ANNALES DE L'I. H. P., SECTION A

A. K. PRYKARPATSKY
J. A. ZAGRODZIŃSKI

Lagrangian and hamiltonian aspects of Josephson type media

Annales de l'I. H. P., section A, tome 70, nº 5 (1999), p. 497-524
<http://www.numdam.org/item?id=AIHPA_1999__70_5_497_0>

Ann. Inst. Henri Poincaré,

Vol. 70, n° 5, 1999, p. 497-524. *Physique théorique*

Lagrangian and Hamiltonian aspects of Josephson type media

by

A.K. PRYKARPATSKY[a, b], **J.A. ZAGRODZIŃSKI**[c, 1]

[a] Department of Nonlinear Mathematical Analysis, Institute for Applied Problems of Mechanics and Mathematics, NAS, Lviv, Ukraine

[b] Institute for Basic Research, Monteroduni, Italy

[c] Institute of Physics, Polish Academy of Sciences, 02-668 Warsaw, Poland

Manuscript received 16 April 1997

ABSTRACT. – Dynamical properties of Josephson media are studied within Lagrangian and Hamiltonian formalisms with gauge type constraints. A geometrical interpretation of the first class gauge type constraints involved is suggested basing on the Marsden–Weinstein momentum map reduction. An equivalent operatorial approach is also considered in detail giving rise to certain two essentially different Poisson structures upon orbits of the Abelian gauge group, the first Poisson structure being nondegenerate in contrast to the second one—degenerated. The consideration of the Poisson structure associated with Josephson media are essentially augmented by means of introducing new Josephson–Vlasov kinetic type equations endowed with the canonical Lie–Poisson bracket. The reduction of the corresponding Hamiltonian flow explains the physical nature of the a priori involved constraints. © Elsevier, Paris

PACS: 3.40.Kf, 04.20Fy, 74.50+r, 74.60G, 74.76-w

RÉSUMÉ. – Nous étudions les propriétés dynamiques des milieux Josephson grâce à des formalismes lagrangiens et hamiltoniens avec des contraintes de type jauge. Une interprétation géométrique des

[1] Email: zagro@ifpan.edu.pl.

contraintes de type jauge de première classe mises en jeu est suggéré par une réduction de la carte des moments de Marsden–Weinstein. Une approche opérationnelle est aussi considérée en détail : elle donne lieu à essentiellement deux structures de Poisson différentes sur les orbites du groupe de jauge abelien, la première étant non dégénérée et la seconde dégénérée. La structure de Poisson associée aux milieux Josephson est renforcée de façon essentielle en introduisant de nouvelles équations du type Josephson–Vlasov munies du crochet canonique de Lie–Poisson. La réduction du flot hamiltonien correspondant explique la nature physique des contraintes mises en jeu à priori.

1. INTRODUCTION

1.1. Physical background

The dynamical properties of so-called Josephson media were recently often discussed [1–3]. Such a specific medium can be considered as a limit of the two-dimensional (2D), plane Josephson junction arrays when the "density" of junctions tends to infinity if the physical background or history is reported as well as there are different motives that such a medium is interesting. The possible approaches are originated either by the spin-glass formalism on lattice or by systems of partial differential equations when continuous dynamic processes are under investigation.

In the first case different collective phenomena are considered, e.g., the existence of vortex fluids [4], the interaction between vortices, [5], topological invariance of arrays [6], and even 3D structures [7], although in the one direction the structure is discrete. As a generic, the Hamiltonian as in XY-Ising model is assumed.

In the second case, usually a phenomenological approach is adopted starting from a discrete picture but equations can be derived also either from a suitable Lagrangian or Hamiltonian. The ultimate form of a Lagrangian is in the spirit of the Ginzburg–Landau free energy functional with modifications introduced by Lawrence and Doniach for layered systems [8]. The discretization coincides with the spin-glass formalism if self-magnetic fields are neglected.

There are, however, some questions when one wants to compare more precisely the results of both Lagrangian and Hamiltonian approaches.

The source of problems is found in a choice of canonical variables and next in the linear dependence of canonical momenta if the mentioned choice was done incorrectly. The point is that adopting the classical choice of variables in the Lagrangian the space derivative of velocity appears that makes a difficulty. To avoid the arisen problem in [3] there was proposed the simplest solution: a reduction of the number of generalized coordinates which is equivalent to the consideration some invariant submanifold, carrying the canonical Poisson structure.

The arguments originating the present paper we summarize briefly below. For details a reader is referred to [3], or [9] and to papers cited there.

Let us consider the canonical Lagrangian in the form

$$\mathcal{L} := \int_{\mathbb{R}^2} L\,dx, \quad \text{where } L := \left(\boldsymbol{\Phi}_t^2 - \operatorname{rot}^2\boldsymbol{\Phi}\right)/2 - \sum_{i=1}^{2}(1-\cos\varphi_i) \tag{1}$$

and $\boldsymbol{\Phi} = (\varphi_1, \varphi_2, 0)$, since we are interested in time evolution in 2D Euclidean space, i.e., $\boldsymbol{\Phi}: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$. Here and later on by $(.)_t$ we understand the derivative with respect to the variable $t \in \mathbb{R}$. Because of the correspondence with the spin-glass formalism we assume that

$$\boldsymbol{\Phi} = \mathbf{A} + \nabla\Theta, \tag{2}$$

and of course $\mathbf{A} = (A_1, A_2, 0): \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$; $\Theta: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$; i.e., $\nabla := (\partial/\partial x_1, \partial/\partial x_2, 0)$, if notation relating to 3D space is preserved.

Since the derivative $\boldsymbol{\Phi}_t$ has the physical interpretation of the electric field, quantities $\mathbf{A}$ and $-\int\Theta\,dt$ as the vector and scalar potential, respectively, it is natural, following standard classical approaches, to choose as generalizes canonical coordinates the scalar potential and components of the vector potential. In our concrete case these coordinates would be Θ, A_1, A_2.

The relevant Euler–Lagrange equations

$$\frac{\delta\mathcal{L}}{\delta\boldsymbol{\Theta}} = 0 \quad \text{and} \quad \frac{\delta\mathcal{L}}{\delta\mathbf{A}} = 0, \tag{3}$$

lead to the following field equations

$$\nabla\cdot\left[(\mathbf{A}+\nabla\Theta)_{tt} + \sin(\mathbf{A}+\nabla\Theta)\right] = 0, \tag{4}$$

and

$$\nabla \times (\nabla \times \mathbf{A}) + (\mathbf{A} + \nabla\Theta)_{tt} + \sin(\mathbf{A} + \nabla\Theta) = 0, \tag{5}$$

respectively, where we are using for brevity the shorthand mnemo notation $\sin\mathbf{A} := (\sin A_1, \sin A_2, 0)$.

The first remark is that (4) follows from (5). Defining momenta via the standard way we obtain

$$p_0 = \frac{\delta\mathcal{L}}{\delta\Theta_t} = -\nabla\cdot(\mathbf{A} + \nabla\Theta)_t \quad \text{and} \quad \mathbf{p} := \frac{\delta\mathcal{L}}{\delta\mathbf{A}_t} = (\mathbf{A} + \nabla\Theta)_t, \tag{6}$$

i.e., once more dependent ones, although the Poisson brackets have the correct form. Finally the Hamiltonian is

$$\mathcal{H} := \int_{\mathbb{R}^2} H\,dx \tag{7}$$

with the density defined as usual $H = \mathbf{p}\mathbf{A}_t + p_0\Theta_t - L$ has the form

$$H = \frac{1}{2}\left[(\nabla\Theta + \mathbf{A})_t^2 + \mathrm{rot}^2\,\mathbf{A}\right] + \sum_{i=1}^{2}\left[1 - \cos(\nabla_i\Theta + A_i)\right], \tag{8}$$

up to full divergence $\mathrm{div}[(\mathbf{A} + \nabla\Theta)_t\Theta_t]$. The field equations calculated either from the Hamiltonian or from the Lagrangian should be the same. The field equations corresponding to the Hamilton functional (7)

$$p_{0,t} = -\frac{\delta\mathcal{H}}{\delta\Theta} \quad \text{and} \quad \mathbf{p}_t := -\frac{\delta\mathcal{H}}{\delta\mathbf{A}}, \tag{9}$$

coincide with those following from the Lagrangian, but surprisingly equations

$$\Theta_t = \frac{\delta\mathcal{H}}{\delta p_0} \quad \text{and} \quad \mathbf{A}_t := \frac{\delta\mathcal{H}}{\delta\mathbf{p}}, \tag{10}$$

lead to the conclusion that $\Theta_t = 0$, which seems at least strange.

As it was proposed in [3] a simplest remedium makes a reduction of a number of canonical coordinates. A situation becomes more clear if instead of Θ, A_1, A_2 as canonical coordinates there are taken two nontrivial components of $\boldsymbol{\Phi} = \mathbf{A} + \nabla\Theta$. Then the component Θ becomes only a gauge parameter and as a result p_0 drops out from the further analysis. The second set of Hamilton Eqs. (10) reduces then to the equivalence $\boldsymbol{\Phi}_t = \delta\mathcal{H}/\delta\mathbf{p} = \delta\mathcal{H}/\delta\boldsymbol{\Phi}_t$.

1.2. Problem statement

From the mathematical point of view such a procedure can be considered as a reduction of a starting manifold to some submanifold but with some nontrivial, additional restrictions.

Firstly let us notice that when the Josephson field $\boldsymbol{\Phi}$ is given physically as a 3D-vector, the condition $\varphi_3 \equiv 0$ should be considered as an outer constraint imposed on the Lagrangian (1). Secondly, let us observe that the decomposition (2) is not alike to the corresponding Helmholtz decomposition commonly used in classical electrodynamics and has quite different meaning. And thirdly, we must take care of the solenoidality condition $\operatorname{div} \boldsymbol{\Phi} = \operatorname{div} \mathbf{A} + \boldsymbol{\Delta}\Theta = 0$. Moreover, the Lagrangian (1) and next the Hamiltonian and field equations are not invariant with respect SO2 group. We need, however, to derive true dynamical equations for fields $\mathbf{A}$ and Θ being compatible with all constraints involved above and to build the corresponding correct additional conditions. To do this program successfully one needs to recast our initial Lagrangian theory into dual Hamiltonian picture performing the standard Legendre transform

$$\mathbf{p} := \frac{\delta \mathcal{L}}{\delta \boldsymbol{\Phi}_t} = \boldsymbol{\Phi}_t,$$

$$H = \langle \mathbf{p}_t, \boldsymbol{\Phi}_t \rangle - L = \frac{1}{2}\left(\langle \mathbf{p}, \mathbf{p} \rangle + \langle \operatorname{rot} \boldsymbol{\Phi}, \operatorname{rot} \boldsymbol{\Phi} \rangle\right) + \sum_{i=1}^{2} (1 - \cos \varphi_i),$$

up to full divergence, i.e., according to (8), where canonical phase variables $\mathbf{p}$ and $\boldsymbol{\Phi}$ pertain to the cotangent space $T^*(C^\infty(\mathbb{R}^2; \mathbb{R}^2 \times \{0\}))$ over the configuration space $C^\infty(\mathbb{R}^2; \mathbb{R}^2 \times \{0\}) \ni \boldsymbol{\Phi}$, and generate the following symplectic structure

$$\omega^{(2)} = \int_{\mathbb{R}^2} \langle d\mathbf{p}, d\boldsymbol{\Phi} \rangle \, dx := \int_{\mathbb{R}^2} \left(\sum_{j=1}^{2} dp_j \wedge d\varphi_j \right) dx.$$

Since this symplectic structure is invariant under the gauge transform (2) but the Hamiltonian (8) is not, we fail to derive in the commonly used straightforwardly manner all necessary reduced boundary conditions on the Josephson fields $\mathbf{A}$ and Θ. To cope with all these problems generated by the special form of Lagrangian (1) in 2D-space, we intend below to discuss a general Lagrangian picture with the degeneracy of the corresponding Legendre transform via Dirac reduction procedure when such reduction is not introduced a priori. This leads further to treating the first class gauge type constraints by means of a geometric momentum map reduction, see, e.g., [16–20]. The price paid in this case is an existence of some multipliers as a consequence of considered constrains.

In order to transform such a problem with a degenerate Lagrangian, but on some invariant submanifold, the second part of the work is devoted to analysis of the related symplectic structures in such cases. In the third part some equivalent operator approach aspects of the problem are discussed.

The last part contains some important corollaries as well prospective sketches of methods used in the case of self-consistent Josephson media in spirit of Vlasov approach.

Many interesting aspects of similar physical problems with a gauge type structure were in the past analyzed in detail in numerous papers [10,11]. Authors made use of the Lie-algebraic theory of Poisson brackets on duals of some naturally built Lie algebras. By this way authors were able to construct gauge invariant Poisson structures for superfluids, spin-glasses, chromo and magnetohydrodynamics. For these physical models there are corresponding gauge invariant Hamiltonians giving rise to related evolution equations with conditions derived from the momentum map reduction.

Concerning the problem treated in the present article, we deal with Josephson type media which are not described by a gauge invariant Hamiltonian. That a natural canonical symplectic structure could not be found via the standard Lie-algebraic theory of Poisson structures on duals of some Lie algebras, even though it is also gauge invariant one. Moreover, some additional obstacle is imposed by a special 2D degeneration of Josephson medium under regard.

2. BASIC SETTING—LAGRANGIAN ANALYSIS

Let us be given the following simple form of the Lagrangian $\mathcal{L}$ on some functional manifold M smoothly parametrized by an evolution parameter $t \in \mathbb{R}$:

$$\mathcal{L} := \int_{\mathbb{R}^k} L[u, u_t]\, dx \quad \text{with } L[u, u_t] := L(u, u_t, u_x, u_{xx}, u_{xt}, u_{tt}, \ldots), \tag{11}$$

where $u \in M \subset C^\infty(\mathbb{R}^k; \mathbb{R}^m)$, $m, k \in \mathbb{Z}_+$, $L : J(\mathbb{R}_t, M) \to \mathbb{R}$ some smooth map on the jet manifold $J(\mathbb{R}_t, M)$.

The Eulerian equation has the form

$$\delta \mathcal{S} = 0 \Rightarrow \operatorname{grad} \mathcal{L}[u, u_t] = 0 = \frac{\delta \mathcal{S}}{\delta u}, \tag{12}$$

where by definition

$$\mathcal{S} := \int_0^t \mathcal{L}\, dt. \tag{13}$$

We see from (13) that the Lagrangian can hold derivatives in time $t \in \mathbb{R}$ of orders greater or equal to one, leading then to the nondegenerate Lagrangian problem if

$$\det \left\| \frac{\partial^2 Lt[u, u_t]}{\partial u_t^{(n)} \partial u_t^{(n)}} \right\| \neq 0 \tag{14}$$

for all $u \in M$; $u_t^{(n)} := \partial^n u / \partial t^n$; $\mathbb{Z}_+ \ni n \geqslant 1$—the maximal degree of the derivative $u_t^{(n)}$ contained in the canonical Lagrangian $L[u, u_t]$. If that is the case, we can construct [12–15] simply a Hamiltonian theory on some extended manifold $M_\pi \subset J^{(n)}(\mathbb{R}_t, M)$ as follows

$$\frac{dv_j}{dt} = \frac{\delta \mathcal{H}[v, \pi]}{\delta \pi_j}, \qquad \frac{d\pi_j}{dt} = -\frac{\delta \mathcal{H}[v, \pi]}{\delta v_j}, \tag{15}$$

where, by definition, $j = 0, \ldots, n-1$,

$$\pi_0 = \frac{\delta \mathcal{L}[u, u_t]}{\delta u_t} = \frac{\partial L}{\partial u_t} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial u_{tt}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial u_{xt}}\right) + \cdots + \frac{\partial^2}{\partial x^2}\left(\frac{\partial L}{\partial u_{xxt}}\right) + \frac{\partial^2}{\partial x \partial t}\left(\frac{\partial L}{\partial u_{xtt}}\right) + \cdots, \tag{16}$$

$$\pi_1 = \frac{\delta \mathcal{L}[u, u_t]}{\delta u_{tt}} = \frac{\partial L}{\partial u_{tt}} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial u_{ttt}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial u_{xtt}}\right) + \frac{\partial^2}{\partial t^2}\left(\frac{\partial L}{\partial u_{tttt}}\right) + \cdots, \tag{17}$$

$$\pi_2 := \frac{\delta \mathcal{L}[u, u_t]}{\delta u_{ttt}} = \frac{\partial L}{\partial u_{ttt}} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial u_{4t}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial u_{xttt}}\right) + \cdots, \tag{18}$$

and so on, the Hamiltonian being built as

$$\mathcal{H} := \int_{\mathbb{R}^k} \left[\sum_{j=0}^{n-1} \left\langle \pi_j, \frac{dv_j}{dt} \right\rangle - L[u, u_t] \right] dx, \tag{19}$$

where

$$v_0 = u,\ v_1 = u_t,\ v_j = u_t^{(j)} = \left(u_1^{(j)}, u_2^{(j)}, \ldots, u_m^{(j)}\right)^{\tau}, \quad j = 0, \ldots, n-1,$$

and $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^m$.

We see, thus that the system (15) has the standard Hamiltonian form upon the extended manifold M_π. When the Lagrangian $\mathcal{L}[u, u_t]$ is degenerate, the above procedure fails, that is one insists to use the standard Dirac theory of reduction [12–16] for the Lagrangian $\mathcal{L}$ with constraints of the first and second class generated by the procedure of constructing the conjugated "impulse" variables in the Hamiltonian theory. To do this we need to introduce the so-called Lagrangian expansion following from (17):

$$c_0 = \pi_0 - \frac{\delta \mathcal{L}}{\delta u_t}, \qquad c_1 = \pi_1 - \frac{\delta \mathcal{L}}{\delta u_{tt}}, \ldots \tag{20}$$

serving as constraints on the Lagrangian (12).

By definition, the impulse variables π_j, $j = 0, \ldots, n-1$, enjoy the canonical Poisson brackets:

$$\{v_j(x), \pi_k(y)\} = -\delta_{j,k}\delta(x - y), \qquad \{v_j, v_k\} = 0 = \{\pi_j, \pi_k\}, \tag{21}$$

for all $j, k = 0, \ldots, n-1$, where $\delta(\cdot)$ is the standard Dirac delta-function distribution and $\delta_{j,k}$ the Kronecker symbol. Now the total Hamiltonian is given by

$$\mathcal{H}_c := \int_{\mathbb{R}^k} H_c[v, \pi]\, dx, \quad \text{with } H_c[v, \pi] = H[v, \pi] + \sum_{j=0}^{n} \langle \lambda_j, c_j[v, \pi] \rangle, \tag{22}$$

where $\lambda_j \in C^\infty(\mathbb{R}^k, \mathbb{R}^m)$, $j = 0, \ldots, n-1$, are Lagrangian vector multipliers, to be found further in an explicit form. The Poisson brackets of the constraints (20) with H_c must vanish and this requirement will determine vector multipliers λ_j, $j = 0, \ldots, n-1$, provided there exist no further independent constraints in the problem under consideration. We have for all $k = 0, \ldots, n-1$

$$\{\mathcal{H}_c, c_k\} = 0 = \{\mathcal{H}, c_k\} + \sum_{j=0}^{n-1} \lambda_j \{c_j, c_k\}. \tag{23}$$

If $\det \|\{c, c\}\|_{j,k=0,\ldots,n-1} \neq 0$, from (23) one can find vector multipliers λ_j, $j = 0, \ldots, n-1$:

$$\lambda_j = -\sum_k \|\{c, c\}\|_{j,k}^{-1} \{c_k, \mathcal{H}\}. \tag{24}$$

Thus, the total Hamiltonian $\mathcal{H}_c$ defined by (22) takes the form:

$$\mathcal{H}_c = \sum_{j=0}^{n-1} \int_{\mathbb{R}^k} \left[\left\langle \pi_j, \frac{dv_j}{dt} \right\rangle - L[u, u_t] \right] dx - \sum_{j,k} c_j \|\{c, c\}\|_{j,k}^{-1} \{c_k, \mathcal{H}\}. \tag{25}$$

When the case $\det \|\{c, c\}\|_{j,k} = 0$ takes place, we need to introduce some new secondary $nm - r$ constraints following from (23), where $r := \operatorname{rank} \|\{c, c\}\|$, and to proceed further by the same way as it was done before.

Josephson medium—Example 1

To be more precise, let us consider the following canonical Lagrangian from [3]:

$$\mathcal{L} = \int_{\mathbb{R}^2} \left[\frac{1}{2} (\langle \boldsymbol{\Phi}_t, \boldsymbol{\Phi}_t \rangle - \langle \operatorname{rot} \boldsymbol{\Phi}, \operatorname{rot} \boldsymbol{\Phi} \rangle) - \sum_{i=1}^{2} (1 - \cos \varphi_i) \right] dx, \tag{26}$$

where $\langle \cdot, \cdot \rangle$ the usual scalar product in $\mathbb{R}^3$. We make the following change of variables

$$\boldsymbol{\Phi} := \nabla \Theta + \mathbf{A}, \quad \boldsymbol{\Phi} := (\varphi_1, \varphi_2, 0). \tag{27}$$

From (26) and (27) we obtain:

$$\mathcal{L} = \int_{\mathbb{R}^2} dx \left\{ \frac{1}{2} [\langle \nabla \Theta_t + \mathbf{A}_t, \nabla \Theta_t + \mathbf{A}_t \rangle - \langle \operatorname{rot} \mathbf{A}, \operatorname{rot} \mathbf{A} \rangle] - \sum_{i=1}^{2} (1 - \cos(A_j + \partial \Theta / \partial x_j)) \right\}, \tag{28}$$

that is a new Lagrangian $\mathcal{L}$ with new independent variables Θ and $(A_1, A_2, 0)$.

Now we are in a position to use the basic theory presented above. At the very beginning we need to introduce the conjugated impulse variables:

$$\pi := \frac{\delta \mathcal{L}}{\delta \Theta_t} = -\Delta\Theta_t - \operatorname{div}\mathbf{A}_t \quad \text{and} \quad \mathbf{p} := \frac{\delta \mathcal{L}}{\delta \mathbf{A}_t} = \nabla\Theta_t + \mathbf{A}_t, \tag{29}$$

where $\Delta := \nabla^2$—the standard Laplacian.

The system (29) gives rise us to the following relationships:

$$\pi = -\operatorname{div}\mathbf{p}, \qquad \mathbf{A}_t = \mathbf{p} - \nabla\Theta_t. \tag{30}$$

As it was mentioned before and once more it is seen from (30) that the impulse π and the vector impulse $\mathbf{p}$ are not independent what is needed for the quantities Θ_t and $\mathbf{A}_t$ to be defined from (29) in a unique way. Thus we must introduce a one additional restriction as follows:

$$\pi + \operatorname{div}\mathbf{p} = c, \tag{31}$$

which can be read as a new constraint on the Lagrangian $\mathcal{L}$ on M.

The nontrivial canonical Poisson brackets are given as

$$\{\Theta(x), \pi(y)\} = -\delta(x-y), \quad \{A_j(x), p_k(y)\} = -\delta_{j,k}\delta(x-y), \tag{32}$$

where $j, k = 1, 2$. The augmented Hamiltonian $\mathcal{H}_c$ takes the form

$$\begin{aligned} \mathcal{H}_c := \int_{\mathbb{R}^2} dx \Big\{ & \pi(x)\Theta_t(x) + \langle \mathbf{p}(x), \mathbf{A}_t(x)\rangle \\ & - \tfrac{1}{2}\langle \nabla\Theta_t + \mathbf{A}_t, \nabla\Theta_t + \mathbf{A}_t\rangle + \tfrac{1}{2}\langle \operatorname{rot}\mathbf{A}, \operatorname{rot}\mathbf{A}\rangle \\ & + \sum_{i=1}^{2}\big(1 - \cos(A_i + \nabla_i\Theta)\big)\Big\} + \int_{\mathbb{R}^2} dx \big[\pi(x) + \operatorname{div}\mathbf{p}(x)\big]\lambda(x), \end{aligned} \tag{33}$$

where $\mathbf{A}_t = \mathbf{p} - \nabla\Theta_t$ and $\lambda(x)$ is some Lagrangian multiplier. As a result from (33) we obtain

$$\begin{aligned} \mathcal{H}_c := \int_{\mathbb{R}^2} dx \Big\{ & \tfrac{1}{2}\big[\langle \mathbf{p}(x), \mathbf{p}(x)\rangle + \langle \operatorname{rot}\mathbf{A}, \operatorname{rot}\mathbf{A}\rangle\big] \\ & + \sum_{i=1}^{2}\big(1 - \cos(A_i + \nabla_i\Theta)\big) + \big[\pi(x) + \operatorname{div}\mathbf{p}(x)\big]\lambda(x)\Big\}. \end{aligned} \tag{34}$$

3. SYMPLECTIC ANALYSIS

We can convince ourselves quite simply that the constraint (31) is of the first class and has gauge nature. Indeed, let us recast the canonical Hamiltonian system (34) into the usual symplectic picture. First we define a functional phase manifold $M_\pi \subset T^*(C^\infty(\mathbb{R}^2, \mathbb{R}^2)) \times T^*(C^\infty(\mathbb{R}^2, \mathbb{R}))$ with coordinates $(\mathbf{A}, \mathbf{p}; \Theta, \pi) \in M_\pi$. Upon the manifold M_π there exists the canonical symplectic structure

$$\omega^{(2)} := \int_{\mathbb{R}^2} [\langle d\mathbf{p}, \wedge d\mathbf{A}\rangle + d\pi \wedge d\Theta]\, dx,$$

where $\langle d\mathbf{p}, \wedge d\mathbf{A}\rangle := \sum_j dp_j \wedge dA_j$, with respect to which the Euler equation $\delta\mathcal{S}[\Theta, \mathbf{A}] = 0$ is equivalent to the corresponding Hamiltonian system stemming from (34):

$$\begin{cases} \pi_t = \operatorname{div}[\sin(\mathbf{A} + \nabla\Theta)], \\ \mathbf{p}_t = -\operatorname{rot}\operatorname{rot}\mathbf{A} - \sin(\mathbf{A} + \nabla\Theta), \\ \mathbf{A}_t = \mathbf{p} - \nabla\lambda(\mathbf{x}), \\ \Theta_t = \lambda(\mathbf{x}), \end{cases} \tag{35}$$

where $\lambda \in C^\infty(\mathbb{R}^2, \mathbb{R})$ is some gauge function. We can see that the condition $\mathbf{A}_t = \mathbf{p} - \nabla\Theta_t$ holds for all $t \in \mathbb{R}$, as well.

Notice that the Hamiltonian (34) is invariant with respect to the following gauge group action $g_\psi : M_\pi \to M_\pi$

$$M_\pi \ni (\mathbf{A}, \mathbf{p}; \Theta, \pi) \xrightarrow{g_\psi} (\mathbf{A} + \nabla\psi, \mathbf{p}; \Theta - \psi, \pi) \in M_\pi \tag{36}$$

of an Abelian Lie group G, consisting of real valued invertible functions on $\mathbb{R}^2$with an addition group operation:

$$G = \left\{\exp\psi \in C^\infty(\mathbb{R}^2, \mathbb{R}) \colon \sup_{\mathbf{x}\in\mathbb{R}^2} |\mathbf{x}^k \times \psi^{(n)}(\mathbf{x})| < \infty,\ k, n \in \mathbb{Z}_+^2\right\}, \tag{37}$$

where $\mathbf{x}^k := x_1^{k_1} x_2^{k_2}$, $\psi^{(n)}(\mathbf{x}) := \partial^{n_1+n_2}\psi(\mathbf{x})/\partial x_1^{n_1}\partial x_2^{n_2}$, by definitions.

The above means that the gauge group G coincides with the standard Schwartz functional space on $\mathbb{R}^2$ and the invariance property $H_c \circ g_\psi = H_c$ takes place on the manifold M_π for all $\exp(\psi) \in G$. Since the Lie algebra $\mathcal{G}$ of the Lie group G coincides as a manifold with the space G, we will interpret further the gauge element $\psi \in \ln G$ in (36) as that belonging to the Lie group G. Notice here that the gauge action (36) is a symplectic one on the symplectic manifold $(M_\pi, \omega^{(2)})$:

$$g_\psi^* \omega^{(2)} := \omega^{(2)} \tag{38}$$

for all $\exp(\psi) \in G$. Indeed,

$$\begin{aligned} g_\psi^* \omega^{(2)} &= \int_{\mathbb{R}^2} dx \left[\langle d\mathbf{p}, \wedge d(\mathbf{A} + \nabla\psi) \rangle + d\pi \wedge d(\Theta - \psi) \right] \\ &= \int_{\mathbb{R}^2} dx \left[\langle d\mathbf{p}, \wedge d\mathbf{A} \rangle - \langle d(\operatorname{div} \mathbf{p}) \wedge d\psi \rangle + d\pi \wedge d\Theta - d\pi \wedge d\psi \right] \\ &= \omega^{(2)} - \int_{\mathbb{R}^2} dx\, d(\operatorname{div} \mathbf{p} + \pi) \wedge d\psi \equiv \omega^{(2)} \end{aligned} \tag{39}$$

because of both the equality $d\psi \wedge dx \equiv 0$ and the compatible constraint condition $\operatorname{div} \mathbf{p} + \pi = c$ on the manifold M_π. Compatibility here means that the condition

$$c := \operatorname{div} \mathbf{p} + \pi \equiv c_0(\mathbf{x})$$

holds for all times $t \in \mathbb{R}$, that is

$$c_t := \{H_c, c\} = 0, \tag{40}$$

which is simply proved to be true. Now we are in a position to use the standard Marsden–Weinstein reduction procedure [17] with respect to the group action (36).

Let us define a momentum map $J : M_\pi \to \mathcal{G}^*$ related with the action (36). By definition we have: for any $\psi \in \mathcal{G}$

$$(J[\mathbf{A}, \mathbf{p}; \Theta, \pi], \psi) = \int_{\mathbb{R}^2} dx \left[\langle \mathbf{p}, \nabla\psi \rangle - \psi\pi \right] = -\int_{\mathbb{R}^2} dx (\operatorname{div} \mathbf{p} + \pi, \psi) \tag{41}$$

whence we obtain the momentum map

$$J[\mathbf{A}, \mathbf{p}; \Theta, \pi] = -\operatorname{div} \mathbf{p} - \pi := -c \in \mathcal{G}^*. \tag{42}$$

The result above is almost obvious since the constraint $c = c_0(\mathbf{x})$ is the even one being involved on the phase manifold M_π for the Hamiltonian flow of (34) to be compatible. From (42) we see that the submanifold

$J^{-1}(c = c_0(\mathbf{x})) \subset M_\pi$ can be reduced [12,16] on some submanifold $M_0 \subset M_\pi$ with respect to the group action (36)

$$\begin{aligned} M_\pi \to M_0 &= J^{-1}(c = c_0(\mathbf{x}))/G \\ &= \{(\mathbf{A}, \mathbf{p}; \theta, \pi) \in M_\pi \colon \operatorname{div} \mathbf{p} + \pi = c_0(\mathbf{x})\} \\ &\subset \{(\mathbf{A}, \mathbf{p}) \in T^*(C^\infty(\mathbb{R}^2, \mathbb{R}^2))\}. \end{aligned} \tag{43}$$

The reduced symplectic structure $\omega_0^{(2)}$ on the submanifold M_0 is given by

$$\omega_0^{(2)} := \int_{\mathbb{R}^2} dx \langle d\mathbf{p}, \wedge d\mathbf{A}\rangle, \tag{44}$$

and the corresponding reduced Hamiltonian $\mathcal{H}_0 :\in \mathcal{D}(M_0)$ has the form:

$$\mathcal{H}_0 = \int_{\mathbb{R}^2} dx \left[\tfrac{1}{2}(\langle \mathbf{p}, \mathbf{p}\rangle + \langle \operatorname{rot} \mathbf{A}, \operatorname{rot} \mathbf{A}\rangle) + \sum_{j=1}^{2}(1 - \cos A_j)\right], \tag{45}$$

where we have fixed the gauge invariance as follows:

$$\mathbf{A}(\mathbf{x}) \to \mathbf{A}(\mathbf{x}) - \nabla\Theta(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2.$$

The resulting Hamiltonian flow on M_0 is completely equivalent to the flow (35) on M:

$$\mathbf{p}_t = -\operatorname{rot} \operatorname{rot} \mathbf{A} - \sin \mathbf{A}, \qquad \mathbf{A}_t = \mathbf{p}, \tag{46}$$

with condition $\operatorname{div} \mathbf{p} + \pi = c_0(\mathbf{x})$. Thus we have built the closed Hamiltonian theory related to the Lagrangian (28), on the symplectic phase space $M_\pi \subset T^*(C^\infty(\mathbb{R}^2, \mathbb{R}^2)) \times T^*(C^\infty(\mathbb{R}^2, \mathbb{R}))$.

4. AN EQUIVALENT OPERATOR APPROACH

Let us be given any dynamical system

$$u_t = K[u] \tag{47}$$

on some functional manifold $M \ni u$, where $K : M \to T(M)$ is a smooth vector field on M, the flow (47) can be recast into the following Eulerian form

$$\frac{\delta S}{\delta u} = 0 \tag{48}$$

with the action functional S to be

$$S = \int_{\mathbb{R}^n} dx \int_0^t dt \{\langle \varphi[u], u_t \rangle - H[u]\} \tag{49}$$

if and only if the dynamical system (47) is Hamiltonian on M. This means that if

$$u_t = K[u] := -\Omega^{-1} \operatorname{grad} \mathcal{H}[u], \tag{50}$$

where the symplectic structure $\Omega : T(M) \to T^*(M)$ is a nondegenerate skew-symmetric operator on the adjoint space $T(M)$, satisfying the following condition:

$$\Omega := \varphi' - \varphi'^*. \tag{51}$$

The element $\varphi[u] \in T^*(M)$ is given by (49) as the definition. It is obvious that the operator Ω in (51) is skew-symmetric one acting on $T(M)$. From (49) we can obtain easily that the equation

$$-\varphi_t + \varphi'^* u_t - \operatorname{grad} \mathcal{H}[u] \Rightarrow \frac{\delta S}{\delta u} = 0, \tag{52}$$

gives rise to the Hamiltonian form as follows: using

$$\varphi_t = \varphi' K[u], \qquad u_t = K[u], \tag{53}$$

we have equivalently

$$\begin{aligned} -\varphi' K + \varphi'^* K &= \operatorname{grad} \mathcal{H}[u], \\ -(\varphi' - \varphi'^*) K &= \operatorname{grad} \mathcal{H}[u], \\ u_t = -\Omega^{-1} \operatorname{grad} \mathcal{H}[u] &= K[u], \end{aligned} \tag{54}$$

that is we have obtained the standard Hamiltonian form of the system (47). Besides we claim that the local functional $\varphi[u] \in T^*(M)$ satisfies the following determining (or characteristic) Lax type equation [13, p. 216]:

$$\varphi_t + K'^* \varphi = \operatorname{grad} \xi[u], \tag{55}$$

where $\xi \in \mathcal{D}(M)$—some canonical transformation generating functional in the Hamiltonian–Jacobi theory. Indeed, from (55) we can get that

$$\varphi_t + K'^* \varphi = \varphi' K - \varphi'^* K + (\varphi'^* K + K'^* \varphi)$$

$$= (\varphi' - \varphi'^{*})K + \operatorname{grad}(\varphi, K) = -\operatorname{grad}\mathcal{H}[u] + \operatorname{grad}(\varphi, K)$$
$$= \operatorname{grad}\big[(\varphi, K) - \mathcal{H}\big][u] := \operatorname{grad}\xi[u], \tag{56}$$

that is,

$$\xi[u] := \langle\varphi, K\rangle - H[u], \tag{57}$$

the generating functional of the Hamiltonian system (54).

The above results are completely valid up to the exact form of the Hamiltonian function $\mathcal{H}$ because of the some ambiguous definition of the local functional $\varphi[u] \in T^*(M)$. Indeed, we have

$$\xi[u] = \langle\varphi, K\rangle - H[u] \sim \langle\varphi + \psi, K\rangle - H[u]$$
$$= \langle\varphi, K\rangle + \langle\psi, K\rangle - H[u], \tag{58}$$

where $\psi \in T^*(M)$ is some gradient-wise kernel of the vector field $K[u]$: $(\psi, K) = 0$, $\psi' = \psi'^{*}$, it means that there exists some functional $Q \in \mathcal{D}(M)$ such that $\operatorname{grad} Q = \psi$ on M. Here and below the sign "$'$" denotes the standard Frechet derivative of a local functional on M. Therefore, for the Hamiltonian density $H[u]$ to be found in exact form, we need to calculate the quantity

$$\varphi' K + K'^{*}\varphi = \operatorname{grad}\xi[u], \tag{59}$$

determining the "Lagrangian" $\xi[u]$. Further we construct the Hamiltonian density $H[u]$ as follows:

$$H[u] := \langle\varphi, K\rangle - \xi[u] \tag{60}$$

stemming from (57). This consideration ends the formal setting of the Hamiltonian theory via the operator approach on the manifold M with a given dynamical system (47).

Josephson medium—Example 2

To use the above results to the Lagrangian (28), we find that its density takes the following form on an extended manifold $M' = (\mathbf{A}, \mathbf{q}; \Theta, \lambda)$

$$L[\mathbf{A}, \mathbf{q}; \Theta, \lambda] = \big\langle\varphi, (\mathbf{A}_t, \mathbf{q}_t; \Theta_t, \lambda_t)^{\tau}\big\rangle - \tfrac{1}{2}\langle\operatorname{rot}\mathbf{A}, \operatorname{rot}\mathbf{A}\rangle$$
$$- \sum_{j=1}^{3}\big(1 - \cos(A_j + \partial\Theta/\partial x_j)\big) + \langle\mu, \mathbf{A}_t - \mathbf{q}\rangle + \nu(\Theta_t - \lambda), \tag{61}$$

where by definition $\mathbf{q} := \mathbf{A}_t$, $\lambda := \Theta_t$, $\varphi \in T^*(M')$, μ, ν are some Lagrangian multipliers, and

$$\varphi := (\mathbf{q}/2,\ 0,\ -\nabla\mathbf{q} - \Delta\lambda/2,\ 0)^{\tau}. \tag{62}$$

(Here the sign "τ" denotes the usual transposition operation.) From (62) using the formula (51) we obtain the corresponding symplectic operator Ω as follows:

$$\Omega = \varphi' - \varphi'^{*} = \left\|\begin{matrix} 0 & 1/2 & 0 & 0 \\ -1/2 & 0 & -\nabla & 0 \\ 0 & -\nabla & 0 & -\Delta/2 \\ 0 & 0 & \Delta/2 & 0 \end{matrix}\right\|. \tag{63}$$

The inverse operator Ω^{-1} is given by the expression:

$$\Omega^{-1} = \left\|\begin{matrix} 0 & 2 & 0 & 4\nabla^{-1} \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\Delta^{-1} \\ 4\nabla^{-1} & 0 & -2\Delta^{-1} & 0 \end{matrix}\right\|. \tag{64}$$

Thus we can represent the Euler dynamical system $\delta S[\mathbf{A}, \mathbf{q}; \Theta, \lambda] = 0$ as a Hamiltonian one:

$$\frac{d}{dt}(\mathbf{A}, \mathbf{q}; \Theta, \lambda)^{\tau} = -\Omega^{-1}\operatorname{grad}\mathcal{H}[\mathbf{A}, \mathbf{q}; \Theta, \lambda], \tag{65}$$

where by definition

$$\mathcal{H} := (\varphi, K) - \xi, \qquad \varphi' K + K'^{*}\varphi = \operatorname{grad}\xi, \tag{66}$$

and

$$\begin{cases} 0 = \delta\mathcal{L}/\delta\mathbf{A} = -\boldsymbol{\mu}_t - \frac{1}{2}\mathbf{q}_t - \operatorname{rot}\operatorname{rot}\mathbf{A} - \sin(\mathbf{A} + \nabla\Theta), \\ 0 = \delta\mathcal{L}/\delta\mathbf{q} = \frac{1}{2}\mathbf{A}_t + \nabla\Theta_t - \boldsymbol{\mu}, \\ 0 = \delta\mathcal{L}/\delta\Theta = \nabla.\mathbf{A}_{tt} + \frac{1}{2}\Delta\lambda_t + \nabla\cdot\sin(\mathbf{A} + \nabla\Theta) - \nu_t, \\ 0 = \delta\mathcal{L}/\delta\lambda = -\frac{1}{2}\Delta\Theta_t - \nu, \\ 0 = \delta\mathcal{L}/\delta\boldsymbol{\mu} = \mathbf{A}_t - \mathbf{q}, \\ 0 = \delta\mathcal{L}/\delta\nu = \Theta_t - \lambda. \end{cases} \tag{67}$$

It is obviously proved that the system of Eqs. (67) is completely equivalent to (35). From (67) one can simply extract the above found vector field $K[\mathbf{A}, \mathbf{q}; \Theta, \lambda]$:

$$\begin{cases} \mathbf{A}_t = \mathbf{q}, \\ \mathbf{q}_t = -\nabla\lambda_t - \operatorname{rot}\operatorname{rot}\mathbf{A} - \sin(\mathbf{A} + \nabla\Theta), \\ \lambda_t = -\nabla^{-1} \cdot [\mathbf{q}_t + \sin(\mathbf{A} + \nabla\Theta) + \operatorname{rot}\operatorname{rot}\mathbf{A}], \\ \Theta_t = \lambda. \end{cases} \tag{68}$$

We see from (68) that phase variables $\mathbf{q}$ and λ are not independent, that compels us to involve new Clebsch type independent phase variables $\pi, \mathbf{p}$ as follows:

$$\pi := -\Delta\lambda - \operatorname{div}\mathbf{q}, \qquad \mathbf{p} := \mathbf{q} + \nabla\lambda. \tag{69}$$

As a result we obtain a new evolution system on a properly extended functional manifold M_π:

$$\begin{cases} \mathbf{A}_t = \mathbf{p} - \nabla\lambda, \\ \mathbf{p}_t = -\operatorname{rot}\operatorname{rot}\mathbf{A} - \sin(\mathbf{A} + \nabla\Theta), \\ \Theta_t = \lambda, \\ \pi_t = \operatorname{div}\sin(\mathbf{A} + \nabla\Theta), \end{cases} \tag{70}$$

that is the system completely identical with (35), the function $\lambda(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^2$, being interpreted as a functional parameter. From (69) we can easily obtain also that the constraint

$$\operatorname{div}\mathbf{p} + \pi = c_0(\mathbf{x}) \tag{71}$$

is satisfied on the manifold M_π identically. The flow (70) persists the constraint (71) as well:

$$\operatorname{div}\mathbf{p}_t + \pi_t = 0 \tag{72}$$

upon all orbits of (70). The important consequence of the consideration performed above is that the Lagrangian $\mathcal{L}$ (28) can be represented in the correct nondegenerate form only on the extended functional manifold M_π, giving rise to the following expression:

$$\begin{aligned} &L[\mathbf{A}, \mathbf{q}; \Theta, \lambda] \Rightarrow L[\mathbf{A}, \mathbf{p}; \Theta, \pi] \\ &\quad = \left\langle \tfrac{1}{2}\mathbf{p} - \alpha, \mathbf{A}_t \right\rangle + \left(-\tfrac{1}{2}\operatorname{div}\mathbf{p} + \operatorname{div}\alpha\right)\Theta_t + \langle \alpha, \mathbf{p}\rangle \\ &\quad\quad - \tfrac{1}{2}\langle \operatorname{rot}\mathbf{A}, \operatorname{rot}\mathbf{A}\rangle - \sum_{j=1}^{3}\left[1 - \cos(A_j + \partial\Theta/\partial x_j)\right] + \lambda(\pi + \operatorname{div}\mathbf{p}), \end{aligned} \tag{73}$$

where the vector $\alpha \in C^{\infty}(\mathbb{R}^2; \mathbb{R}^2)$ serves as a Lagrangian multiplier subject to the constraint $\mathbf{p} = \mathbf{A}_t + \nabla\Theta_t$ on the manifold M_π, the function λ-serves like that subject to the constraint $\pi + \operatorname{div}\mathbf{p} = c_0(\mathbf{x})$ on M_π. From (73) we get that the corresponding vector $\varphi \in T^*(M_\pi)$ is given as follows:

$$\varphi = \left(\tfrac{1}{2}\mathbf{p} - \alpha,\ 0,\ -\tfrac{1}{2}\operatorname{div}\mathbf{p} + \operatorname{div}\alpha,\ 0\right)^\tau. \tag{74}$$

The expression (74) leads us using (51) to the following coimplectic structure:

$$\Omega = \begin{Vmatrix} 0 & 1/2 & 0 & 0 \\ -1/2 & 0 & -\nabla/2 & 0 \\ 0 & -\nabla/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix}. \tag{75}$$

Unfortunately, the operator (75) is strongly degenerate what means the necessity to make a reduction of the flow (70) on some submanifold $M_0 \subset M_\pi$.

To proceed further, let us determine the Hamiltonian function related with the flow (70). To do this we need to calculate only the functional $\xi \in \mathcal{D}(M_\pi)$ using the definition (66):

$$\begin{aligned} &\varphi_t + K'^*\varphi \\ &\quad = \left(\tfrac{1}{2}\mathbf{p}_t,\ 0,\ -\tfrac{1}{2}\operatorname{div}\mathbf{p}_t,\ 0\right)^\tau + K'^*\left(\tfrac{1}{2}\mathbf{p} - \alpha,\ 0,\ -\tfrac{1}{2}\operatorname{div}\mathbf{p} + \operatorname{div}\alpha,\ 0\right)^\tau \\ &\quad = \begin{pmatrix} -\tfrac{1}{2}[\nabla \times (\nabla \times \mathbf{A}) + \sin(\mathbf{A} + \nabla\Theta)] \\ 0 \\ \tfrac{1}{2}\nabla \cdot \sin(\mathbf{A} + \nabla\Theta) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \tfrac{1}{2}\mathbf{p} \\ 0 \\ 0 \end{pmatrix} \\ &\quad = \begin{pmatrix} -\tfrac{1}{2}\nabla \times (\nabla \times A) - \tfrac{1}{2}\sin(A + \nabla\Theta) \\ \tfrac{1}{2}\mathbf{p} \\ \tfrac{1}{2}\nabla \cdot \sin(\mathbf{A} + \nabla\Theta) \\ 0 \end{pmatrix} \Rightarrow \operatorname{grad}\xi := \mathbf{a}. \end{aligned} \tag{76}$$

The necessary condition for the above vector $\mathbf{a} \in T^*(M_\pi)$ to be a gradient is the following Volterra criterium: $\mathbf{a}' = \mathbf{a}'^*$, i.e., selfadjointness of the corresponding Frechet derivative. It is easy to show that this condition holds good on M_π. Thus we are in position to calculate the functional $\xi \in \mathcal{D}(M_\pi)$ via the following homotopy formula:

$$\xi = \int_{\mathbb{R}^2} dx \int_0^1 d\lambda \langle \mathbf{a}[\lambda\mathbf{A}, \lambda\mathbf{p}, \lambda\Theta, \lambda\pi], (\mathbf{A}, \mathbf{p}, \Theta, \pi)^\tau \rangle$$

$$= \frac{1}{2} \int_{\mathbb{R}^2} dx \int_0^1 d\lambda \big[-\langle \nabla \times (\nabla \times \mathbf{A}\lambda), \mathbf{A} \rangle - \langle \sin(\lambda \mathbf{A} + \lambda \nabla \Theta), \mathbf{A} \rangle$$

$$+ \langle \lambda \mathbf{p} - \alpha, \mathbf{p} \rangle + \langle \nabla \cdot \sin(\lambda \mathbf{A} + \lambda \nabla \Theta), \Theta \rangle \big]$$

$$= \frac{1}{4} \int_{\mathbb{R}^2} dx \bigg\{ \langle \operatorname{rot} \mathbf{A}, \operatorname{rot} \mathbf{A} \rangle + 2 \sum_{j=1}^{3} [\cos(A_j + \nabla_j \Theta) - 1]$$

$$+ \langle \mathbf{p}, \mathbf{p} \rangle - 4 \langle \alpha, \mathbf{p} \rangle \bigg\}. \tag{77}$$

Whence the Hamiltonian functional $\mathcal{H} \in \mathcal{D}(M_\pi)$ is given via (66) as follows:

$$\mathcal{H} = \int_{\mathbb{R}^2} \langle \mathbf{Q}_1, \mathbf{Q}_2 \rangle \, dx - \xi, \tag{78}$$

where we denoted

$$\mathbf{Q}_1 := \left(\tfrac{1}{2} \mathbf{p} - \alpha, 0, -\tfrac{1}{2} \operatorname{div} \mathbf{p} + \operatorname{div} \alpha, 0 \right), \tag{79}$$

$$\mathbf{Q}_2 := \big(\mathbf{p} - \nabla \lambda, \nabla \times (\nabla \times \mathbf{A}) - \sin(\mathbf{A} + \nabla \Theta),$$
$$\lambda, \nabla \cdot \sin(\mathbf{A} + \nabla \Theta) \big)^\tau. \tag{80}$$

Thus

$$\mathcal{H} = \int_{\mathbb{R}^2} dx \bigg\{ \frac{1}{2} \langle \mathbf{p}, \mathbf{p} \rangle - \langle \alpha, \mathbf{p} \rangle + \frac{1}{4} \langle \operatorname{rot} \mathbf{A}, \operatorname{rot} \mathbf{A} \rangle \tag{81}$$

$$- \sum_{j=1}^{2} \frac{1}{2} [\cos(A_j + \nabla_j \Theta) - 1] - \frac{1}{4} \langle \mathbf{p}, \mathbf{p} \rangle + \langle \alpha, \mathbf{p} \rangle \bigg\}$$

$$= \frac{1}{2} \int_{\mathbb{R}^2} dx \bigg\{ \frac{1}{2} [\langle \mathbf{p}, \mathbf{p} \rangle + \langle \operatorname{rot} \mathbf{A}, \operatorname{rot} \mathbf{A} \rangle] + \sum_{j=1}^{2} [1 - \cos(A_j + \nabla_j \Theta)] \bigg\}.$$

Thereby, we have got the Hamiltonian functional which up to factor 2 coincides with that given by formula (34) and that given in [3,9]. From the structure of (75) one can see that the dynamical system (70) allows the nondegenerate reduction upon the submanifold $M_0 \subset M''$ with the canonical symplectic structure Ω_0:

$$\Omega_0 = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}, \tag{82}$$

that is

$$\mathbf{A}_t = \{\mathcal{H}, \mathbf{A}\}_0, \qquad \mathbf{p}_t = \{\mathcal{H}, \mathbf{p}\}_0, \tag{83}$$

where by definition we have adopted the following Poisson bracket notation.

$$\{\cdot, \cdot\}_0 := \int_{\mathbb{R}^2} dx \langle \operatorname{grad}(\cdot),\ \Omega_0^{-1} \operatorname{grad}(\cdot) \rangle. \tag{84}$$

Note that for the above functional $\xi \in \mathcal{D}(M_\pi)$ to be obtained via calculations (76), the following expressions for Frechet derivatives K' and K'^* were used

$$K' = \begin{Vmatrix} -(\nabla \times \nabla \times) - \cos(\mathbf{A} + \nabla \Theta) & 0 & -\cos(\mathbf{A} + \nabla \Theta) \cdot \nabla & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \nabla \cdot \cos(\mathbf{A} + \nabla \Theta) & 0 & (\nabla \cdot \cos(\mathbf{A} + \nabla \Theta))\nabla & 0 \end{Vmatrix}, \tag{85}$$

$$K'^* = \begin{Vmatrix} -(\nabla \times \nabla \times) - \cos(\mathbf{A} + \nabla \Theta) & 1 & 0 & -\cos(\mathbf{A} + \nabla \Theta) \cdot \nabla \\ 0 & 0 & 0 & 0 \\ \nabla \cdot \cos(\mathbf{A} + \nabla \Theta) & 0 & 0 & (\nabla \cdot \cos(\mathbf{A} + \nabla \Theta))\nabla \\ 0 & 0 & 0 & 0 \end{Vmatrix}. \tag{86}$$

5. SOME COROLLARIES AND PERSPECTIVES

We have demonstrated two different approaches to general Euler–Lagrangian type dynamical systems on functional manifolds with constraints of a gauge type.

The constraints (31) being found via analysis the Lagrangian problem (28), as was shown by (72), is a single and of the first class. This fact suggests to perform the standard reduction upon some invariant submanifold on which the reduced symplectic structure is nondegenerate. This was the case treated in detail in this study of the special Lagrangian problem (28).

The momentum map $J[\mathbf{A}, \mathbf{p}, \Theta, \pi]$, defined by (42) in the frame of our approach admits a further physical interpretation being important for the study of Josephson media more deeply. Especially, the constraint

$$\operatorname{div} \mathbf{p} + \pi = c_0(\mathbf{x}) \Leftrightarrow \operatorname{div} \mathbf{p} = c_0(\mathbf{x}) - \pi \in C^\infty(\mathbb{R}^2; \mathbb{R}) \tag{87}$$

imposes on the reduced symplectic manifold $M_0 \subset T^*(C^\infty(\mathbb{R}^2;\mathbb{R}^2))$ some functional conditions, which can be interpreted as a charge density a'priori disseminated throughout a Josephson medium under consideration. This constraint imposed on the wanted solutions to the Eqs. (46) should obviously play the role of the physical compatibility with the flow of charged particles in the Josephson medium. Indeed, if these collisionless particles interact with the induced electromagnetic Maxwell field, one can write down the following Josephson–Vlasov (J–V) dynamical equations

$$f_t + \langle \mathbf{v}, \nabla f\rangle + \frac{e}{m}\left\langle \left(-\mathbf{p} + \frac{\mathbf{v}}{c}\times \operatorname{rot}\mathbf{A}\right), \frac{\partial f}{\partial \mathbf{v}}\right\rangle = 0, \tag{88}$$

$$\mathbf{p} = \mathbf{A}_t/c, \tag{89}$$

$$-\operatorname{rot}\operatorname{rot}\mathbf{A} + \frac{e}{c}\int_{\mathbb{R}^2} f(\mathbf{x},\mathbf{v})\mathbf{v}\,dv - \sin\mathbf{A} = \mathbf{p}_t/c, \tag{90}$$

$$\pi = -\operatorname{div}\mathbf{p} - e\int_{\mathbb{R}^2} f(\mathbf{x},\mathbf{v})\,dv, \tag{91}$$

on the functional manifold $M^{(f)} = M \times F$, where the last equation determines the ambient charge density as a compatibility constraint condition. The quantity $f \in F \subset C^\infty(T^*(\mathbb{R}^2;\mathbb{R}_+))$represents the charged particle density in position-velocity space at time $t \in \mathbb{R}$; $e \in \mathbb{R}$ is the particle charge and $m \in \mathbb{R}_+$ is the particle mass. Here we consider the motion of a cloud of charged particles inside a Josephson medium of a single species for simplicity. A generalization introducing several species is pretty elementary. We choose further the natural unit system in which $e = m = c = 1$. Our goal now is to understand a Hamiltonian structure of the J–V Eqs. (88) using the group theoretical momentum map reduction approach of Marsden–Weinstein [17]. To proceed further one needs the investigation of the symmetry properties of the corresponding to (88) Hamiltonian functional

$$\mathcal{H} = \frac{1}{2}\int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dv\left[\langle \mathbf{v},\mathbf{v}\rangle f(\mathbf{x},\mathbf{v})\right] + \frac{1}{2}\int_{\mathbb{R}^2} dx\left[\langle \mathbf{p},\mathbf{p}\rangle + \langle \operatorname{rot}\mathbf{A}, \operatorname{rot}\mathbf{A}\rangle\right] + \int_{\mathbb{R}^2} dx \sum_{j=0}^{2}(1-\cos A_j). \tag{92}$$

Since Hamiltonian (92) is written in improper phase space variables $(\mathbf{x}, \mathbf{v}) \in T^*(\mathbb{R}^2)$ we introduce a proper momentum-position phase space with variables $(\mathbf{x}, \mathbf{y}) \in T^*(\mathbb{R}^2)$ carrying the canonical symplectic structure $\Omega^{(2)} := \sum_j dy_j \wedge dx_j := \langle d\mathbf{y}, \wedge\, d\mathbf{x}\rangle$, the consistency condition $\mathbf{y} = \mathbf{v} + \mathbf{A}$ being satisfied throughout the Josephson medium.

Corresponding to the definition of the phase space as above, we introduce a new particle density distribution function $\tilde{f}(\mathbf{x}, \mathbf{y}) \in F \subset C^\infty(T^*(\mathbb{R}^2); \mathbb{R}_+)$ as follows

$$f(\mathbf{x}, \mathbf{v}) \equiv \tilde{f}(\mathbf{x}, \mathbf{y} = \mathbf{v} + \mathbf{A}) \tag{93}$$

identically upon $T^*(\mathbb{R}^2)$ for all $t \in \mathbb{R}$. As a result we obtain the next expression for the Hamiltonian functional (92)

$$\begin{aligned}\mathcal{H} = \tfrac{1}{2} \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy \big[\langle \mathbf{y} - \mathbf{A}, \mathbf{y} - \mathbf{A}\rangle f(\mathbf{x}, \mathbf{y})\big] \\ + \int_{\mathbb{R}^2} dx \sum_{j=0}^{2} (1 - \cos A_j) + \tfrac{1}{2} \int_{\mathbb{R}^2} dx \big[\langle \mathbf{p}, \mathbf{p}\rangle + \langle \operatorname{rot} \mathbf{A}, \operatorname{rot} \mathbf{A}\rangle\big].\end{aligned} \tag{94}$$

Let us consider the following functional gauge transformation

$$g_\Theta : M^{(f)} \to M^{(f)}$$

of the whole phase space $M^{(f)} := M \times F \subset T^*(C^\infty(\mathbb{R}^2; \mathbb{R}^2)) \times \mathcal{S}^*(\mathbb{R}^2; \mathbb{R})$,

$$\begin{aligned} F \ni \tilde{f}(\mathbf{x}, \mathbf{y}) &\overset{g_\Theta}{\to} \tilde{f}(\mathbf{x}, \mathbf{y} - \nabla\Theta) \in F, \\ M \ni (\mathbf{A}, \mathbf{p}) &\overset{g_\Theta}{\to} (\mathbf{A} + \nabla\Theta, \mathbf{p}) \in M,\end{aligned} \tag{95}$$

where we denoted by $\mathcal{S}^*(\mathbb{R}^2; \mathbb{R}) \subset C^\infty(T^*(\mathbb{R}^2; \mathbb{R}^2))$ the dual space to the Lie algebra $\mathcal{S}(\mathbb{R}^2; \mathbb{R})$ of canonical transformations of the symplectic space $T^*(\mathbb{R}^2)$ introduced above. The gauge function $\exp(\Theta) \in G$, used before, is chosen arbitrary, but satisfying the natural growth at infinity. The Lie algebra $\mathcal{S}(\mathbb{R}^2; \mathbb{R})$ consists of elements isomorphic to Hamiltonian vector fields on $T^*(\mathbb{R}^2)$ with respect to the canonical symplectic structure $\Omega^{(2)} = \langle d\mathbf{y}, \wedge\, d\mathbf{x}\rangle$ mentioned already above. Since the set of these Hamiltonian vector fields is in one to one correspondence to the set of generating Hamiltonian functions $\mathcal{S}(\mathbb{R}^2; \mathbb{R})$, the phase space $F \subset T^*(\mathcal{S}(\mathbb{R}^2; \mathbb{R}))$ describes correctly an infinite-dimensional space of particle density distribution functions $\tilde{f}(\mathbf{x}, \mathbf{y})\, dx\, dy$ at the point $(\mathbf{x}, \mathbf{y}) \in$

$T^*(\mathbb{R}^2)$. The last considerations give rise to getting a canonical Poisson bracket on the phase space F as a Lie–Poisson bracket [16,18] as follows: for any smooth functionals $\alpha, \beta \in \mathcal{D}(F)$ the bracket $\{\alpha, \beta\}_F$ reads as

$$\{\alpha, \beta\}_F := \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy \tilde{f}(\mathbf{x}, \mathbf{y}) \left\{ \frac{\delta \alpha}{\delta \tilde{f}}, \frac{\delta \beta}{\delta \tilde{f}} \right\}_{T^*(\mathbb{R}^2)}, \tag{96}$$

where $\{\cdot, \cdot\}_{T^*(\mathbb{R}^2)}$ denotes the standard Poisson bracket on $T^*(\mathbb{R}^2)$ corresponding to the canonical symplectic structure $\Omega^{(2)} = \langle d\mathbf{y}, \wedge d\mathbf{x} \rangle$ at $(\mathbf{x}, \mathbf{y}) \in T^*(\mathbb{R}^2)$.

In analogous way one can build a symplectic structure naturally connected with the Maxwell–Josephson part of the Hamiltonian (94). Indeed, if as a configuration space for this Hamiltonian part is considered, the corresponding symplectic phase space to the space of vector potential fields $\mathbf{A} \in C^\infty(\mathbb{R}^2; \mathbb{R}^2)$ can be defined. It is some set M pertaining to the cotangent bundle $T^*(C^\infty(\mathbb{R}^2; \mathbb{R}^2))$, carrying the canonical symplectic structure $\omega_M^{(2)} := \int_{\mathbb{R}^2} dx \langle d\mathbf{p}, \wedge d\mathbf{A} \rangle$, where $(\mathbf{A}, \mathbf{p}) \in M$. One can easily be convinced that the Maxwell–Josephson part of the Hamiltonian (94) is just described by the canonical symplectic structure $\omega_M^{(2)}$ that was used intensively in Section 2. Thus we have constructed via the canonical procedure the symplectic structure $\omega^{(2)}; \omega_F^{(2)}$ upon the phase space $M^{(f)} := M \times F$, where by $\omega_F^{(2)}$ is denoted the symplectic structure corresponding to the Lie–Poisson bracket $\{\cdot, \cdot\}_F$ (96) on the distribution function phase space $F \subset \mathcal{S}^*(\mathbb{R}^2; \mathbb{R})$.

Turning now back to the gauge group transformation $g_\Theta : M^{(f)} \to M^{(f)}$, $\Theta \in \mathcal{S}(\mathbb{R}^2; \mathbb{R})$ defined by (95), we claim that:

(1) the Hamiltonian functional (94) is not invariant, but
(2) the symplectic structure $\omega^{(2)}$ is invariant with respect to the mentioned above gauge transformation.

To cope with this situation we consider as before a new properly augmented symplectic functional phase space $M_\pi^{(f)} := M_\pi \times F$, where by the definition $M_\pi^{(f)} \subset M_\pi \times \mathcal{S}^*(\mathbb{R}^2; \mathbb{R})$. This means thereby that $M_\pi \simeq T^*(Or(G \times M \to M))$, where by $Or(G \times M \to M)$ we denoted a one-parameter orbit of the group action (95) on the symplectic manifold M. Since the symplectic structure $\omega_M^{(2)}$ is invariant with respect to this group action, we claim due to the general theory [16,19,20] and [17], that this orbit of M is necessarily a symplectic space with a canonical Lie–Poisson structure. In our case we obtain the following symplectic structure upon this orbit

$$\omega_{M_\pi}^{(2)} := \int_{\mathbb{R}^2} dx\big(\langle d\mathbf{p}, \wedge d\mathbf{A}\rangle + d\pi \wedge d\Theta\big),$$

where due to the construction above $(\Theta, \pi) \in T^*(\mathbb{R}^2)$. As a consequence the induced symplectic structure $\omega_\pi^{(2)} = \omega_{M_\pi}^{(2)} \otimes \omega_F^{(2)}$ is a subject to further considerations.

Let us consider now the Hamiltonian functional (94) properly extended as the one on the augmented phase space $M_\pi^{(f)}$:

$$\begin{aligned}\mathcal{H} = \tfrac{1}{2}\int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy\big[\langle \mathbf{y}-\mathbf{A}, \mathbf{y}-\mathbf{A}\rangle\, \tilde{f}(\mathbf{x},\mathbf{y})\big] \\ + \int_{\mathbb{R}^2} dx \sum_{j=1}^{2}\big[1-\cos(A_j + \partial\Theta/\partial x_j)\big] \\ + \tfrac{1}{2}\int_{\mathbb{R}^2} dx\big[\langle \mathbf{p},\mathbf{p}\rangle + \langle \operatorname{rot}\mathbf{A}, \operatorname{rot}\mathbf{A}\rangle\big].\end{aligned} \tag{97}$$

It is obvious now that both

(1) the Hamiltonian functional (97) is invariant, and

(2) the symplectic structure $\omega_\pi^{(2)}$ is invariant with respect to the extended group action (95) upon $M_\pi^{(f)}$.

This means that

$$g_\psi^*\omega_\pi^{(2)} = \omega_\pi^{(2)}, \qquad \mathcal{H}\cdot g_\psi = \mathcal{H} \tag{98}$$

for all $\exp(\psi) \in G$, where by definition

$$F \ni \tilde{f}(\mathbf{x},\mathbf{y}) \stackrel{g_\psi}{\rightarrow} \tilde{f}(\mathbf{x},\mathbf{y}-\nabla\psi) \in \mathbf{F}, \tag{99}$$

$$M \ni (\mathbf{A},\mathbf{p},\Theta,\pi) \stackrel{g_\psi}{\rightarrow} (\mathbf{A}+\nabla\psi,\ \mathbf{p},\ \Theta-\psi,\ \pi) \in M. \tag{100}$$

Thus, the same as it was in Section 2, we can find the corresponding to (99) momentum map $J : M_\pi^{(f)} \to \mathcal{G}^*$ as follows: for any $\psi \in \mathcal{G}$

$$\begin{aligned}(J[\mathbf{A},\mathbf{p},\Theta,\pi;\tilde{f}],\psi) = \int_{\mathbb{R}^2} dx\Big[\langle \mathbf{p},\nabla\psi\rangle - \psi\pi - \int_{\mathbb{R}^2}\tilde{f}\psi\,dy\Big] \\ \equiv -\Big(\Big[\operatorname{div}\mathbf{p} + \pi + \int_{\mathbb{R}^2}\tilde{f}\,dy\Big],\psi\Big),\end{aligned} \tag{101}$$

whence at each point $(\mathbf{A}, \mathbf{p}, \Theta, \pi; \tilde{f}) \in T^*(Or(M^{(f)})) \simeq M_\pi^{(f)}$

$$J[\mathbf{A}, \mathbf{p}, \Theta, \pi; \tilde{f}] = -\Big(\operatorname{div}\mathbf{p} + \pi + \int\limits_{\mathbb{R}^2} \tilde{f}(\mathbf{x}, \mathbf{y})\, dy\Big) = -c \in \mathcal{G}^*, \quad (102)$$

where by $\mathcal{G} \subset T(G)$ we denote some regular Lie subalgebra of the Lie group G.

As the first inference of the above consideration we can conclude that the Josephson–Vlasov dynamical system (88)–(91) in the properly extended form on the phase space $M_\pi^{(f)}$

$$\tilde{f}_t + \langle(\mathbf{y} - \mathbf{A}), \nabla \tilde{f}\rangle + \langle[-\nabla\lambda + (\mathbf{y} - \mathbf{A}) \times \operatorname{rot}\mathbf{A}], \partial\tilde{f}/\partial\mathbf{v}\rangle = 0, \quad (103)$$

$$\mathbf{A}_t = \mathbf{p} - \nabla\lambda, \quad (104)$$

$$-\operatorname{rot}\operatorname{rot}\mathbf{A} + \int\limits_{\mathbb{R}^2} \tilde{f}(\mathbf{x}, \mathbf{y})(\mathbf{y} - \mathbf{A})\, dy - \sin(\mathbf{A} + \nabla\boldsymbol{\Theta}) = \mathbf{p}_t, \quad (105)$$

$$\pi_t = \operatorname{div}\, \sin(\mathbf{A} + \nabla\boldsymbol{\Theta}), \quad (106)$$

$$\Theta_t = \lambda(\mathbf{x}) \quad (107)$$

with respect to the extended Hamiltonian function

$$\mathcal{H}_c = \mathcal{H} + \int\limits_{\mathbb{R}^2} c(\mathbf{x})\lambda(\mathbf{x})\, dx,$$

admits the invariant Marsden–Weinstein type reduction [17] upon the following reduced phase space $M_0^{(f)} := J^{-1}(c = c_0)/G \subset M_\pi^{(f)}$.

This means that the dynamical system (103)–(107) upon the invariant submanifold $M_0^{(f)} \subset M_\pi^{(f)}$ defined by the constraint

$$\operatorname{div}\mathbf{p} + \int\limits_{\mathbb{R}^2} \tilde{f}(\mathbf{x}, \mathbf{y})\, dy + \pi = c \equiv c_0(\mathbf{x}), \quad (108)$$

takes the form

$$\tilde{f}_t + \langle(\mathbf{y} - \mathbf{A}), \nabla \tilde{f}\rangle + \Big\langle(\mathbf{y} - \mathbf{A}), \sum_j (\partial\tilde{f}/\partial y_j\, \nabla A_j)\Big\rangle +$$

$$+ \langle[-\nabla\boldsymbol{\Theta}_t + (\mathbf{y} - A) \times \operatorname{rot}\mathbf{A}], \partial\tilde{f}/\partial\mathbf{y}\rangle = 0, \quad (109)$$

$$\mathbf{A}_t = \mathbf{p} - \nabla\boldsymbol{\Theta}_t, \tag{110}$$

$$-\mathrm{rot}\,\mathrm{rot}\,\mathbf{A} + \int\limits_{\mathbb{R}^2} \tilde{f}(\mathbf{x}, \mathbf{y})(\mathbf{y} - \mathbf{A})\,dy - \sin(\mathbf{A} + \nabla\boldsymbol{\Theta}) = \mathbf{p}_t, \tag{111}$$

for some arbitrary fixed parameter function $\Theta \in \mathcal{G}$. It is easy to prove also that $c_t = 0$ for all $t \in \mathbb{R}$ with respect to the evolution Eqs. (103)–(107).

Returning back to the Hamiltonian (92), the resulting J–V equations upon the phase space $M_0^{(f)} \ni (\mathbf{A}, \mathbf{p}, f(\mathbf{x}, \mathbf{v}))$ due to the transformation (93) read as follows:

$$f_t + \langle \mathbf{v}, \nabla f \rangle + \left\langle (-\mathbf{p} + \mathbf{v} \times \mathrm{rot}\,\mathbf{A}), \frac{\partial f}{\partial \mathbf{v}} \right\rangle = 0, \tag{112}$$

$$\mathbf{A}_t = \mathbf{p} - \nabla\boldsymbol{\Theta}_t, \tag{113}$$

$$-\mathrm{rot}\,\mathrm{rot}\,\mathbf{A} + \int\limits_{\mathbb{R}^2} f(\mathbf{x}, \mathbf{v})\mathbf{v}\,dv - \sin(\mathbf{A} + \nabla\boldsymbol{\Theta}) = \mathbf{p}_t. \tag{114}$$

Introducing new physical variables $\mathbf{E} := -\mathbf{p}$ and $\mathbf{B} := \mathrm{rot}\,\mathbf{A}$, the J–V system (112) will take the form

$$f_t + \langle \mathbf{v}, \nabla f \rangle + \left\langle (\mathbf{E} + \mathbf{v} \times \mathbf{B}), \frac{\partial f}{\partial \mathbf{v}} \right\rangle = 0, \tag{115}$$

$$\mathbf{B}_t = -\mathrm{rot}\,\mathbf{E}, \tag{116}$$

$$\mathbf{E}_t = \mathrm{rot}\,\mathbf{B} - \int\limits_{\mathbb{R}^2} f(\mathbf{x}, \mathbf{v})\mathbf{v}\,dv + \sin\left(\mathrm{rot}^{-1}\mathbf{B}\right), \tag{117}$$

$$c = \mathrm{div}\,\mathbf{E} - \pi - \int\limits_{\mathbb{R}^2} f(\mathbf{x}, \mathbf{v})\,dv = c_0(\mathbf{x}), \tag{118}$$

$$\mathrm{div}\,\mathbf{B} = 0, \tag{119}$$

on some physical space $M_0^{\mathrm{rot}} \ni (\mathbf{E}, \mathbf{B}, f)$.

The J–V evolution system (115)–(119) admits the Lie–Poisson structure [18] given by $\{\cdot, \cdot\}_{M_0^{\mathrm{rot}}}$ naturally induced from the symplectic structure $\omega_\pi^{(2)}$ built above, i.e., for any $\alpha, \beta \in \mathcal{D}(M_0^{\mathrm{rot}})$

$$\{\alpha,\beta\}_{M_0^{\mathrm{rot}}}=\int_{\mathbb{R}^2}dx\int_{\mathbb{R}^2}dv\, f\left[\left\{\frac{\delta\alpha}{\delta f},\frac{\delta\beta}{\delta f}\right\}+\left\langle \mathbf{B},\left(\frac{\partial}{\partial\mathbf{v}}\frac{\delta\alpha}{\delta f}\times\frac{\partial}{\partial\mathbf{v}}\frac{\delta\beta}{\delta f}\right)\right\rangle\right]$$
$$+\int_{\mathbb{R}^2}dx\int_{\mathbb{R}^2}dv\left(\left\langle\frac{\delta\alpha}{\delta\mathbf{E}},\frac{\delta f}{\delta\mathbf{v}}\right\rangle\frac{\delta\beta}{\delta f}-\left\langle\frac{\delta\beta}{\delta\mathbf{E}},\frac{\delta f}{\delta\mathbf{v}}\right\rangle\frac{\delta\alpha}{\delta f}\right)$$
$$+\int_{\mathbb{R}^2}dx\left(\left\langle\frac{\delta\alpha}{\delta\mathbf{E}},\operatorname{rot}\frac{\delta\beta}{\delta\mathbf{B}}\right\rangle-\left\langle\frac{\delta\beta}{\delta\mathbf{E}},\operatorname{rot}\frac{\delta\alpha}{\delta\mathbf{B}}\right\rangle\right). \qquad (120)$$

As a result, the J–V dynamical system (115)–(119) reads compactly as follows

$$\begin{aligned} f_t&=\{\mathcal{H},f\}_{M_0^{\mathrm{rot}}}, & \mathbf{E}_t&=\{\mathcal{H},\mathbf{E}\}_{M_0^{\mathrm{rot}}},\\ \mathbf{B}_t&=\{\mathcal{H},\mathbf{B}\}_{M_0^{\mathrm{rot}}}, & 0&=\{\mathcal{H},c\}_{M_0^{\mathrm{rot}}}. \end{aligned} \qquad (121)$$

The Hamiltonian structure (120) for the J–V dynamical system (115)–(119) seems to be of great interest for superconductive theories because it enables the application of the powerful techniques of perturbation theory for Hamiltonian systems. The relationship between some classical, semiclassical, even quantum aspects of Josephson media and J–V theory with its various truncations requires further development. In this frame it seems also of interest some modification [9] of J–V type evolution equations as

$$f_t+\langle\mathbf{v},\nabla f\rangle+\langle(-\mathbf{p}+\mathbf{v}\times\operatorname{rot}\mathbf{A}),\partial f/\partial\mathbf{v}\rangle=0, \qquad (122)$$

$$\mathbf{A}_t=\mathbf{p}, \qquad (123)$$

$$-\operatorname{rot}\operatorname{rot}\mathbf{A}+\int_{\mathbb{R}^2}f(\mathbf{x},\mathbf{v})\mathbf{v}\,dv-\frac{\sin|\mathbf{A}|}{|\mathbf{A}|}\mathbf{A}=\mathbf{p}_t, \qquad (124)$$

with compatibility constraints on the ambient charge density $\operatorname{div}\mathbf{p}-\int_{\mathbb{R}^2}f(\mathbf{x},\mathbf{v})\,dv=\pi$, that is $c_t=0$ for $c=\pi+\operatorname{div}\mathbf{p}-\int_{\mathbb{R}^2}f(\mathbf{x},\mathbf{v})\,dv$ upon $M_\pi^{(f)}$. Treating the model (122)–(124) in detail it seems also to shine a new light at some conceptual problems concerning the stability of vortex type solutions upon a special type background [9].

The latter leads to the verification of collective effects theory via artificially built and produced discrete models which mimic Josephson media.

ACKNOWLEDGEMENT

One of authors (A.P.) is grateful for hospitality and support to authorities of Institute of Physics P.A.N. at Warsaw, where the essential part of this work was done. Authors acknowledge also support of Polish Committee Grant (KBN) No. 2-P03B-114-11. Authors are indebted to the referee for remarks elucidating the reduction procedure and gauge invariant Poisson structures.

REFERENCES

[1] K.F. NAKAJIMA and Y. SAWADA, *J. Appl. Phys.* 52 (1991) 2162.
[2] G. FILATRELLA and K. WIESEFELD, in: R.D. Parmentier and N.F. Pedersen (Eds.), *Proc. Conf. Nonlinear Superconducting Devices and HTc Materials*, Capri, World Scientific, 1995, p. 329.
[3] J. ZAGRODZIŃSKI, *Phys. Rev. B* 53 (1995) 59.
[4] B. HOROVITZ, *Phys. Rev. B* 51 (1995) 3989.
[5] M. FRANZ and S. TEITEL, *Physica B* 222 (1996) 287.
[6] M.Y. CHOI, *Phys. Rev. B* 50 (1994) 13875.
[7] T. ONOGI, *Physica B* 222 (1996) 391.
[8] W.E. LAWRENCE and S. DONIACH, in: E. Kanda (Ed.), *Proc. 12th Internat. Conf. on Low Temperature Physics*, Kyoto, Keygaku, Tokyo, 1970, p. 361.
[9] J. ZAGRODZIŃSKI, *Phys. Scripta* 54 (1996) 24.
[10] J. GIBBONS, D.D. HOLM and B. KUPERSHMIDT, *Physica* 6D (1983) 179.
[11] D.D. HOLM and B. KUPERSHMIDT, *Phys. Rev. A* 36 (1987) 3947; also *J. Math. Phys.* 29 (1988) 21.
[12] P.A.M. DIRAC, *Quantum Mechanics of Curved Subspace*, Oxford, 1962.
[13] A.K. PRYKARPATSKY and I.V. MYKYTIUK, *Algebraic Aspects of Nonlinear Dynamical Systems on Manifolds*, Nauk. Dumka, Kiev, 1991 (in Russian).
[14] A.K. PRYKARPATSKY, D. BLACKMORE, W. STRAMP, Yu. SYDORENKO and R. SAMULIAK, *Cond. Matter. Phys.* 6 (1995) 72.
[15] A.K. PRYKARPATSKY, D. BLACKMORE, W. STRAMP, Yu. SYDORENKO and R. SAMULIAK, *J. Math. Phys.* (1996) will be published.
[16] R. ABRAHAM and J. MARSDEN, *Foundation of Mechanics*, W.A. Benjamin, 1978.
[17] J. MARSDEN and A. WEINSTEIN, *Rept. on Math. Phys.* 5 (1974) 129.
[18] J. MARSDEN, *Canad. Math. Bull.* 24 (1982) 129.
[19] V. GUILLEMIN and S. STERNBERG, *Ann. Phys.* 127 (1980) 220.
[20] V.I. ARNOLD, *Mathematical Methods of Classical Mechanics*, Grad. Texts in Math., Vol. 60, Springer, New York, 1968.