to \infty$ in probability to $\frac{1}{-\log\{1 - h(0)\}}$ and $\frac{1}{-\log\{1 - h(1)\}}$, respectively.

It is noted that if $B(t) = 1 - e^{-t/\beta}$ for $t > 0$ then $\varepsilon\beta = -(1-a)$, $b_1 = (1-a)$, $b_2 = a(1-a)$, $b_3 = a^{-1}(1-a)$, $1 - \alpha\varepsilon = a^{-1}$,

$$
\begin{aligned}
f(0) &= \frac{1 - e^{-(1-a)K/\beta}}{1 - ae^{-(1-a)K/\beta}}, & h(0) &= \frac{1-a}{1 - ae^{-(1-a)K/\beta}}, & a &< 1,\\
&= \frac{K/\beta}{1 + K/\beta}, & &= \frac{1}{1 + K/\beta}, & a &= 1,\\
f(1) &= \frac{1 - a^{K}}{1 - a^{K+1}}, & h(1) &= \frac{1-a}{1 - a^{K+1}}, & a &< 1,\\
&= \frac{K}{1 + K}, & &= \frac{1}{1 + K}, & a &= 1.
\end{aligned}
$$

3. EXTREME VALUE DISTRIBUTIONS FOR G/M/1

Denote by $A(t)$ the distribution function of the interarrival times for the queueing system G/M/1 ;

$$\alpha(\rho) = \int_0^\infty e^{-\rho t} dA(t), \quad \text{Re } \rho \geqq 0, \quad A(0+) = 0, \quad \alpha = \int_0^\infty t\, dA(t) < \infty.$$

For the system G/M/1 the variables $\underline{v}_{\max}$, $\underline{w}_{\max}$ and $\underline{x}_{\max}$ will have the same meaning as those for the system M/G/1, and similarly for $\underline{V}_n$, $\underline{W}_n$ and $\underline{X}_n$.

For $a \leqq 1$ we have (cf. Cohen [3], [5], [6]),

$$
\begin{aligned}
(3.1)\quad 1 - \Pr\{\underline{v}_{\max} < v\} &= 0, & v &< 0,\\
&= \left\{\frac{1}{2\pi i}\int_{C_\xi} e^{\xi v}\frac{\beta d\xi}{\alpha(\xi) + \beta\xi - 1}\right\}^{-1}, \ \text{Re } \xi > \psi, & v &> 0,
\end{aligned}
$$

$$
\begin{aligned}
(3.2)\quad 1 - \Pr\{\underline{w}_{\max} < w\} &= 0, & w &< 0,\\
&= \left\{\frac{1}{2\pi i}\int_{C_\xi} \frac{e^{\xi v}}{1 - \beta\xi}\frac{\beta d\xi}{\alpha(\xi) + \beta\xi - 1}\right\}^{-1}, \ \frac{1}{\beta} > \text{Re } \xi > \psi, & w &> 0,
\end{aligned}
$$

$$
\begin{aligned}
(3.3)\quad &1 - \Pr\{\underline{x}_{\max} \leqq x\}\\
&= \left\{\frac{1}{2\pi i}\int_{D_\omega} \frac{d\omega}{\omega^x}\frac{1}{\alpha\left\{\frac{1}{\beta}(1-\omega)\right\} - \omega}\right\}^{-1}, \ |\omega| < \varphi, \quad x = 1, 2, \ldots,
\end{aligned}
$$

here Ψ is the larger zero of $\alpha(\xi) + \beta\xi - 1$ with Re $\xi \geqq 0$, and φ is the smaller zero of $\alpha\left\{\frac{1}{\beta}(1-\omega)\right\} - \omega$ with $|\omega| \leqq 1$. If $a = 1$ then $\psi = 0$, $\varphi = 1$, whereas for $a < 1$ both φ and ψ are positive with multiplicity one. Put

$$\begin{aligned} \mathrm{N}(t) &\overset{\text{def}}{=} 0, && t < 0, \\ &= \int_0^t \{1 - \mathrm{A}(u)\} \frac{du}{\alpha}, && t > 0, \end{aligned}$$

and

$$\mathrm{M}(t) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \{\mathrm{N}(t)\}^{n*}, \qquad t > 0,$$

so that M(t) is the renewal function of the renewal process with N(t) as renewal distribution. As in section 1 (cf. the derivation of (1.10), ..., (1.12)) we have from (3.1), ..., (3.3) for $a = 1$

$$\begin{aligned} 1 - \Pr\{\underline{v}_{\max} < v\} &= \{\mathrm{M}(v)\}^{-1}, && v > 0, \\ 1 - \Pr\{\underline{w}_{\max} < w\} &= \left\{\int_0^\infty \mathrm{M}(w + t) e^{-t/\beta} \frac{dt}{\beta}\right\}^{-1}, && w > 0, \\ 1 - \Pr\{\underline{x}_{\max} \leqq x\} &= \left\{\int_0^\infty \frac{(t/\beta)^{x-1}}{(x-1)!} e^{-t/\beta} \mathrm{M}(t) \frac{dt}{\beta}\right\}^{-1}, && x = 1, 2, \ldots \end{aligned}$$

If the second moment α_2 of A(t) is finite then from renewal theory

$$\lim_{t \to \infty} \frac{\mathrm{M}(t)}{t} = \frac{2\alpha}{\alpha_2}.$$

The same argumentation as used in the proof of theorem 1 leads immediately to

THEOREM 4. — If $a = 1$ and $\alpha_2 < \infty$ then the distribution functions of $\frac{2\alpha}{n\alpha_2}\underline{\mathrm{V}}_n$, of $\frac{2\alpha}{n\alpha_2}\underline{\mathrm{W}}_n$ and of $\frac{2\alpha^2}{n\alpha_2}\underline{\mathrm{X}}_n$ all converge to G(x) for $n \to \infty$.

Further

THEOREM 5. — If $a < 1$ then for $-\infty < x < \infty$,

$$\begin{aligned} \lim_{n\to\infty} \Pr\left\{\frac{1}{\beta}\underline{\mathrm{V}}_n < \frac{x + \log(nc_1)}{\psi\beta}\right\} &= e^{-e^{-x}}, \\ \lim_{n\to\infty} \Pr\left\{\frac{1}{\beta}\underline{\mathrm{W}}_n < \frac{x + \log(nc_2)}{\psi\beta}\right\} &= e^{-e^{-x}}, \\ \lim_{n\to\infty} \Pr\left\{\underline{\mathrm{X}}_n < \frac{x + \log(nc_1)}{-\log(1 - \alpha\psi)}\right\} &= e^{-e^{-x}}, \end{aligned}$$

with

$$c_1 = \frac{\alpha'(\psi) + \beta}{\beta}, \qquad c_2 = \frac{\alpha'(\psi) + \beta}{\beta}(1 - \beta\psi),$$

$$\alpha'(\rho) = -\int_0^\infty te^{-\rho t}dA(t), \qquad \text{Re}\ \rho \geqq 0.$$

Proof. From (3.1) we have for Re $\xi > \psi >$ Re $\eta > 0$, $v > 0$

$$\frac{1}{2\pi i}\int_{C_\xi} e^{\xi v}\frac{\beta d\xi}{\alpha(\xi) + \beta\xi - 1} = \frac{\beta e^{\psi v}}{\alpha'(\psi) + \beta} + \frac{1}{2\pi i}\int_{C_\eta} e^{\eta v}\frac{\beta d\eta}{\alpha(\eta) + \beta\eta - 1},$$

so that, since $\alpha(\eta) + \eta\beta - 1$ has no zeros for $0 <$ Re $\eta < \psi$, it immediately follows from (3.1) that

$$\lim_{v\to\infty} e^{\psi v}\{1 - \Pr\{\underline{v}^{(j)}_{\max} < v\}\} = \frac{\alpha'(\psi) + \beta}{\beta} = \frac{1}{\beta}\left\{\alpha'(\psi) + \frac{1 - \alpha(\psi)}{\psi}\right\} > 0.$$

From this relation the statement for $\underline{\mathbf{V}}_n$ follows as in the proof of theorem 2. The proof of the statement for $\underline{\mathbf{W}}_n$ is similar. To prove the statement for $\underline{\mathbf{X}}_n$ move the path of integration D_ω to a circle with radius $|\zeta|$ and such that $\varphi < |\zeta| < 1$, and observe that $\varphi = 1 - \alpha\psi$. The statement for $\underline{\mathbf{X}}_n$ is now easily derived.

COROLLARY to theorem 5. For $a < 1$ the variables $\frac{1}{\beta}\frac{\underline{\mathbf{V}}_n}{\log n}$, $\frac{1}{\beta}\frac{\underline{\mathbf{W}}_n}{\log n}$ and $\frac{\underline{\mathbf{X}}_n}{\log n}$ converge for $n \to \infty$ in probability to $\frac{1}{\psi\beta}$, $\frac{1}{\psi\beta}$ and $\frac{1}{-\log(1 - \alpha\psi)}$, respectively.

The proof is analogous to that of the corollary of theorem 2 in the preceding section.

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