ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 71, nº 2 (1999), p. 129-198
<http://www.numdam.org/item?id=AIHPA_1999__71_2_129_0>

Ann. Inst. Henri Poincaré,

Vol. 71, n° 2, 1999, p. 129-198. *Physique théorique*

Infrared analysis of the tridimensional Gross–Neveu model: Pointwise bounds for the effective potential

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Article received on 14 May 1998, revised 30 June 1998

ABSTRACT. – Within the context of renormalization group analysis, we describe how to get a very detailed control of the effective potential theory for some fermionic systems using the tree expansion technique. We consider the tridimensional Gross–Neveu model (with smooth ultraviolet cut-off) and we prove that the kernels of the effective potential can be written in terms of a convergent perturbative expansion in the initial interaction parameter (with an upper bound for the convergence radius independent on the volume). Moreover, we obtain pointwise bounds for these kernels showing that they decay polynomially (in a well precise sense) as the distance between points becomes large. © Elsevier, Paris

RÉSUMÉ. – Du point de vue de l'analyse par le groupe de renormalization, nous présentons une façon d'obtenir un contrôle très fin de la théorie du potentiel effectif pour quelques systèmes fermioniques, en utilizant

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la technique d'expansion en arbres. Nous étudions le modèle de Gross–Neveu en trois dimensions (avec cutoff ultraviolet lisse) et nous démonstrons que les noyaux du potentiel effectif peuvent être écrits comme un developpement convergent dans le paramètre initial d'interaction (avec une borne supérieure pour le rayon de convergence, indépendante du volume). En plus, nous obtenons des bornes ponctuelles pour ces noyaux, ce qui démontre leur décroissance polynomiale (dans un sens trés précis) lorsque la distance entre les points devient grande.

1. INTRODUCTION

In the last fifteen years a considerable effort has been spent in order to apply the renormalization group (RG) method to many popular problems in mathematical physics, ranging from classical mechanics to field theory and quantum many body systems. The basic idea of RG, i.e., roughly speaking, the analysis of the problem through a splitting in many scales of lenght, has been made rigorous and applied in many different ways.

In the present paper, within the framework of fermionic interacting systems, we aim to show how to obtain a very detailed description and control of the effective potential theory, i.e., of the changes of the interactions with the RG flow, using the Gallavotti–Nicoló tree expansion technique. We study the infrared limit of the tridimensional Gross–Neveu model (with a smooth ultraviolet cut-off which regularizes the theory at short distances), and we obtain pointwise bounds for all the k-point kernels of the effective potential after n steps of the renormalization group transformation, showing that their long distance behavior is given, as $n \to \infty$, in terms of polynomially decaying functions. We still prove that they are analytic functions of the initial interaction parameter (with an upper bound for the convergence radius independent on the volume). The pointwise behaviour of the kernels of effective potential is an important information to the knowledge of the correlation decay, which has a direct physics interest. In many place it is possible to find detailed RG analysis to get "integral" (respect to some norm) bounds (e.g., [6,8, 15,18] and see Remark 5 after Theorem 3.1 below). On the other hand, we could not be able to find in the literature a complete and rigorous analysis for pointwise bounds (this problem was in some sense understimated in [6], see Remark 6 after Theorem 3.1).

Let us now introduce the model and some notations.

The Gross–Neveu model is a relativistc Fermi system described by the action (in Euclidean formalism) $H = H_0 + V^{(0)}$ with H_0 and $V^{(0)}$ beeing the free action and the perturbation, respectively, given by:

$$H_0 = \int_\Lambda dx \overline{\psi}_x \, \mathrm{i} \, \partial\!\!\!/^{(\geqslant 0)} \psi_x, \qquad V^{(0)} = \lambda \int_\Lambda \left[\overline{\psi}_x \psi_x\right]^2. \tag{1.1}$$

Here we will consider just the tridimensional case ($d = 3$), so $x \in \Lambda \subset \mathbb{R}^{2+1}$ ($\mathbb{R}^{2+1}$ is the tridimensional Euclidean space-time), Λ is a periodic box in $\mathbb{R}^{2+1}$, λ a small real paramenter; $\overline{\psi}_x$, ψ_x represent spinors with entries $\overline{\psi}_x^{\beta}$, ψ_x^{α} ($\alpha, \beta = 1, 2, 3, 4$), with ψ_x^{α} and $\overline{\psi}_x^{\beta}$ standard Grassmann fields. We name the model Gross–Neveu but take the number of flavours $N = 1$. The notation $\partial\!\!\!/$ means $\partial\!\!\!/ = \partial_\mu \gamma^\mu$ (sum over $\mu = 0, 1, 2$, $\partial_\mu = \frac{\partial}{\partial x^\mu}$) with γ^0, γ^1, γ^2 being 4×4 antihermitian traceless matrices such that $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = -2\delta^{\mu\nu}$. Expressions like $\overline{\psi}\psi$ or $\overline{\psi} \, \partial\!\!\!/ \psi$ or $\psi\overline{\psi}$ have to be interpreted in their matricial sense, i.e.,

$$\overline{\psi}\psi = \sum_{\alpha=1}^{4} \overline{\psi}^{\alpha} \psi^{\alpha}, \quad \overline{\psi} \, \partial\!\!\!/ \psi = \sum_{\alpha,\beta=1}^{4} \overline{\psi}^{\alpha} \, \partial\!\!\!/^{\alpha\beta} \psi^{\beta}, \quad \psi\overline{\psi} = \psi^{\alpha}\overline{\psi}^{\beta}$$

(the last is a 4×4 matrix as the product of a 4-column spinor with a 4-row spinor). The free propagator of the model (with a smooth U.V. cut-off) is, by definition (1.1)

$$\begin{aligned}\left[i \, \partial\!\!\!/^{(\geqslant 0)}\right]^{-1}(x - y) &= g^{(\geqslant 0)}(x - y) \\ &= \frac{1}{(2\pi)^3} \int d^3 p \, \mathrm{e}^{\mathrm{i}p(x-y)} \frac{p\!\!\!/}{p^2} \, \mathrm{e}^{-p^2}.\end{aligned} \tag{1.2}$$

The notation ($\geqslant 0$) indicates that the U.V. cut-off is at the scale $L^0 = 1$ ($L > 1$ being a constant with dimensions of a lenght). The reasons for this notation will be clear in the next section. $p(x - y)$ means $p_\mu (x - y)^\mu$.

The free propagator $g^{(\geqslant 0)}$ satisfies the pointwise (asymptotic) bound:

$$\left|g^{(\geqslant 0)}(x - y)\right| \lesssim \frac{C}{1 + |x - y|^2}, \quad C > 0. \tag{1.3}$$

The action of the model is invariant under a (discret) Chiral transformation: $\psi \to \gamma^5 \psi$, $\overline{\psi} \to \overline{\psi}\gamma^5$. Thus, in the I.R. analysis, this model has just a local marginal term $\int dx \overline{\psi}_x (\mathrm{i} \, \partial\!\!\!/) \psi_x$. The local quartic term $V^{(0)}$ in H is irrelevant. Local quadratic terms as $\delta m \int dx \overline{\psi}_x \psi_x$ or

$\delta m_\mu \int dx \overline{\psi}_x \gamma^\mu \psi_x$ are forbidden in the effective potential by Chiral symmetry and euclidean symmetry, respectively. Thus, in the study of the renormalization group (RG) flow there will be just a running coupling constant, the "wavefunction" renormalization constant related to the marginal term.

The structure of the RG flow for this model is therefore rather simple, but the model is far from being trivial. Roughly speaking, it can be viewed as a sort of fermionic version of the dipole gas in $d \geqslant 2$ dimensions. The latter is a very studied problem of statistical mechanics (including rigorous RG analysis [8,14]), consisting in a gas of classical particles interacting through a two-body stable but not absolutely integrable potential. The rigorous RG analysis of the dipole gas is performed, in general, by mapping the model (throught a Sine–Gordon transformation) into a bosonic field theory. The action of this bosonic model is formed by a kinetic marginal term, $\phi \partial^2 \phi$ plus a small irrelevant perturbation term given by a function of $\partial \phi$. The relevant mass term ϕ^2 cannot be generated in the RG flow due to the symmetry of the initial action (its dependence on derivative fields). Hence, the parallel with our model is made clear: the action of our fermionic model has also the structure of a kinetic marginal term $\overline{\psi}(\mathrm{i}\ \not{\partial})\psi$ plus an irrelevant (quartic) perturbation, and the relevant mass term $\overline{\psi}\psi$ cannot be generated during the RG flow because of the symmetry properties of the initial action (i.e., discret Chiral symmetry).

Still concerning the non-triviality of the model to be considered here, we recall that to obtain some rigorous results such as the absolute convergence of the perturbative expansion in λ (uniform in the volume Λ) for the pressure, effective potential kernels, etc., a treatment involving just one step integration (all scales at once) does not work, as in the case of the dipole gas, unless one is able to exploit suitable cancellations without introducing dangerous combinatorial factors. Due to the difficulty of the latter task, a direct proof of the analicity of the pressure for the dipole gas is still missing [8].

In relation to our fermionic model, the machinery of the scale per scale RG analysis (and the consequent resummation) provides the standard (and, as far as we know, unique) tool to handle these kind of cancellations, while the Brydges–Battle–Federbush tree equality [7], and the good combinatorial behaviour of fermionic expectations allow to keep under control the combinatorics.

Our multiscale RG analysis is based on the Gallavotti–Nicoló tree expansion algorithm adapted to Fermi systems (e.g., [3–5] and [6]), and includes an important technical device: the rescaling is performed by introducing into the free measure, step by step, the wavefunction term extracted at each step. This a very general way to perform the rescaling, especifically suitable (and probably necessary [12]) to treat non-canonical scaling models [6], but also widely used for canonical scaling or even asymptotically free theories [8,14] (more comments at the end of Section 2.3 below).

We think that the results obtained here, and also the method used in order to obtain these results, can be useful for further purpose, e.g., the study of the k-points correlation or Schwinger functions (with $k > 2$) for fermionic systems with anomalous scaling, like the one-dimensional Fermi liquid and the Thirring model.

Finally, we list some rigorous constructive results available in the literature on the general Gross–Neveu model in its various versions (N flavours, d dimensions, massive or massless, IR or UV). In particular, we mention the construction of the $d = 2$, N large, massless IR case (which is a highly non-trivial model due to the mass generation mechanism) [16] and of the two-dimensional massive, $N \geqslant 2$, UV (which is asymptotically free) case [11]. The tridimensional, N large, UV case (again a very untrivial model due to its nonrenormalizability) has also been rigorousely studied in [10]. In all these references the model is mapped in a purely bosonic theory and consequentely, beyond the multiscale analysis, technical tools like, e.g., polymer expansion and large field small field analysis are heavely used. In [15] one may find an alternative construction of the two-dimensional massive, $N \geqslant 2$, UV case, based just on a purely fermionic formalism (i.e., more similar to the techniques used in this paper). Constructive results on fermionic models using just fermionic approach can also be found in [6] ($d = 1 + 1$ Fermi liquid, which should include also the IR Gross–Neveu model in $d = 2$ with $N = 1$ and the Luttinger model) and in [18] (Yukawa model). Finally we remark a recent renewed interest on purely fermionic constructive field theories, e.g., [1] and [9].

We will try to be as self-contained and pedagogical as possible. In this spirit, Sections 2 and 3 of the paper will be devoted to introduce all the notations and definitions indispensable to understand the proofs of Sections 4 and 5. In particular, in Section 2 we present the multiscale decomposition leading to the RG mechanism and we define the tree expansion algorithm for the multiscale perturbation theory. In Section 3 we define the (anomalous type) renormalization prescription, set up the RG flow for the effective potential and introduce the main theorem of the article (Theorem 3.1) concerning the pressure and the effective potential kernels of the present model. Section 4 is devoted to preliminary estimates including the bound on the wave function renormalization constant and the consequent proof of analiticity of the pressure (which is the "easy" part of the main theorem, involving just "integral" bounds). In Section 5, we complete the the proof of the hard part of the main theorem, i.e., the analiticity in λ of the k-point kernels of effective potential and, especially, their pointwise bounds.

2. THE RG MECHANISM AND THE BARE TREE EXPANSION

2.1. Basic definitions

Before describing the RG mechanism to be used here, we introduce some previous structures and definitions. The generating functional of the correlation functions is written as

$$\exp\left[-S(\bar{h},h)\right] = \int P^{(\geqslant 0)}(d\psi)\, \mathrm{e}^{V^{(0)}(\overline{\psi},\psi)-\int dx \bar{h}_x \psi_x - \int dx \overline{\psi}_x h_x}. \tag{2.1}$$

In this formula (as also in (1.1)) ψ_x, $\overline{\psi}_x$, h_x and $\bar{h}_x$ (h and $\bar{h}$ are the external fields), with $x \in \Lambda$, are generators of a Grassmann algebra, i.e., they are anticommuting numbers (e.g., $\psi_x\psi_y + \psi_y\psi_x = 0$, etc.). The symbol $P^{(\geqslant 0)}(d\psi)$ represents a normalized Gaussian Fermionic Measure (GFM) (formally) given by:

$$P^{(\geqslant 0)}(d\psi) = \frac{\prod_{x\in\Lambda} d\overline{\psi}_x\, d\psi_x\, \mathrm{e}^{\int_\Lambda dx \overline{\psi}_x (\mathrm{i}\partial\!\!\!/^{(\geqslant 0)})\psi_x}}{\int \prod_{x\in\Lambda} d\overline{\psi}_x\, d\psi_x\, \mathrm{e}^{\int_\Lambda dx \overline{\psi}_x (\mathrm{i}\partial\!\!\!/^{(\geqslant 0)})\psi_x}},$$

$$\int (\ldots) P^{(\geqslant 0)}(d\psi) = \mathcal{E}_{g^{(\geqslant 0)}}(\ldots). \tag{2.2}$$

$P^{(\geqslant 0)}(d\psi)$ works like a Gaussian measure ruled by a Fermionic Wick Theorem with covariance $g^{(\geqslant 0)}(x-y)$. Hence, the simple expectation $\mathcal{E}_{g^{(\geqslant 0)}}$ acts on field monomials as follows:

$$\mathcal{E}_{g^{(\geqslant 0)}}\left(\psi_x^{\alpha}\overline{\psi}_y^{\beta}\right) = g_{\alpha\beta}^{(\geqslant 0)}(x-y),$$

$$\mathcal{E}_{g^{(\geqslant 0)}}\left(\psi_{x_1}^{\alpha_1}\cdots\psi_{x_n}^{\alpha_m}\overline{\psi}_{y_1}^{\beta_1}\cdots\psi_{y_m}^{\beta_m}\right) = \begin{cases} 0 & \text{if } n \neq m, \\ \det[G_{ij}] & \text{if } n = n, \end{cases} \tag{2.3}$$

where $[G_{ij}]$ is the $n \times n$ matrix with entries $G_{ij} = g_{\alpha_i\beta_j}^{(\geqslant 0)}(x_i - y_j)$.

The reader worried about the rigorous meaning of the formulas above must recall that Λ is periodic cube of size $|\Lambda|^{1/3}$, and so the integral over momenta p in (1.2) is actually a sum over a discrete set of values of $p = 2\pi|\Lambda|^{-1/3}(n_1, n_2, n_3)$. Hence, defining $\psi_x = |\Lambda|^{-1}\sum_p \widetilde{\psi}_p \mathrm{e}^{\mathrm{i}px}$ we may reduce ourselves to a countable set ($\widetilde{\psi}_p$ and c.c.) of Grassmann generators. Following this scheme we can give a precise mathematical meaning to Grassmann integrals as Berezin integrals and also to infinite sums of Grassmann monomials. This is a commun practice (see, e.g., [4,6,15]). Here and throughout this paper we interpretate the expressions involving Grassmann variables in the sense of [4,6].

Towards the analysis of correlations, it is very useful (and a widely adopted tool) to study a functional $V_{\text{eff}}(\bar{h}, h)$, called effective potential or effective action, defined by

$$V_{\text{eff}}(\bar{h}, h) = \log \int P^{(\geqslant 0)}(d\psi)\, \mathrm{e}^{V^{(0)}(\bar{\psi}+\bar{h}, \psi+h)}. \tag{2.4}$$

The relation between $V_{\text{eff}}(\bar{h}, h)$ and $S(\bar{h}, h)$ is, via a change of variables [4],

$$S(\bar{h}, h) = (\bar{h}, g^{(\geqslant 0)} * h) - V_{\text{eff}}(g^{(\geqslant 0)} * h), \tag{2.5}$$

where $(f, g) = \int dx f(x) g(x)$ and $f * g(x) = \int dy f(x-y) g(y)$.

The expansion of $V_{\text{eff}}(\bar{h}, h)$ in terms of the perturbative potential $V^{(0)}$ is immediate. Namely, by the cumulant expansion formula, right hand side of (2.4) can be rewritten as

$$V_{\text{eff}}(\bar{h}, h) = \sum_{N=1}^{\infty} \frac{1}{N!} \mathcal{E}^T_{g^{(\geqslant 0)}}\big[\underbrace{V^{(0)}(\bar{\psi}+\bar{h}, \psi+h), \ldots, V^{(0)}(\bar{\psi}+\bar{h}, \psi+h)}_{N \text{ times}}\big], \tag{2.6}$$

where $\mathcal{E}^T_{g^{(\geqslant 0)}}$ is the truncated expectation relatively to the Gaussian Fermi measure $P^{(\geqslant 0)}(d\psi)$, defined by

$$\mathcal{E}^T_{g^{(\geqslant 0)}}(V^{(0)}, \ldots, V^{(0)}) = \left(\frac{\partial}{\partial\lambda_1 \cdots \partial\lambda_N} \log \mathcal{E}_{g^{(\geqslant 0)}}\big[\mathrm{e}^{(\lambda_1 V^{(0)} + \cdots + \lambda_N V^{(0)})}\big]\right)\Big|_{\lambda_1 = \cdots = \lambda_N = 0}. \tag{2.7}$$

Recalling the definition of $V^{(0)}$ (1.1), it is clear that the expansion (2.6) is actually an expansion in power on λ. In particular, the partition function (which coincides with $\exp[V_{\text{eff}}(\bar{h} = 0, h = 0)]$) is given explicitly, as a power series in λ, by

$$\Xi_\Lambda(\lambda) = \int P^{(\geqslant 0)}(d\psi)\, e^{V^{(0)}(\overline{\psi},\psi)}$$
$$= \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} \int_\Lambda dx_1 \cdots \int_\Lambda dx_N \mathcal{E}_{g(\geqslant 0)} [\underbrace{(\overline{\psi}_{x_1}\psi_{x_1})^2 \cdots (\overline{\psi}_{x_N}\psi_{x_N})^2}_{N \text{ times}}], \quad (2.8)$$

while the "pressure" of the model $p_\Lambda(\lambda) = |\Lambda|^{-1} V_{\text{eff}}\ (\bar{h} = 0, h = 0)$ is given explicitly by

$$|\Lambda| p_\Lambda(\lambda)$$
$$= \log \Xi_\Lambda(\lambda)$$
$$= \sum_{N=1}^{\infty} \frac{\lambda^N}{N!} \int_\Lambda dx_1 \cdots \int_\Lambda dx_N \mathcal{E}^T_{g(\geqslant 0)} [\underbrace{(\overline{\psi}_{x_1}\psi_{x_1})^2, \ldots, (\overline{\psi}_{x_N}\psi_{x_N})^2}_{N \text{ times}}]. \quad (2.9)$$

The expansions in power of λ (2.6) and (2.9) are analytic in λ but the Λ dependence of their convergence radius is, for the time being, out of control and the radius may shrink to zero as $\Lambda \to \infty$. Of course, the RG analysis below will provide a bound for the convergence radius uniform in Λ. We want to stress also that (2.8) and (2.9) make even stronger the analogy of the present model with the dipole gas. Actually, (2.8) and (2.9) can be viewed (*cum grano salis*) as the partition function and the Mayer series of a system of classical particles in the gran canonical ensamble at inverse temperature $\beta = 1$ and fugacity λ, enclosed in a volume Λ. The factor

$$\mathcal{E}_{g(\geqslant 0)} [(\overline{\psi}_{x_1}\psi_{x_1})^2 \cdots (\overline{\psi}_{x_N}\psi_{x_N})^2]$$

can be interpreted (modulo a sign) as the Gibbs factor

$$\exp[-U(x_1, \ldots, x_N)],$$

and

$$\mathcal{E}^T_{g(\geqslant 0)} [(\overline{\psi}_{x_1}\psi_{x_1})^2, \ldots, (\overline{\psi}_{x_N}\psi_{x_N})^2]$$

is interpreted as the N order Ursell coefficient of the Mayer series [7]. The potential $U(x_1, \ldots, x_N)$ is actually stable (i.e., $U(x_1, \ldots, x_N) \geqslant -BN$, with B constant), due to the Hadamard inequality, which provides a bound of type C^N for the simple expectations of Fermionic fields, as in (2.8) (see later (4.21)). But $U(x_1, \ldots, x_N)$ is not a tempered interaction [24], i.e., if $|x_i - x_j| \to \infty$ and all others x's are fixed, then

$$|U(x_1, \ldots, x_N)| \approx \text{const}/|x_i - x_j|^2,$$

which is a too slow decay rate, a consequence of the fact that $g^{(\geqslant 0)}$ is not absolutely integrable (see (1.3)). This fact make extremely hard to obtain a $|\Lambda|$-independent C^N bound on the N order Mayer coefficent in (2.9) via a one step integration: one must be able to exhibit suitable cancellations without destroying stability (i.e., without producing dangerous combinatorial factors), exactly as in the case of the dipole gas. The way out, of course, is to analyze the series (2.6) and (2.8) using the RG multiscale analysis, which we describe in next section.

2.2. Multiscale decomposition of the free covariance and tree expansion definition and notations

Now we describe the free propagator multiscale decomposition which will lead us to the multiscale tree expansion of the effective potential. We write

$$g^{(\geqslant 0)}(x-y)=\sum_{j=0}^{\infty} g^{(j)}(x-y), \tag{2.10}$$

where

$$g^{(j)}(x-y)=\int \frac{d^3p}{(2\pi)^3}\left(\mathrm{e}^{-L^{2j}p^2}-\mathrm{e}^{-L^{2j+2}p^2}\right)\frac{\not p}{p^2}\,\mathrm{e}^{\mathrm{i}p(x-y)},\qquad L>1, \tag{2.11}$$

$$g^{(j)}(x)=L^{-2j}C\left(L^{-j}x\right),\ \left|g^{(j)}(x)\right|\leqslant CL^{-2j}\exp\left[-\alpha L^{-j}|x|\right], \tag{2.12}$$

$$C(x)=\int \frac{d^3p}{(2\pi)^3}\left(\mathrm{e}^{-p^2}-\mathrm{e}^{-L^2p^2}\right)\frac{\not p}{p^2}\,\mathrm{e}^{\mathrm{i}px},\qquad |C(x)|\leqslant C\exp[-\alpha|x|], \tag{2.13}$$

$$g^{(\geqslant j)}(x-y)=\sum_{k=j}^{\infty} g^{(k)}(x-y)=\int \frac{d^3p}{(2\pi)^3}\,\mathrm{e}^{-L^{2j}p^2}\frac{\not p}{p^2}\,\mathrm{e}^{\mathrm{i}p(x-y)}, \tag{2.14}$$

where α and C are positive constants. Here and throughout this paper C always denotes a generic constant (i.e., the notation C may be used for different constants). The decomposition (2.10) induces a decomposition of the free Gaussian fermionic measure (2.2)

$$\psi^{(\geqslant 0)}=\psi=\sum_{j=0}^{\infty}\psi^{(j)}\quad\text{and}\quad P^{(\geqslant 0)}(d\psi)=\prod_{j=0}^{\infty}P\left(d\psi^{(j)}\right), \tag{2.15}$$

where $\psi^{(j)}$ are Grassmann independent fields (respect to j) on Λ, with Gaussian Fermionic Measure (GFM) $P(d\psi^{(j)})$, with covariance $g^{(j)}(x-y)$. Analogously, $P(d\psi^{(\geqslant j)})$ will indicate the Gaussian fermionic measure with covariance $g^{(\geqslant j)}(x-y)$, acting on Grassmann random fields on Λ, indicated by $\psi^{(\geqslant j)} = \sum_{k=j}^{\infty} \psi^{(k)}$.

The "running" effective potential $V_{\text{eff}}^{(j)}(\psi^{(\geqslant)})$ at scale j is defined by

$$e^{V_{\text{eff}}^{(j+1)}(\psi^{(\geqslant j+1)})} = \int P(d\psi^{(j)})\, e^{V_{\text{eff}}^{(j)}(\psi^{(\geqslant j+1)}+\psi^{(j)})} \tag{2.16}$$

and

$$\Xi_\Lambda(\lambda) = \lim_{n\to\infty} \prod_{j=0}^{n} Z_j \quad \text{with } Z_j = \int P(d\psi^{(j)})\, e^{V_{\text{eff}}^{(j)}(\psi^{(j)})}. \tag{2.17}$$

By a cumulant expansion (2.16) implies

$$V_{\text{eff}}^{(j+1)}(\psi^{(\geqslant j)}1)$$
$$= \sum_{N=1}^{\infty} \frac{1}{N!} \mathcal{E}_{g^{(j)}}^{T}\big[V_{\text{eff}}^{(j)}(\psi^{(j)} + \psi^{(\geqslant j+1)}), \ldots, V_{\text{eff}}^{(j)}(\psi^{(j)} + \psi^{(\geqslant j+1)})\big]. \tag{2.18}$$

We may represent graphically the truncated expecation in the formula (see Fig. 1). The n steps iteration of (2.18), through the graphical identification above, produces the so called Gallavotti–Nicoló tree expansion representation of the effective potential $V_{\text{eff}}^{(n)}$ at scale n.

We now review the main ingredients and the basic notations of this expansion (general treatments can be found in [12,13] and [4]).

Let us indicate with the symbol θ^N a *rooted Cayley tree* with N end points. θ^N is organized hierarchically in a natural way. Namely, a rooted Cayley tree starts with a single vertex, named the *root* of the tree, followed by a line which bifurcates at the vertex v_0 into $s_{v_0} > 1$

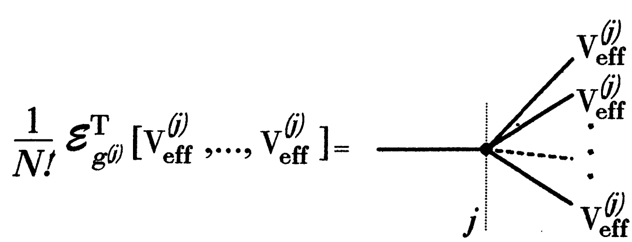

Fig. 1. Graphical representation of a truncated expectation.

branches, each one of these branches bifurcates again, and so on, until an end point is reached. A *non-trivial vertex* v of θ^N is a point where a bifurcation occurs (we may consider, sometimes, the root and the end points as non-trivial vertices too). Non-trivial vertices in a rooted Cayley tree form a partial ordered set in a natural way. I.e., given two non-trivial vertices v and w, we say that $w < v$ if w can be reached when we climb the tree starting from the vertex v. If $w < v$, we will also say that w *precedes* v, or either, v *follows* w. Throughout the paper, we denote v_0 the greatest non-trivial vertex of θ^N (i.e., the first preceding the root). For any non-trivial vertex v, we will indicate as s_v the number of branches in which v bifurcates. We also will indicate as $v^1, v^2, \ldots, v^{s_v}$ the s_v non-trivial vertices immediately preceding v (remark that some of them may be end points). We denote v' the unique non-trivial vertex immediately following v (if $v = v_0$ then v' is the root). Two rooted Cayley trees are said *topologically identical* (and they will be considered as the *same* rooted Cayley tree) if they can be superimposed exactly just streching/shortening lines between non-trivial vertices or increasing/reducing angles between lines starting from the same non-trivial vertex, without creating or destroying non-trivial vertices and without overlapping lines. So, a rooted Cayley tree will be univocally determinated once the sequence of its non-trivial vertices, hierarchically organized in clusters according to the natural partial order above, has been given. It is an easy combinatorial exercise to bound the number of all topologically different rooted Cayley trees with N end points (e.g., [6])

$$\sum_{\theta^N} 1 \leqslant (4\mathrm{e})^N. \tag{2.19}$$

A *labelled tree* $\theta_{\text{lab}}^{N,n}$ with N end points and root at scale n is a rooted Cayley tree for which scale labels $n_v = 0, 1, 2, \ldots,$ have been assigned at each non-trivial vertex v, compatibly with the natural partial order of the rooted Cayley tree. Namely, if $w < v$, then $n_w < n_v$. In a line of a labelled tree among two successive non-trivial vertices v and v', we place $n_{v'} - n_v - 1$ points, called *trivial vertices*, and assign to them scale labels $n_v + 1, n_v + 2, \ldots, n_{v'} - 1$, respectively. For later use, we also use a specific symbol v^* for the greater among them (i.e., the one with scale label equal to $n_{v'} - 1$).

End points in a labelled tree $\theta_{\text{lab}}^{N,n}$ are numbered, from top to bottom, as $1, 2, \ldots, N$. The factor $\lambda \int dx_i [\overline{\psi}_{x_i} \psi_{x_i}]^2$ is attached to the ith end point. We also denote n_i the scale label of the first non-trivial vertex

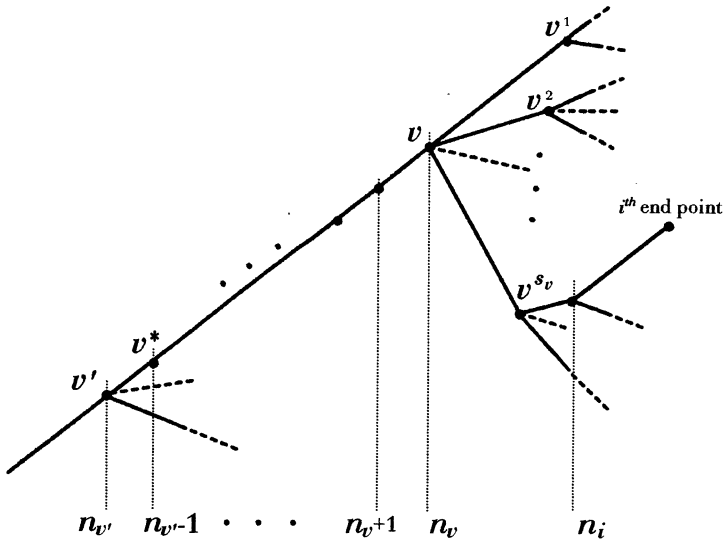

Fig. 2. The neighborhood of a non-trivial vertex v in a labelled tree.

following the end point i (see Fig. 2). A non-trivial vertex v in a labelled tree represents, through the graphical identification of Fig. 1, a factor $1\backslash s_v!$ times a truncated expectation, ruled by the covariance $g^{(n_v)}(x-y)$, of s_v objects; a trivial vertex u represents in a natural way a simple expectation ruled by the covariance $g^{(n_u)}(x-y)$. Then it is natural to associate to a given labelled tree $\theta_{\text{lab}}^{N,n}$ a hierarchical organized sequence of simple and truncaded expectations at different scales of N objects, say $\mathcal{E}_{\text{lab}}^{\theta^{(N,n)}}(\cdot,\cdot,\ldots,\cdot)$, i.e., explicitly

$$\mathcal{E}_{\text{lab}}^{\theta^{(N,n)}}(\cdot,\cdot,\ldots,\cdot) = \left[\prod_{v\leqslant v_0} \mathcal{E}_{n_{v'}-1}\cdots\mathcal{E}_{n_v+1}\frac{1}{s_v!}\mathcal{E}_{n_v}^T\right] \times \left[\mathcal{E}_{n_1+1}\cdots\mathcal{E}_0(\cdot),\ldots,\mathcal{E}_{n_N+1}\cdots\mathcal{E}_0(\cdot)\right], \quad (2.20)$$

where the productory $\prod_{v\leqslant v_0}$ runs over all *non-trivial vertices* (end point and root excluded). The right hand side of (2.20), evalutaded when the N objects are N copies of $V^{(0)}$, is easily recognized as a single term, say $V^{(n)}(\theta_{\text{lab}}^{N,n})$, in the multiscale expansion of $V_{\text{eff}}^{(n)}$ obtained by iteration of (2.18). Hence, the tree expansion of $V_{\text{eff}}^{(n)}$ is simply

$$V_{\text{eff}}^{(n)} = \sum_{N=1}^{\infty}\sum_{\theta^N}\sum_{\theta_{\text{lab}}^{N,n}:\,\theta^N \text{ fixed}} V^{(n)}(\theta_{\text{lab}}^{N,n}). \quad (2.21)$$

For N fixed, the second sum in formula above is over all possible (topologically different) rooted Cayley trees with N end points. For θ^N fixed, the third sum is over all the possible ways of labelling θ^N (i.e., all possible attributions of scale labels n_v in non-trivial vertices v compatibly with the hierarchical structure of the tree θ^N), in such way that the root carries the scale label n. $V^{(n)}(\theta_{\text{lab}}^{N,n}, \psi^{(\geqslant n)})$ is explicitly given by

$$\begin{aligned}V^{(n)}(\theta_{\text{lab}}^{N,n}, \psi^{(\geqslant n)}) &= \mathcal{E}_{\text{lab}}^{\theta(N,n)}(V^{(0)}, \dots, V^{(0)}) \\ &= \lambda^N \int_\Lambda dx_1 \cdots \int_\Lambda dx_N \left[\prod_{v \leqslant v_0} \mathcal{E}_{n_{v'}-1} \cdots \mathcal{E}_{n_v+1} \frac{1}{s_v!} \mathcal{E}_{n_v}^T \right] \\ &\quad \times \Big[\mathcal{E}_{n_1+1} \cdots \mathcal{E}_0 \big[\overline{\psi}_{x_1}^{(\geqslant 0)} \psi_{x_1}^{(\geqslant 0)} \big]^2, \dots, \\ &\qquad \mathcal{E}_{n_N+1} \cdots \mathcal{E}_0 \big[\overline{\psi}_{x_N}^{(\geqslant 0)} \psi_{x_N}^{(\geqslant 0)} \big]^2 \Big]. \end{aligned} \tag{2.22}$$

Remark also that, due to the symmetries of our model ($g^{(j)}(0) = 0$ for all j)

$$\mathcal{E}_{n_i+1} \cdots \mathcal{E}_0 \Big[\big[\overline{\psi}_{x_i}^{(\geqslant 0)} \psi_{x_i}^{(\geqslant 0)} \big]^2 \Big] = \big[\overline{\psi}_{x_i}^{(\geqslant n_i)} \psi_{x_i}^{(\geqslant n_i)} \big]^2, \tag{2.23}$$

i.e., everything goes as if $[\overline{\psi}\psi]^2$ were Wick ordered.

2.3. The renormalization

Of course a "bare" multiscale analysis as the one presented above is not enough to obtain convergence of the perturbative expansion. Due to the problem of lack of absolute integrability of the free covariance, we also need to perform some resummation of the perturbative series to look for suitable cancellations. This resummation is provided by the scale per scale renormalization prescription for the marginal term $\overline{\psi}\, \partial\!\!\!/ \psi$ of the model. We follow the scheme of [6], and [15]. We define $\mathcal{L}$ and $\mathcal{R} = 1 - \mathcal{L}$ operations acting on field monomials (no matter the scale), which split the effective potential into its relevant/marginal part plus the irrelevant part (in the sense of dimensional power counting)

$$V_{\text{eff}}^{(n)} = \mathcal{L} V_{\text{eff}}^{(n)} + \mathcal{R} V_{\text{eff}}^{(n)} \tag{2.24}$$

and $\mathcal{L}$, $\mathcal{R}$ are defined directly acting on field monomials (we drop scale indices here) as

$$\begin{aligned} &\mathcal{L}[\overline{\psi}_{x_1} \cdots \overline{\psi}_{x_m} \psi_{y_1} \cdots \psi_{y_m}] = 0, \\ &\mathcal{R}[\overline{\psi}_{x_1} \cdots \overline{\psi}_{x_m} \psi_{y_1} \cdots \psi_{y_m}] = \overline{\psi}_{x_1} \cdots \overline{\psi}_{x_m} \psi_{y_1} \cdots \psi_{y_m} \quad \text{if } m > 2, \end{aligned}$$

$$\mathcal{L}[\overline{\psi}_{x_1}\psi_{x_2}] = \overline{\psi}_{x_1}\psi_{x_1} + (x_2 - x_1)^\mu \overline{\psi}_{x_1}\partial_\mu \psi_{x_1},$$

$$\mathcal{R}[\overline{\psi}_{x_1}\psi_{x_2}] = \int_0^1 dt(1-t)(x_2 - x_1)^2 \overline{\psi}_{x_1}\partial^2 \psi_{x_{12(t)}}, \tag{2.25}$$

where $x_{12(t)} = x_1 + t(x_2 - x_1)$, and $(x_2 - x_1)^2\partial^2 = (x_2 - x_1)^\mu (x_2 - x_1)^\nu \partial_\mu \partial_\nu$. Note that the definitions (2.25) are consistent with $\mathcal{L} + \mathcal{R} = 1$, and that (2.25) is a consequence of the Taylor formula

$$\psi_{x_2} = \psi_{x_1} + (x_2 - x_1)\partial\psi_{x_1} + (x_2 - x_1)^2 \int_0^1 dt(1-t)\partial^2 \psi_{x_{12(t)}}.$$

Now, for our model, due to chiral and Euclidean symmetries, and translational invariance, it is easy to see that

$$\mathcal{L}V_{\text{eff}}^{(n)} = b_n \int dx \overline{\psi}_x \,\mathrm{i}\, \partial\!\!\!/ \psi_x, \tag{2.26}$$

where b_n is a scalar constant (wavefunction constant). As a matter of fact, we have

$$\mathcal{L}V_{\text{eff}}^{(n)} = \int dx\, dy W_{[2]}^{(n)}(x-y)\overline{\psi}_x \psi_x + \int dx\, dy (x-y)^\mu W_{[2]}^{(n)}(x-y)\overline{\psi}_x \partial_\mu \psi_x. \tag{2.27}$$

Observe that $\int dz W_{[2]}^{(n)}(z) = 0$: $W_{[2]}^{(n)}(z)$ is necessarely a sum of product of an odd number of covariances $g^{(j)}$ at various scales $g^{(\geqslant 0)}(x)$ and $g^{(j)}(x)$ are odd functions of x for any j. Moreover, by Euclidean symmetry, $\int dz z^\mu W_{[2]}^{(n)}(z) = \text{const}\, \gamma^\mu$. Hence, recalling that in $d = 3$ $\text{tr}[\gamma_\mu \gamma^\mu] = -12$, we have that the running coupling b_n in (2.26) is explicitly given by:

$$b_n = \frac{1}{12}\text{tr}\left[\int dz W_{[2]}^{(n)}(z)\,\mathrm{i}\, z\!\!\!/\right]. \tag{2.28}$$

The irrelevant part of the effective potential is given by

$$\mathcal{R}V_{\text{eff}}^{(n)} = \int_0^1 dt(1-t)\int dx_1\, dx_2 (x_1 - x_2)^2 \times W_{[2]}^{(n)}(x_1 - x_2)\overline{\psi}_{x_1}\partial^2 \psi_{x_{12(t)}} + \sum_{l\geqslant 2} V_{[l]}^{(n)}(\psi), \tag{2.29}$$

where $V_{[l]}^{(n)}(\psi)$ contains the terms with more than two fields (or two fields with one $\partial^2\psi$).

The scale per scale renormalization can be done following, at least, two ways.

(1) At each step we split

$$V_{\text{eff}}^{(j)} = \mathcal{L}V_{\text{eff}}^{(j)} + \mathcal{R}V_{\text{eff}}^{(j)}$$

and we regard b_j in $\mathcal{L}V_{\text{eff}}^{(j)}$ as a new expansion parameter which appears at scale j. So, generally speaking $V_{\text{eff}}^{(j+1)}$ can be regarded as a power series in $\lambda, b_1, \ldots, b_j$. Remark that, with this procedure, the factor $\mathcal{L}V_{\text{eff}}^{(j)}$ *is left* in the effective potential, thus we are actually defining the *same* sequence $V_{\text{eff}}^{(1)}, \ldots, V_{\text{eff}}^{(j)}, \ldots$ of running effective potential, but the expansion is reorganized collecting toghether, step by step, an amount of terms forming new expansion parameters $b_1, b_2, \ldots, b_j$ (i.e., the running coupling constants). Hence $V_{\text{eff}}^{(j+1)}$ now is a power expansion not just in terms of λ, but also in terms of $b_1, b_2, \ldots, b_j$. This method is very well illustrated in [12] and [4]. It has the great advantage to leave unchanged the effective potential during the RG analysis, so leading directly to correlation functions via (2.5); but it seems to work just in the case of asymptotically free theories and in general it is expected to fail in anomalous scaling cases [3].

(2) At each step we split again $V^{(j)} = \mathcal{L}V^{(j)} + \mathcal{R}V^{(j)}$, then we *remove* $\mathcal{L}V^{(j)} = \delta b_j \overline{\psi}\,\mathrm{i}\,\not{\partial}\psi$ from the effective potential and put it into the measure $P(d\psi^{(\geqslant j)})$ which has still to be integrated out. This, roughly speaking, will change the constant in front of the covariance by an amount δb_j, the correction at scale j of the wave function constant. The *new* effective potential $V^{(j)}$ is indeed a power series in λ, and parameters δb_j appear implicitly inside the covariances (remark that $V^{(j)}$ is now a *different object* respect to $V_{\text{eff}}^{(j)}$ defined before; that is why we are using a different symbol). In this way a *new* sequence of running effective potentials $V^{(1)}, V^{(2)}, \ldots, V^{(j)}, \ldots$ is contructed, which is obviously different from the sequence on point (1).

This method is more general, since it can be also adopted for non-asymptotically free or even non-canonical scaling models. Moreover the analysis of the effective potential is expected to be simpler, since it does not depend explicitly on all running coupling at lower scales. An unpleasent (and not always remarked in the literature) consequence of this procedure is that $\lim_{n\to\infty} V^n \neq V_{\text{eff}}$ so that (2.5) cannot be used in order to calculate correlation functions, and the relation between the renormalized effective potential and the generating functional of correlation becomes in general more involved. Anyway, an explicit formula, with a structure very similar to (2.5), does exist. It has been furnished for the first time in [19] (see also [21]) for bosonic lattice systems using the block spin RG transformation, and it has been extended in [20,22] for lattice fermions. The generalization of the formula to continuous formalism has been obtained in [23], where it is used to perform a pointwise analysis of the correlation functions of the present model.

As said in the introduction, having also in mind to provide a model-independet algorithm which can be in principle applied also to non-asymptotically free fermionic models, we will adopt the latter and more general method and we will follow the scheme of [6].

3. THE RG FLOW AND THE RENORMALIZED TREE EXPANSION

Now we describe the RG flow. We will generate a sequence of running effective potential $V^{(j)}$ and a sequence of running coupling constants b_j ($j=0,1,\ldots$, and $b_0=1$). We will indicate as $P_{b_j}(d\psi^{(\geqslant j)})$ the normalized GFM with covariance $b_j^{-1}g^{(\geqslant j)}$ and $P_{b_j}(d\psi^{(j)})$ the normalized GFM with covariance $b_j^{-1}g^{(j)}$ (see (2.11)–(2.14)). Consider the partition function (2.8), where $P_{b_0}(d\psi^{(\geqslant 0)})$ may replace $P(d\psi^{(\geqslant 0)})$ (since $b_0 = 1$). We start integrating out the fluctuation field $\psi^{(0)}$ using

$$P_{b_0}(d\psi^{(\geqslant 0)}) = P_{b_0}(d\psi^{(\geqslant 1)})\, P_{b_0}(d\psi^{(0)}).$$

Thus,

$$\begin{aligned}\Xi_\Lambda(\lambda) &= \int P_{b_0}(d\psi^{(\geqslant 1)})\, P_{b_0}(d\psi^{(0)})\, \mathrm{e}^{V^{(0)}(\psi^{(\geqslant 0)})}\\ &= \int P_{b_0}(d\psi^{(\geqslant 1)})\, \mathrm{e}^{V^{(1)}(\psi^{(\geqslant 0)})}\, \mathrm{e}^{|\Lambda| T_1},\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}&V^{(1)}(\psi^{(\geqslant 1)})\\ &\quad= \sum_{N=1}^{\infty} \frac{1}{N!} \mathcal{E}^T_{g^{(0)}/b_0}\left[V^{(0)}(\psi^{(\geqslant 1)}+\psi^{(0)}), \ldots, V^{(0)}(\psi^{(\geqslant 1)}+\psi^{(0)})\right],\end{aligned} \tag{3.2}$$

$$|\Lambda|T_1 = \log \int P_{b_0}(d\psi^{(0)})\, e^{V^{(0)}(\psi^{(0)})}$$
$$= \sum_{N=1}^{\infty} \frac{1}{N!} \mathcal{E}^T_{g^{(0)}/b_0}\left[V^{(0)}(\psi^{(0)}), \ldots, V^{(0)}(\psi^{(0)})\right]. \tag{3.3}$$

The Fermionic truncated expectations are performed using as covariance $b_0^{-1} g^{(0)}(x-y)$ acting on fields $\psi^{(0)}$ and considering the fields $\psi^{(\geqslant 1)}$ as constants. The general form for $V^{(1)}(\psi^{(\geqslant 1)})$ is

$$V^{(1)}(\psi^{(\geqslant 1)})$$
$$= \int dx_1\, dx_2\, W^{(1)}_{[2]}(x_1 - x_2) \overline{\psi}^{(\geqslant 1)}_{x_1} \psi^{(\geqslant 1)}_{x_2} + \sum_{l>2} V^{(1)}_{[l]}(\psi^{(\geqslant 1)}), \tag{3.4}$$

where $\sum_{l>2} V^{(1)}_{[l]}(\psi^{(\geqslant 1)})$ contains the terms with more than two fields. Now we split $V^{(1)}$ into its marginal and irrelevant part using definitions (2.25) of operations $\mathcal{L}$ and $\mathcal{R}$ and using the symmetries of the model

$$V^{(1)}(\psi^{(\geqslant 1)}) = \mathcal{L}V^{(1)}(\psi^{(\geqslant 1)}) + \mathcal{R}V^{(1)}(\psi^{(\geqslant 1)})$$
$$= \delta b_0 \int dx \overline{\psi}^{(\geqslant 1)}_x \mathrm{i}\, \partial\!\!\!/ \psi^{(\geqslant 1)}_x + \mathcal{R}V^{(1)}(\psi^{(\geqslant 1)}) \tag{3.5}$$

with

$$\mathcal{R}V^{(1)}(\psi^{(\geqslant 1)}) = \int_0^1 dt(1-t) \int dx_1\, dx_2 (x_1 - x_2)^2 W^{(1)}_{[2]}(x_2 - x_1)$$
$$\times \overline{\psi}^{(\geqslant 1)}_{x_1} \partial^2 \psi^{(\geqslant 1)}_{x_{12}(t)} + \sum_{l>2} V^{(1)}_{[l]}(\psi^{(\geqslant 1)}) \tag{3.6}$$

and

$$\delta b_0 = \frac{1}{12} \mathrm{tr}\left[\int dz\, W^{(1)}_{[2]}(z)\, \mathrm{i}\, z\!\!\!/\right]. \tag{3.7}$$

Hence, the partition function (2.8) can be written as

$$\Xi_\Lambda(\lambda) = e^{T_1|\Lambda|} \int P_{b_0}(d\psi^{(\geqslant 1)})\, e^{\delta b_0 \int dx \overline{\psi}^{(\geqslant 1)}_x \mathrm{i}\partial\!\!\!/ \psi^{(\geqslant 1)}_x}\, e^{\mathcal{R}V^{(1)}(\psi^{(\geqslant 1)})}. \tag{3.8}$$

Note that

$$P_{b_0}(d\psi^{(\geqslant 1)})\, e^{\delta b_0 \int dx \overline{\psi}^{(\geqslant 1)}_x \mathrm{i}\partial\!\!\!/ \psi^{(\geqslant 1)}_x} = \overline{P}_{b_0}(\psi^{(\geqslant 1)})\, e^{|\Lambda|t_1}, \tag{3.9}$$

where $\overline{P}_{b_0}(\psi^{(\geqslant 1)})$ is the normalized GFM with covariance $[b_0 \mathrm{i}\, \partial\!\!\!/^{(\geqslant 1)} + \delta b_0 \mathrm{i}\, \partial\!\!\!/]^{-1}(x-y)$ and

$$
\begin{aligned}
|\Lambda| t_1 &= \log \int P_{b_0}(d\psi^{(\geqslant 1)})\, e^{\delta b_0 \int \overline{\psi}_x \, i \partial\!\!\!/ \psi_x} \\
&= \sum_{N=1}^{\infty} \frac{(\delta b_0)^N}{N!} \int_\Lambda dx_1 \cdots \int_\Lambda dx_N \mathcal{E}^T_{C^{(\geqslant 1)}/b_0}\left[\overline{\psi}_{x_1}\psi_{x_1}, \ldots, \overline{\psi}_{x_N}\psi_{x_N}\right]
\end{aligned}
\tag{3.10}
$$

with

$$
C^{(\geqslant 1)}(x-y) = \int \frac{d^3p}{(2\pi)^3} e^{-L^{2\cdot 1}p^2} e^{ip(x-y)} = i\, \partial\!\!\!/ g^{(\geqslant 1)}(x-y). \tag{3.11}
$$

Consider now, following [6], the Fourier tranform of the $[b_0 \, i \, \partial\!\!\!/^{(\geqslant 1)} + \delta b_0 \, i \, \partial\!\!\!/]^{-1}(x-y)$

$$
\left[b_0 \, i \, \partial\!\!\!/^{(\geqslant 1)} + \delta b_0 \, i \, \partial\!\!\!/\right]^{-1}(p) = \frac{p\!\!\!/}{p^2}\left[b_0 \, e^{L^2 p^2} + \delta b_0\right]^{-1} \tag{3.12}
$$

and define

$$
b_1 = b_0 + \delta b_0. \tag{3.13}
$$

Then it is easy to check that

$$
\begin{aligned}
&\left[b_0 \, e^{L^2 p^2} + \delta b_0\right]^{-1} \\
&= \frac{1}{b_1} e^{-L^4 p^2} + \frac{1}{b_1}\left[e^{-L^2 p^2} - e^{-L^4 p^2}\right] + \frac{1}{b_1} \frac{e^{-L^2 p^2} - e^{-2L^2 p^2}}{b_0 + \delta b_0 \, e^{-L^2 p^2}},
\end{aligned}
\tag{3.14}
$$

$$
\begin{aligned}
&\left[b_0 \, i \, \partial\!\!\!/^{(\geqslant 1)} + \delta b_0 \, i \, \partial\!\!\!/\right]^{-1}(x-y) \\
&= \frac{1}{b_1} g^{(\geqslant 2)}(x-y) + \frac{1}{b_1}\left[g^{(1)}(x-y) + r^{(1)}(x-y)\right],
\end{aligned}
\tag{3.15}
$$

where

$$
r^{(1)}(x-y) = \delta b_0 \int \frac{d^3p}{(2\pi)^3} e^{ip(x-y)} \frac{p\!\!\!/}{p^2} \frac{\left[e^{-L^2 p^2} - e^{-2L^2 p^2}\right]}{b_0 + \delta b_0 \, e^{-L^2 p^2}}. \tag{3.16}
$$

Defining now

$$
\tilde{g}^{(1)}(x-y) = g^{(1)}(x-y) + r^{(1)}(x-y). \tag{3.17}
$$

(3.15) implies that

$$
\overline{P}_{b_0}(\psi^{(\geqslant 1)}) = P_{b_1}(\psi^{(\geqslant 2)})\, \widetilde{P}_{b_1}(\psi^{(1)}), \tag{3.18}
$$

where $\widetilde{P}_{b_1}(\psi^{(1)})$ is the normalized GFM with propagator $\frac{1}{b_1}\widetilde{g}^{(1)}(x-y)$ defined in (3.16)–(3.17). Note that the bound

$$\left|\widetilde{g}^{(1)}(x-y)\right| \leqslant \text{const}\, L^{-2\cdot 1} \exp\left[-\alpha L^{-1}|x-y|\right] \tag{3.19}$$

holds, with $\alpha > 0$ constant, if δb_0 is sufficiently small (actually, we will show that $\delta b_0 \approx \mathrm{O}(\lambda^2)$, where λ is the interaction parameter).

Therefore, using (3.18), we obtain

$$\Xi_\Lambda(\lambda) = \mathrm{e}^{(T_1+t_1)|\Lambda|} \int P_{b_1}\left(d\psi^{(\geqslant 2)}\right) \widetilde{P}_{b_1}\left(d\psi^{(1)}\right) \mathrm{e}^{\mathcal{R}V^{(1)}(\psi^{(\geqslant 1)})} \tag{3.20}$$

which completes the first RG step. In order to perform the next step, remark that $\mathcal{R}V^{(1)}(\psi^{(\geqslant 1)})$ contains now a term of the form $\overline{\psi}\partial^2\psi$. Thus, $V^{(2)}(\psi^{(\geqslant 2)})$ will contain infinitely many terms with fields of the type $\partial^2\psi$, and so, we must extend the definition on $\mathcal{R}$ and $\mathcal{L}$ to monomials $\widetilde{\psi}(P)$ which contain fields of type $\partial^2\psi$. The extension is obvious, since a monomial containing double derivative fields is, by power counting, irrelevant:

$$\mathcal{R}\widetilde{\psi}(P) = \widetilde{\psi}(P), \quad \mathcal{L}\widetilde{\psi}(P) = 0, \quad \text{if } \widetilde{\psi}(P) \text{ contains terms } \partial^2\psi. \tag{3.21}$$

Iterating we generate a sequence $V^{(n)}$ of *renormalized* running effective potentials, a sequence δb_n of running coupling constants, and a sequence of constants T_n, t_n defined as follows

$$\begin{aligned} &V^{(n+1)}\left(\psi^{(\geqslant n+1)}\right) \\ &\quad = \log \int \widetilde{P}_{b_n}\left(d\psi^{(n)}\right) \mathrm{e}^{\mathcal{R}V^{(n)}(\psi^{(\geqslant n)})} = \sum_{N=1}^{\infty} \frac{1}{N!} \mathcal{E}^T_{\frac{\widetilde{g}^{(n)}}{b_n}} \\ &\quad \times \left[\mathcal{R}V^{(n)}\left(\psi^{(>n)} + \psi^{(n)}\right), \ldots, \mathcal{R}V^{(n)}\left(\psi^{(>n)} + \psi^{(n)}\right)\right] \end{aligned} \tag{3.22}$$

with

$$\begin{aligned} &\widetilde{g}^{(n)}(x-y) = g^{(n)}(x-y) + r^{(n)}(x-y), \\ &r^{(n)}(x-y) = \delta b_{n-1} \int \frac{d^3p}{(2\pi)^3} \mathrm{e}^{\mathrm{i}p(x-y)} \frac{\not{p}}{p^2} \frac{[\mathrm{e}^{-L^{2n}p^2} - \mathrm{e}^{-2L^{2n}p^2}]}{[b_{n-1} + \delta b_{n-1}\,\mathrm{e}^{-L^{2n}p^2}]}. \end{aligned} \tag{3.23}$$

And, if δb_n is sufficiently small,

$$\left|\widetilde{g}^{(n)}(x-y)\right| \leqslant \text{const}\, L^{-2n} \exp\left[-\alpha L^{-n}|x-y|\right]. \tag{3.24}$$

The running coupling constants b_n are defined by

$$b_n = b_{n-1} + \delta b_{n-1}, \tag{3.25}$$

$$\delta b_{n-1} = \frac{1}{12} \operatorname{tr}\left[\int dz W^{(n)}_{[2,0]}(z)\, \mathrm{i}\, \not{z} \right], \tag{3.26}$$

where, $W^{(n)}_{[2,0]}(x-y)$ is the kernel in front of the monomial $\overline{\psi}_x^{(\geqslant 2)} \psi_y^{(\geqslant 2)}$ in the expansion of $V^{(n)}$ (remark that $V^{(n)}$ has also an irrelevant quadratic term proportional to $\overline{\psi}_x^{(\geqslant 2)} \partial^2 \psi_{xy(t)}^{(\geqslant 2)}$ whose kernel does not contribute to δb_{n-1}). Finally, the pressure (or free energy) of the system is:

$$p_\Lambda(\lambda) = \sum_{j=0}^{\infty} (T_j + t_j), \tag{3.27}$$

with

$$\begin{aligned} |\Lambda| T_{j+1} &= \log \int \widetilde{P}_{b_j}(d\psi^{(j)})\, \mathrm{e}^{\mathcal{R}V^{(j)}(\psi^{(j)})} \\ &= \sum_{N=1}^{\infty} \frac{1}{N!} \mathcal{E}^T_{\widetilde{g}^{(j)}/b_j} \left[\mathcal{R}V^{(j)}(\psi^{(j)}), \ldots, \mathcal{R}V^{(j)}(\psi^{(j)}) \right], \end{aligned} \tag{3.28}$$

$$\begin{aligned} |\Lambda| t_{j+1} &= \log \int P_{b_j}(d\psi^{(>j)})\, \mathrm{e}^{\delta b_j \int \overline{\psi}_x^{(>j)} \mathrm{i} \not{\partial} \psi_x^{(>j)}} \\ &= \sum_{N=1}^{\infty} \frac{(\delta b_j)^N}{N!} \int_\Lambda dx_1 \cdots \int_\Lambda dx_N \mathcal{E}^T_{C^{(\geqslant j+1)}/b_j} \\ &\quad \times \left[\overline{\psi}_{x_1} \psi_{x_1}, \ldots, \overline{\psi}_{x_N} \psi_{x_N} \right], \end{aligned} \tag{3.29}$$

where

$$C^{(\geqslant j)}(x-y) = \int \frac{d^3 p}{(2\pi)^3} \mathrm{e}^{-L^{2j} p^2} \mathrm{e}^{\mathrm{i} p(x-y)} = \mathrm{i}\, \not{\partial} g^{(\geqslant j)}(x-y). \tag{3.30}$$

As a consequence of renormalization procedure described above, a labelled tree $\theta^{N,n}_{\text{lab}}$ in the renormalized tree expansion has to be interpreted in a slightly different way. Actually, comparing (2.18) with (3.22), we see that a renormalized labelled tree $\theta^{N,n}_{\text{lab}}$ differs from the old one just by two facts: (1) Each non-trivial (trivial) vertex v with label scale n_v means now a truncated (simple) expectation with propagator $\frac{1}{b_{n_v}} \widetilde{g}^{(n_v)}$. (2) Now, immediately after each vertex of type v^* (see remark below) of $\theta^{N,n}_{\text{lab}}$ the $\mathcal{R}$ operation is applied.

Remark. – By (3.22), the $\mathcal{R}$ operation $\theta^{N,n}_{\text{lab}}$, should be applied after *any* expectation (simple or truncated), i.e., in terms of trees, after *any* tree vertex (trivial or non-trivial), but this is not necessary. As a matter of fact, let v and v' be two subsequent non-trivial vertices with scale indices n_v and $n_{v'} > n_v$, respectively, and let $\widetilde{\psi}^{(>n_v)}(P)$ be a monomial of fields at scales greater than n_v. Then, using the definition of $\mathcal{R}$ operation, it is easy to check that

$$\begin{aligned}&\mathcal{R}\mathcal{E}_{n_{v*}}\big[\mathcal{R}\mathcal{E}_{n_{v*}-1}\big[\cdots\big[\mathcal{R}\mathcal{E}_{n_v+1}\widetilde{\psi}^{(>n_v)}(P)\big]\cdots\big]\big]\\ &\quad=\mathcal{R}\mathcal{E}_{n_{v*}}\big[\mathcal{E}_{n_{v*}-1}\big[\cdots\big[\mathcal{E}_{n_v+1}\widetilde{\psi}^{(>n_v)}(P)\big]\cdots\big]\big],\end{aligned}\tag{3.31}$$

where v^* is the greatest vertex (see Fig. 2) in the line from v to v': it is the trivial vertex at scale $n_{v*} = n_{v'} - 1$ if $n_{v'} - n_v > 1$, and in the case that $n_{v'} - n_v = 1$, then $v^* = v$. Thus, due to (3.31), the vertices v^* can be though as the particular vertices of $\theta_{\rm lab}^{N,n}$ in which the $\mathcal{R}$ is applied ($\mathcal{R}$ operation is applied to *what is coming out* of v^*).

Thus, analogously to (2.20), we can associate to each label tree $\theta_{\rm lab}^{N,n}$ a *renormalized* hierachically organized sequence of simple and truncated expectations at different scales of N objects, say $\widetilde{\mathcal{E}}_{\rm lab}^{\theta(N,n)}(\cdot,\cdot,\ldots,\cdot)$, given by

$$\begin{aligned}\widetilde{\mathcal{E}}_{\rm lab}^{\theta(N,n)}(\cdot,\cdot,\ldots,\cdot)=&\left[\prod_{v\leqslant v_0}\mathcal{O}_v\mathcal{E}_{n_{v'}-1}\cdots\mathcal{E}_{n_v+1}\frac{1}{s_v!}\mathcal{E}_{n_v}^T\right]\\ &\times\big(\mathcal{E}_{n_1+1}\cdots\mathcal{E}_0(\cdot),\ldots,\mathcal{E}_{n_N+1}\cdots\mathcal{E}_0(\cdot)\big),\end{aligned}\tag{3.32}$$

where $\mathcal{O}_v = \mathcal{R}$ if $v < v_0$ and $\mathcal{O}_{v_0} = 1$, $\mathcal{E}_m^T$ stands for $\mathcal{E}_{\tilde{g}^{(m)}/b_m}^T$, $\mathcal{E}_m$ stands for $\mathcal{E}_{\tilde{g}^{(m)}/b_m}$, and note that $n_{v'} - 1 = n_{v*}$. The tree expansion of the renormalized effective potential is (compare with (2.21))

$$V^{(n)}(\overline{\psi},\psi,\partial^2\psi)=\sum_{N=1}^{\infty}\sum_{\theta^N}\sum_{\theta_{\rm lab}^{N,n}:\,\theta^N\text{ fixed}}V^{(n)}(\theta_{\rm lab}^{N,n},\overline{\psi},\psi,\partial^2\psi).\tag{3.33}$$

By the definitions above, $V^{(n)}(\theta_{\rm lab}^{N,n})$ is given explicitly by (compare with (2.22))

$$\begin{aligned}V^{(n)}(\theta_{\rm lab}^{N,n},\psi)=&\lambda^N\int dx_1\,dx_2\cdots dx_N\prod_{v\leqslant v_0}\mathcal{O}_v\mathcal{E}_{n_{v'}-1}\cdots\mathcal{E}_{n_v+1}\frac{1}{s_v!}\mathcal{E}_{n_v}^T\\ &\times\left[\prod_{i=1}^{N}\big([\overline{\psi}_{x_i}^{(\geqslant n_i)}\psi_{x_i}^{(\geqslant n_i)}]^2\big)\right],\end{aligned}\tag{3.34}$$

where we also use (2.23).

Observe that $V^{(n)}(\theta_{\rm lab}^{N,n},\psi)$ is proportional to λ^N, but, of course, the proportionality coefficent is not the Nth order coefficient of the power expansion of $V^{(n)}$ in terms of the parameter λ. Actually, a residual dependence on λ is also hidden in the covariances $\frac{1}{b_{n_v}}\widetilde{g}^{(n_v)}$ (which are used to calculate truncated and simple expectations in the (3.34)), since b_{n_v} and $\widetilde{g}^{(n_v)}$ both depend on λ (through δb_{n-1}, see (3.23)).

We conclude this section giving the main theorem of the paper. First, we need to introduce some notations concerning connected tree graphs in finite sets which we will use below and throughout the rest of the paper.

Connected tree graphs. Whenever A denotes a finite set, we denote by $|A|$ the number of elements of A. Given a finite set A, we define a *connected tree graph* τ in A as a collection $\tau = \{\rho_1, \rho_2, \ldots, \rho_{|A|-1}\}$ such that: $\rho_i \subset A$: $|\rho_i| = 2$; $\rho_i \neq \rho_j$ for all i, j; for any pair B, C of subsets of A such that $B \cup C = A$ and $B \cap C = \emptyset$, there is a $\rho_i \in \tau$ such that $\rho_i \cap B \neq \emptyset$ and $\rho_i \cap C \neq \emptyset$ (connection). Given a connected tree graph $\tau = \{\rho_1, \rho_2, \ldots, \rho_{|A|-1}\}$ in a finite set A, $\rho_1, \rho_2, \ldots, \rho_{|A|-1}$ are called *links* of τ. We denote with $\mathcal{T}_A$ the set of all connected tree graphs of A. Whenever $A \equiv \{1, 2, \ldots, n\}$, we will abbreviate $\mathcal{T}_{\{1,2,\ldots,n\}} \equiv \mathcal{T}_n$.

THEOREM 3.1. – *There exist $\varepsilon > 0$ and $D > 0$ such that:*
A. *The effective potential at scale n, $V^{(n)}$ defined inductively by* (3.22), *can be written, for $|\lambda| \leqslant \varepsilon$, in the following way*

$$
\begin{aligned}
V^{(n)}(\psi) = {} & \lambda \int_{\Lambda} dx \left[\overline{\psi}_x \psi_x\right]^2 + \int_{\Lambda} dx_1 \int_{\Lambda} dx_2 W^{(n)}_{2,0}(x_1 - x_2)\overline{\psi}_{x_1}\psi_{x_2} \\
& + \int_{\Lambda} dx_1 \int_{\Lambda} dx_2 W^{(n)}_{2,1}(x_1 - x_2)\overline{\psi}_{x_1}\partial^2\psi_{x_2} \\
& + \sum_{m=2}^{\infty}\sum_{k=0}^{m}\sum_{p=\max\{m-1+k,2\}}^{2m} \int_{\Lambda} dx_1 \cdots \int_{\Lambda} dx_p \\
& \times \sum_{r_1,\ldots,r_{p-k}}\sum_{s_1,\ldots,s_{p-k}} W^{(n)\{r_j,s_j\}}_{2m,k}(x_1, x_2, \ldots, x_p) \\
& \times \overline{\psi}^{r_1}_{x_1} \cdots \overline{\psi}^{r_{p-k}}_{x_{p-k}} \psi^{s_1}_{x_1} \cdots \psi^{s_{p-k}}_{x_{p-k}} \partial^2\psi_{x_{p-k+1}} \cdots \partial^2\psi_{x_p},
\end{aligned}
\tag{3.35}
$$

where (for $j = 1, 2, \ldots, p-k$), $0 \leqslant r_j \leqslant 2$, and analogously for s_j. Moreover, for $j = 1, 2, \ldots, p-k$, $1 \leqslant r_j + s_j \leqslant 3$, and $\sum_j r_j = m$, $\sum_j s_j = m - k$. The kernels $W^{(n)\{r_j,s_j\}}_{2m,k}$, $W^{(n)}_{2,0}(x_1 - x_2)$ and $W^{(n)}_{2,1}(x_1 - x_2)$ are analytic in λ, if $|\lambda| < \varepsilon$, uniformely in Λ, and satisfy the following pointwise estimates

$$\begin{aligned}&\left|W_{2m,k}^{(n)}(x_1, x_2, \ldots, x_p)\right| \\ &\quad\leqslant D^p \lambda^p \sum_{\tau \in \mathcal{T}_p} B_\tau \sum_{\{\beta_\rho\}_{\rho\in\tau}}{}^{*} \prod_{\rho\in\tau} \left[\sum_{n_\rho=0}^{n-1} L^{-\beta_\rho n_\rho}\, \mathrm{e}^{-\alpha L^{-n_\rho}|x_\rho|}\right],\end{aligned} \tag{3.36}$$

$$\left|W_{2,0}^{(n)}(x_1 - x_2)\right| \leqslant D^2 \lambda^2 L^{-n} \sum_{n_1=0}^{n-1} L^{-6n_1}\, \mathrm{e}^{-\alpha L^{-n_1}|x_1 - x_2|}, \tag{3.37}$$

$$\left|W_{2,1}^{(n)}(x_1 - x_2)\right| \leqslant D^2 \lambda^2 \sum_{n_1=0}^{n-1} L^{-4n_1}\, \mathrm{e}^{-\alpha L^{-n_1}|x_1 - x_2|}, \tag{3.38}$$

where:

$$W_{2m,k}^{(n)}(x_1, x_2, \ldots, x_p) = \sum_{\{r_j, s_j\}} W_{2m,k}^{(n)\{r_j,s_j\}}(x_1, x_2, \ldots, x_p);$$

$\mathcal{T}_p$ denotes the set of all connected tree graphs τ in $\{1, 2, \ldots, p\}$; ρ denotes the generic link of τ (recall that the number of links in $\tau \in \mathcal{T}_p$ is $p - 1$), and if $\rho = \{i, j\}$, then $x_\rho = x_i - x_j$; B_τ are combinatorial positive factors such that $\sum_\tau B_\tau \leqslant C^p$ where $C > 0$ is some constant; β_ρ is an integer $\geqslant 1$, and $\sum^{*}_{\{\beta_\rho\}_{\tau\in p}}$ runs over all possible choices of $p - 1$ positive integers (one for each link $\rho \in \tau$) such that (recall that $d = 3$)

$$\sum_{\rho\in\tau} \beta_\rho = (d+1)p - \left[2m\frac{(d-1)}{2} + 2k\right] = 4p - 2(m+k). \tag{3.39}$$

In the limit $n \to \infty$ (3.36) becomes:

$$\begin{aligned}&\lim_{n\to\infty} \left|W_{2m,k}^{(n)}(x_1, x_2, \ldots, x_p)\right| \\ &\quad\leqslant (D|\lambda|)^p \sum_{\tau\in\mathcal{T}_p} B_\tau \sum_{\{\beta_\rho\}_{\rho\in\tau}}{}^{*} \prod_{\rho\in\tau} \frac{1}{|x_\rho|^{\beta_\rho}}.\end{aligned} \tag{3.40}$$

Thus, the kernels of effective potential decay polynomially at large distances, in general, in a non-integrable way.

B. *The pressure of the model can be written as*

$$\frac{1}{|\Lambda|} \log \Xi_\Lambda(\lambda) = \sum_{j=0}^{\infty} (T_j + t_j) \tag{3.41}$$

with T_j and t_j given by (3.28)*, analytic functions of λ and satisfying the bound*

$$|T_j| \leqslant L^{-2j} D\lambda^2, \qquad |t_j| \leqslant L^{-(4-\varepsilon)j} D\lambda^2. \tag{3.42}$$

Remark 1. – The notation $\overline{\psi}_{x_i}^{r_i}$ means $\overline{\psi}_{x_i}^{0} = 1$, $\overline{\psi}_{x_i}^{1} = \overline{\psi}_{x_i}$ and $\overline{\psi}_{x_i}^{2} = \overline{\psi}_{x_i}^{\alpha}\overline{\psi}_{x_i}^{\beta}$; in particular $\overline{\psi}_{x_i}^{2}$ carries two spinor indices. Analogously for $\psi_{x_j}^{s_j}$. We stress that the factor

$$\overline{\psi}_{x_1}^{r_1} \cdots \overline{\psi}_{x_{p-k}}^{r_{p-k}} \psi_{x_1}^{s_1} \cdots \psi_{x_{p-k}}^{s_{p-k}} \partial^2 \psi_{x_{p-k+1}} \cdots \partial^2 \psi_{x_p}$$

always contains exactly $2m$ fields and so carries $2m$ spinor indices, which are contracted with the $2m$ spinor indices attached to $W_{m,k}^{\{r_j\},\{s_j\}}$. Moreover, recall that ∂^2 means $\partial_\mu \partial_\nu$ and indices μ, ν are again contracted to the correspondent μ, ν indices in $W_{m,k}^{(n)\{r_j,s_j\}}$ (we do not make explicit the spinor and euclidean indices in order to avoid a heavy notation).

Remark 2. – $W_{2m,k}^{(n)\{r_j,s_j\}}$ is the kernel in $V^{(n)}$ related to the monomial with $2m$ fields, m of which are of type $\overline{\psi}$, $m-k$ of type ψ, and k of type $\partial^2\psi$. It is a p point contribution to the $2m$ point truncated correlation function (see [23]). Indices r_i and s_i are just telling us which fields among ψ and $\overline{\psi}$ are sitting in the same point x_i. Observe that the number of all possible combinations in which r_i and s_i numbers can be distributed cannot exceed 3^{2p}.

Remark 3. – Note that, by construction, if a field of type $\partial^2\psi$ appears at a point y in a monomial in $V^{(n)}$, then this monomial cannot contain a field $\overline{\psi}_y$ or ψ_y *in the same point.*

Remark 4. – Note that

$$\sum_{\{\beta_\rho\}_{\tau\in p}}^{*} 1 = \sum_{n_1+\cdots+n_{p-1}=4p-2(m+k)} 1 \leqslant C^p$$

for some constant C. In other words, this sum is not combinatorially dangerous.

Remark 5. – By a standard analysis, one may obtain "integral" bounds on the kernels of the effective potential of the form

$$\begin{aligned} \|W_{m,k}^{(n)}(x_1,\ldots,x_p)\| &= \int_\Lambda dx_1 \cdots \int_\Lambda dx_p |W_{m,k}^{(n)}(x_1,\ldots,x_p)| \\ &\leqslant |\Lambda| \mathrm{const}\, L^{n[2m\frac{d-1}{2}+2k-d-1]}. \end{aligned} \tag{3.43}$$

Note that $L^{n[2m\frac{d-1}{2}+2k-d]}$ grows up exponentially as $n \to \infty$, which means that the polinomial decay of the kernels of effective potential is too slow in order to keep $\frac{1}{|\Lambda|}\|W^{(n)}_{m,k}(x_1,x_2,\ldots,x_p)\|$ finite as $\Lambda \to \infty$. This kind of divergence is expected since the free covariance is not integrable. Such integral bounds are not useful in order to get pointwise bounds on the correlation functions (at least we were not able to use it, see [23] for bounds on truncated correlations using the pointwise bounds (3.36) of the theorem above).

Remark 6. – It is very interesting to derive from (3.36) the pointwise behaviour of the kernels of the so called "adimensional" potential at scale n (see [4,6]; in [14,21] and [15] it is called potential in the "thin lattice"), which can be obtained from the usual effective potential through the replacement

$$\psi_x \to L^{\frac{-(d-1)}{2}n}\psi_{L^{-n}x}.$$

The relation between the kernels $W^{(n),\mathrm{ad.}}_{2m,k}$ of the "adimensional" potential and the usual ones $W^{(n)}_{2m,k}$ is therefore, setting $d=3$

$$\begin{aligned} &W^{(n),\mathrm{ad.}}_{2m,k}(x_1,x_2,\ldots,x_p)\\ &\quad= L^{n[3p-2(m+k)]}W^{(n)}_{2m,k}\big(L^n x_1, L^n x_2,\ldots,L^n x_p\big). \end{aligned} \tag{3.44}$$

Hence, using (3.36), (3.39) and the fact that $[3p-2(m+k)] \geqslant p$, the bound on $W^{(n),\mathrm{ad.}}_{2m,k}(x_1,x_2,\ldots,x_p)$ becomes after some algebraic manipulations,

$$\begin{aligned} &\big|W^{(n),\mathrm{ad.}}_{2m,k}(x_1,\ldots,x_p)\big|\\ &\quad\leqslant L^{-n(1-\varepsilon)p}(D|\lambda|)^p\\ &\qquad\times \sum_{\tau\in\mathcal{T}_p} B_\tau \sum_{\{\beta_\rho\}_{\rho\in\tau}}{}^{*}\prod_{\rho\in\tau}\left[\sum_{n_\rho=0}^{n-1} L^{-\varepsilon n_\rho}L^{(\beta_\rho-\varepsilon)(n-n_\rho)}\,\mathrm{e}^{-\alpha L^{n-n_\rho}|x_\rho|}\right], \end{aligned}$$

where ε is a small positive constant. Considering the following inequality, which holds uniformely *if and only if* the distance $|x_\rho| \geqslant 1$

$$L^{\beta j}\,\mathrm{e}^{-\alpha L^j|x_\rho|} = L^{\beta j}\,\mathrm{e}^{-\alpha(L^j-1)|x_\rho|}\,\mathrm{e}^{-\alpha|x_\rho|} \leqslant C\,\mathrm{e}^{-\alpha|x_\rho|} \tag{3.45}$$

(for $|x_\rho|$ small, C is actually of the order C^j), we get (use also Remark 4 to bound $\sum^*_{\{\beta_\rho\}_{\rho\in\tau}} 1$)

$$\big|W^{(n),\mathrm{ad.}}_{2m,k}(x_1,x_2,\ldots,x_p)\big| \leqslant (D|\lambda|)^p C^p L^{-n(1-\varepsilon)p}\,\mathrm{e}^{-\alpha d(x_1,\ldots,x_p)}, \tag{3.46}$$

where $d(x_1, \ldots, x_p) = \min_\tau \sum_{\rho\in\tau} |x_\rho|$ is the shortest tree distance between points $x_1, x_2, \ldots, x_p$.

The formula above is the well known claim that kernels of the adimensional potential decay exponentially fast (with a rate O(1)) as distances between points become large (see, e.g., [12], or (10.24) in [4], or (5.56) in [6]). By the considerations above, (3.46) is correct just for distances between points O(1) or greater: the bound (3.45) (which is the key point in order to get (3.46)) simply *is not true* when the distance $|x_\rho|$ is small compared with 1. However, we recall that we are interested in the region where the distance between points is O(1) or smaller: note that distances of order O(1) in the kernels of the *adimensional* potential at scale n correspond to distances of order $O(L^n)$ in the kernels of the usual effective potential at scale n (see (3.44)).

In [6] (see there Remark 2 after Theorem 2 of Section 5, and specially formula (5.55)) formula (3.46) was derived from a pointwise bound for kernels of effective potential different from the one stated in Theorem 3. Namely, *mutatis mutandis* (in the scaling factor $4p - 2(m+k) = \sum_j \beta_j$), this alternative bound is

$$\left|W^{(n)}_{2m,k}(x_1, x_2, \ldots, x_p)\right| \leqslant D^p\lambda^p C^p \sum_{m=0}^{n-1} \cdots L^{-[4p-2(m+k)]m} \, \mathrm{e}^{-\alpha L^{-m} d(x_1,\ldots,x_p)}.$$

The inequality was suggested as a consequence of the analysis performed to get integral bounds on the effective potential (Theorem 2 p. 141 in [6]). One can easely check that (3.36) implies, for distances greater than L^n, the inequality above, which, on the other hand, appears as a much sharper bound respect to (3.36), for distances smaller than L^n. Indeed, the analysis of Section 5 will make clear that the bound above cannot be obtained for all distances. Anyway, it is easy to understand intuitively why it cannot work: if we suppose it true for all distances, in the $n \to \infty$ limit, we get a pointwise bound as

$$\lim_{n\to\infty} \left|W^{(n)}_{2m,k}(x_1, x_2, \ldots, x_p)\right| \leqslant D^p\lambda^p C^p \left[d(x_1, \ldots, x_p)\right]^{-4p+2(m+k)}.$$

This is not consistent compared to what one should expect from perturbation theory. By inequality above, if, say $|x_i - x_j| \to \infty$ and all others x's are kept fixed, the polynomial decay of the p-point kernels should be like $|x_i - x_j|^{-cp}$ (with c some constant), so increasing proportionally to p. On the other hand, one can easely produce many contributions from perturbation theory to $W^{(n)}_{2m,k}(x_1, x_2, \ldots, x_p)$ which decay with a power not proportional to p. Note that (3.40), for this specific case, gives a more realistic p independent decay as $|x_i - x_j|^{-1}$.

4. BOUNDS ON THE KERNELS OF EFFECTIVE POTENTIAL

4.1. Notations and preliminary bounds

We want now to write down explicitly the the contribution $V^{(n)}(\psi, \theta^N)$ to the effective potential at scale n coming from a given rooted Cayley tree θ^N with N end points.

Using the notations of Theorem 3.1, we can make the following ansatz (from now on, unless necessary, we will omit scale apices in the fields)

$$\begin{aligned}
&V^{(n)}(\psi, \theta^N)\\
&\quad= \sum_{\theta^{N,n}_{\text{lab}}:\ \theta^N \text{ fixed}} V^{(n)}(\psi, \theta^{N,n}_{\text{lab}}) = \sum_{m=1}^{N+1}\sum_{k=0}^{m}\sum_{p=\max\{m-1+k,2\}}^{2m} \int dx_1 \cdots dx_p\\
&\qquad \times \sum_{\underline{r},\underline{s}}\Bigg[\sum_{\theta^{N,n}_{\text{lab}}:\ \theta^N \text{ fixed}} W^{(n),\underline{r},\underline{s}}_{2m,k}(\theta^{N,n}_{\text{lab}}, x_1, x_2, \ldots, x_p)\Bigg]\\
&\qquad \times \big[\overline{\psi}\big]^{\underline{r}}\big[\psi\big]^{\underline{s}}\big[\partial^2\psi\big]^{k}(x_1, x_2, \ldots, x_p),
\end{aligned} \tag{4.1}$$

where we wrote $\underline{r} = \{r_1, \ldots, r_{p-k}\}$, $\underline{s} = \{s_1, \ldots, s_{p-k}\}$, and

$$\begin{aligned}
&\big[\overline{\psi}\big]^{\underline{r}}\big[\psi\big]^{\underline{s}}\big[\partial^2\psi\big]^{k}(x_1, x_2, \ldots, x_p)\\
&\quad= \overline{\psi}^{\,r_1}_{x_1} \cdots \overline{\psi}^{\,r_{p-k}}_{x_{p-k}} \psi^{s_1}_{x_1} \cdots \psi^{s_{p-k}}_{x_{p-k}} \partial^2\psi_{x_{p-k+1}} \cdots \partial^2\psi_{x_p}.
\end{aligned}$$

On the other hand, (3.34) we expect

$$\begin{aligned}
&V^{(n)}(\psi, \theta^{N,n}_{\text{lab}})\\
&\quad= \sum_{m=1}^{N+1}\sum_{k=0}^{m} \sum_{\substack{P_{v_0}\in P_N:\\ |P_{v_0}|=2m;\ k(P^{\text{ren}}_{v_0})=k}} \ \sum_{\substack{\{P_v\}_{v<v_0}:\\ P^{\text{ren}}_{v_0}\text{ fixed}}}\\
&\qquad \times \int d\underline{X}_{v_0} V^{(n)}_{2m,k}(\theta^{N,n}_{\text{lab}}, P^{\text{ren}}_{v_0}, \{P_v\}_{v<v_0}, \underline{X}_{v_0})\widetilde{\psi}(P^{\text{ren}}_{v_0}).
\end{aligned} \tag{4.2}$$

The notations in (4.2) are the following.

$$P_N = \{ \overline{\psi}_{x_1}^{\alpha_1}, \overline{\psi}_{x_1}^{\beta_1}, \psi_{x_1}^{\alpha_1}, \psi_{x_1}^{\beta_1}, \ldots, \overline{\psi}_{x_N}^{\alpha_N}, \overline{\psi}_{x_N}^{\beta_N}, \psi_{x_N}^{\alpha_N}, \psi_{x_N}^{\beta_N} \}$$

is the set formed by the $4N$ fields in the end points (recall that each end point j carries the factor $\lambda \int_\Lambda d^3x_j \sum_{\alpha_j\beta_j} \overline{\psi}_{x_j}^{\alpha_j} \overline{\psi}_{x_j}^{\beta_j} \psi_{x_j}^{\alpha_j} \psi_{x_j}^{\beta_j}$);

- P_{v_0} indicates the subset of P_N surviving at the root of $\theta_{\text{lab}}^{N,n}$. We stress that P_{v_0} indicates the *bare* subset, i.e., the one formed with fields $\overline{\psi}$ and ψ without considering that a field of type ψ may have been changed due to the renormalization (namely, a field ψ_y can turn into $\partial^2\psi_{xy(t)}$, where x is some other point and $xy(t) = x - t(y - x)$ is an interpolated point in the line which joins x to y);
- $P_{v_0}^{\text{ren}}$ indicates the set of fields constructed with the fields in P_{v_0} once the $\mathcal{R}$ operation has acted in each vertex of $\theta_{\text{lab}}^{N,n}$ of type v^* *preceding* v_0, and $\widetilde{\psi}(P_{v_0}^{\text{ren}})$ is the monomial constructed with the set $P_{v_0}^{\text{ren}}$ with fields organized in a prefixed order. Recall that the last renormalization in v_0^* (i.e., the root of $\theta_{\text{lab}}^{N,n}$, see (3.22)) has not been applied yet when we write $P_{v_0}^{\text{ren}}$. Thus, $\widetilde{\psi}(P_{v_0}^{\text{ren}})$ may be also a monomial like $\overline{\psi}_x\psi_y$;
- $\underline{X}_{v_0}$ are the set of space-time variables attached to the end points which can be reached from v_0, thus, since v_0 is the first non-trivial vertex of $\theta_{\text{lab}}^{N,n}$, we have $\underline{X}_{v_0} = \{x_1, \ldots, x_N\}$ and $d\underline{X}_{v_0} = dx_1 \cdots dx_N$.

Further notations and definitions. Given any non-trivial vertex $v \in \theta_{\text{lab}}^{N,n}$, we indicate by $\theta_{\text{lab}}^{N,n}(v)$ the subtree of $\theta_{\text{lab}}^{N,n}$ obtained by disconnecting from $\theta_{\text{lab}}^{N,n}$ the vertex v', so that v' is the root of $\theta_{\text{lab}}^{N,n}(v)$ and v is its first non-trivial vertex.

- P_v indicates the (*bare*) subset of P_N surviving at the root v' of $\theta_{\text{lab}}^{N,n}(v)$; $\underline{X}_v$ is the set of space-time variables attached to the end points which can be reached from v;
- P_v^{ren} indicates the set of fields constructed with the fields in P_v once the $\mathcal{R}$ operation has acted in each vertex of $\theta_{\text{lab}}^{N,n}(v)$ of type v^* *preceding* v, and $\widetilde{\psi}(P_v^{\text{ren}})$ is the monomial constructed with the set P_v^{ren}. Thus, $\sum_{\{P_v\}_{v<v_0}}$ means to sum over all possible choice of P_v in non-trivial vertices $v < v_0$ of $\theta_{\text{lab}}^{N,n}$ which leads to a fixed $\widetilde{\psi}(P_{v_0}^{\text{ren}})$.

Moreover, for any $v \in \theta_{\text{lab}}^{N,n}$ we define

$$|P_v| = |P_v^{\text{ren}}| = \text{number of fields in } P_v,$$
$$\|P_v^{\text{ren}}\| = |P_v| + 2 \times \left[\text{number of fields of type } \partial^2\psi \text{ in } P_v^{\text{ren}}\right]. \quad (4.3)$$

Note that $-\|P_v^{\text{ren}}\|$ represents the dimensional scaling exponent of $\widetilde{\psi}(P_v^{\text{ren}})$ (recall that $d = 3$), and $\frac{1}{2}(\|P_v^{\text{ren}}\| - |P_v|)$ is the number of fields of type $\partial^2\psi$ in P_v^{ren}. In particular, for the vertex v_0,

$$|P_{v_0}| = 2m, \quad \|P_{v_0}\| = 2m + 2k \quad \text{and} \quad \frac{1}{2}\left(\|P_{v_0}^{\text{ren}}\| - |P_{v_0}|\right) = k.$$

Recall now that (see (3.22)), we need to apply the $\mathcal{R}$ operation on $\widetilde{\psi}(P_v^{\text{ren}})$ emerging from v^* before the truncated expectation at vertex v' of $\theta_{\text{lab}}^{N,n}$. So we write (see (2.25))

$$\mathcal{R}\widetilde{\psi}(P_v^{\text{ren}}) = [\xi_v]^{z_v} L^{z_v n_v} \widetilde{\psi}(P_v^*), \tag{4.4}$$

with

$$\widetilde{\psi}(P_{v_0}^*) = \begin{cases} \widetilde{\psi}(P_{v_0}^{\text{ren}}) & \text{if } \|P_v^{\text{ren}}\| > 2, \\ \int_0^1 dt(1-t)\overline{\psi}_x \partial^2 \psi_{xy(t)} & \text{if } \|P_v^{\text{ren}}\| = 2, \\ & \text{i.e., if } \widetilde{\psi}(P_{v_0}^{\text{ren}}) = \overline{\psi}_x \psi_y, \end{cases} \tag{4.5}$$

with $\{x, y\} \in \underline{X}_v$ and

$$\xi_v = \begin{cases} 1 & \text{if } \|P_v^{\text{ren}}\| > 2, \\ L^{-n_v}(x-y) & \text{if } \|P_v^{\text{ren}}\| = 2, \text{ i.e., if } \widetilde{\psi}(P_{v_0}^{\text{ren}}) = \overline{\psi}_x \psi_y, \end{cases} \tag{4.6}$$

and

$$z_v = \begin{cases} 0 & \text{if } \|P_v^{\text{ren}}\| > 2, \\ 2 & \text{if } \|P_v^{\text{ren}}\| = 2. \end{cases} \tag{4.7}$$

Again, observe that we indicate as P_v^{ren} the set of fields in P_v after the $\mathcal{R}$ operation has acted in all vertices $w < v$ (thus v^* *excluded*), and P_v^* is the set of fields in P_v after the $\mathcal{R}$ operation has acted in all vertices $w < v'$ (thus, v^* *included*). Remark that $\widetilde{\psi}(P_{v_0}^{\text{ren}})$ is univocally determinated once P_{v_0} *and* all P_v, for $v < v_0$, are given. In other words, $P_{v_0}^{\text{ren}}$ depends on the choice $\{P_v\}_{v<v_0}$, thus, to be more precise, we should write

$$\widetilde{\psi}(P_{v_0}^{\text{ren}}) = \widetilde{\psi}_{\{P_v\}_{v<v_0}}(P_{v_0}^{\text{ren}}). \tag{4.8}$$

The notation (4.8) is consistent: once $\{P_v\}_{v<v_0}$ are given, the the structure of $\widetilde{\psi}(P^{\text{ren}}_{v_0})$ is completely determined since $\mathcal{R}$ operation transforms in a unique way $\overline{\psi}_x\psi_y$ into $\overline{\psi}_x\partial^2\psi_{xy(t)}$. We will not use the heavy notation (4.8) but, of course, we tacitely assume it.

Define now

$$K^{(n)}_{2m,k}(\theta^{N,n}_{\text{lab}}, P^{\text{ren}}_{v_0}, \{P_v\}_{v<v_0}, \underline{X}_{P^{\text{ren}}_{v_0}})$$
$$= \int d(\underline{X}_{v_0}\backslash\underline{X}_{P^{\text{ren}}_{v_0}})\, V^{(n)}_{2m,k}(\theta^{N,n}_{\text{lab}}, P^{\text{ren}}_{v_0}, \{P_v\}_{v<v_0}, \underline{X}_{v_0}), \qquad (4.9)$$

where $\underline{X}_{P_{v_0}}$ ($\underline{X}_{P^{\text{ren}}_{v_0}}$) is the set of space time variables attached to fields in P_{v_0}, and $|\underline{X}_{P_{v_0}}|$ ($|\underline{X}_{P^{\text{ren}}_{v_0}}|$) is the number of elements in $\underline{X}_{P_{v_0}}$ ($\underline{X}_{P^{\text{ren}}_{v_0}}$). Observe that $|\underline{X}_{P^{\text{ren}}_{v_0}}| = |\underline{X}_{P_{v_0}}| + k$, where k is the number of fields of type $\partial^2\psi$ in $\widetilde{\psi}(P^{\text{ren}}_{v_0})$.

In the monomial $\widetilde{\psi}(P^{\text{ren}}_{v_0})$ exactly k fields are sitting in interpolated points. Let these k interpolated points be $x_1+t_1(y_1-x_1), \ldots, x_k+t_k(y_k-x_k)$, with $\{x_i, y_i\} \subset \underline{X}_{P^{\text{ren}}_{v_0}}$ some subset of $2k$ elements of $\underline{X}_{v_0}$ such that $y_i \in \underline{X}_{P_{v_0}}$ and $x_i \notin \underline{X}_{P_{v_0}}$ for all i.

Define

$$W^{(n)}_{2m,k}(\theta^{N,n}_{\text{lab}}, P^{\text{ren}}_{v_0}, \{P_v\}_{v<v_0}, \underline{X}_{P_{v_0}})$$
$$= \int_0^1 \frac{dt_1}{t_1^3} \cdots \int_0^1 \frac{dt_k}{t_k^3} \int d^3x_1 \cdots d^3x_k K^{(n)}_{2m,k}$$
$$\times \left(\theta^{N,n}_{\text{lab}}, P^{\text{ren}}_{v_0}, \{P_v\}_{v<v_0}, \underline{X}_{P^{\text{ren}}_{v_0}}, y_r \to x_r + \frac{(y_r - x_r)}{t_r}\right), \qquad (4.10)$$

where $y_r \to x_r + \frac{(y_r-x_r)}{t_r}$ means that we have to replace the variables y_r by the variables $x_r + \frac{(y_r-x_r)}{t_r}$ in $K^{(n)}_{2m,k}$. The apparent UV singularity in dt_i-integrals in right hand side of (4.10) is compensated by the d^3x_i-integrals, see ahead formula (5.25) and the remark below. By definition (4.10) we can write

$$\int d(\underline{X}_{P^{\text{ren}}_{v_0}})K^{(n)}_{2m,k}(\theta^{N,n}_{\text{lab}}, P^{\text{ren}}_{v_0}, \{P_v\}_{v<v_0}, \underline{X}_{P^{\text{ren}}_{v_0}})\widetilde{\psi}(P^{\text{ren}}_{v_0})$$
$$= \int d(\underline{X}_{P_{v_0}})W^{(n)}_{2m,k}(\theta^{N,n}_{\text{lab}}, P^{\text{ren}}_{v_0}, \{P_v\}_{v<v_0}, \underline{X}_{P_{v_0}})\widetilde{\psi}'(P^{\text{ren}}_{v_0}), \qquad (4.11)$$

where $\widetilde{\psi}'(P^{\text{ren}}_{v_0}) = \widetilde{\psi}(P^{\text{ren}}_{v_0}, [z_r + t_r(y_r - z_r)] \to y_r)$. Finally, fixing $|X_{P_{v_0}}| = p$ and noting that p is the number of space-time points in $X_{P_{v_0}}$

(and so, for fixed m and k, we have $\max\{m-1+k,2\} \leqslant p \leqslant 2m$), we can write

$$V^{(n)}(\psi, \theta_{\text{lab}}^{N,n}) = \sum_{m=1}^{N+1} \sum_{k=0}^{m} \sum_{p=\max\{m-1+k,2\}}^{2m} \sum_{\substack{P_{v_0} \in P_N:\ |P_{v_0}|=2m;\\ k(P_{v_0}^{\text{ren}})=k,\ |X_{P_{v_0}}|=p}} \sum_{\substack{\{P_v\}_{v<v_0}:\\ P_{v_0}^{\text{ren}}\ \text{fixed}}} \int d\underline{X}_{P_{v_0}}$$
$$\times W_{2m,k}^{(n)}(\theta_{\text{lab}}^{N,n}, P_{v_0}^{\text{ren}}, \{P_v\}_{v<v_0}, \underline{X}_{P_{v_0}}) \widetilde{\psi}'(P_{v_0}^{\text{ren}}). \tag{4.12}$$

Now rename $\underline{X}_{P_{v_0}} = \{x_1, x_2, \ldots, x_p\}$ so that

$$\widetilde{\psi}'(P_{v_0}^{\text{ren}}) = (-1)^{\pi(P_{v_0}^{\text{ren}})} [\overline{\psi}]^{\underline{r}} [\psi]^{\underline{s}} [\partial^2 \psi]^k (x_1, x_2, \ldots, x_p), \tag{4.13}$$

where $\pi(P_{v_0}^{\text{ren}})$ represents the number of permutations that are necessary to rearrange $\widetilde{\psi}'(P_{v_0}^{\text{ren}})$ as

$$[\overline{\psi}]^{\underline{r}} [\psi]^{\underline{s}} [\partial^2 \psi]^k (x_1, x_2, \ldots, x_p),$$

once the substitution $\underline{X}_{P_{v_0}} = \{x_1, x_2, \ldots, x_p\}$ has been done. Note that $\underline{s}$ and $\underline{r}$ are determinated univocally by $P_{v_0}^{\text{ren}}$ and

$$|\underline{r}| = \sum_{j=1}^{p-k} r_j = \frac{|P_{v_0}|}{2} = m, \qquad |\underline{s}| = \sum_{j=1}^{p-k} s_j = \frac{|P_{v_0}|}{2} - k = m - k.$$

Hence,

$$V^{(n)}(\psi, \theta_{\text{lab}}^{N,n})$$
$$= \sum_{m=1}^{N+1} \sum_{k=0}^{m} \sum_{p=\max\{m-1+k,2\}}^{2m} \sum_{\substack{P_{v_0} \in P_N:\ |P_{v_0}|=2m,\\ k(P_{v_0}^{\text{ren}})=k,\ |X_{P_{v_0}}|=p}} (-1)^{\pi(P_{v_0}^{\text{ren}})} \sum_{\substack{\{P_v\}_{v<v_0}:\\ P_{v_0}^{\text{ren}}\ \text{fixed}}}$$
$$\times \int dx_1 \cdots dx_p W_{2m,k}^{(n)}(\theta_{\text{lab}}^{N,n}, P_{v_0}^{\text{ren}}, \{P_v\}_{v<v_0}, \underline{X}_{P_{v_0}} \to \{x_1, \ldots, x_p\})$$
$$\times [\overline{\psi}]^{\underline{r}} [\psi]^{\underline{s}} [\partial^2 \psi]^k (x_1, \ldots, x_p). \tag{4.14}$$

Comparing with (4.1)

$$W_{2m,k}^{(n),\underline{r},\underline{s}}(\theta_{\text{lab}}^{N,n}, x_1, x_2, \ldots, x_p) = \sum_{\substack{P_{v_0} \in P_N:\ |P_{v_0}|=2m,\\ k(P_{v_0}^{\text{ren}})=k,\ |X_{P_{v_0}}|=p}} (-1)^{\pi(P_{v_0}^{\text{ren}})}$$

$$\times \left[\sum_{\substack{\{n_v\}_{v\leqslant v_0}\\ n \text{ fixed}}} \sum_{\substack{\{P_v\}_{v<v_0}:\\ P_{v_0}^{\text{ren}} \text{ fixed}}} W_{2m,k}^{(n)}\big(\theta_{\text{lab}}^{N,n}, P_{v_0}^{\text{ren}}, \{P_v\}_{v<v_0}, \underline{X}_{P_{v_0}} \right.$$
$$\left. \to \{x_1, \dots, x_p\}\big)\right]. \qquad (4.15)$$

Note that

$$\sum_{\{n_v\}_{v\leqslant v_0}:\, n \text{ fixed}} = \sum_{\theta_{\text{lab}}^{N,n}:\, \theta^N \text{ fixed}},$$

i.e., to sum over all $\theta_{\text{lab}}^{N,n}$ for a fixed θ^N is the same as to sum over all possible scale labels n_v in the non-trivial vertices of θ^N, keeping fixed the root scale label at the value n. Moreover we note that

$$\sum_{\substack{P_{v_0}\in P_N:\ |P_{v_0}|=2m,\\ k(P_{v_0}^{\text{ren}})=k,\ |X_{P_{v_0}}|=p}} \leqslant \binom{4N}{2m} \leqslant 2^{4N}. \qquad (4.16)$$

Finally we can obtain the kernels of effective potential as they appear in the main theorem by summing over all trees. Thus,

$$W_{2m,k}^{(n),\underline{r},\underline{s}}(x_1, x_2, \dots, x_p)$$
$$= \sum_{N=p}^{\infty} \sum_{\theta^N} \sum_{\theta_{\text{lab}}^{N,n}:\, \theta^N \text{ fixed}} W_{2m,k}^{(n),\underline{r},\underline{s}}\big(\theta_{\text{lab}}^{N,n}, x_1, x_2, \dots, x_p\big). \qquad (4.17)$$

4.2. Bound on $V_{2m,k}^{(n)}$

By definition of $\theta_{\text{lab}}^{N,n}$ (recall (3.34)), the function $V_{2m,k}^{(n)}(\theta_{\text{lab}}^{N,n})$ appearing in (4.2) is explicitly given by

$$V_{2m,k}^{(n)}\big(\theta_{\text{lab}}^{N,n}, P_{v_0}^{\text{ren}}, \{P_v\}_{v<v_0}, \underline{X}_{v_0}\big)$$
$$= \lambda^N \Big\{ \prod_{v<v_0} [\xi_v]^{z_v} L^{z_v n_v} \Big\} \sum_{\{Q_v\}_{v<v_0}} \prod_{v\leqslant v_0}$$
$$\times \Big\{ \sum_{R_{v1},\dots,R_{vl}} \mathcal{E}_{(n_{v'}-1)}\big[\widetilde{\psi}(R_{vl})\big] \mathcal{E}_{(n_{v'}-2)}\big[\widetilde{\psi}(R_{vl-1}\backslash R_{vl})\big]$$
$$\cdots \mathcal{E}_{(n_v+2)}\big[\widetilde{\psi}(R_{v1}\backslash R_{v2})\big] \mathcal{E}_{(n_v+1)}\big[\widetilde{\psi}(q_v\backslash P_v^{\text{ren}}\backslash R_{v1})\big] \Big\}$$
$$\times \prod_{v\leqslant v_0} \frac{1}{s_v!} \mathcal{E}_{(n_v)}^T\big[\widetilde{\psi}(P_{v^1}^*\backslash Q_{v^1}), \dots, \widetilde{\psi}(P_{v^{s_v}}^*\backslash Q_{v^{s_v}})\big]. \qquad (4.18)$$

Here Q_v is a subset of P_v^* and $\sum_{\{Q_v\}_{v<v_0}}$ runs over all possible ways to choose Q_v as a (proper) subset of P_{v^*} for any vertex $v < v_0$. Moreover, for any non-trivial vertex $v \leqslant v_0$

$$q_v = \bigcup_{i=1}^{s_v} Q_{v^i}, \tag{4.19}$$

R_{v1} is a subset of $q_v \backslash P_v^{\mathrm{ren}}$, and R_{vj} is a subset of R_{vj-1} for $j = 2, 3, \ldots, l = n_{v'} - n_v - 1$. In short, the inclusion relations between the sets appearing in (4.18) are

$$R_{vl} \subset R_{vl-1} \subset \cdots \subset R_{v1} \subset q_v \backslash P_v^{\mathrm{ren}}. \tag{4.20}$$

Remark. – Observe the renormalization scheme implicit in (4.18). At the vertex v the set of fields $q_v = \bigcup_{i=1}^{s_v} Q_{v^i}$ emerges, after the contraction occurred in the truncated expectation at scale n_v. This set contains the set that we are calling P_v^{ren}, and it is effectively reduced to P_v^{ren} after $n_{v'} - n_v - 1$ simple expectations, where the sets $R_{v1} \backslash R_{v2}, \ldots, R_{vl-1} \backslash R_{vl}, R_{vl}$ are successively contracted. Then, at v^*, the set of fields P_v^{ren} emerges, and it is converted by the $\mathcal{R}$ operation (acting exactly after v^*) into the set P_v^*. The $\mathcal{R}$ operation also produce the factor $\{\prod_{v<v_0} [\xi_v]^{z_v} L^{z_v n_v}\}$ in the right hand side of (4.18).

We now get a bound directly on the kernels $V_{2m,k}^{(n)}$ defined in (4.18). We use the following well known bounds for simple and truncated expectations of fermionic fields.

LEMMA 4.1. – *Suppose that $\mathcal{E}_{(n)}$ represents a simple expectation respect to a GFM with covariance $G_n(x-y)$ acting on Grassmann fields $\overline{\psi}$, ψ, such that*

$$\left|G_n(x-y)\right| \leqslant C L^{-2n} \exp\left[-L^{-n}|x-y|\right].$$

Then, for any monomial $\widetilde{\psi}(P)$ containing $|P|/2$ fields of type $\overline{\psi}$, $|P|/2 - k$ fields of type ψ and k fields of type $\partial^2\psi$, the following inequality holds

$$\left|\mathcal{E}_n\left[\widetilde{\psi}^{(n)}(P)\right]\right| \leqslant C^{|P|} L^{-n\|P\|}, \tag{4.21}$$

where

$$\|P\| = \frac{(3-1)}{2}|P| + 2k. \tag{4.22}$$

LEMMA 4.2. – *In the hypothesis of Lemma* 4.1, *let* $\widetilde{\psi}^{(n)}(P_1), \ldots, \widetilde{\psi}^{(n)}(P_s)$ *be s monomials sitting in clusters of space-time points* $\underline{X}_1, \ldots, \underline{X}_s$, *respectively (with* $\underline{X}_i \cap \underline{X}_j = \emptyset$ *for all* $\{i, j\} \subset \{1, 2, \ldots, s\}$*), then the following inequality holds pointwisely*

$$|\mathcal{E}^T_{(n)}[\widetilde{\psi}^{(n)}(P_1), \ldots, \widetilde{\psi}^{(n)}(P_s)]| \leqslant C^{\sum_{i=1}^s |P_i|} L^{-n\sum_{i=1}^s \|P_i\|} \sum_{\vartheta \in \mathcal{T}_{\underline{X}_1,\ldots,\underline{X}_s}} e^{-\alpha L^{-n} d_\vartheta(\underline{X}_1,\ldots,\underline{X}_s)}, \quad (4.23)$$

where $\mathcal{T}_{\underline{X}_1,\ldots,\underline{X}_s}$ *is the set of all possible graphs* ϑ *(not necessarely connected) between points in* $\underline{X} = \underline{X}_1 \cup \cdots \cup \underline{X}_s$ *such that:* ϑ *contains exactly* $s - 1$ *links; each link* $\rho \in \vartheta$ *(unordered pair of space-time points* $\rho = \{x, y\}$*) is formed by a point in a cluster* $\underline{X}_i$ *and a point in a different cluster* $\underline{X}_j$*; for any pair B, C of subsets of* $\{1, 2, \ldots, s\}$ *such that* $B \cup C = \{1, 2, \ldots, s\}$ *and* $B \cap C = \emptyset$, *there is a* $\rho_i \in \vartheta$ *such that* $\rho_i \cap [\bigcup_{j\in B} \underline{X}_j] \neq \emptyset$ *and* $\rho_i \cap [\bigcup_{j\in C} \underline{X}_j] \neq \emptyset$ *(connection modulo clusters). The factor* $d^\vartheta(\underline{X}_1, \ldots, \underline{X}_s)$ *is the lenght of* τ, *namely,* $d_\vartheta(\underline{X}_1, \ldots, \underline{X}_s) = \sum_{\rho\in\vartheta} |\rho|$, *where, if* $\rho \equiv \{x, y\}$, *then* $|\rho| = |x - y|$ *indicates the Euclidean distance between the two points x, y in different clusters.*

Remark 1. – The number of space-time points in $\underline{X}_i$ cannot exceed $|P_i|$, but, of course, can be less then $|P_i|$ if some of the fields in $\widetilde{\psi}(P_i)$ is sitting in the same space-time point.

Remark 2. – Again we stress that ϑ is *not* in general a connected tree graph in $\underline{X} = \underline{X}_1 \cup \cdots \cup \underline{X}_s$, but only realizes the connection between the clusters $X_1, \ldots, X_s$. So further on we will use the symbole τ to denote connected tree graphs in some set X, while we use the symbol ϑ to denote graphs *connected modulo cluster* in the sense specified above.

The first lemma, known since a very long time, is the origin of the claim that purely fermionic field theories have perturbative expansions with better convergence properties than purely bosonic field theories. It is a trivial consequence of the Gramm–Hadamard inequality for determinants (recall that a fermionic simple expectation can be written as a determinant, see (2.3)). The second lemma is more recent (mid-eighties) and its proof requires the use of the Brydges–Battle–Federbush tree graph equality [2,7]. The reader may find the original proof in [15] (see there Appendix 3), and a successive and simpler proof in [18] (see there Appendix A). A detailed proof of both lemmas, with notations very similar to ours, can be found in [6] (see there Appendix 2).

To show that the sum $\sum_{\vartheta}$ in right hand side of (4.23) does not introduce dangerous combinatorial factors, we have the following proposition.

PROPOSITION 4.3. – *With the notations of Lemma 4.2, there exists a constant B such that*

$$\sum_{\vartheta \in \mathcal{T}_{\underline{X}_1,\dots,\underline{X}_s}} 1 \leqslant (s-2)! B^{\sum_{i=1}^{s} |P_i|}. \tag{4.24}$$

Proof. – By definition, we can associate to any graph $\vartheta \in \mathcal{T}_{\underline{X}_1,\dots,\underline{X}_s}$ a connected tree graph $\tau \in \mathcal{T}_s$ in $\{1, 2, \dots, s\}$ by shrinking each cluster $\underline{X}_i$ to a point i, so that, if $d_i(\tau)$ is the incidence number of the vertex i in τ (i.e., the number of links of τ attached to i), then we have at most $\prod_{i=1}^{s} |P_i|^{d_i(\tau)}$ graphs ϑ which correspond to τ. Thus, using the Cayley formula

$$\begin{aligned}
\sum_{\vartheta \in \mathcal{T}_{\underline{X}_1,\dots,\underline{X}_s}} 1 &\leqslant \sum_{\tau \in \mathcal{T}_s} \prod_{i=1}^{s} |P_i|^{d_i(\tau)} = \sum_{d_1+d_2+\cdots+d_s = 2(s-1)} \frac{(s-2)!}{\prod_{i=1}^{s}(d_i-1)!} \prod_{i=1}^{s} |P_i|^{d_i} \\
&= \prod_{i=1}^{s} |P_i| \left[\sum_{i=1}^{s} |P_i| \right]^{s-2} = (s-2)! \prod_{i=1}^{s} |P_i| \frac{[\sum_{i=1}^{s} |P_i|]^{s-2}}{(s-2)!} \\
&\leqslant (s-2)!\, \mathrm{e}^{\sum_{i=1}^{s} \log |P_i|}\, \mathrm{e}^{\sum_{i=1}^{s} |P_i|} \leqslant (s-2)!\, \mathrm{e}^{2\sum_{i=1}^{s} |P_i|}. \qquad \square
\end{aligned}$$

Using Lemmas 4.1 and 4.2 we get, after some algebraic manipulations, a bound on $V_{2m,k}^{(n)}$ defined in (4.18)

$$\begin{aligned}
&\left| V_{2m,k}^{(n)}\left(\theta_{\mathrm{lab}}^{N,n}, P_{v_0}^{\mathrm{ren}}, \{P_v\}_{v<v_0}, \underline{X}_{v_0}\right) \right| \\
&\quad \leqslant \lambda^N \left\{ \prod_{v<v_0} [\xi_v]^{z_v} L^{n_v z_v} \right\} \sum_{\{Q_v\}_{v<v_0}} \prod_{v \leqslant v_0} \\
&\qquad \times \left\{ \sum_{R_{v1},\dots,R_{vl}} C^{|q_v| - |P_v|} L^{-n_v(\|q_v\| - \|P_v^{\mathrm{ren}}\|)} L^{-\|R_{vl}\|} L^{-\|R_{vl-1}\|} \right. \\
&\qquad\qquad \left. \cdots L^{-\|R_{v1}\|} L^{-(\|q_v\| - \|P_v^{\mathrm{ren}}\|)} \right\} \\
&\qquad \times \prod_{v \leqslant v_0} \left\{ C^{\sum_{i=1}^{s_v}(|P_{v^i}| - |Q_{v^i}|)} L^{-n_v[\sum_{i=1}^{s_v}(\|P_{v^i}^*\| - \|Q_{v^i}\|)]} \right\} \\
&\qquad \times \prod_{v \leqslant v_0} \left\{ \frac{1}{s!} \sum_{\vartheta_v \in \underline{X}_v} \mathrm{e}^{-\alpha L^{-n_v} d_{\vartheta_v}(\underline{X}'_v)} \right\}.
\end{aligned} \tag{4.25}$$

Here $\sum_{\vartheta_v \in \underline{X}_v}$ is a short notation. ϑ_v is a graph between the points in the set $\bigcup_{i=1}^{s_v} \underline{X}_{P^*_{vi} \backslash Q_{vi}}$, formed by $s_v - 1$ links, which realizes the connection between clusters

$$\underline{X}_{P^*_{v1} \backslash Q_{v1}}, \ldots, \underline{X}_{P^*_{vs_v} \backslash Q_{vs_v}}$$

in the sense of Lemma 4.2; thus, by $\vartheta_v \in \underline{X}_v$ we mean actually

$$\vartheta_v \in \mathcal{T}_{\underline{X}_{P^*_{v1} \backslash Q_{v1}}, \ldots, \underline{X}_{P^*_{vs_v} \backslash Q_{vs_v}}}.$$

Recall that ϑ_v in general is a non-connected graph (which, however, becomes connected if one shrinks each cluster $\underline{X}_{P^*_{v1} \backslash Q_{v1}}$ to a single point z_i). $d_{\vartheta_v}(\underline{X}'_v)$ is also a short notation for

$$d_{\vartheta_v}\left(\underline{X}'_{P^*_{v1} \backslash Q_{v1}}, \ldots, \underline{X}'_{P^*_{vs_v} \backslash Q_{vs_v}}\right)$$

(see again Lemma 4.2). Note the index $'$ stressing that some points in d_{ϑ_v} may be interpolated points.

Recalling that (see (4.19))

$$\sum_{i=1}^{s_v} |Q_{v^i}| = |q_v|, \qquad \sum_{i=1}^{s_v} \| Q_{v^i} \| = \|q_v\|,$$

and observing that $\|R_{vi}\| \geqslant |R_{vi}|$, and $\|q_v\| - \|P_v^{\text{ren}}\| \geqslant |q_v| - |P_v|$, we get

$$\sum_{R_{v1},\ldots,R_{vl}} L^{-\|R_{vl}\|} \ldots L^{-\|R_{v1}\|} L^{-(\|q_v\| - \|P_v^{\text{ren}}\|)} \leqslant \sum_{R_{v1},\ldots,R_{vl}} L^{-|R_{vl}|} \ldots L^{-|R_{v1}|} L^{-(|q_v| - |P_v|)}.$$

With the inclusion relations (4.20), we obtain

$$\sum_{R_{v1},\ldots,R_{vl}} L^{-|R_{vl}|} L^{-|R_{vl-1}|} \ldots L^{-|R_{v1}|} L^{-[|q_v| - |P_v|]} \leqslant \left[1 + L^{-1} + L^{-2} + \cdots + L^{-(n_{v'} - n_v - 1)}\right]^{|q_v| - |P_v|} \leqslant C^{|q_v| - |P_v|}. \quad (4.26)$$

Thus, (4.25) may be written as

$$\left| V_{2m,k}^{(n)}\left(\theta_{\text{lab}}^{N,n}, P_{v_0}^{\text{ren}}, \{P_v\}_{v<v_0}, \underline{X}_{v_0}\right)\right| \leqslant \lambda^N \left\{ \prod_{v<v_0} [\xi_v]^{z_v} L^{n_v z_v} \right\}$$

$$\times \sum_{\{Q_v\}_{v<v_0}} \prod_{v \leqslant v_0} \left[C^{\sum_{i=1}^{s_v} |P_{v_i}| - |P_v|} L^{-n_v (\sum_{i=1}^{s_v} \|P_{v_i}^*\| - \|P_v^{\text{ren}}\|)} \right]$$

$$\times \prod_{v \leqslant v_0} \left[\frac{1}{s_v!} \sum_{\vartheta_v \in \underline{X}_v} e^{-\alpha L^{-n_v} d_{\vartheta_v}(\underline{X}'_v)} \right]. \tag{4.27}$$

By definition of renormalization,

$$\|P_v^*\| = \|P_v^{\text{ren}}\| + z_v \tag{4.28}$$

and recalling that in v_0^* we are not performing any renormalization, it follows

$$\prod_{v \leqslant v_0} L^{-n_v (\sum_{i=1}^{s_v} \|P_{v_i}^*\| - \|P_v^{\text{ren}}\|)}$$

$$= L^{-n_{v_0} (\sum_{i=1}^{s_{v_0}} \|P_{v_0^i}^*\| - \|P_{v_0}^{\text{ren}}\|)} \prod_{v < v_0} L^{-n_v (\sum_{i=1}^{s_v} \|P_{v_i}^*\| - \|P_v^*\| + z_v)} \tag{4.29}$$

and so

$$\left| V_{2m,k}^{(n)} \left(\theta_{\text{lab}}^{N,n}, P_{v_0}^{\text{ren}}, \{P_v\}_{v<v_0}, \underline{X}_{v_0} \right) \right|$$

$$\leqslant \lambda^N \prod_{v \leqslant v_0} C^{\sum_{i=1}^{s_v} |P_{v_i}| - |P_v|} L^{-n_{v_0} (\sum_{i=1}^{s_{v_0}} \|P_{v_0^i}^*\| - \|P_{v_0}^{\text{ren}}\|)}$$

$$\times \sum_{\{Q_v\}_{v<v_0}} \prod_{v < v_0} L^{-n_v (\sum_{i=1}^{s_v} \|P_{v_i}^*\| - \|P_v^*\|)}$$

$$\times \left[\prod_{v < v_0} [\xi_v]^{z_v} \prod_{v \leqslant v_0} \frac{1}{s_v!} \sum_{\vartheta_v \in \underline{X}_v} e^{-\alpha L^{-n_v} d_{\vartheta_v}(\underline{X}'_v)} \right]. \tag{4.30}$$

It is now easy to check (exponents in left hand side of equations below are telescopic sums) that

$$C^{\sum_{v \leqslant v_0} (\sum_{i=1}^{s_v} |P_{v_i}| - |P_v|)} \leqslant C^{4N}, \tag{4.31}$$

$$L^{\sum_{v \leqslant v_0} [-n_v (\sum_{i=1}^{s_v} \|P_{v_i}^*\| - \|P_v^*\|)]}$$

$$= L^{-n_{v_0} (4N - \|P_{v_0}^*\|)} L^{\sum_{v < v_0} (n_{v'} - n_v)(4N_v - \|P_v^*\|)}. \tag{4.32}$$

In right hand side of (4.32) N_v denotes the number of end points of $\theta_{\text{lab}}^{N,n}$ that can be reached starting from v (thus $N_{v_0} = N$). From these formulas, it follows

$$L^{-n_{v_0}(\sum_{i=1}^{s_{v_0}}\|P^*_{v_0^i}\|-\|P^{\rm ren}_{v_0}\|)}\prod_{v<v_0}L^{-n_v(\sum_{i=1}^{s_v}\|P^*_{v^i}\|-\|P^*_v\|)}$$
$$=L^{-n_{v_0}(4N-\|P^{\rm ren}_{v_0}\|)}\prod_{v<v_0}L^{(n_{v'}-n_v)(4N_v-\|P^*_v\|)}, \tag{4.33}$$

whence

$$\left|V^{(n)}_{2m,k}(\theta^{N,n}_{\rm lab},P^{\rm ren}_{v_0};\{P_v\}_{v<v_0},\underline{X}_{v_0})\right|$$
$$\leqslant C^N\lambda^N L^{-n_{v_0}(4N-\|P^{\rm ren}_{v_0}\|)}\sum_{\{Q_v\}_{v<v_0}}\prod_{v<v_0}L^{(n_{v'}-n_v)(4N_v-\|P^*_v\|)}$$
$$\times\left[\prod_{v<v_0}[\xi_v]^{z_v}\prod_{v\leqslant v_0}\frac{1}{s_v!}\sum_{\vartheta_v\in\underline{X}_v}e^{-\alpha L^{-n_v}d_{\vartheta_v}(\underline{X}'_v)}\right]. \tag{4.34}$$

In right hand side of (4.34) nothing depends on sets $\{Q_v\}_{v<v_0}$, so we can bound directly this sum, of course with the condition that all set $\{P_v\}_{v\leqslant v_0}$ are kept fixed. Recalling the definition (4.19) and inclusion relations (4.20), each set Q_{v^i} must contain the set $\widetilde{P}_{v^i}=P_v\bigcap P_{v^i}$, so defining $R_{v^i}=Q_{v^i}\backslash\widetilde{P}_{v^i}$, noting that $\bigcup_{i=1}^{s_v}\widetilde{P}_{v^i}=P_v$ (thus also $\sum_{i=1}^{s_v}|\widetilde{P}_{v^i}|=|P_v|$) and also using (4.31), we have

$$\sum_{\substack{\{Q_v\}_{v<v_0}\\ P_v\ \text{fixed}\ \forall v\leqslant v_0}}\leqslant\prod_{v\leqslant v_0}\left\{\prod_{i=1}^{s_v}\left[\sum_{|R_{v^i}|=0}^{|P_{v^i}|-|\widetilde{P}_{v^i}|}\binom{|P_{v^i}|-|\widetilde{P}_{v^i}|}{|R_{v^i}|}\right]\right\}$$
$$=\prod_{v\leqslant v_0}2^{\sum_{i=1}^{s_v}|P_{v^i}|-|P_v|}\leqslant 2^{4N}. \tag{4.35}$$

Hence the factor $\sum_{\{Q_v\}_{v<v_0}}$ in right hand side of (4.34) yields at worst a contribution 2^{4N} which can be included in the factor C^N.

We now make explicit the product $\prod_{v<v_0}[\xi_v]^{z_v}$. It is a product which contains two kind of factors ξ_v: the k factors ξ_v associated to the k double derivative in $\widetilde{\psi}(P^{\rm ren}_{v_0})$ (see (2.25)), and all the others which belong to double derivative fields created and then contracted along the tree $\theta^{N,n}_{\rm lab}$. Let us separate these two kind of ξ_v as

$$\prod_{v<v_0}[\xi_v]^{z_v}=\prod_{v<v_0}{}^{*}[\xi_v]^{z_v}\prod_{r=1}^{k}[\xi_{v_r}]^2. \tag{4.36}$$

In the formula above the first productory (with the apix $*$) is over all vertices v in which double derivative fields are created and contracted later (i.e., ξ_v, for these vertices, links two space-time points which do not appear in $\underline{X}_{P_{v_0}^{\text{ren}}}$ as interpolated points). The other productory is over the vertices v_r in which double derivative fields are created but not contracted later, thus elements of $P_{v_0}^{\text{ren}}$. The factors ξ_{v_r} links couples of points which appear in $\underline{X}_{P_{v_0}^{\text{ren}}}$ as interpolated points. □

We now estabilish the following propositions

PROPOSITION 4.4. – *The function*

$$f(\underline{X}_{v_0}) = \prod_{v \leqslant v_0} \frac{1}{s_v!} \sum_{\vartheta_v \in \underline{X}_v} \mathrm{e}^{-\alpha L^{-n_v} d_{\vartheta_v}(\underline{X}'_v)}$$

in (4.34) *contains intermediate points which are not among the k intermediate points in* $\underline{X}_{P_{v_0}^{\text{ren}}}$.

Proof. – The only way in which interpolated points can appear in $f(\underline{X}_{v_0})$ is by contracting double derivative fields. The interpolated points in $\underline{X}_{P_{v_0}^{\text{ren}}}$ are precisely those attached to double derivative fields which are not contracted, whence the proposition. □

PROPOSITION 4.5. – *With the notations of formulas* (4.34) *and* (4.36), *there exists a constant* $C > 0$ *such that the following inequality holds*

$$\begin{aligned}\Big[\prod_{v<v_0}{}^{*}[\xi_v]^{z_v} \prod_{v \leqslant v_0} \frac{1}{s_v!} \sum_{\vartheta_v \in \underline{X}_v} \mathrm{e}^{-\alpha L^{-n_v} d_{\vartheta_v}(\underline{X}'_v)}\Big] \\ \leqslant C^N \Big[\prod_{v \leqslant v_0} \frac{1}{s_v!} \sum_{\vartheta \in \underline{X}_v} \mathrm{e}^{-\tilde{\alpha} L^{-n_v} d^*_{\vartheta_v}(\underline{X}_v)}\Big],\end{aligned} \tag{4.37}$$

where

$$\tilde{\alpha} \geqslant \frac{1}{2}\alpha\left(1 - \frac{1}{L}\right) \quad \textit{and} \quad d^*_{\vartheta_v}(\underline{X}_v) = d_{\vartheta_v}\big(\underline{X}'_v\colon xy(t) \to y\big),$$

i.e., $d^*_{\vartheta_v}(\underline{X}_v)$ *is obtained from* $d_{\vartheta_v}(\underline{X}'_v)$ *by replacing* $xy(t)$ *appearing in* $\underline{X}'_v$ *by* y.

The proof is given some lines below.

Remark. – The function in the right hand side of (4.37) can be rewritten as

$$\left[\prod_{v\leqslant v_0}\frac{1}{s_v!}\sum_{\vartheta_v} \mathrm{e}^{-\tilde{\alpha}L^{-n_v}d^*_{\vartheta_v}(\underline{X}_v)}\right] = \left[\prod_{v\leqslant v_0}\frac{1}{s_v!}\right] \sum_{\substack{\tau\in\mathcal{T}_{\underline{X}_{v_0}}:\\ \tau \text{ comp } \theta^N}} \prod_{\rho\in\tau} \mathrm{e}^{-\tilde{\alpha}L^{-n_\rho}|\rho|}, \tag{4.38}$$

where τ is a *connected tree graph* in the set $\underline{X}_{v_0}$ (hence a link $\rho \in \tau$ is an unordered pair $\{x, y\} \subset \underline{X}_{v_0}$ and $|\rho| = |x - y|$), which is *compatible with the topological structure of* $\theta^{N,n}_{\text{lab}}$ (hence θ^N), and the scale label n_ρ associated to the link ρ in τ is fixed univocally by $\theta^{N,n}_{\text{lab}}$. The definition of compatibility of a connected tree graph $\tau \in \mathcal{T}_{\underline{X}_{v_0}}$ respect to the topological structure of a tree $\theta^{N,n}_{\text{lab}}$ is as follows:

DEFINITION 4.6. – *Let $\tau \in \mathcal{T}_{\underline{X}_{v_0}}$ and, for any $\underline{X}_v \subset \underline{X}_{v_0}$, denote by τ_v the subgraph of τ obtained by cancelling all links $\rho = \{x, y\}$ in τ such that $\{x, y\} \cap \underline{X}_{v_0} \backslash \underline{X}_v \neq \emptyset$. Then τ is compatible with the topological structure of $\theta^{N,n}_{\text{lab}}$ if, $\forall v \in \theta^{N,n}_{\text{lab}}$, $\tau_v \in \mathcal{T}_{\underline{X}_v}$ (i.e., τ_v is a connected tree graph between the points $\underline{X}_v$). The scale label n_ρ of a given link in τ is such that $n_\rho = n_{v_\rho}$, where v_ρ is the minimum vertex of $\theta^{N,n}_{\text{lab}}$ which contains the link ρ.*

Remark 1. – Note that τ_v is a connected tree graph in $\underline{X}_v$ which has $|\underline{X}_v| - 1$ links, among which, $s_v - 1$ are at scale n_v, while all the others are at scales lower than n_v.

Remark 2. – Recalling the definition of ϑ_v above (4.25), it is easy to see that τ can be written is a unique way as $\tau = \bigcup_{v\leqslant v_0} \vartheta_v$ for some choice of ϑ_v in any non-trivial vertex of $\theta^{N,n}_{\text{lab}}$. In other words, the correspondence $\{\vartheta_v\}_{v\leqslant v_0} \to \tau$ is one to one. Moreover we also have $\tau_v = \bigcup_{w\leqslant v} \vartheta_w$.

Proof of Proposition 4.5. – First note that

$$\prod_{v\leqslant v_0} \mathrm{e}^{-\alpha L^{-n_v}d^*_{\vartheta_v}(\underline{X}_v)} = \prod_{\rho\in\tau} \mathrm{e}^{-\alpha L^{-n_\rho}|\rho|},$$

where $\tau = \bigcup_{v\leqslant v_0} \vartheta_v$ is the connected tree graph in $\mathcal{T}_{\underline{X}_{v_0}}$ compatible with the topological structure of $\theta^{N,n}_{\text{lab}}$ between points $x_1, x_2, \ldots, x_N$ univocally determinated by the sequence $\{\vartheta_v\}_{v\leqslant v_0}$. Thus we can also write

$$\prod_{v\leqslant v_0} \mathrm{e}^{-\alpha L^{-n_v}d_{\vartheta_v}(\underline{X}'_v)} = \prod_{\rho\in\tau} \mathrm{e}^{-\alpha L^{-n_\rho}|\rho'|},$$

where ρ' can be a link between intermediate points, univocally determinated by $\rho \in \tau$ once $\theta_{\text{lab}}^{N,n}$, τ and the sequence $\{P_v\}$ are chosen. Due to the hierachical cluster structure of $\theta_{\text{lab}}^{N,n}$, it is not difficult to check that:

$$\sum_{\rho\in\tau} L^{-n_\rho}|\rho'| \geqslant \left(1-\frac{1}{L}\right)\sum_{\rho\in\tau} L^{-n_\rho}|\rho|, \tag{4.39}$$

whence

$$\prod_{v\leqslant v_0} \mathrm{e}^{-\alpha L^{-n_v} d_{\vartheta_v}(\underline{X}'_v)} \leqslant \prod_{v\leqslant v_0} \mathrm{e}^{-\alpha(1-\frac{1}{L})L^{-n_v} d^*_{\vartheta_v}(\underline{X}_v)}. \tag{4.40}$$

On the other hand, each ξ_v in left hand side of (4.37) can be written as

$$|\xi_v| = L^{-n_v}|x_i - x_j|, \tag{4.41}$$

where x_i and x_j are two points surely in $\underline{X}_v$, thus

$$|x_i - x_j| \leqslant \sum_{\rho\in\tau_v} |\rho| \tag{4.42}$$

(see Definition 4.6). Note $\vartheta_v \subset \tau_v$ (see Remark 2 below Definition 4.6), i.e., $s_v - 1$ links in τ_v are at scale n_v (they are the links of ϑ_v defined below (4.25)) and all the other links in τ_v are at scale lower than n_v. Thus left hand side of (4.41) can be bounded as

$$|\xi_v| \leqslant L^{-n_v}\sum_{\rho\in\tau_v}|\rho| = \sum_{\rho\in\tau_v} L^{-n_\rho}|\rho| L^{-(n_v-n_\rho)}. \tag{4.43}$$

We call $p_\rho^v = n_v - n_\rho$. Remark that p_ρ^v is surely a non negative number for all $\rho \in \tau_v$.

Hence

$$\begin{aligned}
\prod_{v<v_0}^{*} |\xi_v|^{z_v} &\leqslant \prod_{v<v_0}^{*}\left(\sum_{\tau\in\tau_{v_r}} L^{-n_\rho}|\rho|L^{-p_\rho^v}\right)^{z_v} \\
&= \exp\left(\sum_{v<v_0}^{*} z_v \log\left[\sum_{\rho\in\tau_v} L^{-n_\rho}|\rho|L^{-p_\rho^v}\right]\right) \\
&\leqslant \exp\left(\sum_{v<v_0}^{*} z_v \left[\sum_{\rho\in\tau_v} L^{-n_\rho}|\rho|L^{-p_\rho^v}\right]^{1/2}\right) \\
&\leqslant \exp\left(\sum_{v<v_0}^{*} z_v \sum_{\rho\in\tau_v} L^{-n_\rho}|\rho|^{1/2} L^{\frac{-p_\rho^v}{2}}\right).
\end{aligned}$$

$$\leqslant \exp\left(2\sum_{\rho\in\tau}\left(L^{-n_\rho}|\rho|\right)^{1/2}\sum_{p=1}^{\infty}L^{-\frac{p}{2}}\right)$$

$$= \exp\left[2\frac{\sqrt{L}}{\sqrt{L}-1}\sum_{\rho\in\tau}\left(L^{-n_\rho}|\rho|\right)^{1/2}\right] = \prod_{\rho\in\tau}\mathrm{e}^{2\frac{\sqrt{L}}{\sqrt{L}-1}|L^{-n_\rho}\rho|^{1/2}}$$

In the third line we use that $\log x \leqslant x^{1/2}$ uniformely in $x > 0$ and also that

$$\sqrt{a_1+\cdots+a_l} \leqslant \sqrt{a}_1 + \cdots + \sqrt{a}_l$$

(with $a_i \geqslant 0$, $\forall i$). In the fifth line we use the fact that, for any ρ, $p_\rho^v \neq p_\rho^w$ if $\rho \in \underline{X}_v$ and simultaneously $\rho \in \underline{X}_w$, and, due to the cluster structure of $\theta_{\mathrm{lab}}^{N,n}$, if $w < v$ then $p_\rho^w < p_\rho^v$.

By the inequality above and (4.40) we have

$$\left[\prod_{v<v_0}{}^{*}[\xi_v]^{z_v}\prod_{v\leqslant v_0}\frac{1}{s_v!}\sum_{\vartheta_v}\mathrm{e}^{-\alpha L^{-n_v}d_{\vartheta_v}(\underline{X}'_v)}\right]$$

$$\leqslant \prod_{\rho\in\tau}\mathrm{e}^{2\frac{\sqrt{L}}{\sqrt{L}-1}(L^{-n_\rho}|\rho|)^{1/2}}\prod_{\rho\in\tau}\mathrm{e}^{-\alpha(1-\frac{1}{L})L^{-n_\rho}|\rho|}$$

$$\leqslant C^N\prod_{\rho\in\tau}\mathrm{e}^{-\tilde{\alpha}L^{-n_\rho}|\rho|} = C^N\prod_{v\leqslant v_0}\mathrm{e}^{-\tilde{\alpha}L^{-n_v}d^*_{\vartheta_v}(\underline{X}_v)},$$

where, for example, one can take

$$C = \max_{x>0}\left\{\mathrm{e}^{-\alpha(1-\frac{1}{L})x+2\frac{\sqrt{L}}{\sqrt{L}-1}\sqrt{x}}\right\} \quad \text{and} \quad \tilde{\alpha} = \frac{1}{2}\left(1-\frac{1}{L}\right)\alpha. \qquad \square$$

Thus, using also Proposition 4.5 and formula (4.38) it follows that (4.34) can be written as

$$\left|V_{2m,k}^{(n)}\left(\theta_{\mathrm{lab}}^{N,n}, P_{v_0}^{\mathrm{ren}}, \{P_v\}_{v<v_0}, \underline{X}_{v_0}\right)\right|$$

$$\leqslant C^N\lambda^N L^{n_{v_0}(4N-\|P_{v_0}^{\mathrm{ren}}\|)}\prod_{v<v_0}L^{(n_{v'}-n_v)(4N_v-\|P_v^*\|)}$$

$$\times\left[\prod_{r=1}^{k}[\xi_{v_r}]^2\left[\prod_{v\leqslant v_0}\frac{1}{s_v!}\right]\sum_{\substack{\tau\in\mathcal{T}_{\underline{X}_{v_0}}:\\ \tau\ \mathrm{comp}\ \theta^N}}\prod_{\rho\in\tau}\mathrm{e}^{-\alpha L^{-n_\rho}|\rho|}\right], \qquad (4.44)$$

where we renamed the constant $\tilde{\alpha}$ as α.

Remark 1. – The factor $\prod_{v\leqslant v_0}\frac{1}{s_v!}\sum_{\tau\in\underline{X}_{v_0}}$ is not combinatorially dangerous. Actually, by Proposition 4.3 and Remark 2 below Definition 4.6, we have

$$\sum_{\substack{\tau\in\underline{X}_{v_0}:\\ \tau\ \mathrm{comp}\ \theta^N}} 1=\prod_{v\leqslant v_0}\sum_{\vartheta_v}1\leqslant\prod_{v\leqslant v_0}C^{\sum_{i=1}^{s_v}(|P_{v^i}|-|Q_{v^i}|)}(s_v-2)!$$

$$\leqslant\prod_{v\leqslant v_0}C^{(\sum_{i=1}^{s_v}|P_{v^i}|)-|P_v|}(s_v-2)!,\tag{4.45}$$

hence, recalling (4.31)

$$\left[\prod_{v\leqslant v_0}\frac{1}{s_v!}\right]\sum_{\substack{\tau\in\underline{X}_{v_0}:\\ \tau\ \mathrm{comp}\ \theta^N}}1\leqslant\prod_{v\leqslant v_0}\frac{C^{(\sum_{i=1}^{s_v}|P_{v^i}|)-|P_v|}}{s_v(s_v-1)}\leqslant C^{4N}.\tag{4.46}$$

Remark 2. – The factor $\sum_{\{Q_v\}_{v<v_0}}\leqslant 2^{4N}$ (see (4.35)) has been incorporated in the factor C^N.

4.3. The bound on δb_n

From (4.44) it is not difficult to obtain a bound for δb_n. Actually, the contribution of the labelled tree $\theta_{\mathrm{lab}}^{N,n}$ to δb_n is obtained by considering the kernel

$$V_{2,0}^{(n)}\big(\theta_{\mathrm{lab}}^{N,n},P_{v_0}^{\mathrm{ren}},\{P_v\}_{v<v_0},\underline{X}_{v_0}\big).$$

Note that, in this particular case where $2m=2$ and $k=0$, we have that $P_{v_0}=P_{v_0}^{\mathrm{ren}}=\{\bar\psi_x,\psi_y\}$, and so $\|P_{v_0}^{\mathrm{ren}}\|=|P_{v_0}|=2$. We name x and y the space-time coordinates (just two) in $\underline{X}_{P_{v_0}^{\mathrm{ren}}}=\underline{X}_{P_{v_0}}$. Then, by definition (3.26), we have

$$\begin{aligned}&\big|\delta b_n\big(\theta_{\mathrm{lab}}^{N,n},P_{v_0},\{P_v\}_{v<v_0}\big)\big|\\&=\frac{1}{|\Lambda|}\left|\int_\Lambda d\underline{X}_{v_0}\frac{1}{9}\,\mathrm{tr}\big[V_{2,0}^{(n)}\big(\theta_{\mathrm{lab}}^{N,n},P_{v_0}^{\mathrm{ren}},\{P_v\}_{v<v_0},\underline{X}_{v_0}\big)\gamma_\mu\big](x-y)^\mu\right|\\&\leqslant\frac{1}{|\Lambda|}\int_\Lambda d\underline{X}_{v_0}\big|V_{2,0}^{(n)}\big(\theta_{\mathrm{lab}}^{N,n},P_{v_0}^{\mathrm{ren}},\{P_v\}_{v<v_0},\underline{X}_{v_0}\big)\big||x-y|.\end{aligned}\tag{4.47}$$

Using bound (4.44)

$$\big|\delta b_n\big(\theta_{\mathrm{lab}}^{N,n},P_{v_0},\{P_v\}_{v<v_0}\big)\big|$$

$$\leqslant \frac{1}{|\Lambda|} C^N \lambda^N L^{-n_{v_0}(4N - \|P^{\rm ren}_{v_0}\|)} \prod_{v<v_0} L^{(n_{v'}-n_v)(4N_v - \|P^*_v\|)} L^{n_{v_0}}$$
$$\times \int d\underline{X}_{v_0} (L^{-n_{v_0}}|x-y|) \left[\prod_{v\leqslant v_0} \frac{1}{s_v!}\right] \sum_{\substack{\tau\in\mathcal{T}_{\underline{X}_{v_0}}: \\ \tau \text{ comp } \theta^N}} \prod_{\rho\in\tau} \mathrm{e}^{-\tilde{\alpha} L^{-n_\rho}|\rho|}. \quad (4.48)$$

Note that there is no factor $\prod_{r=1}^{k}[\xi_{v_r}]^2$ in this case.

Now, using (4.46) and recalling Definition 4.6, we have

$$\frac{1}{|\Lambda|} \int d\underline{X}_{v_0} (L^{-n_{v_0}}|x-y|) \left[\prod_{v\leqslant v_0} \frac{1}{s_v!}\right] \sum_{\substack{\tau\in\mathcal{T}_{\underline{X}_{v_0}}: \\ \tau \text{ comp } \theta^N}} \prod_{\rho\in\tau} \mathrm{e}^{-\tilde{\alpha} L^{-n_\rho}|\rho|}$$
$$\leqslant \left[\prod_{v\leqslant v_0} \frac{1}{s_v!}\right] \sum_{\substack{\tau\in\mathcal{T}_{\underline{X}_{v_0}}: \\ \tau \text{ comp } \theta^N}} \prod_{\rho\in\tau} L^{3n_\rho} \int d\xi (N-1)|\xi|\,\mathrm{e}^{-\tilde{\alpha}|\xi|}$$
$$\times \left[\int d\xi |\xi|\,\mathrm{e}^{-\tilde{\alpha}|\xi|}\right]^{N-2} \leqslant C^N \prod_{\rho\in\tau} L^{3n_\rho} = C^N \prod_{v\leq v_0} L^{3n_v(s_v-1)}. \quad (4.49)$$

Moreover observe (just developing the telescopic sums in left hand side of (4.50) and (4.51) below) that

$$\prod_{v\leqslant v_0} L^{3n_v(s_v-1)} = L^{3n_{v_0}(N-1)} \prod_{v<v_0} L^{-3(n_{v'}-n_v)(N_v-1)}, \quad (4.50)$$

$$L^{-n_{v_0}N} \prod_{v<v_0} L^{(n_{v'}-n_v)N_v} = \prod_{i=1}^{N} L^{-n_i}. \quad (4.51)$$

Recall that n_i is the scale label of the first non-trivial vertex in $\theta^{N,n}_{\rm lab}$ which follows the ith end point.

Hence, using (4.49), (4.50) and (4.51) and recalling also that in the present case $\|P^{\rm ren}_{v_0}\| = 2$, we obtain

$$\left|\delta b_n\left(\theta^{N,n}_{\rm lab}, P_{v_0}, \{P_v\}_{v<v_0}\right)\right|$$
$$\leqslant C^N \lambda^N \prod_{v<v_0} L^{(n_{v'}-n_v)(3-\|P^*_v\|)} \prod_{i=1}^{N} L^{-n_i}. \quad (4.52)$$

Now we define

$$\delta b_n(\theta^N) = \sum_{\substack{\{n_v\}_{v\leqslant v_0} \\ n \text{ fixed}}} \sum_{\substack{\{P_v\}_{v\leqslant v_0}: \\ \|P^{\rm ren}_{v_0}\|=2}} \delta b_n\left(\theta^{N,n}_{\rm lab}, P_{v_0}, \{P_v\}_{v<v_0}\right) \quad (4.53)$$

as the contribution to δb_n of all tree topologically identical, but with different attributions of scale labels $\{n_v\}$ and sets $\{P_v\}$ (chosen such that $\|P_{v_0}^{\text{ren}}\| = 2$). Once again we stress that $\sum_{\{n_v\}_{v\leqslant v_0}:\, n \text{ fixed}}$ and $\sum_{\{\theta_{\text{lab}}^{N,n}:\, \theta^N \text{ fixed}\}}$ means exactly the same thing. I.e., the sum over all possible scale labels n_v (compatibly with the cluster structure of θ^N keeping fixed the root scale at n) is equivalent to the sum over all possible labelled trees $\theta_{\text{lab}}^{N,n}$.

Thus, we have

$$|\delta b_n(\theta^N)| \leqslant C^N \lambda^N \sum_{\substack{\{n_v\}_{v\leqslant v_0} \\ n \text{ fixed}}} \sum_{\substack{\{P_v\}_{v\leqslant v_0}: \\ \|P_{v_0}^{\text{ren}}\|=2}} \prod_{v<v_0} L^{(n_{v'}-n_v)(3-\|P_v^*\|)} \prod_{i=1}^{N} L^{-n_i}, \tag{4.54}$$

where $(3 - \|P_v^*\|) < 0$ in any case. This is the key point of renormalization: in any case $\|P_v^*\| \geqslant |P_v|$, and when $|P_v| = 2$ then $\|P_v^*\| = 4$. Thus, $(3 - \|P_v^*\|) \leqslant -1$ for all non-trivial vertices v.

We can safely perform all sums in the right hand side of (4.54) by using the following theorem.

THEOREM 4.7. – *For any θ^N, with the notation of this section, there exists a constant C such that the following inequality holds*

$$\sum_{\substack{\{n_v\}_{v<v_0}: \\ n_{v_0} \text{ fixed}}} \sum_{\substack{\{P_v\}_{v<v_0}: \\ P_{v_0} \text{ fixed}}} \prod_{v<v_0} L^{(n_{v'}-n_v)(3-\|P_v^*\|)} \prod_{i=1}^{N} L^{-n_i} \leqslant C^N L^{-s_{v_0} n_{v_0}}. \tag{4.55}$$

The proof follows from the three lemmas below.

LEMMA 4.8. – *For any fixed $\theta_{\text{lab}}^{N,n}$, with the notations of this section, let $D(P_v)$ be a function of P_v such that*
(1) *It is possible to find a constant $\varepsilon > 0$ such that $D(P_v) \geqslant \varepsilon |P_v|$, for all $v < v_0 \in \theta_{\text{lab}}^{N,n}$, and*
(2) *$D(P_v) \geqslant 1$, for all $v < v_0$,*
then the following inequality holds

$$\prod_{v<v_0} L^{-D(P_v)(n_v'-n_v)} \leqslant L^{N-1} \prod_{v<v_0} L^{-\varepsilon|P_v|} \prod_{v<v_0} L^{-(n_{v'}-n_v)}. \tag{4.56}$$

Proof. –

$$\prod_{v<v_0} L^{-D(P_v)(n_{v'}-n_v)} = \prod_{v<v_0} L^{-D(P_v)} \prod_{v<v_0} L^{-D(P_v)(n_{v'}-n_v-1)}$$

$$\leqslant \prod_{v<v_0} L^{-\varepsilon|P_v|} \prod_{v<v_0} L^{-(n_{v'}-n_v-1)}$$
$$\leqslant \prod_{v<v_0} L^{-\varepsilon|P_v|} \prod_{v<v_0} L^{-(n_{v'}-n_v)} \prod_{v_0} L$$
$$\leqslant L^{N-1} \prod_{v<v_0} L^{-\varepsilon|P_v|} \prod_{v<v_0} L^{-(n_{v'}-n_v)}. \qquad \square$$

Remark that, in the proof above, we used that $n_{v'} - n_v \geqslant 1$, for all v.

LEMMA 4.9. – *For any fixed $\theta_{\text{lab}}^{N,n}$, any fixed $|P_{v_0}|$ and any $\varepsilon > 0$, with the notations of this section, there exists C_ε such that*:

$$\sum_{\substack{\{P_v\}_{v<v_0}: \\ P_{v_0} \text{ fixed}}} \prod_{v<v_0} L^{-\varepsilon|P_v|} \leqslant C_\varepsilon^N. \tag{4.57}$$

Proof. – We can overestimate the sum over sets P_v as follows

$$\sum_{\substack{\{P_v\}_{v<v_0}: \\ P_{v_0} \text{ fixed}}} \prod_{v<v_0} L^{-\varepsilon|P_v|} \leqslant \prod_{v\leqslant v_0} \left[\sum_{|P_v|=0}^{\sum_{i=1}^{s_v}|P_{v_i}|} \binom{\sum_{i=1}^{s_v}|P_{v_i}|}{|P_v|} L^{-\varepsilon|P_v|} \right].$$

The last sum can be performed explicitly, starting by summing first $|P_{v_0}|$, then $|P_{v_0^1}|, \ldots, |P_{v_0^{s_{v_0}}}|$ and so on, always following the cluster structure of $\theta_{\text{lab}}^{N,n}$. Using also the fact that

$$\sum_{k=0}^{n} \binom{n}{k} L^{-\varepsilon k} = (1 + L^{-\varepsilon})^n,$$

and that $1 + L^{-\varepsilon} + L^{-2\varepsilon} + \cdots \leqslant (1 - L^{-\varepsilon})^{-1}$, one can easily bound the sum as

$$\prod_{v\leqslant v_0} \left[\sum_{|P_v|=0}^{\sum_{i=1}^{s_v}|P_{v_i}|} \binom{\sum_{i=1}^{s_v}|P_{v_i}|}{|P_v|} L^{-\varepsilon|P_v|} \right] \leqslant \left[(1 - L^{-\varepsilon})^{-1} \right]^S,$$

where S is the number of non-trivial vertices of $\theta_{\text{lab}}^{N,n}$. Since, for any rooted Cayley tree, $S \leqslant N - 1$, the lemma is proved with $C_\varepsilon = (1 - L^{-\varepsilon})^{-1}$. $\square$

LEMMA 4.10. – *For any $\theta_{\text{lab}}^{N,n}$ fixed, there exists a constant C such that*

$$\sum_{\substack{\{n_v\}_{v<v_0}:\\ n_{v_0}\ \text{fixed}}} \prod_{v<v_0} L^{-(n_{v'}-n_v)} \prod_{i=1}^{N} L^{-n_i} \leqslant C^N L^{-s_{v_0}n_{v_0}}. \tag{4.58}$$

Proof. – Due to the cluster structure of $\theta_{\text{lab}}^{N,n}$, the follows indentity holds

$$\prod_{v<v_0} L^{-(n_{v'}-n_v)} = \prod_{v<v_0} L^{-n_v(\tilde{s}_v-1)} L^{-n_{v_0}\tilde{s}_{v_0}},$$

where $\tilde{s}_v$ is the number of branches in v which do not terminate with end points. In other words, $\tilde{s}_v$ = number of v^i which are not end points (i.e., v^i is a real non-trivial vertex). Hence

$$\prod_{v<v_0} L^{-(n_{v'}-n_v)} \prod_{i=1}^{N} L^{-n_i} = \prod_{v<v_0} L^{-n_v(s_v-1)} L^{-n_{v_0}s_{v_0}}$$

and, since v is non-trivial, $s_v - 1 \geqslant 1$, we have

$$\sum_{\substack{\{n_v\}_{v<v_0}:\\ n_{v_0}\ \text{fixed}}} \prod_{v<v_0} L^{-n_v(s_v-1)} \leqslant \left(\frac{1}{1-L^{-1}}\right)^{N-1}. \qquad \square$$

Proof of Theorem 4.7. – Just note that the function $D(P_v) = \|P_v^*\| - 3$ satisfies the condition of Lemma 4.9 with, e.g., $\varepsilon = 1/4$. Thus, applying Lemmas 4.8–4.10, the proof is straightfoward. $\square$

Using Theorem 4.7 and the inequality $\sum_{P_{v_0}} 1 \leqslant 2^{4N}$, we get

$$\left|\delta b_n(\theta^N)\right| \leqslant C^N \lambda^N \sum_{n_{v_0}=0}^{n} L^{-s_{v_0}n_{v_0}}. \tag{4.59}$$

This bound is not enough to control the wavefunction flow. It implies that b_n diverges logarithmically, which is not expected (summing over n_{v_0} in (4.59) we will get a bound for $|\delta b_n|$ independent on n). However, it can be improved. Consider in the formula (4.18) the factor

$$\sum_{R_{v_0 1},\ldots,R_{v_0 l}} \mathcal{E}_{(n_{v_0'}-1)}\left[\widetilde{\psi}'(R_{v_0 l})\right]\mathcal{E}_{(n_{v_0'}-2)}\left[\widetilde{\psi}'(R_{v_0 l-1}\backslash R_{v_0 l})\right]$$
$$\cdots \mathcal{E}_{(n_{v_0}+2)}\left[\widetilde{\psi}'(R_{v_0 1}\backslash R_{v_0 2})\right]\mathcal{E}_{(n_{v_0}+1)}\left[\widetilde{\psi}'(q_{v_0}\backslash P_{v_0}^{\text{ren}}\backslash R_{v_0 1})\right] \tag{4.60}$$

for the special case in which $v = v_0$ and $\|P_{v_0}^{\text{ren}}\| = 2$. By definition of renormalization it is not difficult to convince oneself that the further restriction must be imposed in factor above

$$\|R_{v_0 l}\| \geqslant 2, \quad \text{if } \left\|P_{v_0}^{\text{ren}}\right\| = 2. \tag{4.61}$$

Using the constraint (4.61) we improve the estimate on the factor (4.26) (for the special case $v = v_0$), bounded previously by constant, in the following way

$$\sum_{R_{v_0 1},\ldots,R_{v_0 l}} L^{-|R_{v_0 l}|} L^{-|R_{v_0 l-1}|} \cdots L^{-|R_{v_0 1}|} \leqslant L^{-2(n-n_{v_0})} C^{|q_{v_0}|-|P_{v_0}|}. \tag{4.62}$$

Thus, (4.59) becomes

$$\left|\delta b_n(\theta^N)\right| \leqslant C^N \lambda^N \sum_{n_{v_0}=0}^{n} L^{-s_{v_0} n_{v_0}} L^{-2(n-n_{v_0})}. \tag{4.63}$$

Considering that, for any θ^N which contributes to δb_n we must have $s_{v_0} \geqslant 2$ (the trivial tree with $s_{v_0} = 1$ does not contributes to δb_n), performing the sum over the scale n_{v_0}, we obtain at worst

$$\left|\delta b_n(\theta^N)\right| \leqslant C^N \lambda^N n L^{-2n}. \tag{4.64}$$

We sum over all topologically distinct trees θ^N with N end points, so we get the contribution of the Nth order in the perturbation theory to δb_n. Name this contribution $\delta b_n^{[N]}$. Hence

$$|\delta b_n^{[N]}| \leqslant \sum_{\theta^N} C^N \lambda^N n L^{-2n} \leqslant \widetilde{C}^N \lambda^N n L^{-2n}, \tag{4.65}$$

where we used (2.19). Finally, summing over N we have

$$|\delta b_n| \leqslant \sum_{N \geqslant 2} |\delta b_n^{[N]}| \leqslant |f(\lambda)| n L^{-2n}, \tag{4.66}$$

where $f(\lambda)$ is analytic in λ in a suitable convergence radius ε, which does not depend on Λ and n, and

$$|f(\lambda)| \leqslant D\lambda^2$$

for some $D > 0$. Thus, we have proved the following theorem.

THEOREM 4.11. – *The number (wavefunction renormalization) δb_n defined by* (3.26) *is analytic in λ in a convergence radius $R \geqslant \varepsilon$, where $\varepsilon > 0$ is a constant independent on Λ and n. Moreover, there exists a number $D > 0$ such that*

$$|\delta b_n| \leqslant D\, n\, L^{-2n}\lambda^2. \tag{4.67}$$

As a trivial corollary of this theorem we have that the running coupling constant b_n defined iteratively by (3.25), for $b_0 = 1$ (as in the present model), admits the estimate

$$|b_n - 1| \leqslant C\lambda^2. \tag{4.68}$$

We can now prove part B of Theorem 3.1. We recall that the pressure of the model can be written as

$$p_\Lambda(\lambda) = \sum_{j=0}^{\infty}(T_j + t_j),$$

where T_j and t_j are given by (3.28). Of course, T_n can be written in term of a tree expansion, i.e.,

$$T_n = \sum_{N=2}^{\infty}\sum_{\theta^N} T_n(\theta^N),$$

where $T_n(\theta^N)$ is given by

$$T_n(\theta^N) = \sum_{\{n_v\}_{v\leqslant v_0}:\, n_{v_0'}=n}\ \sum_{\substack{\{P_v\}_{v\leqslant v_0}:\\ \|P^{\rm ren}_{v_0}\|=0}} \frac{1}{|\Lambda|}\int_\Lambda d\underline{X}_{v_0} V^{(n)}_{0,0}$$
$$\times \left(\theta^{N,n}_{\rm lab}, P^{\rm ren}_{v_0} = \emptyset, \{P_v\}_{v<v_0}, \underline{X}_{v_0}\right).$$

Using right hand side of (4.44) with $\|P^*_{v_0}\| = |P_{v_0}| = 0$ to bound

$$\left|V^{(n)}_{0,0}\left(\theta^{N,n}_{\rm lab}, P^{\rm ren}_{v_0} = \emptyset, \{P_v\}_{v<v_0}, \underline{X}_{v_0}\right)\right|$$

and then performing an analysis completely analogous to the one presented above for δb_n, it is an easy exercise to check that $|T_n(\theta^N)| \leqslant C^N\lambda^N L^{-2n}$, hence $|T_n| \leqslant L^{-2n}D(\lambda)$ with $D(\lambda)$ analytic in λ, with a lower bound for the convergence radius independent from the volume, and $D(\lambda) = O(\lambda^2)$. In relation to t_n, recalling its definition (3.29), observing that, by definition (3.30)

$$\left|C^{(\geqslant n)}(x-y)\right| \leqslant C\, L^{-2n}\, e^{-\alpha L^{-n}|x-y|}$$

(C and α positive contants), and using (4.23), (4.67), (4.68), we obtain

$$t_n \leqslant \sum_{N=1}^{\infty} D^N \lambda^{2N} L^{-n(N-\varepsilon+3)}$$

concluding the proof of (3.42).

5. POINTWISE BOUNDS ON THE KERNELS OF EFFECTIVE POTENTIAL

5.1. Preliminaries

Now we turn to the pointwise estimates. We need the following definition.

DEFINITION 5.1. – *Given $\theta_{\rm lab}^{N,n}$ and a set of space-time coordinates $\underline{Y} \subset \underline{X}_{v_0}$, a non-trivial vertex $w \in \theta_{\rm lab}^{N,n}$ is called fixed scale label vertex (f.s.l. vertex) respect to the set $\underline{Y}$, if $|\underline{X}_w \cap \underline{Y}| > |\underline{X}_{\tilde{w}} \cap \underline{Y}|$ for all $\tilde{w} < w$.*

Further on, we will indicate with the symbol v a generic f.s.l. vertex for a given $\underline{Y}$, and {v} will represent the set of all f.s.l. vertices of $\theta_{\rm lab}^{N,n}$. The set {v} is naturally ordered by the tolological structure of $\theta_{\rm lab}^{N,n}$ (since each v is a non-trivial vertex of $\theta_{\rm lab}^{N,n}$) and we denote as v_0 the maximum of {v}. We also define, for any f.s.l. vertex v of $\theta_{\rm lab}^{N,n}$

$$\begin{aligned}\bar{s}_{\rm v} = \#\text{ of vertices } \mathrm{v}^i \text{ in the set } \{\mathrm{v}^1, \ldots, \mathrm{v}^{s_{\rm v}}\}\\ \text{such that } |\underline{X}_{\mathrm{v}^i} \cap \underline{Y}| \geqslant 1.\end{aligned} \tag{5.1}$$

Intuitively, in a f.s.l. vertex v of $\theta_{\rm lab}^{N,n}$, $\bar{s}_{\rm v}$ points of the set $\underline{Y}$ become connected at the scale $n_{\rm v}$. Note that $\mathrm{v}_0 \leqslant v_0$, i.e., the non-trivial vertex on $\theta_{\rm lab}^{N,n}$ at which *all* points in $\underline{Y}$ becomes connected can be smaller than v_0. Tipically, the set $\underline{Y}$ will be $\underline{X}_{P_{v_0}^{\rm ren}}$ (and so it contains $p+k$ elements), or $\underline{X}_{P_{v_0}}$ (with p elements).

DEFINITION 5.2. – *Given $\theta_{\rm lab}^{N,n}$, a set of space-time coordinates $\underline{Y} \subset \underline{X}_{v_0}$ and a connected tree graph $\tau^*(\underline{Y}) \equiv \{\rho_1, \ldots, \rho_{|\underline{Y}|-1}\}$ in $\underline{Y}$, we say* *that* $\tau^*(\underline{Y})$ *is compatible with the topological structure of* $\theta_{\text{lab}}^{N,n}$ *if, for all f.s.l. vertices* $\text{v} \in \theta_{\text{lab}}^{N,n}$ *repect to* $\underline{Y}$, τ_{v}^* *is a connected tree graph between the points* $\underline{X}_{\text{v}} \cap \underline{Y}$, *where* τ_{v}^* *is defined as the graph between the points* $\underline{X}_{\text{v}} \cap \underline{Y}$ *obtained from* $\tau^*(\underline{Y})$ *by cancelling all links* $\rho = \{x, y\}$ *of* $\tau^*(\underline{Y})$ *such that* $\{x, y\} \cap [\underline{Y} \backslash (\underline{X}_{\text{v}} \cap \underline{Y})] \neq \emptyset$ *(compare with Definition* 4.6).

Remark. – Recall that, given a tree $\tau \in \underline{X}_{v_0}$ compatible with $\theta_{\text{lab}}^{N,n}$, to each link ρ of τ is associated a scale label $n_\rho = n_v$, where v_ρ is the minimum non-trivial vertex in $\theta_{\text{lab}}^{N,n}$ such that $\rho \subset \underline{X}_v$. Analogously, to each link ρ of a tree $\tau^*(\underline{Y})$ compatible with $\theta_{\text{lab}}^{N,n}$ we associate a scale label $n_\rho = n_{\text{v}_\rho}$, where v is the minimum f.s.l. vertex in $\theta_{\text{lab}}^{N,n}$ respect to $\underline{Y}$ such that $\rho \subset \underline{Y} \cap \underline{X}_{\text{v}}$ and

$$\prod_{\rho \in \tau^*(\underline{Y})} L^{-n_\rho} = \prod_{\text{v} \leqslant \text{v}_0} L^{-n_{\text{v}}(\bar{s}_{\text{v}}-1)}. \tag{5.2}$$

We now enunciate a result which will be useful in order to get the pointwise bounds.

LEMMA 5.3. – *With the notations of Section* 4, *Definitions* 5.1 *and* 5.2, *let* $\tau \in \mathcal{T}_{\underline{X}_{v_0}}$ *(i.e.,* τ *is a connected tree graph in* $\underline{X}_{v_0}$*) be compatible with a given* $\theta_{\text{lab}}^{N,n}$ *and let*

$$f_\tau(\underline{X}_{v_0}) = \prod_{\rho \in \tau} \mathrm{e}^{-\alpha L^{-n_\rho} |\rho|},$$

where, for $\rho \in \tau$, $n_\rho = n_{v_\rho}$ *with* v_ρ *being the minimum non-trivial vertes in* $\theta_{\text{lab}}^{N,n}$ *such that such that* $\rho \subset \underline{X}_v$. *Then we can find a constant C such that the following inequality holds*

$$\int d(\underline{X}_{v_0} \backslash \underline{X}_{P_{v_0}^{\text{ren}}}) f_\tau(\underline{X}_{v_0})$$
$$\leqslant C^N \prod_{v \leqslant v_0} L^{3n_v(s_v-1)} \left[\prod_{\rho \in \tau^*(\tau, \underline{X}_{P_{v_0}^{\text{ren}}})} L^{-3n_\rho} \mathrm{e}^{-\frac{\alpha}{4} L^{-n_\rho} |\rho|} \right], \tag{5.3}$$

where $\tau^*(\tau, \underline{X}_{P_{v_0}^{\text{ren}}})$ *is a tree graph between points* $\underline{X}_{P_{v_0}^{\text{ren}}}$ *compatible with* $\theta_{\text{lab}}^{N,n}$ *in the sense of Definition* 5.2, *which can be obtained univocally from* τ, *and, for* $\rho \in \tau^*(\tau, \underline{X}_{P_{v_0}^{\text{ren}}})$, $n_\rho = n_{\text{v}_\rho}$ *where* v_ρ *is the minimum f.s.l. vertex of* $\theta_{\text{lab}}^{N,n}$ *(respect to* $\underline{X}_{P_{v_0}^{\text{ren}}}$*) such that* $\rho \subset \underline{X}_{\text{v}} \cap \underline{X}_{P_{v_0}^{\text{ren}}}$.

Proof. – First observe that, since $\tau^*(\tau, \underline{X}_{P^{\text{ren}}_{v_0}})$ is compatible with the topological structure of $\theta^{N,n}_{\text{lab}}$, by Definition 5.2,

$$\prod_{\rho\in\tau^*(\tau,\underline{X}_{P^{\text{ren}}_{v_0}})} L^{-3n_\rho} = \prod_{\mathrm{v}\leqslant \mathrm{v}_0} L^{-3n_{\mathrm{v}_h}(\bar{s}_{\mathrm{v}_h}-1)}, \tag{5.4}$$

where $\{\mathrm{v}\}$ is the set of f.s.l. vertices of $\theta^{N,n}_{\text{lab}}$ when $\underline{Y} = \underline{X}_{P^{\text{ren}}_{v_0}}$.

Then we write

$$f_\tau(\underline{X}_{v_0}) = \left[\prod_{\rho\in\tau} \mathrm{e}^{-\frac{\alpha}{2}L^{-n_\rho}|\rho|}\right]\left[\prod_{\rho\in\tau} \mathrm{e}^{-\frac{\alpha}{2}L^{-n_\rho}|\rho|}\right], \tag{5.5}$$

hence,

$$\int d(\underline{X}_{v_0}\backslash\underline{X}_{P^{\text{ren}}_{v_0}}) f_\tau(\underline{X}_{v_0})$$
$$\leqslant \sup_{\underline{X}_{v_0}\backslash\underline{X}_{P^{\text{ren}}_{v_0}}} \left[\prod_{\rho\in\tau} \mathrm{e}^{-\frac{\alpha}{2}L^{-n_\rho}|\rho|}\right] \int d(\underline{X}_{v_0}\backslash\underline{X}_{P^{\text{ren}}_{v_0}}) \left[\prod_{\rho\in\tau} \mathrm{e}^{-\frac{\alpha}{2}L^{-n_\rho}|\rho|}\right], \tag{5.6}$$

where the sup is taken over all possible configurations of the points in $\underline{X}_{v_0}\backslash\underline{X}_{P^{\text{ren}}_{v_0}}$ for a fixed configuration of the points in $\underline{X}_{P^{\text{ren}}_{v_0}}$ so that the non-integrated term in the left hand side of formula above is a function only of the space coordinates in $\underline{X}_{P^{\text{ren}}_{v_0}}$.

Now, it is not difficult to see that

$$\int d(\underline{X}_{v_0}\backslash\underline{X}_{P^{\text{ren}}_{v_0}}) \left[\prod_{\rho\in\tau} \mathrm{e}^{-\frac{\alpha}{2}L^{-n_\rho}|\rho|}\right]$$
$$\leqslant C^N \prod_{v\leqslant v_0} L^{3n_v(s_v-1)} \prod_{\mathrm{v}\leqslant\mathrm{v}_0} L^{-3n_\mathrm{v}(\bar{s}_\mathrm{v}-1)}, \tag{5.7}$$

where $\{\mathrm{v}\}$ is the set the f.s.l. vertices of $\theta^{N,n}_{\text{lab}}$ when $\underline{Y} = \underline{X}_{P^{\text{ren}}_{v_0}}$. As a matter of fact, consider the tree graph $\tau = \{\rho_1, \ldots, \rho_{N-1}\}$. Since τ is compatible with $\theta^{N,n}_{\text{lab}}$, for any $v \in \theta^{N,n}_{\text{lab}}$ we know that $s_v - 1$ links in τ are at scale n_v, so that $\sum_{v\leqslant v_0}(s_v - 1) = N - 1$. If v is also a f.s.l vertex, i.e., $v \equiv \mathrm{v}$, then we can individuate $\bar{s}_\mathrm{v} - 1$ links in τ among the $s_v - 1$ links, which realize the connection between s_v points in $\underline{X}_{P^{\text{ren}}_{v_0}}$ at scale n_v. The total number of these links is $\sum_{\mathrm{v}\leqslant\mathrm{v}_0}(\bar{s}_\mathrm{v} - 1) = p + k - 1$. Let us cut out these links from τ, obtaining thus a new graph, say $\tilde{\tau}$, between the points in $\underline{X}_{v_0}$; the graph $\tilde{\tau}$ has exactly $N - p - k = |\underline{X}_{v_0}\backslash\underline{X}_{P^{\text{ren}}_{v_0}}|$ links and, by construction, each one of the $N-p-k$ space-time coordinates $y \in \underline{X}_{v_0} \backslash \underline{X}_{P_{v_0}^{\text{ren}}}$ appears in at least one link of $\tilde{\tau}$. Thus using translational invariance we have

$$\int d(\underline{X}_{v_0} \backslash \underline{X}_{P_{v_0}^{\text{ren}}}) \left[\prod_{\rho \in \tilde{\tau}} \mathrm{e}^{-\frac{\alpha}{2} L^{-n_\rho} |\rho|} \right]$$
$$= \left[\frac{2}{\alpha} \int d^3 z \, \mathrm{e}^{-|z|} \right]^{N-p-k} \prod_{\rho \in \tilde{\tau}} L^{3n_\rho}$$
$$\leqslant C^N \prod_{v \leqslant v_0} L^{3n_v(s_v - 1)} \prod_{\mathrm{v} \leqslant \mathrm{v}_0} L^{-3n_{\mathrm{v}}(\bar{s}_{\mathrm{v}} - 1)} \tag{5.8}$$

and since $\prod_{\rho \in \tilde{\tau}} \mathrm{e}^{-\frac{\alpha}{2} L^{-n_\rho} |\rho|} \leqslant \prod_{\rho \in \tau} \mathrm{e}^{-\frac{\alpha}{2} L^{-n_\rho} |\rho|}$, (5.7) follows.

We now note that, given $n_1 \geqslant n_2$ non-negative integers and x_1, x_2, y points in $\mathbb{R}^3$, the following inequality holds pointwisely

$$\sup_{y \in \mathbb{R}^3} \mathrm{e}^{-\frac{\alpha}{2} L^{-n_1} |x_1 - y|} \mathrm{e}^{-\frac{\alpha}{2} L^{-n_2} |x_2 - y|} \leqslant \mathrm{e}^{-\frac{\alpha}{2} L^{-n_1} |x_1 - x_2|}. \tag{5.9}$$

So, considering the non-integrated term in the right hand side of (5.6), by using repeatedly the inequality (5.9), we can bound it as

$$\sup_{\underline{X}_{v_0} \backslash \underline{X}_{P_{v_0}^{\text{ren}}}} \prod_{\rho \in \tau} \mathrm{e}^{-\frac{\alpha}{2} L^{-n_\rho} |\rho|} \leqslant \prod_{\rho \in \tau^*(\underline{X}_{P_{v_0}^{\text{ren}}})} \mathrm{e}^{-\frac{\alpha}{4} L^{-n_\rho} |\rho|}, \tag{5.10}$$

where τ^* is a connected tree graph between points $\underline{X}_{P_{v_0}^{\text{ren}}}$, and for a given link $\rho = \{x, x'\} \in \tau^*$, calling τ_ρ the unique path of τ which joins x to x', $n_\rho = \max_{\rho' \in \tau_\rho} \{n_{\rho'}\}$. It is now not difficult to convince oneself that we can always chose τ^* in such way that it is compatible with $\theta_{\text{lab}}^{N,n}$. Of course, the association $\tau \to \tau^*$ is not unique in the sense that there are various τ^* compatible with $\theta_{\text{lab}}^{N,n}$ which satisfy (5.10), but as we will see later, what is really important in relation to the structure of τ^* is just the fact that τ^* is compatible with $\theta_{\text{lab}}^{N,n}$. So, for each τ, we take just one of the possible τ^* compatible with $\theta_{\text{lab}}^{N,n}$ which satisfy (5.10) and call it $\tau^*(\tau, \underline{X}_{P_{v_0}^{\text{ren}}})$. Inserting (5.7) and (5.10) in (5.6), we get (5.3) $\square$

Let us analyze in detail the structure of $\tau^*(\tau, \underline{X}_{P_{v_0}^{\text{ren}}})$. It involves $|\underline{X}_{P_{v_0}}| + k$ points. Here k is the number of fields of type $\partial^2 \psi$ in $\widetilde{\psi}(P_{v_0}^{\text{ren}})$, and $|\underline{X}_{P_{v_0}}|$ is the number of space-time points in the set $\underline{X}_{P_{v_0}}$ (which can be at most $|P_{v_0}|$, but, of course, can be less if some fields in P_{v_0} are sitting in the same space-time point). We write the set of space-time points in the following way

$$\underline{X}_{P_{v_0}^{\text{ren}}} = \{z_1, z_2, \dots, z_l, x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k\} \tag{5.11}$$

and make explicit $\widetilde{\psi}(P_{v_0}^{\text{ren}})$, given by (after eventually some permutations of the fields, which can just affect it by a signal)

$$\widetilde{\psi}(P_{v_0}^{\text{ren}}) = \overline{\psi}_{z_1} \cdots \overline{\psi}_{z_s} \psi_{z_{s+1}} \cdots \psi_{z_l} \partial^2 \psi_{x_1 y_1(t_1)} \cdots \partial^2 \psi_{x_k y_k(t_k)}. \tag{5.12}$$

Points z are attached to fields of $P_{v_0}^{\text{ren}}$ which did not suffer renormalization (in particular, $\overline{\psi}$ cannot suffer renormalization). Points x_i $(i = 1, 2, \dots, k)$ belong to fields $\overline{\psi}$ which appears in some cluster in a couple $\overline{\psi}_{x_i}\psi_{y_i}$. In other words, there must be a v in $\theta_{\text{lab}}^{N,n}$ such that $\widetilde{\psi}(P_v^{\text{ren}}) = \overline{\psi}_{x_i}\psi_{y_i}$. This monomial have been replaced by the renormalized one

$$\widetilde{\psi}(P_v^*) = \mathcal{R}\widetilde{\psi}(P_v^{\text{ren}}) = \overline{\psi}_{x_i} \partial^2 \psi_{x_i y_i(t_i)}.$$

Then the field $\overline{\psi}_{x_i}$ has been contracted, but its space-time coordinate still survives in $P_{v_0}^{\text{ren}}$. Points y_i belongs to fields ψ_{y_i} which are in $P_{v_0}^{\text{ren}}$, but they have been transformed by the $\mathcal{R}$ operation into fields $\partial^2 \psi_{x_i y_i(t_i)}$.

From the discussion above, the definition of renormalization and the cluster structure of $\theta_{\text{lab}}^{N,n}$, the set of points (5.11) has the following properties:

(i) Some points $z_1, z_2, \dots, z_l$ may be coincident.

(ii) All points $x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k$ are distinct.

(iii) $\{z_1, z_2, \dots, z_l\} \cap \{x_1, x_2, \dots, x_k, y_1, y_2\} = 0$, i.e., any z is distinct from any x and any y.

Moreover, due *only* to the fact that $\tau^*(\tau, \underline{X}_{P_{v_0}^{\text{ren}}})$ is compatible with $\theta_{\text{lab}}^{N,n}$, $\tau^*(\tau, \underline{X}_{P_{v_0}^{\text{ren}}})$ has the following properties:

(1) If x_r, y_r is a couple in $\underline{X}_{P_{v_0}^{\text{ren}}}$ which forms the interpolated point $x_r y_r(t_r)$ and v_r is the non-trivial vertex of $\theta_{\text{lab}}^{N,n}$ for which $\widetilde{\psi}(P_{v_r}^{\text{ren}}) = \overline{\psi}_{x_r}\psi_{y_r}$, then one of the links ρ in $\tau^*(\tau, \underline{X}_{P_{v_0}^{\text{ren}}})$ is surely the line $\rho_r = \{x_r, y_r\}$ and $n_{\rho_r} \leqslant n_{v_r}$;

(2) Each link $\{x_r, y_r\}$ in $\tau^*(\tau, \underline{X}_{P_{v_0}^{\text{ren}}})$ is connected to the rest of the tree $\tau^*(\tau, \underline{X}_{P_{v_0}^{\text{ren}}})$ just by a link of the type $\{x_r, z_j\}$, for some z_j and *nothing* but x_r is connected to y_r. I.e., the vertices of $\tau^*(\tau, \underline{X}_{P_{v_0}^{\text{ren}}})$ sitting in points of the type y_r have incidence number equal to one. This is a trivial consequence of the fact that in the monomial $\overline{\psi}_{x_r}\psi_{y_r} \to \overline{\psi}_{x_r} \partial^2 \psi_{x_r y_r(t_r)}$ just the field $\overline{\psi}_{x_r}$ can be contracted, and

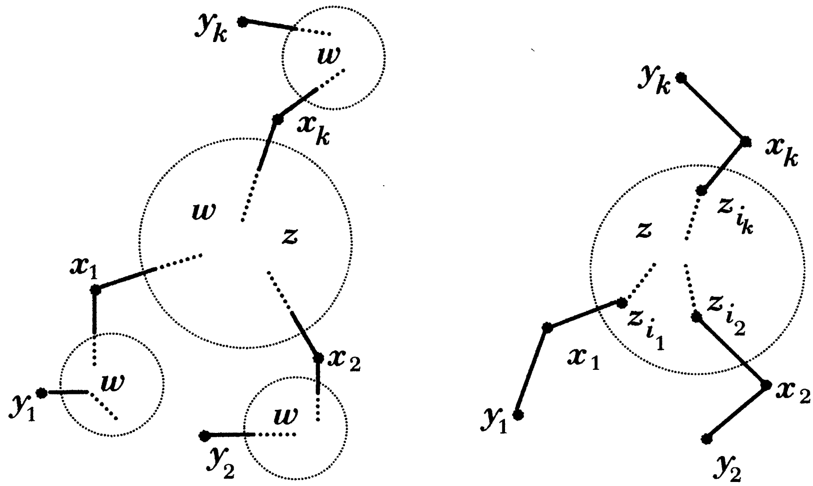

Fig. 3. The structure of the tree graphs τ and $\tau^*(\tau, \underline{X}_{P^{\text{ren}}_{v_0}})$.

it cannot contract with another $\overline{\psi}_{x_s}$ field (even if the contraction is modulo fields ψ_w, where w represents a generic point in the set $\underline{X}_{v_0} \backslash \underline{X}_{P^{\text{ren}}_{v_0}}$).

These properties yields the following structure of $\tau^*(\tau, \underline{X}_{P^{\text{ren}}_{v_0}})$ (see Fig. 3)

$$\begin{aligned}\tau^*(\tau, \underline{X}_{P^{\text{ren}}_{v_0}}) = \big\{&(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k), (x_1, z_{i_1}), (x_2, z_{i_2}), \ldots,\\ &(x, z_{i_k}),\ \text{and other links between } z \text{ points such that}\\ &\tau^*(\tau, \underline{X}_{P^{\text{ren}}_{v_0}}) \text{ is a connected tree graph}\big\}.\end{aligned}$$

Remark. – Since each y_i $(i = 1, \ldots, k)$ is connected to $\tau^*(\tau, \underline{X}_{P^{\text{ren}}_{v_0}})$ just by a link, the subset of $\tau^*(\tau, \underline{X}_{P^{\text{ren}}_{v_0}})$ obtained by eliminating all vertices y and all links $\{x, y\}$ is still a tree between the points x and z; let us call this tree $\tau^{**}(\tau)$.

Using the bound (4.44) for $V^{(n)}_{2m,k}$, the definition (4.9) of $K^{(n)}_{2m,k}$ and the bound (5.3) we get

$$\begin{aligned}&\big|K^{(n)}_{2m,k}(\theta^{N,n}_{\text{lab}}, P^{\text{ren}}_{v_0}, \{P_v\}_{v<v_0}, \underline{X}_{P^{\text{ren}}_{v_0}})\big|\\ &\leqslant C^N \lambda^N L^{-n_{v_0}(3-\|P^{\text{ren}}_{v_0}\|)} \prod_{v<v_0} L^{(n_{v'}-n_v)(3-\|P^*_v\|)} \prod_{i=1}^{N} L^{-n_i} \Bigg[\prod_{v\leqslant v_0} \frac{1}{s_v!}\Bigg]\end{aligned}$$

$$\times \sum_{\substack{\tau \in \mathcal{T}_{\underline{X}_{v_0}} \\ \tau \text{ comp } \theta_{\text{lab}}^{N,n}}} F_\tau(x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k, z_1, \dots, z_l), \quad (5.13)$$

where

$$F_\tau(x_1, \dots, x_k, y_1, \dots, y_k, z_1, \dots, z_l) = \prod_{r=1}^{k} \left[L^{-n_{v_r}}(x_r - y_r)\right]^2 \prod_{\rho \in \tau^*(\tau, \underline{X}_{P_{v_0}^{\text{ren}}})} L^{-3n_\rho} \, \mathrm{e}^{-\frac{\alpha}{4} L^{-n_\rho} |\rho|} \quad (5.14)$$

and recall that v_r is the vertex of $\theta_{\text{lab}}^{N,n}$ where the factor $[\xi_{v_r}]^{z_{v_r}} = [L^{-n_{v_r}}(x_r - y_r)]^2$ is generated by the renormalization operation $\mathcal{R}$.

From (4.10), the kernel of the effective potential $W_{2m,k}^{(n)}$ is

$$W_{2m,k}^{(n)}(\theta_{\text{lab}}^{N,n}, P_{v_0}^{\text{ren}}, \{P_v\}_{v<v_0}, \underline{X}_{P_{v_0}}) = \int_0^1 \frac{dt_1}{t_1^3} \cdots \int_0^1 \frac{dt_k}{t_k^3} \int dx_1 \cdots dx_k \times K_{2m,k}^{(n)}(\theta_{\text{lab}}^{N,n}, P_{v_0}^{\text{ren}}, \{P_v\}_{v<v_0}, \underline{X}'_{P_{v_0}^{\text{ren}}}), \quad (5.15)$$

where $\underline{X}'_{P_{v_0}^{\text{ren}}}$ is the set of space-time points obtained from $\underline{X}_{P_{v_0}^{\text{ren}}}$ replacing y_r by $x_r + \frac{1}{t_r}(y_r - x_r)$ (for $r = 1, 2, \dots, k$).

Using (5.13), an estimate for this kernel is

$$\left|W_{2m,k}^{(n)}(\theta_{\text{lab}}^{N,n}, P_{v_0}^{\text{ren}}, \{P_v\}_{v<v_0}, \underline{X}_{P_{v_0}})\right| \leqslant C^N \lambda^N L^{-n_{v_0}(3-\|P_{v_0}^{\text{ren}}\|)} \prod_{v<v_0} L^{(n_{v'}-n_v)(3-\|P_v^*\|)} \prod_{i=1}^{N} L^{-n_i} \prod_{v \leqslant v_0} \frac{1}{s_v!}$$
$$\times \sum_{\substack{\tau \in \mathcal{T}_{\underline{X}_{v_0}} \\ \tau \text{ comp } \theta_{\text{lab}}^{N,n}}} \int dx_1 \cdots dx_k \int_0^1 \frac{dt_1}{t_1^3} \cdots \int_0^1 \frac{dt_k}{t_k^3}$$
$$\times F_\tau\left(x_1, \dots, x_k, x_1 + \frac{y_1 - x_1}{t_1}, \dots, x_k + \frac{y_k - x_k}{t_k}, z_1, \dots, z_l\right). \quad (5.16)$$

But, by (5.14) and the structure of $\tau^*(\tau)$ and still recalling the definition of $\tau^{**}(\tau)$ and that $n_{v_r} \geqslant n_{v_{\rho_r}}$, where v_r is the vertex at which the factor ξ_r is created while v_{ρ_r} is the vertex where x_r, y_r become connected, we have

$$
\begin{aligned}
F_\tau&\left(x_1,\ldots,x_k,x_1+\frac{y_1-x_1}{t_1},\ldots,x_k+\frac{y_k-x_k}{t_k},z_1,\ldots,z_l\right)\\
&\leqslant\prod_{r=1}^{k}L^{-3n_{v_r}}\frac{1}{t_r^2}\left[L^{-n_{v_r}}(x_r-y_r)\right]^2\mathrm{e}^{-\frac{\alpha}{4}L^{-n_{v_r}}\frac{|x_r-y_r|}{t_r}}\\
&\quad\times\prod_{\rho\in\tau^{**}(\tau)}L^{-n_\rho}\,\mathrm{e}^{-\frac{\alpha}{4}L^{-n_\rho}|\rho|}.
\end{aligned}
\tag{5.17}
$$

Using again the structure of $\tau^{**}(\tau)$

$$
\begin{aligned}
&\prod_{\rho\in\tau^{**}(\tau)}L^{-3n_\rho}\,\mathrm{e}^{-\frac{\alpha}{4}L^{-n_\rho}|\rho|}\\
&=\prod_{r=1}^{k}L^{-3n_{\rho_{i_r}}}\,\mathrm{e}^{-\frac{\alpha}{4}L^{-n_{\rho_{i_r}}}|x_r-z_{i_r}|}\prod_{\rho\in\tau^{**}(\tau):\ x_r\notin\rho}L^{-3n_\rho}\,\mathrm{e}^{-\frac{\alpha}{4}L^{-n_\rho}|\rho|},
\end{aligned}
\tag{5.18}
$$

where the second productory in the right hand side of (5.18) above is over the links ρ in $\tau^{**}(\tau)$ of the type $\{z,z'\}$ and $\rho_{i_r}=\{x_r,z_{i_r}\}$.

Observe that here we are using the fact that each vertex x_r of $\tau^{**}(\tau)$ is connected to the corresponding y_r, and then to a z, but *never* to another $x_{r'}$.

Now note that

$$
\begin{aligned}
\prod_{r=1}^{k}\mathrm{e}^{-\frac{\alpha}{4}L^{-n_{\rho_{i_r}}}|x_r-z_{i_r}|}&=\prod_{r=1}^{k}\mathrm{e}^{-\frac{\alpha}{4}L^{-n_{\rho_{i_r}}}|x_r-y_r+y_r-z_{i_r}|}\\
&\leqslant\prod_{r=1}^{k}\mathrm{e}^{+\frac{\alpha}{4}L^{-n_{\rho_{i_r}}}|x_r-y_r|}\,\mathrm{e}^{-\frac{\alpha}{4}L^{-n_{\rho_{i_r}}}|y_r-z_{i_r}|}.
\end{aligned}
\tag{5.19}
$$

Observe that the graph obtained from $\tau^{**}(\tau)$ replacing every x_r by the corresponding y_r is a connected tree graph in $\underline{X}_{P_{v_0}}$ which is, by construction, compatible with $\theta_{\mathrm{lab}}^{N,n}$. We denote this tree graph as $\tau^*(\tau,\underline{X}_{P_{v_0}})$ (see Fig. 4).

Hence, we have

$$
\begin{aligned}
&\prod_{\rho\in\tau^{**}(\tau)}L^{-3n_\rho}\,\mathrm{e}^{-\frac{\alpha}{4}L^{-n_\rho}|\rho|}\\
&\leqslant\prod_{r=1}^{k}\mathrm{e}^{+\frac{\alpha}{4}L^{-n_{\rho_{i_r}}}|x_r-y_r|}\prod_{\rho\in\tau^*(\tau,\underline{X}_{P_{v_0}})}L^{-3n_\rho}\,\mathrm{e}^{-\frac{\alpha}{4}L^{-n_\rho}|\rho|}.
\end{aligned}
\tag{5.20}
$$

And so,

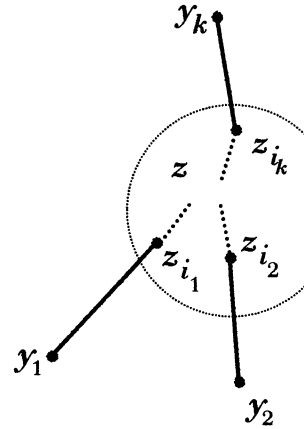

Fig. 4. The structure of the tree graph $\tau^*(\tau, \underline{X}_{P_{v_0}})$.

$$F_\tau\left(x_1, \ldots, x_k, x_1 + \frac{y_1 - x_1}{t_1}, \ldots, x_k + \frac{y_k - x_k}{t_k}, z_1, \ldots, z_l\right)$$
$$\leqslant \prod_{r=1}^{k} L^{-3n_{v_r}} t_r^{-2} \left[L^{-n_{v_r}}(x_r - y_r)\right]^2 e^{-\frac{\alpha}{4}(t_r^{-1} L^{-n_{v_r}}|x_r - y_r| - L^{-n_{\rho_{i_r}}}|x_r - y_r|)}$$
$$\times \prod_{\rho \in \tau^*(\tau, \underline{X}_{P_{v_0}})} L^{-3n_\rho} e^{-\frac{\alpha}{4} L^{-n_\rho}|\rho|}. \tag{5.21}$$

Observe that $n_{\rho_{i_r}} > n_{v_r}$, since $n_{\rho_{i_r}}$ is the scale where x_r becomes connected with z_{i_r}, which is a scale greater or equal to the scale where $\overline{\psi}_{x_r}$ is contracted, while n_{v_r} is the scale where the monomial $\overline{\psi}_{x_r} \psi_{y_r}$ appears. Thus, considering also that $0 \leqslant t_r \leqslant 1$,

$$t_r^{-1} L^{-n_{v_r}}|x_r - y_r| - L^{-n_{\rho_{i_r}}}|x_r - y_r| \geqslant t_r^{-1} L^{-n_{v_r}}|x_r - y_r|\left[1 - L^{-1}\right]. \tag{5.22}$$

Performing the change of variables

$$s_r = t_r^{-1}, \qquad \xi_r = L^{-n_{v_r}}(x_r - y_r), \tag{5.23}$$

we get

$$\int dx_1 \cdots dx_k \int_0^1 \frac{dt_1}{t_1^3} \cdots \int_0^1 \frac{dt_k}{t_k^3}$$
$$\times F_\tau\left(x_1, \ldots, x_k, x_1 + \frac{y_1 - x_1}{t_1}, \ldots, x_k + \frac{y_k - x_k}{t_k}, z_1, \ldots, z_l\right)$$

$$\leqslant \left(\prod_{r=1}^{k} \int d^3\xi_r \int_1^\infty ds_r s_r^3 \xi_r^2 \,\mathrm{e}^{-\frac{\alpha}{4}(1-L^{-1})s_r|\xi_r|} \right)$$
$$\times \prod_{\rho\in\tau^*(\tau,\underline{X}_{P_{v_0}})} L^{-3n_\rho}\,\mathrm{e}^{-\frac{\alpha}{4}L^{-n_\rho}|\rho|}.$$
$$\leqslant C^k \prod_{\rho\in\tau^*(\tau,\underline{X}_{P_{v_0}})} L^{-3n_\rho}\,\mathrm{e}^{-\frac{\alpha}{4}L^{-n_\rho}|\rho|} \tag{5.24}$$

The last inequality in the right hand side of (5.24) follows from the fact that

$$\int d^3\xi_r \int_1^\infty ds_r s_r^3 \xi_r^2 \,\mathrm{e}^{-\frac{\alpha}{4}(1-L^{-1})s_r|\xi_r|}$$
$$= 4\pi \int_0^\infty u^4 du \int_1^\infty dv v^3\,\mathrm{e}^{-\bar{\alpha}uv} \leqslant C, \tag{5.25}$$

where $\bar{\alpha} = \frac{\alpha}{4}(1 - L^{-1})$. Observe how the integration over $d^3\xi$ takes care of the outward U.V. divergence of Kernels (4.10) due to the dt_i-integrals.

Thus, the bound on the kernels of the effective potential becomes

$$\left| W_{2m,k}^{(n)}\left(\theta_{\mathrm{lab}}^{N,n}, P_{v_0}^{\mathrm{ren}}, \{P_v\}_{v<v_0}^{jk}, \underline{X}_{P_{v_0}}\right)\right|$$
$$\leqslant C^N \lambda^N L^{-n_{v_0}(3-\|P_{v_0}^{\mathrm{ren}}\|)} \prod_{v<v_0} L^{(n_{v'}-n_v)(3-\|P_v^*\|)} \prod_{i=1}^{N} L^{-n_i}$$
$$\times \left[\prod_{v\leqslant v_0} \frac{1}{s_v!}\right] \sum_{\substack{\tau\in\mathcal{T}_{\underline{X}_{v_0}} \\ \tau \text{ comp } \theta_{\mathrm{lab}}^{N,n}}} \prod_{\rho\in\tau^*(\tau,\underline{X}_{P_{v_0}})} L^{-3n_\rho}\,\mathrm{e}^{-\frac{\alpha}{4}L^{-n_\rho}|\rho|}. \tag{5.26}$$

Remark that, by definition of $\tau^*(\tau, \underline{X}_{P_{v_0}})$, we have

$$\prod_{\rho\in\tau^*(\tau,\underline{X}_{P_{v_0}})} L^{-3n_\rho} = \prod_{\mathrm{v}\leqslant\mathrm{v}_0} L^{-3(\bar{s}_{\mathrm{v}}-1)n_{\mathrm{v}}}, \tag{5.27}$$

where now {v} is the set of f.s.l. vertices when (see Definition 5.1) $\underline{Y} = \underline{X}_{P_{v_0}}$.

5.2. Proof of part B of Theorem 3.1

Consider the factor $D(P_v) = -(3 - \|P_v^*\|)$ which appears in the formula (5.26). We can write the set P_v^* as

$$P_v^* = \widetilde{P}_v^* \cup R_v^*, \tag{5.28}$$

where $\widetilde{P}_v^*$ is the subset of P_v^* constructed with those fields which are among the P_{v_0} ones (eventually modified by the renormalization getting some double derivative; and so $\widetilde{P}_v$ will indicate the bare subset of P_{v_0} contained in P_v). $R_v^* = P_v^* \backslash \widetilde{P}_v^*$ is the subset of P_v^* made by those fields which are contracted in some vertex $w > v$, i.e., those which are not among the P_{v_0} ones. Using (5.28), we define

$$D(P_v) = D^{(1)}(R_v) + D^{(2)}(\widetilde{P}_v), \tag{5.29}$$

where

$$D^{(1)}(R_v) = \begin{cases} -\|R_v^*\| & \text{if } \|\widetilde{P}_v^*\| \neq 0, \\ -(3 - \|R_v^*\|) & \text{if } \|\widetilde{P}_v^*\| = 0, \end{cases} \tag{5.30}$$

$$D^{(2)}(\widetilde{P}_v) = \begin{cases} 0 & \text{if } \|\widetilde{P}_v^*\| = 0, \\ -(3 - \|\widetilde{P}_v^*\|) & \text{if } \|\widetilde{P}_v^*\| \neq 0. \end{cases} \tag{5.31}$$

Observe that $D^{(1)}(R_v)$ satisfies the hypothesis of Lemma 4.8, i.e.,

$$D^{(1)}(R_v) \geqslant 1 \quad \text{and} \quad D^{(1)}(R_v) \geqslant \varepsilon |R_v| \quad \text{for some } \varepsilon > 0. \tag{5.32}$$

Using the definitions (5.28)–(5.31), the productory $\prod_{v<v_0} L^{(n_{v'}-n_v)(3-\|P_v^*\|)}$ appearing in the right hand side of (5.26) can be rewritten as

$$\prod_{v<v_0} L^{(n_{v'}-n_v)(3-\|P_v^*\|)} = \prod_{v<v_0} L^{-(n_{v'}-n_v)D^{(1)}(R_v)} \prod_{v<v_0} L^{-(n_{v'}-n_v)D^{(2)}(\widetilde{P}_v)}. \tag{5.33}$$

Using (5.30), (5.32) and Lemma 4.8, we bound the first productory in the right hand side of (5.33) by

$$\prod_{v<v_0} L^{-(n_{v'}-n_v)D^{(1)}(R_v)} \leqslant C^N \prod_{v<v_0} L^{-\varepsilon|R_v|} \prod_{v<v_0} L^{-(n_{v'}-n_v)}.$$

Towards the pointwise bounds of the effective potential, we have to sum the left hand side of (5.26) over all possible scale labels in the non-trivial vertices of $\theta_{\text{lab}}^{N,n}$. We make this sum keeping fixed, in a first moment, the scale labels $\{n_{\text{v}}\}$ of the f.s.l vertices (whence the name!) $\{\text{v}\}$ of $\theta_{\text{lab}}^{N,n}$ for $\underline{Y} = \underline{X}_{P_{v_0}}$, which are the scale labels n_ρ appearing in the factor

$$\prod_{\rho \in \tau^*(\tau, \underline{X}_{P_{v_0}})} L^{-3n_\rho} \mathrm{e}^{-\tilde{\alpha} L^{-n_\rho}|\rho|} = \prod_{\text{v} \leqslant \text{v}_0} L^{-3n_{\text{v}}(\bar{s}_{\text{v}}-1)} \prod_{\rho \in \tau^*(\tau, \underline{X}_{P_{v_0}})} \mathrm{e}^{-\tilde{\alpha} L^{-n_\rho}|\rho|}.$$

Note that the total number of the f.s.l. vertices, for any $\theta_{\text{lab}}^{N,n}$, cannot be less than 1 and cannot exceed $p-1$ (where $p = |\underline{X}_{P_{v_0}}|$ is the number of points in the set $\underline{X}_{P_{v_0}}$).

Now we calculate the following sum

$$\mathcal{F} = \sum_{\substack{\{n_v\}_{v<v_0}: \\ n_{v_0}, \{n_{\text{v}}\} \text{ fixed}}} \sum_{\substack{\{P_v\}_{v<v_0}: \\ P_{v_0}, \{\widetilde{P}_{\text{v}}\} \text{ fixed}}} \prod_{v<v_0} L^{-\varepsilon|R_v|} \prod_{v<v_0} L^{-(n_{v'}-n_v)} \prod_{i=1}^{N} L^{-n_i}, \tag{5.34}$$

where the first sum (over the scale labels in the non-trivial vertices) is performed, as said, keeping fixed the scales of vertices $\{\text{v}\}$ and v_0. Analogously, the second sum is over all choices of sets $\{P_v\}$ keeping fixed the sets P_{v_0} and $\widetilde{P}_{\text{v}}$. The latter one can be rewritten as

$$\sum_{\substack{\{P_v\}_{v<v_0}: \\ P_{v_0}, \{\widetilde{P}_{\text{v}}\} \text{ fixed}}} = \sum_{\{R_v\}_{v<v_0}}.$$

Thus, we obtain

$$\mathcal{F} = \sum_{\{R_v\}_{v<v_0}} \prod_{v<v_0} L^{-\varepsilon|R_v|} \sum_{\substack{\{n_v\}_{v<v_0}: \\ n_{v_0}, \{n_{\text{v}}\} \text{ fixed}}} \prod_{v<v_0} L^{-(n_{v'}-n_v)} \prod_{i=1}^{N} L^{-n_i}$$

and, by Lemmas 4.9 and and 4.10,

$$\mathcal{F} \leqslant C^N L^{-n_{v_0} s_{v_0}} \sum_{\substack{\{n_v\}_{v<v_0}: \\ n_{v_0}, \{n_{\text{v}}\} \text{ fixed}}} \prod_{v<v_0} L^{-n_v(s_v-1)}$$
$$\leqslant C^N L^{-n_{v_0} s_{v_0}} \prod_{\text{v}<v_0} L^{-n_{\text{v}}(s_{\text{v}}-1)}. \tag{5.35}$$

Hence, we get finally

$$\sum_{\substack{\{n_v\}_{v<v_0}:\\ n_{v_0},\{n_{\mathrm{v}}\}\text{ fixed}}} \sum_{\substack{\{P_v\}_{v<v_0}:\\ P_{v_0}\text{ fixed}}} \prod_{v<v_0} L^{-(n_{v'}-n_v)D^{(1)}(R_v)} \prod_{i=1}^{N} L^{-n_i}$$
$$\leqslant C^N L^{-n_{v_0}s_{v_0}} \prod_{\mathrm{v}<v_0} L^{-n_{\mathrm{v}}(s_{\mathrm{v}}-1)}, \tag{5.36}$$

where we used also that $\sum_{\widetilde{P}_{\mathrm{v}}} 1 \leqslant C^N$.

Consider now the second productory appearing in the right hand side of (5.33). Recalling the definition (5.31), it can be written as

$$\prod_{v<v_0} L^{-(n_{v'}-n_v)D^{(2)}(\widetilde{P}_v)} = \prod_{\mathrm{v}<v_0} L^{(n_{\mathrm{v}''}-n_{\mathrm{v}})(3-\|\widetilde{P}_{\mathrm{v}}^*\|)},$$

where, for v fixed, v$''$ denote the smaller f.s.l. vertex *following* v. Note that in general v$''$ > v$'$. Recalling that v_0 is maximum among the f.s.l. vertices, i.e., the vertex where *all* points in $\underline{X}_{P_{v_0}}$ become connected, we have that $P_{v_0}^{\mathrm{ren}} = \widetilde{P}_{\mathrm{v}_0}^*$. Noting also that $\mathrm{v}_0 \leqslant v_0$ we get

$$L^{-(3-\|P_{v_0}^{\mathrm{ren}}\|)n_{v_0}} \prod_{v<v_0} L^{-(n_{v'}-n_v)D^{(2)}(\widetilde{P}_v)}$$
$$= L^{-(3-\|P_{v_0}^{\mathrm{ren}}\|)n_{\mathrm{v}_0}} \prod_{\mathrm{v}<\mathrm{v}_0} L^{(n_{\mathrm{v}''}-n_{\mathrm{v}})(3-\|\widetilde{P}_{\mathrm{v}}^*\|)}. \tag{5.37}$$

Using (5.36) and the definition (5.29) we can bound the expression (5.26) summed over labels n_v (keeping fixed n_{v_0} and n_{v}) and over sets P_v by

$$\left| \sum_{\substack{\{n_v\}_{v<v_0}:\\ n_{v_0},\{n_{\mathrm{v}}\}\text{ fixed}}} \sum_{\substack{\{P_v\}_{v<v_0}:\\ P_{v_0}\text{ fixed}}} W_{2m,k}^{(n)}\left(\theta_{\mathrm{lab}}^{N,n}, P_{v_0}^{\mathrm{ren}}, \{P_v\}_{v<v_0}, \underline{X}_{P_{v_0}}\right) \right|$$
$$\leqslant C^N \lambda^N L^{-n_{v_0}s_{v_0}} \prod_{\mathrm{v}<v_0} L^{-n_{\mathrm{v}}(s_{\mathrm{v}}-1)} L^{-n_{\mathrm{v}_0}(3-\|P_{v_0}^{\mathrm{ren}}\|)} \prod_{\mathrm{v}<\mathrm{v}_0} L^{(n_{\mathrm{v}''}-n_{\mathrm{v}})(3-\|\widetilde{P}_{\mathrm{v}}^*\|)}$$
$$\times \left[\prod_{v\leqslant v_0} \frac{1}{s_v!}\right] \sum_{\substack{\tau\in\mathcal{T}_{\underline{X}_{v_0}}\\ \tau\text{ comp }\theta_{\mathrm{lab}}^{N,n}}} \prod_{\rho\in\tau^*(\tau,\underline{X}_{P_{v_0}})} L^{-3n_\rho} e^{-\frac{\alpha}{4}L^{-n_\rho}|\rho|}. \tag{5.38}$$

Now, recall that $\{n_\rho\}_{\rho\in\tau^*} = \{n_{\mathrm{v}}\}_{\mathrm{v}\leqslant\mathrm{v}_0}$ and that τ^* is compatible with $\theta_{\mathrm{lab}}^{N,n}$. Thus, for each f.s.l. vertex $v \in \theta_{\mathrm{lab}}^{N,n}$, we have exactly $\bar{s}_v - 1$ links of

$\tau^*(\tau, \underline{X}_{P_{v_0}})$ at the scale n_v, where $\bar{s}_v$ is the number of space-time points in the set $\underline{X}_{P_{v_0}}$ which become connected at the vertex v.

Let us now consider the factor

$$L^{-3n_{v_0}} \prod_{v<v_0} L^{3(n_{v''}-n_v)} \tag{5.39}$$

which appears in the right hand side of (5.38). One may check (see proof of Lemma 4.10) that

$$L^{-3n_{v_0}} \prod_{v<v_0} L^{3(n_{v''}-n_v)} = L^{3n_{v_0}(\bar{\bar{s}}_{v_0}-1)} \prod_{v<v_0} L^{3n_v(\bar{\bar{s}}_v-1)}, \tag{5.40}$$

where $\bar{\bar{s}}_v$ is the number of f.s.l. vertices among the set $\{v^1, \ldots, v^{\bar{s}_v}\}$ which are not end points.

Recalling (5.27), we have

$$L^{-3n_{v_0}} \prod_{v<v_0} L^{3(n_{v''}-n_v)} \prod_{\rho\in\tau^*(\tau,\underline{X}_{P_{V_0}})} L^{-3n_\rho} = \prod_{v\leqslant v_0} L^{-3n_v(\bar{s}_v-\bar{\bar{s}}_v)}. \tag{5.41}$$

Note that $\bar{s}_v - \bar{\bar{s}}_v$ are exactly the number of end points *directly* attached to v whose space-time coordinates are in the set $\underline{X}_{P_{v_0}}$.

Let us analyze the factor

$$L^{n_{v_0}\|P_{v_0}^{\text{ren}}\|} \prod_{v<v_0} L^{(n_{v''}-n_v)(-\|\widetilde{P}_v^*\|)},$$

which also appears in (5.38). The productory is over the f.s.l. vertices {v} of $\theta_{\text{lab}}^{N,n}$ and does not include end points. If v were allowed to be an end point (say, the ith endpoint), the factor $(n_{v''} - n_v)(-\|\widetilde{P}_v^*\|)$ would be

$$(n_{v''} - n_v)(-\|\widetilde{P}_v^*\|) = -\mathrm{n}_i\alpha_i \quad \text{if v were the end point } i, \tag{5.42}$$

where n_i is the scale label of the first f.s.l. vertex immediately following the end point i (observe that $\mathrm{n}_i \geqslant n_i$) and we define α_i as the number of fields of P_{v_0} attached to the end point i *plus* the number of derivatives. As a matter of fact, a field in the set P_{v_0} may get a double derivative just in a non-trivial vertex of $\theta_{\text{lab}}^{N,n}$ which *precedes* a f.s.l. vertex at scale n_i for some i. This is because two fields must emerge in a vertex where a double derivative field is created, and if one of them is among P_{v_0}, the other cannot be in P_{v_0} and must be contracted later. Thus if v is a fixed scale label vertex such that there exist other fixed scale vertices preceding it, then at least 4 fields must emerge from v and no renormalization can occur. Hence,

$$\prod_{\substack{\mathrm{v}<\mathrm{v}_0: \\ \mathrm{v\ not\ e.p.}}} L^{(n_{\mathrm{v}''}-n_{\mathrm{v}})(-\|\widetilde{P}_{\mathrm{v}}^*\|)} = \prod_{i=1}^{N} L^{\mathrm{n}_i \alpha_i} \prod_{\substack{\mathrm{v}<\mathrm{v}_0: \\ \mathrm{v\ also\ e.p.}}} L^{(n_{\mathrm{v}''}-n_{\mathrm{v}})(-\|\widetilde{P}_{\mathrm{v}}^*\|)}.$$

Moreover, due to the cluster structure of $\theta_{\mathrm{lab}}^{N,n}$

$$\prod_{\substack{\mathrm{v}<\mathrm{v}_0: \\ \mathrm{v\ also\ e.p.}}} L^{(n_{\mathrm{v}''}-n_{\mathrm{v}})(-\|\widetilde{P}_{\mathrm{v}}^*\|)} = L^{-n_{\mathrm{v}_0}\|P_{\mathrm{v}_0}\|} \prod_{\substack{\mathrm{v}<\mathrm{v}_0: \\ \mathrm{v\ not\ e.p.}}} L^{n_{\mathrm{v}}(\|\widetilde{P}_{\mathrm{v}}\|-\sum_{i=1}^{\bar{s}_{\mathrm{v}}}\|\widetilde{P}_{\mathrm{v}^i}\|)}$$

$$= L^{-n_{\mathrm{v}_0}\|P_{\mathrm{v}_0}\|},$$

where we also used that

$$\|\widetilde{P}_{\mathrm{v}}\| - \sum_{i=1}^{\bar{s}_{\mathrm{v}}} \|\widetilde{P}_{\mathrm{v}^i}\| = 0. \tag{5.43}$$

The reader may convince himself that the formula (5.43) above is true by recalling that now v^i can be end-points and recalling the comments following the formula (5.42). Finally, we obtain

$$L^{n_{\mathrm{v}_0}\|P_{v_0}^{\mathrm{ren}}\|} \prod_{\mathrm{v}<\mathrm{v}_0} L^{(n_{\mathrm{v}''}-n_{\mathrm{v}})(-\|\widetilde{P}_{\mathrm{v}}^*\|)} = \prod_{i=1}^{N} L^{\mathrm{n}_i\alpha_i}, \tag{5.44}$$

where we emphasize that α_i is the contribution to $\|P_{v_0}^{\mathrm{ren}}\|$ coming from the end point i and also that $\sum_{i=1}^{N}\alpha_i = \|P_{v_0}^{\mathrm{ren}}\|$. Now, using (5.41) and (5.44), we can bound the left hand side of (5.38) by

$$\left| \sum_{\substack{\{n_v\}_{v<v_0}: \\ n_{v_0},\{n_{\mathrm{v}}\}\ \mathrm{fixed}}} \sum_{\substack{\{P_v\}_{v<v_0}: \\ P_{v_0}\ \mathrm{fixed}}} W_{2m,k}^{(n)}\left(\theta_{\mathrm{lab}}^{N,n}, P_{v_0}^{\mathrm{ren}}, \{P_v\}_{v<v_0}, \underline{X}_{P_{v_0}}\right) \right|$$

$$\leqslant \prod_{\mathrm{v}\leqslant\mathrm{v}_0} L^{-3n_{\mathrm{v}}(\bar{s}_{\mathrm{v}}-\bar{\bar{s}}_{\mathrm{v}})} \prod_{i=1}^{N} L^{\alpha_i \mathrm{n}_i} L^{-n_{v_0}s_{v_0}} \prod_{\mathrm{v}<v_0} L^{-n_{\mathrm{v}}(s_{\mathrm{v}}-1)}$$

$$\times \left[\prod_{v\leqslant v_0} \frac{1}{s_v!}\right] \sum_{\substack{\tau\in\mathcal{T}_{\underline{X}_{v_0}} \\ \tau\ \mathrm{comp}\ \theta_{\mathrm{lab}}^{N,n}}} \prod_{\rho\in\tau^*(\tau,\underline{X}_{P_{v_0}})} \mathrm{e}^{-\frac{\alpha}{4}L^{-n_\rho}|\rho|}. \tag{5.45}$$

Recalling that $\bar{s}_{\mathrm{v}} - \tilde{\bar{s}}_{\mathrm{v}}$ are the number of end points of $\theta_{\mathrm{lab}}^{N,n}$ which contain fields in P_{v_0} and are directly attached to v, we have

$$\prod_{\mathrm{v}\leqslant \mathrm{v}_0} L^{-3n_{\mathrm{v}}(\bar{s}_{\mathrm{v}}-\tilde{\bar{s}}_{\mathrm{v}})} = \prod_{j=1}^{p} L^{-3\mathrm{n}_j}. \tag{5.46}$$

Moreover, we have that

$$\prod_{i=1}^{N} L^{\alpha_i \mathrm{n}_i} = \prod_{j=1}^{p} L^{\alpha_j \mathrm{n}_j}, \tag{5.47}$$

where j is labelling just the end points with space-time coordinate in $\underline{X}_{P_{v_0}}$. Hence

$$\prod_{\mathrm{v}\leqslant \mathrm{v}_0} L^{-3n_{\mathrm{v}}(\bar{s}_{\mathrm{v}}-\tilde{\bar{s}}_{\mathrm{v}})} \prod_{i=1}^{N} L^{\alpha_i \mathrm{n}_i} = \prod_{j=1}^{p} L^{-\mathrm{n}_j(3-\alpha_i)}. \tag{5.48}$$

Note that $3 - \alpha_i \geqslant 0$ in any case. Using (5.48) we can write (5.45) as

$$\begin{aligned} &\left| \sum_{\substack{\{n_v\}_{v<v_0}: \\ n_{v_0}, \{n_{\mathrm{v}}\} \text{ fixed}}} \sum_{\substack{\{P_v\}_{v<v_0}: \\ P_{v_0} \text{ fixed}}} W_{2m,k}^{(n)}\left(\theta_{\mathrm{lab}}^{N,n}, P_{v_0}^{\mathrm{ren}}, \{P_v\}_{v<v_0}, \underline{X}_{P_{v_0}}\right) \right| \\ &\leqslant C^N \lambda^N L^{-n_{v_0} s_{v_0}} \prod_{\mathrm{v}<v_0} L^{-n_{\mathrm{v}}(s_{\mathrm{v}}-1)} \prod_{j=1}^{p} L^{-\mathrm{n}_j(3-\alpha_i)} \\ &\times \left[\prod_{v\leqslant v_0} \frac{1}{s_v!} \right] \sum_{\substack{\tau \in \mathcal{T}_{\underline{X}_{v_0}} \\ \tau \text{ comp } \theta_{\mathrm{lab}}^{N,n}}} \prod_{\rho \in \tau^*(\tau, \underline{X}_{P_{v_0}})} \mathrm{e}^{-\frac{\alpha}{4} L^{-n_\rho} |\rho|}. \end{aligned} \tag{5.49}$$

The total scaling factor of this bound is given by

$$L^{-n_{v_0} s_{v_0}} \prod_{\mathrm{v}<v_0} L^{-n_{\mathrm{v}}(s_{\mathrm{v}}-1)} \prod_{j=1}^{p} L^{-\mathrm{n}_j(3-\alpha_i)}. \tag{5.50}$$

By the standard tree identity

$$\sum_{\mathrm{v}\leqslant \mathrm{v}_0} (s_{\mathrm{v}} - 1) \geqslant p - 1 \tag{5.51}$$

(where the inequality $\geqslant$ appears because $s_{\rm v}$ is the total number of branches from v, i.e., $s_{\rm v} \geqslant \bar{s}_{\rm v}$ for all v) and recalling that $\sum_i \alpha_i = \|P^{\rm ren}_{v_0}\|$, the total scale factor is bounded (independently of $\theta^{N,n}_{\rm lab}$) by

$$s_{v_0} + \sum_{{\rm v}<v_0}(s_{\rm v}-1) + \sum_j (3-\alpha_j) \geqslant \delta_{v_0{\rm v}_0} + p + 3p - \|P^{\rm ren}_{v_0}\|$$
$$= \delta_{v_0{\rm v}_0} + 4p - 2(m+k), \quad (5.52)$$

where $\delta_{v_0{\rm v}_0} = 1$ if $v_0 \neq {\rm v}_0$ and $\delta_{v_0{\rm v}_0} = 0$ if $v_0 = {\rm v}_0$. We also used that $s_{v_0} \geqslant 2$. Thus, we may rewrite the factor (5.50) as

$$L^{-n_{v_0}s_{v_0}} \prod_{{\rm v}<v_0} L^{-n_{\rm v}(s_{\rm v}-1)} \prod_{j=1}^{p} L^{-{\rm n}_j(3-\alpha_i)} = L^{-\delta_{v_0{\rm v}_0}n_{v_0}} \prod_{{\rm v}\leqslant {\rm v}_0} L^{-\beta_{\rm v}n_{\rm v}}, \quad (5.53)$$

where the first factor in the right hand side of (5.53) is present if ${\rm v}_0 \neq v_0$ (otherwise it is equal to one), and it controls the sum over the scale label n_{v_0}. The productory in right hand side of (5.53) runs over all f.s.l. vertices, and $\beta_{\rm v}$ are a sequence of integers, univocally determinated once $\theta^{N,n}_{\rm lab}$ and τ are fixed, such that $\beta_{\rm v} \geqslant 1$ for all v, and, with $d = 3$

$$\sum_{{\rm v}\leqslant {\rm v}_0} \beta_{\rm v} \geqslant (d+1)p - \left[2m\frac{(d-1)}{2} + 2k\right]. \quad (5.54)$$

Recalling that the set {v} of all f.s.l. vertices is at most $p-1$ and at least 1 (obviously $p-1$ is the worst case), we have, for any $\theta^{N,n}_{\rm lab}$ and τ, pointwisely

$$\sum_{n_{v_0},\{n_{\rm v}\}_{{\rm v}\leqslant {\rm v}_0}} \prod_{{\rm v}\leqslant {\rm v}_0} L^{-\beta_{\rm v}n_{\rm v}} \prod_{\rho\in\tau^*(\tau,\underline{X}_{P_{v_0}})} e^{-\frac{\alpha}{4}L^{-n_\rho}|\rho|}$$
$$\leqslant C \sum_{n_1=0}^{n} \cdots \sum_{n_{p-1}=0}^{n} \sum_{\substack{\beta_1+\cdots+\beta_{p-1}\\ =4p-2(m+k)}} L^{-\beta_1 n_1} \ldots L^{-\beta_{p-1}n_{p-1}}$$
$$\times\, e^{-\frac{\alpha}{4}(L^{-n_1}|\rho_1|+\cdots+L^{-n_{p-1}}|\rho_{p-1}|)}, \quad (5.55)$$

where $\{\rho_1, \ldots, \rho_{p-1}\} = \tau^*(\tau, \underline{X}_{P_{v_0}})$. Using the formula (5.55) above we can finally perform the sum over the scale labels of the vertices {v} and v_0 in the left hand side of (5.49) and obtain the pointwise bound

$$\left| \sum_{\{n_v\}_{v\leqslant v_0}} \sum_{\substack{\{P_v\}_{v<v_0}:\\ P_{v_0}\ \text{fixed}}} W^{(n)}_{2m,k}(\theta^{N,n}_{\rm lab}, P^{\rm ren}_{v_0}, \{P_v\}_{v<v_0}, \underline{X}_{P_{v_0}}) \right|$$

$$\leqslant C^N\lambda^N \sum_{\substack{t\in\mathcal{T}_{\underline{X}_{P_{v_0}}}\\ \tau \text{ comp } \theta_{\text{lab}}^{N,n}}} B_t(\theta^N, P_{v_0}) \sum_{n_1=0}^{n}\cdots\sum_{n_{p-1}=0}^{n}\sum_{\substack{\beta_1+\cdots+\beta_{p-1}\\ =4p-2(m+k)}} L^{-\beta_1 n_1}\ldots L^{-\beta_{p-1}n_{p-1}} e^{-\frac{\alpha}{4}(L^{-n_1}|\rho_1|+\cdots L^{-n_{p-1}}|\rho_{p-1}|)}. \tag{5.56}$$

The first sum in the right hand side of (5.56) runs over all connected tree graphs $t\equiv\{\rho_1,\ldots,\rho_{p-1}\}$ between the points in $\underline{X}_{P_{v_0}}$ which are compatible with θ^N, in the sense that $t=\tau^*(\tau,\underline{X}_{P_{v_0}})$ for some tree graph τ between $x_1,\ldots,x_N$ compatible with θ^N, and

$$B_{t(\underline{X}_{P_{v_0}})}(\theta^N)=\left[\prod_{v\leqslant v_0}\frac{1}{s_v!}\right]\sum_{\substack{\tau \text{ comp. } \theta^N:\\ \tau^*(\tau,\underline{X}_{P_{v_0}})=t(\underline{X}_{P_{v_0}})}} 1.$$

We are finally in the position to show the bound on $W_{2m,k}^{(n),\underline{r},\underline{s}}$ presented in Theorem 5.1. Renominating the space-time coordinates $\underline{X}_{P_{v_0}}$ as $x_1,\ldots,x_p$, such that

$$\widetilde{\psi}(P_{v_0}^{\text{ren}})=(-1)^{\pi(P_{v_0}^{\text{ren}})}\left[\overline{\psi}\right]^{\underline{r}}[\psi]^{\underline{s}}[\partial^2\psi]^k(x_1,x_2,\ldots,x_p).$$

Comparing (5.56) with (4.15) and using also (4.16), we obtain the bound

$$\left|W_{2m,k}^{(n),\underline{r},\underline{s}}(\theta^{N,n},x_1,x_2,\ldots,x_p)\right| \leqslant C^N\lambda^N\sum_{\substack{t\in\mathcal{T}_{\{x_1,\ldots,x_p\}}\\ \text{comp. } \theta^N}} B_t(\theta^N)\sum_{n_1=0}^{n-1}\cdots\sum_{n_{p-1}=0}^{n-1}\sum_{\beta_1,\ldots,\beta_{p-1}} L^{-\beta_1 n_1}\ldots L^{-\beta_{p-1}n_{p-1}} e^{-\alpha L^{-n_1}|\rho_1|}\ldots e^{-\alpha L^{-n_{p-1}}|\rho_{p-1}|}, \tag{5.57}$$

where the first sum in the right hand side of (5.57) is over all the tree graphs $t\equiv\{\rho_1,\ldots,\rho_{p-1}\}$ between the points $\{x_1,\ldots,x_p\}$ compatible with θ^N, in the sense that $t=\tau^*(\tau,\underline{X}_{P_{v_0}})$ for some τ tree graph in $\{x_1,\ldots,x_N\}$ compatible with θ^N and some $\underline{X}_{P_{v_0}}$ (modulo a renomination of variables), and

$$B_t(\theta^N)=\sum_{\substack{P_{v_0}:\ t(\underline{X}_{P_{v_0}})=t\\ \text{when } \underline{X}_{P_{v_0}}\to\{x_1,\ldots,x_p\}}} B_t(\theta^N,P_{v_0}), \tag{5.58}$$

where the sum over sets P_{v_0} is carried out with the constraints that $|P_{v_0}| = 2m$, k and p are fixed. Note that, by Proposition 4.3 and formula (4.31)

$$\sum_{\substack{t\in\mathcal{T}_{\{x_1,\dots,x_p\}} \\ \text{comp. } \theta^N}} B_t(\theta^N) \leqslant \sum_{P_{v_0}:\ |\underline{X}_{P_{v_0}}|=p} \left[\prod_{v\leqslant v_0} \frac{1}{s_v!}\right] \sum_{\substack{\tau\in\mathcal{T}_{\underline{X}_{v_0}} \\ \tau \text{ comp. } \theta^{N,n}_{\text{lab}}}} 1 \leqslant C^N. \quad (5.59)$$

Summing finally over $\theta^{N,n}$ and N we obtain the formula (3.36) in the part A of Theorem 5.1. Namely,

$$\begin{aligned} &\left|W^{(n)\{r_j,s_j\}}_{2m,k}(x_1,x_2,\dots,x_p)\right| \\ &\leqslant D^p\lambda^p \sum_{t\in\mathcal{T}_p} B_t \sum_{n_1=0}^{n-1}\cdots\sum_{n_{p-1}=0}^{n-1} \sum_{\substack{\beta_1+\dots+\beta_{p-1} \\ =4p-2(m+k)}} L^{-\beta_1 n_1}\dots L^{-\beta_{p-1}n_{p-1}} \\ &\quad\times \mathrm{e}^{-\alpha L^{-n_1}|\rho_1|}\dots \mathrm{e}^{-\alpha L^{-n_{p-1}}|\rho_{p-1}|}, \end{aligned} \quad (5.60)$$

where

$$B_t = \sum_{M=0}^{\infty} C^M\lambda^M \left[\sum_{\substack{\theta^N:\ N=M+p \\ t \text{ comp. } \theta^N}} B_t(\theta^N)\right],$$

and

$$\sum_{t\in\mathcal{T}_p} B_t \leqslant C^p$$

due to (5.59) and the bound (2.19) on the number of all topologically different rooted Cayley trees with N end points.

The proofs of (3.37) and (3.38) are just simple exercises. In particular, the exponentially decaying factor L^{-n} in (3.37) follows by the same argument of Section 4.3 (see (4.63) and comments above).

ACKNOWLEDGEMENT

We want to thank Giosi Benfatto and Benedetto Scoppola for discussions and reading the manuscript. This work was partially supported by CNPq and FAPEMIG (Brazil).

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