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G. GIELIS
C. MAES
K. VANDE VELDE

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Randomly interacting particle systems: The uniqueness regime *

by

G. GIELIS[1], C. MAES[2], K. VANDE VELDE[3]

Instituut voor Theoretische Fysica, K.U. Leuven, Celestijnenlaan 200D, B-3001 Leuven, Belgium

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ABSTRACT. – We introduce spatial disorder in a large system of interacting particles that evolve according to a non-reversible dynamical law. We show that if the regions where the components strongly interact are scarce, several general properties of the discrete and continuous time dynamics remain unaffected by the disorder.

For the discrete time dynamics we prove that the unique invariant measure is Gibbsian, its two-point spatial correlation function decays exponentially fast for increasing distances and, for a restricted class of models (i.e., directed probabilistic cellular automata), we prove almost sure and disorder-averaged upper bounds for the rate of relaxation towards equilibrium. Moreover we show, by an example, that under our conditions these bounds are (almost) optimal.

For the continuous time dynamics, after showing the existence of the infinite volume limit, we derive approximations by a discrete time updating system, valid uniformly in time. © Elsevier, Paris

Key words: Quenched disorder, spinflip dynamics, ergodicity, Gibbs measures

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[1] Present address: Research Centre, King's College, Cambridge CB2 1ST, United Kingdom. E-mail: guy.gielis@kings.cam.ac.uk.

[2] Onderzoeksleider N.F.W.O. Belgium. E-mail: Christian.Maes@fys.kuleuven.ac.be.

[3] E-mail: Koen VandeVelde@fys.kuleuven.ac.be.

RÉSUMÉ. – Nous introduisons un désordre spatial dans un système étendu de particules en interaction, évoluant suivant une loi dynamique non reversible. Dans le cas ou les régions de fortes interactions sont rares, nous montrons que plusieurs propriétés générales ne sont pas affectées par le désordre.

Pour la dynamique discrète, nous montrons que l'unique mesure invariante est une mesure de Gibbs, dont les functions de corrélation décroissent exponentiellement vite avec la distance. Pour une classe réduite de modèles (les automates cellulaires stochastiques orientés) nous déduisons des bornes supérieures valides, presque sûrement et en moyenne sur le désordre, pour la vitesse de convergence vers l'équilibre. De plus, nous montrons au moyen d'un exemple que sous les conditions que nous avons imposées, ces bornes sont presque optimales.

Quant à la dynamique continue, après avoir montré l'existence de la limite hydrodynamique, nous déduisons des approximations par des systèmes dynamiques à temps discrèt, uniformément dans le temps.

1. INTRODUCTION

One of the interesting aspects of travelling is to find people interacting differently in other places. Looking back in time one realizes that these spatial variations are often much more pronounced than the changes that have occurred as time passed. The same can be said of ecosystems in which various species are interacting. The strength and/or nature of these interactions sometimes very much depends on the spatial location in the system but remains there unaltered over long times. Similarly in physics, the study of disordered systems (amorphous materials, doped semiconductors, spin glasses) concentrates on the effect of spatial non-homogeneity. It is therefore also natural to ask what remains of various general and specific studies in the theory of interacting particle systems (IPS) when the parameters governing the local interaction between the various components is varying spatially. One standard way of incorporating this is to choose random jump intensities (transition probabilities) whose realization remains fixed in time.

In the present paper we have investigated the effect of this modification on the standard theory of IPS close to independence. More precisely, we consider the IPS in a regime where the particles on most places hardly feel each other, but on the other hand, the randomness implies that there are finite, but arbitrary large regions of all shapes, where the particles are strongly interacting (low noise, almost deterministic updating, low temperature, ...). Looking at the system in space-time these regions become infinite cylinders in which the relaxation may be much slower than for the uniform system in the uniqueness regime (high noise, ...).

We will restrict ourselves to discrete time (probabilistic cellular automata = PCA) and continuous time spin flip processes. However we wish to confront the dynamical problem directly and do not therefore consider the stochastic Ising model for which, via arguments based on reversibility, the theory can benefit from strong results clarifying the equilibrium statistical mechanics of disordered Gibbs systems in the Griffiths' regime. Relaxation behavior of the disordered stochastic Ising model is the subject of a series of other papers [4,5,9,10,17]. Hence we are obliged to look here for quite general dynamical arguments to answer quite general questions related to the uniqueness regime of randomly interacting particle systems (RIPS).

The questions we address have to do with the almost sure existence of a unique invariant measure for the RIPS, the characterization of that measure and the convergence to it, starting from arbitrary initial data. For this we concentrate mostly on PCA but we also discuss the existence and the construction of the continuous time analogues. Finally, we wonder how well a RIPS can be approached by a random PCA.

Our methods are based (naturally) on a combination of coupling techniques and domination arguments which lead to the analysis of percolation processes with random parameters.

Our results give criteria under which the RIPS share most of the properties of the corresponding IPS in the uniqueness regime. The most important difference concerns the bounds on the relaxation speed. Generally speaking, we both improve and extend the results that appeared in [8].

Our main results are stated in Section 3. Proofs can be found in Section 4. We start however in the following Section 2 by giving examples and by discussing the general set-up.

2. EXAMPLES OF RIPS

We consider the evolution of spin configurations $\sigma = \{\sigma(x) = \pm 1,\ x \in \mathbb{Z}^d\}$ on the regular d-dimensional lattice $\mathbb{Z}^d$. The set of all possible configurations is denoted by Ω. A Probabilistic Cellular Automaton (PCA) is a parallel updating dynamics σ_n, $n = 0, 1, \ldots$, starting from some initial data $\sigma_0 = \zeta \in \Omega$. It is a Markov process defined by the transition probabilities $p_x(\sigma(x)|\zeta)$, $\zeta \in \Omega$. For any finite $\Lambda \subset \mathbb{Z}^d$,

$$\text{Prob}\left[\sigma_n(z) = \xi(z), \forall z \in \Lambda | \sigma_{n-1} = \zeta\right] = \prod_{z\in\Lambda} p_z\left(\xi(z)|\zeta\right), \tag{1}$$

for $\xi, \zeta \in \Omega$. We get a *random* PCA, when the transition probabilities are themselves random variables. By **Q** we denote the distribution function of these random variables and **E** is the corresponding expectation value.

For the simplest illustration of the problem at hand we turn to one of the oldest examples that have appeared in the study of IPS. It is the so called light bulb PCA or also called

Stavskaya's example. On the linear chain $\mathbb{Z}$ we have lamps $\sigma(x)$, $x \in \mathbb{Z}$, which can be 'off' $(\sigma(x) = 1)$ or 'on' $(\sigma(x) = -1)$. The transition probabilities are given by

$$p_x(+1|\zeta) = \begin{cases} 1 & \text{if } \zeta(x) = \zeta(x+1) = +1, \\ \mathrm{e}^{-\beta\gamma_x} & \text{otherwise.} \end{cases} \tag{2}$$

Here, $\beta > 0$ is a fixed parameter and $\{\gamma_x,\ x \in \mathbb{Z}\}$ is a family of independent and identically distributed non-negative random variables whose distribution **Q** can be chosen freely. If we choose $\gamma_x = \gamma$ fixed (deterministic case) we recover the original Stavskaya PCA for which there is a phase transition as $\beta\gamma$ gets large.

The question we ask here is what happens for β small in case the $\{\gamma_x\}$ are not uniformly bounded (e.g., exponentially distributed).

A similar modification can be made to

Toom's PCA. We now have a two-dimensional PCA and the configuration space is $\Omega = \{-1, +1\}^{\mathbb{Z}^2}$. The only further change with respect to (2) above is that now

$$p_x(+1|\zeta) = \begin{cases} \frac{1}{2}(2 - \mathrm{e}^{-\beta\gamma_x}) & \text{if } \operatorname{sgn}(\zeta(x) + \zeta(x+e_1) \\ & \qquad + \zeta(x+e_2)) = +1, \\ \frac{1}{2}\mathrm{e}^{-\beta\gamma_x} & \text{otherwise.} \end{cases} \tag{3}$$

e_1 and e_2 are the unit vectors in $\mathbb{Z}^2$. The non-disordered (or regular) Toom model $(\gamma_x = \gamma)$ shows a phase transition. Again, we can ask when there is an almost sure (with respect to the distribution of the $\{\gamma_x\}$) unique invariant measure, what is the speed of convergence to that measure and how to characterize it?

The above two examples are well-known PCA, sharing an important feature which we will use in our first Theorem 3.1 below; they are spatially asymmetric. More generally, let $\mathcal{A}_x$ be the set of sites $y \in \mathbb{Z}^d$ such that the single site transition probability $p_x(\sigma(x)|\zeta) = p_x(\sigma(x)|\zeta(y),\ y \in \mathcal{A}_x)$ only depends on the $\zeta(y)$. Note that we restrict ourselves to nearest neighbor updating. Then, we call the PCA *directed* if $\mathcal{A}_x = \{x + \alpha e_i,\ \alpha = 0, 1,\ i = 1, \ldots, d\}$, for e_i the unit vectors of $\mathbb{Z}^d$ in the positive i-direction.

The continuous time spin flip dynamics is defined in terms of transition rates $c(x, \zeta)$ such that for the spin $\sigma_t(x)$ at time t, and $\zeta \in \Omega$,

$$\text{Prob}\big[\sigma_t(x) \neq \zeta(x)|\sigma_0 = \zeta\big] = c(x, \zeta)t + \mathrm{o}(t). \tag{4}$$

We refer to Liggett [11] for details. Analogous to the random PCA the continuous time RIPS has transition rates $c(x, \zeta)$ determined by random variables with a distribution function that we will denote by $\mathbf{Q}$. Also the definitions of $\mathbf{E}$, $\mathcal{A}_x$ and the concept of *directed* are adopted from the PCA.

The Majority Vote Process. This random version of the Majority Vote IPS takes into account that some voters are more inclined to take over the opinion of their neighbors than others are. It is defined by

$$c(x, \zeta) = \begin{cases} \frac{1}{2}\mathrm{e}^{-\beta\gamma_x} & \text{if } \zeta(x) = \mathrm{sgn}(\sum_{z\sim x} \zeta(z)), \\ \frac{1}{2}(2 - \mathrm{e}^{-\beta\gamma_x}) & \text{otherwise.} \end{cases} \tag{5}$$

The sum runs over all nearest neighbors $z \sim x$, i.e., z and x are connected via a bond of the regular lattice $\mathbb{Z}^d$. Also here $\beta > 0$ is a positive number and $\{\gamma_x,\ x \in \mathbb{Z}^d\}$ is a set of independent and identically distributed nonnegative random variables. These are finite with $\mathbf{Q}$-probability one, but they are not uniformly bounded.

This is all also true for

The disordered contact process. This model describes in a conceptually simple way the spread of an infection (or of a particular feature), taking into account the dependence of the spreading rate on the local environment:

$$c(x,\zeta)=\begin{cases} e^{-\beta\gamma_x} & \text{if } \zeta(x)=1, \\ (1-e^{-\beta\gamma_x})\sum_{z\sim x}(1+\zeta(z)) & \text{if } \zeta(x)=-1. \end{cases} \tag{6}$$

Conditions that imply the survival (or extinction) of the population are already intensively studied, e.g., in [7,1,13] and the references you can find there.

Stochastic Ising model. A disordered version of the stochastic Ising model consists in taking a random interaction potential determining the spin flip rates. For example,

$$c(x,\zeta)=\min\Big\{\exp\Big(-2\sum_{y\sim x}J_{xy}\zeta(x)\zeta(y)\Big),1\Big\}, \tag{7}$$

corresponds to the Metropolis algorithm for simulating the Ising model. Here the $\{J_{xy}\}$ are independent identically distributed random variables associated to the bonds (the nearest neighbor pairs) of $\mathbb{Z}^d$. The sum runs over the nearest neighbors $y\sim x$ of x.

Stochastic Ising models were considered in [4,5,9,10,17] and we refer to them for the detailed results on the relaxation behavior.

From these examples it should be clear what we mean by adding disorder to IPS. In what follows we will not always be explicit about the precise way in which the interaction is random but we will continue to write the transition probabilites of PCA as $p_x(\cdot|\cdot)$ without even indicating explicitly that these contain random parameters.

The following random variables $k_x\in[0,1]$ play an important role in the statement of our results;

$$k_x=\max_{\zeta,\tilde\zeta}\big|p_x(+1|\zeta)-p_x(+1|\tilde\zeta)\big|, \tag{8}$$

$$q_x=\max_{\substack{\zeta_i,\tilde\zeta_i\\ i=0,1,2}}\bigg|\frac{1}{N_x(\zeta)}p_x(+1|\zeta_0)\prod_{z:\,x\in A_z}p_z\big(\zeta_2(z)|\zeta_1^{x,+1}\big) - \frac{1}{N_x(\tilde\zeta)}p_x\big(+1|\tilde\zeta_0\big)\prod_{z:\,x\in A_z}p_z\big(\tilde\zeta_2(z)|\zeta_1^{x,+1}\big)\bigg| \tag{9}$$

with $\zeta^{x,a}$ the configuration such that $\zeta^{x,a}(z)=\zeta(z)$ when $z\neq x$ and $\zeta^{x,a}(x)=a$, $a=\pm1$.

$$N_x(\zeta)=\sum_{a=\pm1}p_x(a|\zeta_0)\prod_{z:\,x\in A_z}p_z\big(\zeta_2(z)|\zeta_1^{x,a}\big) \tag{10}$$

is a normalizing constant.

Now we can formulate our main assumptions for the discrete time dynamics

(i) locality; $\mathcal{A}_x$ contains only nearest neighbors of x,

(ii) the $\{k_x\}$ is a set of jointly independent and stationary random variables,

(iii) $k_x < 1$, but $\sup_x k_x = 1$ is not excluded.

In the same way, for continuous time processes we will simply write the spin flip rates as $c(x, \zeta)$ even though they are determined by realizations of an underlying random field. Again we introduce some positive random variables $\lambda_x, \delta_x \in [0, \infty)$ that are derived from the transition rates.

$$\delta_x = \inf_{\zeta,\tilde{\zeta}} \{ c(x, \zeta) + c(x, \tilde{\zeta}),\ \zeta(x) \neq \tilde{\zeta}(x) \}, \tag{11}$$

$$\lambda_x = \sup_{\zeta,\tilde{\zeta}} \{ |c(x, \zeta) - c(x, \tilde{\zeta})|,\ \zeta(x) = \tilde{\zeta}(x) \}. \tag{12}$$

We suppose that

(iv) the $c(x, \zeta)$ are local; $\mathcal{A}_x$ contains only nearest neighbors of x,

(v) $\{\delta_x\}$ and $\{\lambda_x\}$ are both sets of jointly independent and stationary random variables, moreover, δ_x and λ_y are independent when $x \neq y$,

(vi) $\lambda_x < \infty$ and $\delta_x > 0$, but $\sup_x \lambda_x = \infty$ and $\inf_x \delta_x = 0$ is not excluded.

Note that for the stochastic Ising model condition (v) is not verified, because the spin flip rates are not independent for nearest neighbors. But, because they are independent for next nearest neighbors, if wished, a slight modification sufffices to include also this example in the results.

3. DEFINITIONS AND MAIN RESULTS

We endow the configuration space Ω with the product discrete topology. By $\mathcal{C}(\Omega)$ we denote the set of continuous functions on Ω. A function f is called local if it depends only on a finite number of spin variables. The location of these spins is then denoted by $\operatorname{supp} f$. $\|f\| = \sup_\eta f(\eta)$. The oscillation at site x is given by $\Delta_x = \sup_\eta |f(\eta^x) - f(\eta)|$, with $\eta^x \in \Omega$ such that $\eta^x(z) = \eta(z)$ if $z \neq x$ and $\eta^x(x) = -\eta(x)$. $|||f||| = \sum_{z\in\mathbb{Z}^d} \Delta_z f$ is the total oscillation. The distance between two sites $x = (x^i),\ y = (y^i) \in \mathbb{Z}^d$ is

$$d(x, y) = \sum_{i=1}^{d} |x^i - y^i|. \tag{13}$$

x and y are nearest neighbors ($x \sim y$) if $d(x, y) = 1$. The distance $\mathrm{dist}(A, B)$ between two subsets $A, B \subset \mathbb{Z}^d$ is defined as

$$\mathrm{dist}(A, B) = \inf\{d(x, y),\ x \in A,\ y \in B\} \tag{14}$$

and for two functions $f, g,\ \mathrm{dist}(f, g) = \mathrm{dist}(\mathrm{supp}\, f, \mathrm{supp}\, g)$.

The boundary $\partial\Lambda$ of a volume $\Lambda \subset \mathbb{Z}^d$ is the set

$$\partial\Lambda = \{z \in \Lambda^c,\ z \sim x,\ x \in \Lambda\}, \tag{15}$$

with $\Lambda^c = \mathbb{Z}^d \setminus \Lambda$.

The transfer operator P of a PCA is defined by

$$P^N f(\zeta) = P P^{N-1} f(\zeta), \tag{16}$$

with for a local function f

$$Pf(\zeta) = \sum_{\xi\in\Omega} f(\xi) \prod_{z\in \mathrm{supp}\, f} p_z(\xi(z)|\zeta). \tag{17}$$

A measure μ on Ω is invariant with respect to this dynamics if

$$\mu P = \mu. \tag{18}$$

In [8] conditions are given on the k_x implying that for $\mathbf{Q}$-almost every realization of the disorder, the PCA has a unique invariant measure μ. Moreover, for some $m > 0,\ v > 1$ and for every local function f, there exists $\mathbf{Q}$-a.s. a finite constant $N_0 = N_0(\{k_x\}, f)$ such that for $N > N_0$

$$\|P^N f - \mu(f)\| \leqslant \exp[-m(\log(1 + N))^v]. \tag{19}$$

For directed PCA, we can improve this result (Theorem 1) and in addition, we can bound from above the disorder averaged relaxation (Theorem 2).

Introducing a constant $\beta \geqslant 0$, we find it convenient to make the following change of variables:

$$k_x = 1 - \mathrm{e}^{-\beta\gamma_x} \tag{20}$$

where the $\{\gamma_x\}$ are now jointly independent and form a stationary field of possibly unbounded non-negative random variables.

THEOREM 1. – *Consider a directed PCA. Let* $0 < \theta < 1$ *and suppose that*

$$\mathbf{E}\left[\frac{k_x^\theta}{1 - k_x^\theta}\right] < \frac{1}{d} \tag{21}$$

and

$$\mathbf{Q}\{\gamma_x > R\} < \exp(-\alpha R^\nu), \tag{22}$$

with $\nu \geqslant 1$ *and* $\alpha > 0$.

For every $v > 1/\nu$ *or for* $v = 1/\nu$ *and* α/β *large enough, there is a constant* $0 < \lambda < 1$ *so that for every local function* f *there exists* $\mathbf{Q}$*-a.s. a finite constant* $N_0 = N_0(f, \theta, v, \beta, \{\gamma_x\})$, *such that for* $N > N_0$

$$\|P^N f - \mu(f)\| \leqslant \exp(-N \exp(-\lambda(\log N)^v)). \tag{23}$$

THEOREM 2. – *Consider a directed PCA.*
Suppose

$$\mathbf{E}\left[\frac{k_x}{1 - k_x}\right] < \frac{1}{d} \tag{24}$$

and

$$\mathbf{Q}\{\gamma_x > R\} < \exp(-\alpha R^\nu), \tag{25}$$

for some $\nu \geqslant 1,\ \alpha > 0$.

When $\nu > 1$ *or when* $\nu = 1$ *and* α/β *is large enough, there exists a* $\lambda > 0$ *such that for all local functions* f *there is a finite constant* $N_0 = N_0(f, \beta, \alpha, \nu, \lambda)$ *such that when* $N > N_0$,

$$\mathbf{E}\|P^N f - \mu(f)\| \leqslant \exp(-\lambda(\log N)^\nu). \tag{26}$$

Remarks. –

1. (23) is faster than any stretched exponential decay

$$e^{-\lambda N^\delta}, \quad \lambda > 0,\ \delta < 1.$$

2. The hypotheses of Theorems 1 and 2 do not allow for a much better upperbound. Indeed, consider Stavskaya's PCA starting with as initial conditions σ_0 the all -1-configuration. Note that $\sigma_N(x) = -1$ impies that $\sigma_{N-1}(x) = -1$ or $\sigma_{N-1}(x+1) = -1$. Hence,

$$\begin{aligned}
&\mathrm{Prob}\big[\sigma_N(x) = -1 \,|\, \sigma_0(z) = -1, \forall z\big] \\
&\quad \geqslant \mathrm{Prob}\big[\sigma_N(x) = -1, \sigma_{N-1}(x) = -1, \ldots, \sigma_1(x) = -1 \,|\, \sigma_0(z) = -1, \forall z\big] \\
&\quad \geqslant \inf_{\zeta} \mathrm{Prob}\big[\sigma_N(x) = -1 \,|\, \sigma_{N-1}(x) = -1, \sigma_{N-1}(x+1) = \zeta\big] \\
&\qquad \times \mathrm{Prob}\big[\sigma_{N-1}(x) = -1, \ldots, \sigma_1(x) = -1 \,|\, \sigma_0(z) = -1, \forall z\big].
\end{aligned} \tag{27}$$

This is larger then $(1 - \mathrm{e}^{-\beta\gamma_x})^N$. Hence, when the $\{\gamma_x,\ x \in \mathbb{Z}^d\}$ are not uniformly bounded, an exponentially fast relaxation of the form $\mathrm{e}^{-\lambda N}$, with fixed $\lambda > 0$, is not possible.

In the same way, we can bound the disorder-averaged relaxation from below. Suppose that we consider a distribution $\mathbf{Q}$ such that

$$\mathbf{Q}(\gamma_x > R) > \mathrm{e}^{-\lambda R^\nu}, \tag{28}$$

$\lambda > 0,\ \nu > 1$, then

$$\begin{aligned}
&\mathbf{E}\big[(1 - \mathrm{e}^{-\beta\gamma_x})^N\big] \geqslant \varepsilon \mathbf{Q}\big\{(1 - \mathrm{e}^{-\beta\gamma_x})^N > \varepsilon\big\} \\
&\quad \geqslant \varepsilon \exp\left(-\frac{\lambda}{\beta^\nu}\left(\log\left(\frac{1}{1 - \varepsilon^{1/N}}\right)\right)^\nu\right) \geqslant \varepsilon \exp\big(-\tilde{\alpha}(\log N)^\nu\big),
\end{aligned} \tag{29}$$

for any $0 < \varepsilon < 1$, $\tilde{\alpha} > \lambda/\beta^\nu$, when N is large enough.

Note that this argument does not exclude a faster decay for some other model in the class of directed PCA. It only shows that under the conditions stated no better bounds than (23) and (26) are possible in general.

3. The estimates (23) and (26) are similar to the bounds that are obtained in [4] and [5] for the stochastic Ising model. In the case that the interactions J_{xy}, $x \sim y$, are uniformly bounded the dynamics decays, for intermediate temperatures, $\mathbf{Q}$-a.s. as (23) with $v = 1 - 1/d$, or if we average over the disorder as (26) with $\nu = d/(d-1)$.

4. We expect that the bounds for the relaxation of directed continuous time dynamics in [8] can be improved in the same way.

Once we know that there is a unique invariant measure (and we know how fast it is reached by the dynamics) it is natural to ask for a detailed characterization of this state. This will be the subject of Theorem 3. The results are also valid for non-directed PCA.

To state the theorem we need the notion of a Gibbs measure. Denote by Ω_A the restriction of the configuration Ω to the finite set $A \in \mathbb{Z}^d$. Ω_A contains the configurations $\sigma_A = \{\sigma(x) = \pm 1,\ x \in A\}$. An interaction $\boldsymbol{\Phi}$ on Ω is a collection $\{\Phi_A\}_{A \subset \mathbb{Z}^d}$ of real functions indexed by finite subsets

$A \subset \mathbb{Z}^d$,

$$\Phi_A : \Omega_A \mapsto \mathbb{R}, \tag{30}$$

with

$$\sum_{A \ni x} \sup_{\eta} |\Phi_A(\eta)| < \infty \quad \forall x. \tag{31}$$

A measure μ on Ω is called a Gibbs measure with respect to the interaction $\boldsymbol{\Phi}$, if its conditional distributions satisfy (μ-almost surely)

$$\mu[\sigma_\Lambda = \xi_\Lambda | \sigma_{\Lambda^c} = \xi_{\Lambda^c}] = \frac{1}{\mathcal{Z}_\Lambda} \exp\Bigg(\sum_{A \cap \Lambda \neq \emptyset} \Phi_A(\xi) \Bigg), \tag{32}$$

with $\mathcal{Z}_\Lambda$ the normalizing partition sum for every finite $\Lambda \subset \mathbb{Z}^d$.

THEOREM 3. – *Take f, g local functions on Ω and a constant $K > 8d^2$. If*

$$\mathbf{E}\left[\left\{ \log\left(\frac{1}{1-q_x}\right)\right\}^K\right] < \infty, \tag{33}$$

then we can find for every $m > 0$ constants $0 < c_1, c_2 < 1$ such that when

$$\mathbf{Q}\{q_x > c_1\} < c_2 \tag{34}$$

there exists $\mathbf{Q}$-a.s. a finite constant $C = C(f, g, \{q_x\})$ such that

$$|\mu(fg) - \mu(f)\mu(g)| \leqslant C \exp(-m \operatorname{dist}(f, g)). \tag{35}$$

Moreover, if

$$0 < p_x(+1|\zeta) < 1 \tag{36}$$

for all $x \in \mathbb{Z}^d$ and every configuration $\zeta \in \Omega$, then the unique invariant measure μ is Gibbsian for $\mathbf{Q}$-almost every realization of the disorder.

Remarks. –

1. In [8], under the same conditions ((34) and (36)), but then for the variables $\{k_x,\ x \in \mathbb{Z}^d\}$ the uniqueness of the equilibrium state is proven. However, since $k_x \leqslant q_x$, the conditions of Theorem 3 are more restrictive.
2. Note that Stavskaya's PCA does not satisfy (36)
3. In [12] and [15] similar results are proven for non-disordered discrete and continuous time dynamics. There the decay of the spatial correlation function (35) is immediate from the exponential decay of the time correlations.

The remaining two theorems are dealing with continuous time RIPS. First of all we consider conditions on the spin flip rates $c(x, \zeta)$ implying that the infinite volume dynamics P_t constructed with these random rates is well defined.

Define the positive numbers λ_{xy}, E_1, E_2 and E_3 as

$$\lambda_{xy} = \lambda_x + \lambda_y, \tag{37}$$

$$E_1 = \mathbf{E}\Big[\Big(\log\Big(1 + \max_\zeta c(x, \zeta)\Big)\Big)^{d+1}\Big], \tag{38}$$

$$E_2 = \mathbf{E}\big[\big(\log(1 + \lambda_{xy})\big)^K\big], \tag{39}$$

$$E_3 = \mathbf{E}\bigg[\bigg(\log\bigg(1 + \frac{1}{\delta_x}\bigg)\bigg)^K\bigg], \tag{40}$$

with $K > 8d^2$.

Let $\{\Lambda_r\}_{r=1,2,\ldots}$ be a sequence of d-dimensional boxes with diameter $2r$, centered at the origin. Consider the infinite volume semigroup P_r^t with generator

$$L_r f(\zeta) = \sum_z c_r(z, \zeta)\big(f(\zeta^z) - f(\zeta)\big), \quad \zeta \in \Omega,$$

with rates $c_r(x, \zeta)$ such that $c_r(x, \zeta) = c(x, \zeta)$ if $x \in \Lambda_r$ and zero otherwise.

THEOREM 4. – *Suppose that*

$$\max\{E_1, E_2, E_3\} < \infty, \tag{41}$$

then there exists a constant C_1 *such that when*

$$\mathbf{E}\bigg[\bigg(\log\bigg(1 + \frac{\lambda_{xy}}{\delta_x}\bigg)\bigg)^K\bigg] < C_1, \tag{42}$$

there exist for any $m > 0$ *and every local function* f, $\mathbf{Q}$*-a.s. a finite constant* $B = B(f, \{\lambda_{xy}\}, \{\delta_x\})$ *so that*

$$\|P_k^t f - P_l^t f\| \leqslant B e^{-ml}, \tag{43}$$

for all $k > l > l_0$, *with* $\operatorname{supp} f \subset \Lambda_{l_0}$.

Remark. – This bound implies that the infinite volume dynamics with semigroup $\lim_{r\uparrow\infty} P_r^t = P^t$ exists and is *Feller*, i.e., $P^t f(\zeta) \in \mathcal{C}(\Omega)$ for every time $t \geqslant 0$ for every $f \in \mathcal{C}(\Omega)$.

In the final theorem we show that in the regime that we consider the continuous time RIPS can be very well approximated by a discrete time parallel updating random PCA. The difference between the two dynamics can be bounded uniformly in time, which means that the bound also applies to the difference between the invariant measures. An application of such results (for non-disordered systems) can be found in the context of constructive criteria for ergodicity, see, e.g., [6].

The random PCA we have in mind, can be constructed as follows. First we define a new RIPS with spin flip rates

$$d(x,\tilde{\zeta}) = \begin{cases} c(x,\zeta) & \text{if } \tilde{\zeta}(x) = \zeta(x), \\ c(x,\zeta^x) & \text{if } \tilde{\zeta}(x) \neq \zeta(x), \end{cases} \tag{44}$$

depending on some fixed ζ.

Denote by $\overline{P_t}$ the corresponding semigroup with kernel $\overline{p}_t(d\sigma|\zeta)$ corresponding to the probability to find configuration σ at time t when the process was started in configuration ζ:

$$\overline{P}^t f(\zeta) = \int f(\xi)\overline{p}_t(d\xi|\zeta). \tag{45}$$

$\overline{p}_t(d\xi|\zeta)$ is a product measure as there are single spin transition probabilities $\{p_{t,z}(\sigma(z)|\zeta),\ z \in \mathbb{Z}^d\}$ so that formally

$$\overline{p}_t(\sigma|\zeta) = \prod_z p_{t,z}(\sigma(z)|\zeta). \tag{46}$$

We consider now the PCA with transition probabilities

$$\{p_{\delta,z}(\cdot|\cdot),\ z \in \mathbb{Z}^d\}, \quad \delta > 0,$$

and with corresponding transition operator P_δ defined by

$$P_\delta f(\zeta) = \overline{P}^\delta f(\zeta). \tag{47}$$

THEOREM 5. – *Suppose that* $\max\{E_1, E_2, E_3\} < \infty$, *then there exists a constant* C_1 *such that when*

$$\mathbf{E}\left[\left(\log\left(1+\frac{\lambda_{xy}}{\delta_x}\right)\right)^K\right] < C_1, \tag{48}$$

then there exists for every local function f *with* $\mathbf{Q}$*-probability one a finite constant* $C = C(f, \{\lambda_{xy}\}, \{\delta_x\})$ *such that*

$$\left\|P^t f - P_\delta^{\lfloor t/\delta \rfloor} f\right\| \leqslant \delta C \tag{49}$$

uniformly in t, *for* δ *small enough.*

$\lfloor r \rfloor$ is the largest integer smaller than r.

4. PROOF OF THE MAIN RESULTS

4.1. The Toolbox

We start by summarizing some of the standard techniques dealing with the uniqueness regime of IPS and by discussing their applications in the case of RIPS.

4.1.1. Various percolation processes

Two kinds of percolation processes are essential in our study of the uniqueness regime of the discrete and continuous time spin flip dynamics: an independent site percolation on the so-called space–time graph of the PCA and a 'continuous' percolation process on $\mathbb{Z}^d \times \mathbb{R}$.

The space–time graph $\mathcal{L}$ of a PCA has vertices $(x, N) \in \mathbb{Z}^d \times \mathbb{Z}$ and edges between (x, N) and $(y, N-1)$ with $y \in \mathcal{A}_x$, and between (x, N) and (z, N) if there is a $v \in \mathbb{Z}^d$ such that both $x, z \in \mathcal{A}_v$. Consider a set of densities $\{0 < p_x < 1,\ x \in \mathbb{Z}^d\}$. We independently put every vertex $(x, N) \in \mathbb{Z}^d \times \mathbb{Z}$ 'open' with probability p_x and 'closed' with probability $1 - p_x$. Note that the density p_x at a point (x, N) is independent of the time coordinate N. We say that there is an open path from (x, N) to (y, M) on $\mathcal{L}$ if there is a sequence of open vertices

$$(x, N) = (z_0, n_0), (z_1, n_1), \ldots, (z_k, n_k) = (y, M), \quad k > 0,$$

that, consecutively, are connected via an edge of $\mathcal{L}$. The probability that this happens will be denoted by $G((x, N), (y, M))$. $G^{N-M}(x, y)$ is the probability that the points (x, N) and (y, M) are connected by a time oriented path, this means that $n_{i+1} = n_i - 1$.

When the densities $\{p_x,\ x \in \mathbb{Z}^d\}$ are independent and identically distributed random variables with distribution $\mathbf{Q}$, Klein [7] and Campanino and Klein [3] proved (a slight modification of) the following result:

PROPOSITION 1. – *Let* $K > 8d^2$. *If*

$$\Gamma = \mathbf{E}\left[\left\{\log\left(\frac{1}{1-p_x}\right)\right\}^K\right] < \infty, \tag{50}$$

then there exists a $v(K, d, \Gamma) > 1$ *such that for every* $1 < v < v(K, d, \Gamma)$ *and* $m > 0$, *we can find constants* $0 < c_1, c_2 < 1$ *such that when*

$$\mathbf{Q}\{p_x > c_1\} < c_2, \tag{51}$$

there exists, with $\mathbf{Q}$*-probability one, a finite constant* $L = L(x, \{p_x\})$ *such that*

$$G\big((x, N), (y, M)\big) \leqslant \exp\big(-m\big[d(x, y) + \big(\log(1 + |N - M|)\big)^v\big]\big) \tag{52}$$

when $|x - y| > L$ *or* $|N - M| > \mathrm{e}^{L^{1/v}}$.

Note that if we want to use Proposition 1 with the densities q_x as defined in (9), we should extend this proposition to the case where the densities at nearest neighbor sites are correlated. The necessary modifications to the proofs are mentioned in [8]. In the rest of the paper we will always refer to Proposition 1, even when we deal with non-independent densities.

The continuous percolation process on $\mathbb{Z}^d \times \mathbb{R}$ is constructed as follows. Consider two sets of positive numbers

$$\{\delta_x,\ x \in \mathbb{Z}^d\} \quad \text{and} \quad \{\lambda_{xy},\ x \sim y,\ x, y \in \mathbb{Z}^d\}.$$

On each line $\{x\} \times \mathbb{R}$, $x \in \mathbb{Z}^d$, we put cuts according to a Poisson process with intensity δ_x, and place arrows from $y \sim x$ to x with intensity λ_{xy}. A path is a connected set of uncut segments of vertical lines and arrows. $G^{t-s}(x, y)$ is the probability that we can walk from (x, t) to (y, s) via a time oriented path (with non-increasing time coordinates) following the direction of the arrows. Suppose that the δ_x and the λ_{xy} are both sets of independent identically distributed random variables with distribution $\mathbf{Q}$, that in addition are mutually independent, then Klein [7] has proven the following result.

PROPOSITION 2. – *Let $K > 8d^2$. If*

$$\Gamma' = \max\left\{\mathbf{E}\left[\left(\log(1+\lambda_{xy})\right)^K\right], \mathbf{E}\left[\left(\log\left(1+\frac{1}{\delta_x}\right)\right)^K\right]\right\} < \infty, \quad (53)$$

then there exists a $v(K, d, \Gamma') > 1$ such that for every $1 < v < v(K, d, \Gamma')$ and $m > 0$, we can find constants C_1 such that when

$$\mathbf{E}\left[\left(\log\left(1+\frac{\lambda_{xy}}{\delta_x}\right)\right)^K\right] < C_1, \quad (54)$$

there exists, with **Q***-probability one, a finite constant*

$$L = L\left(x, \{\lambda_{xy}\}, \{\delta_x\}\right)$$

such that

$$G^{t-s}(x, y) \leqslant \exp\left(-m\left[d(x, y) + \left(\log(1+|t-s|)\right)^v\right]\right) \quad (55)$$

when $|x-y| > L$ or $|t-s| > e^{L^{1/v}}$.

Note that if we want to use Proposition 2 with intensities δ_x and λ_{xy} as defined in (11) and (37), we should extend the result to the case where the intensities are correlated when they have one lattice site common in their indices. The necessary, but small modifications to the proof of Klein are mentioned in [8]. In the rest of the paper we will always refer to Proposition 2, even when we deal with non-independent intensities.

4.1.2. Various couplings

The major key to the proofs of our results is the connection between the coupling of two spin systems and the percolation processes we defined above. Let $P_1(\sigma)$, $P_2(\sigma)$ be two probability measures on Ω_Λ, $\Lambda \subset \mathbb{Z}^d$. A coupling of the spin systems (P_1, Ω_Λ) and (P_2, Ω_Λ) is given by a probability measure $\text{Prob}(\sigma, \sigma')$ on the product space $\Omega_\Lambda \times \Omega_\Lambda$, with marginals P_1 and P_2. In this paragraph we will define three different couplings that, abusing the notation, we will all denote by Prob. To which one we refer will be clear from the context.

(i) Let $\zeta, \zeta' \in \Omega$ be two configurations coinciding outside some $A \subset \mathbb{Z}^d$, i.e., $\zeta(z) = \zeta'(z)$, when $z \in A^c$. The basic coupling between two copies σ_n and σ_n' of the same PCA with as initial conditions ζ and ζ' gives rise to a new PCA with the properties that

$$\begin{aligned}\text{Prob}\big[\sigma_N(x) \neq \sigma_N'(x) | \sigma_{N-1} = \zeta, \sigma_{N-1}' = \zeta'\big] \\ \leqslant \text{var}\big(p_x(\cdot|\zeta), p_x(\cdot|\zeta')\big),\end{aligned} \tag{56}$$

with, var(· , ·) the variational distance and

$$\text{Prob}\big[\sigma_N(x) \neq \sigma_N'(x) | \zeta, \zeta'\big] \leqslant \sum_{z\in A} G^N(x, z). \tag{57}$$

Here $G^N(x, z)$ is the probability to find a time oriented open path from (x, N) upto $(y, 0)$ in the independent site percolation process on the space–time graph $\mathcal{L}$ with densities k_x as defined in (8).

(ii) When we start a PCA with as initial configuration $\sigma_0 = \zeta$ and remember the configuration σ_n on every time step n we get a configuration $\omega = \{\sigma_n\}_{n=0,1,\ldots}$ on the space–time graph $\mathcal{L}$. If we choose the initial configuration according to any invariant measure μ, we obtain a measure $\boldsymbol{\mu}$ on the space–time configurations. Take $\mathsf{V} \subset \mathbb{Z}^{d+1}$ and $\mathsf{W}, \tilde{\mathsf{W}} \subset \partial(\mathsf{V}^c)$ such that $\mathsf{W} \cup \tilde{\mathsf{W}} = \partial(\mathsf{V}^c)$. Let ω, ω' be two configurations on the space–time graph $\mathcal{L}$, such that $\omega(z, n) = \omega'(z, n)$ when $(z, n) \in \tilde{\mathsf{W}}$. Van den Berg and Maes [2] constructed a coupling between the conditional measures $\boldsymbol{\mu}(\cdot|\omega \text{ on } \mathsf{V})$ and $\boldsymbol{\mu}(\cdot|\omega' \text{ on } \mathsf{V})$ such that for every $(x, N) \in \mathsf{V}^c$,

$$\begin{aligned}\text{Prob}\big[\sigma_N(x) \neq \sigma_N'(x) | \omega \text{ on } \mathsf{V}, \omega' \text{ on } \mathsf{V}\big] \\ \leqslant \sum_{(z,n)\in \mathsf{W}} G\big((x, N), (z, n)\big).\end{aligned} \tag{58}$$

$G((x, N), (y, M))$ is the probability to find an open path from (x, N) to (y, M) in the independent site percolation process on the space–time graph $\mathcal{L}$ with densities q_x as defined in (9).

(iii) Finally we consider the basic coupling between two copies of a continuous time RIPS σ_t, σ_t' with initial conditions ζ, ζ', such that $\zeta(z) = \zeta'(z)$, when $z \in A^c$ for some $A \subset \mathbb{Z}^d$. In this way we obtain a RIPS with the property that ,

$$\text{Prob}\big[\sigma_t(x) \neq \sigma_t'(x) | \zeta, \zeta'\big] \leqslant \sum_{z\in A} G^t(x, z). \tag{59}$$

$G^t(x, z)$ is the probability to find an open path from (x, t) to $(z, 0)$ in the continuous percolation process on $\mathbb{Z}^d \times \mathbb{R}$ with intensities δ_x and λ_{xy} as defined in (11) and (37).

4.2. Proofs

In the proof of Theorem 1 we will use the following proposition.

PROPOSITION 3. – *For every local function f*

$$\|P^N f - \mu(f)\| \leqslant \sum_{y \in \operatorname{supp} f} \Delta_y f \sum_{z \in \mathbb{Z}^d} G^N(y,z), \tag{60}$$

with μ any invariant measure for the dynamics.

Proof. – In the basic coupling we have that for all initial conditions ζ, ζ'

$$|P^N f(\zeta) - P^N f(\zeta')| \leqslant \sum_z \Delta_z f \operatorname{Prob}[\sigma_N(z) \neq \sigma'_N(z) | \zeta, \zeta']. \tag{61}$$

Combining this with (57) and using that there is at least one invariant measure for the dynamics yields Proposition 3. □

Proof of Theorem 1. – Define the box $\Lambda_x^N, x \in \mathbb{Z}^d$, as

$$\Lambda_x^N = \{z,\ x^i \leqslant z^i \leqslant x^i + N,\ i = 1, \ldots, d\}. \tag{62}$$

A time oriented path ω on the space–time graph $\mathcal{L}$ is uniquely determined by its projection ω' on $\mathbb{Z}^d$ and the number of steps $l_i = 1, 2, \ldots$ the path spends on each site $x_i \in \omega'$. Note that ω' is spatially directed. Let ω be any path from (x, N) to (y, M), with $d(x, y) = m$, then $y \in \Lambda_x^{N-M}$ and the number of sites in ω' is $|\omega'| = m + 1$.

Hence,

$$G^N(x,y) \leqslant \sum_{\substack{\omega' : x \to y \\ |\omega'| = m+1}} \sum_{\substack{l_0 + \cdots + l_m = N \\ l_0, \ldots, l_m \geqslant 1}} \prod_{i=0}^m k_{\omega'_i}^{l_i} \leqslant k_{\max}^{(1-\theta)N} \sum_{\substack{\omega' : x \to y \\ |\omega'| = m+1}} \prod_{i=0}^m \frac{k_{\omega'_i}^\theta}{1 - k_{\omega'_i}^\theta}, \tag{63}$$

with $k_{\max} = \max_z \{k_z,\ z \in \Lambda_x^N\}$ and $0 < \theta < 1$.

We will use the Borel–Cantelli lemma to prove that for some $\delta > 0$ there exists $\mathbf{Q}$-a.s. a finite time N_1 such that for all $N > N_1$ and $\forall z \in \Lambda_x^N$

$$k_z^{(1-\theta)N} \leqslant \exp(-N \exp(-\delta (\log N)^v)), \tag{64}$$

whenever $\nu > 1/v$, or $\nu = 1/v$ and α/β is large enough.

Consider indeed,

$$
\begin{aligned}
&\mathbf{Q}\{\exists z \in \Lambda_x^N,\ k_z^{1-\theta} > \exp(-\exp(-\delta(\log N)^v))\}\\
&\quad\leqslant 1 - \left[1 - \mathbf{Q}\{\beta\gamma_x > \delta(\log N)^v\}\right]^{(N+1)^d}\\
&\quad\leqslant 1 - \left[1 - \exp\left(-\alpha\left(\tfrac{\delta}{\beta}(\log N)^v\right)^\nu\right)\right)^{(N+1)^d}, \qquad (65)
\end{aligned}
$$

for all $0 < \theta < 1$. This is summable when

$$
\lim_{N\uparrow\infty} N^{d+1} \exp\left(-\alpha\left(\tfrac{\delta}{\beta}\right)^\nu (\log N)^{v\nu}\right) = 0, \qquad (66)
$$

e.g., when $v\nu > 1$ or when $v\nu = 1$ and α/β is large enough.

Using this together with Proposition 3, we see that there exists $\mathbf{Q}$-a.s. a finite N_2, such that for $N > N_2$

$$
\begin{aligned}
\left\|P^N f - \mu(f)\right\| &\leqslant |||f|||\exp(-N\exp(-\delta(\log N)^v))\\
&\quad\times \sup_{x\in\operatorname{supp} f} \sum_{z\in\mathbb{Z}^d} \sum_{\substack{\omega' : x\to z\\ |\omega'|=d(x,z)+1}} \prod_{i=0}^{d(x,z)} \frac{k_{\omega_i'}^\theta}{1-k_{\omega_i'}^\theta}. \qquad (67)
\end{aligned}
$$

Finally, we observe that we can use condition (21) and the Chebychev inequality to show that

$$
\mathbf{Q}\left\{ \sum_{\substack{\omega' : x\to z\\ |\omega'|=d(x,z)+1}} \prod_{i=0}^{d(x,z)} \frac{k_{\omega_i'}^\theta}{1-k_{\omega_i'}^\theta} > \mathrm{e}^{-\alpha d(x,z)} \right\} \leqslant \mathrm{e}^{-\alpha' d(x,z)}, \qquad (68)
$$

for some α, α'. Hence, we can apply the Borel–Cantelli lemma to conclude that the sum in (67) is $\mathbf{Q}$-a.s. finite.

Theorem 1 follows for $0 < \delta < \lambda < 1$, when N_0 is large enough. □

Proof of Theorem 2. – We first calculate an upperbound for $\mathbf{E}(k_x^N)$. Therefore we define $k^* = k^*(N) = 1 - \mathrm{e}^{-\beta\gamma^*}$ as follows

$$
(k^*)^N = \left(1 - \mathrm{e}^{-\beta\gamma^*}\right)^N = \mathrm{e}^{-\delta(\log N)^\nu}, \qquad (69)
$$

for some $\delta > 0$.

Then,

$$
\begin{aligned}
\mathbf{E}[k_x^N] &= \mathbf{E}\left[(1-\mathrm{e}^{-\beta\gamma})^N | \gamma \leqslant \gamma^*\right]\mathbf{Q}\{\gamma \leqslant \gamma^*\}\\
&\quad + \mathbf{E}\left[(1-\mathrm{e}^{-\beta\gamma})^N | \gamma > \gamma^*\right]\mathbf{Q}\{\gamma > \gamma^*\}
\end{aligned}
$$

$$\leqslant \exp\bigl(-\delta(\log N)^{\nu}\bigr) + \mathbf{Q}\{\gamma > \gamma^*\}$$
$$\leqslant \exp\bigl(-\delta'(\log N)^{\nu}\bigr), \tag{70}$$

for $\delta' < \min\{\delta, \alpha/\beta^{\nu}\}$ and N large enough. Later on in the proof we will use the fact that we can take $\delta > 1$, and hence, when $\beta > 0$ is small enough, $\delta' > 1$.

To prove Theorem 2 we rewrite Proposition 3 as follows

$$\mathbf{E}\bigl[\|P^N f - \mu(f)\|\bigr] \leqslant |||f|||\Bigl(\sum_{z:\ d(x_0,z)\leqslant(\log N)^{\nu}} \mathbf{E}\bigl[G^N(x_0,z)\bigr] + \sum_{z:\ d(x_0,z)>(\log N)^{\nu}} \mathbf{E}\bigl[G^N(x_0,z)\bigr]\Bigr), \tag{71}$$

with x_0 a fixed site in supp f.

We now follow the method of the proof of Theorem 1 and consider the projection ω' of the space–time path ω.

In the first case where $d(x_0,z) \leqslant (\log N)^{\nu}$, we know that in each possible space–time path ω from (x_0, N) to $(z, 0)$ there is at least one site $x_i \in \omega'$ where the path stays at least during $l_i > N/(\log N)^{\nu}$ timesteps. Hence,

$$\sum_{z:\ d(x_0,z)\leqslant(\log N)^{\nu}} \mathbf{E}\bigl[G^N(x_0,z)\bigr]$$
$$\leqslant \sum_{z:\ d(x_0,z)\leqslant(\log N)^{\nu}} \ \sum_{\substack{\omega':\, x_0\to z\\ |\omega'|=d(x_0,z)+1}} \ \sum_{\substack{l_0+\cdots+l_{|\omega'|}=N\\ l_0,\ldots,l_{|\omega'|}\geqslant 1}} \ \prod_{i=0}^{d(x_0,z)} \mathbf{E}\bigl[k_{\omega_i'}^{l_i}\bigr]$$
$$\leqslant \sum_{z:\ d(x_0,z)\leqslant(\log N)^{\nu}} \bigl(d(x_0,z)+1\bigr) \sum_{l>N/(\log N)^{\nu}} \mathbf{E}\bigl[k_x^l\bigr]$$
$$\times \sum_{\substack{\omega':\, x_0\to z\\ |\omega'|=d(x_0,z)+1}} \left(\mathbf{E}\left[\frac{k_x}{1-k_x}\right]\right)^{d(x_0,z)}$$
$$\leqslant \exp -\left(\delta''\left(\log \frac{N}{(\log N)^{\nu}}\right)^{\nu}\right), \tag{72}$$

for $\delta'' < \delta'$ and N large enough. Note that condition (24) guarantees that the sum over the paths is finite and that in the case $\nu = 1$ the sum over l converges when $\delta' > 1$.

Finally, we again use condition (24) to find a bound for the second term in the RHS of (71)

$$\sum_{z:\, d(x_0,z)>(\log N)^\nu} \mathbf{E}\big[G^N(x_0,z)\big]$$

$$\leqslant \sum_{z:\, d(x_0,z)>(\log N)^\nu} \sum_{\substack{\omega':x_0\to z\\ |\omega'|=d(x_0,z)+1}} \sum_{\substack{l_0+\cdots+l_m=N\\ l_0,\dots,l_m\geqslant 1}} \prod_{i=0}^{d(x_0,z)} \mathbf{E}\big[k_{\omega'_i}^{l_i}\big]$$

$$\leqslant \sum_{z:\, d(x_0,z)>(\log N)^\nu} \sum_{\substack{\omega':x_0\to z\\ |\omega'|=d(x_0,z)+1}} \left(\mathbf{E}\left[\frac{k_x}{1-k_x}\right]\right)^{(d(x_0,z)+1)}$$

$$\leqslant \exp\big(-\delta'''(\log N)^\nu\big), \tag{73}$$

for some $\delta''' > 0$ when N is large enough. The combination of (72) and (73) proves Theorem 2. □

Proof of Theorem 3. – Consider the space–time measure $\boldsymbol{\mu}$ (as introduced in paragraph 4.1.2(ii)). The projection of this measure to any time layer is again the invariant measure μ.

Denote by $V_N = V \times \{N\}$ a copy of the set $V \subset \mathbb{Z}^d$ on the time N-layer of the space–time graph $\mathcal{L}$ and by f_N a copy of the function f on this time layer.

Using the space–time coupling (see paragraph 4.1.2(ii)) we can estimate the truncated correlation function as follows:

$$\big|\mu(fg) - \mu(f)\mu(g)\big|$$

$$= \left|\int d\mu(\xi) f(\xi)\big[\mu(g|\xi \text{ on } \operatorname{supp} f) - \mu(g)\big]\right|,$$

$$\leqslant \|f\| \sup_{\xi,\xi'} \big|\boldsymbol{\mu}(g_N|\xi \text{ on } \operatorname{supp} f_N) - \boldsymbol{\mu}(g_N|\xi' \text{ on } \operatorname{supp} f_N)\big|,$$

$$\leqslant \|f\| \|\|g\|\| \sup_{x\in\operatorname{supp} g} \sum_{y\in\operatorname{supp} f} G\big((x,N),(y,N)\big), \tag{74}$$

for any N. Applying Proposition 1 gives (35).

To prove that μ is a Gibbs measure it suffices to show that for every $x \in \mathbb{Z}^d$ there is a version of the conditional probability

$$\mu\big[\sigma(x) = \xi(x)|\sigma(y) = \xi(y), \text{ for } y \neq x\big] \tag{75}$$

which is strictly positive and continuous in ξ. (See, e.g., [16].)

The positivity is guaranteed by (36). To prove the continuity we define a sequence of d-dimensional sets $\Lambda_r \subset \mathbb{Z}^d$, $r = 1, 2, \dots$, containing the site x, such that $\Lambda_r \subset \Lambda_{r+1}$ and filling up whole $\mathbb{Z}^d$. We also consider two associated sets of configurations $\{\xi_r\}_{r=1,2,\dots}$ and $\{\xi'_r\}_{r=1,2,\dots}$ such that $\xi_r(z) = \xi'_r(z)$ for $z \in \Lambda_r$.

Then for any time N, for all finite $V \supset \Lambda_r$,

$$\begin{aligned}
&\big|\mu\big[\sigma(x) = \xi(x) | \sigma(y) = \xi(y) \text{ on } V, \text{ for } y \neq x\big] \\
&\quad -\mu\big[\sigma(x) = \xi'(x) | \sigma(y) = \xi'(y) \text{ on } V, \text{ for } y \neq x\big]\big| \\
&\qquad = \big|\boldsymbol{\mu}\big[\sigma_N(x) = \xi(x) | \sigma_N(y) = \xi(y) \text{ on } V_N, \text{ for } y \neq x\big] \\
&\qquad\quad - \boldsymbol{\mu}\big[\sigma_N(x) = \xi'(x) | \sigma_N(y) = \xi'(y) \text{ on } V_N, \text{ for } y \neq x\big]\big|.
\end{aligned} \tag{76}$$

Using again the space–time coupling we see that (76) is bounded by

$$\begin{aligned}
&2\,\mathrm{Prob}\big[\sigma_N(x) \neq \sigma'_N(x) | \sigma_N(y) = \xi_r(y), \sigma'_N(y) = \xi_r(y) \text{ for } y \neq x\big] \\
&\quad \leqslant 2 \sum_{z \in \Lambda_r^c \times \{N\}} G\big((x, N), (z, N)\big),
\end{aligned} \tag{77}$$

uniformly in V. Proposition 1 implies that this tends to zero when Λ_r approaches $\mathbb{Z}^d$. □

Proof of Theorem 4. – Let $G_r^t(x, y)$ be the connectivity function that corresponds with the infinite volume dynamics P_r^t (cf. paragraph 4.1.2(iii)). To prove Theorem 4, we show that the sequence $P_r^t f(\zeta)$, $r = 1, 2, \dots,$ is a Cauchy sequence that converges uniformly in the Banach space of continuous functions $\mathcal{C}(\Omega)$, $\|\cdot\|$.

Take l_0 such that $\operatorname{supp} f \subset \Lambda_{l_0}$ and let $k > l > l_0$.

$$\begin{aligned}
\big|P_k^t f(\zeta) - P_l^t f(\zeta)\big| &= \bigg|\int_0^t ds\, P_l^{t-s}(L_k - L_l) P_k^s f(\zeta)\bigg| \\
&\leqslant \int_0^t ds \sum_{z \in \Lambda_k \setminus \Lambda_l} c(z, \zeta) \big|P_k^s f(\zeta) - P_k^s f(\zeta^z)\big| \\
&\leqslant |||f||| \int_0^t ds \sum_{z \in \Lambda_k \setminus \Lambda_l} \sup_{y \in \operatorname{supp} f} \sup_{\xi} c(z, \xi) \\
&\quad \times \mathrm{Prob}\big[\sigma_s(y) \neq \sigma'_s(y) | \zeta, \zeta^z\big].
\end{aligned} \tag{78}$$

In the last step, we introduced the basic coupling between two copies of the semigroup P_k^t, σ_t and σ'_t with as initial conditions ζ and ζ^z. Now, note that $G_r^t(x, y) \leqslant G^t(x, y)$ when $x \neq y$. Then, (59) and Proposition 2 tell us, that for any $m > 0$ there exists **Q**-a.s. a finite constant $B = B(f)$ such that for all times $t \geqslant 0$ (78) is bounded by

$$B \sum_{r \geqslant l} \mathrm{e}^{-m \operatorname{dist}(\partial \Lambda_r, \operatorname{supp} f)} \sum_{z \in \partial \Lambda_r} \max_{\zeta} c(z, \zeta)$$
$$\leqslant B \mathrm{e}^{-m(l-l_0)} \sum_{r \geqslant 0} \mathrm{e}^{-mr} \sum_{z \in \partial \Lambda_{l-l_0+r}} \max_{\zeta} c(z, \zeta). \tag{79}$$

So, when the sum over r is **Q**-a.s. finite, the sequence P_r^t, $r = 0, 1, \dots$, is a Cauchy sequence and the infinite volume dynamics P^t is Feller **Q**-a.s. Therefore we are left with estimating

$$\mathbf{Q}\Big\{ \max_{z \in \partial \Lambda_{l-l_0+r}} \max_{\zeta} c(z, \zeta) > \mathrm{e}^{mr} \Big\}$$
$$\leqslant 1 - \mathbf{Q}\Big\{ \max_{\zeta} c(z, \zeta) \leqslant \mathrm{e}^{mr}, \forall z \in \partial \Lambda_{l-l_0+r} \Big\}$$
$$\leqslant 1 - \Big[1 - \mathbf{Q}\Big\{ \Big(\log\Big(1 + \max_{\zeta} c(z, \zeta)\Big)\Big)^K > (mr)^K \Big\} \Big]^{2d(2(l-l_0+r)+1)^{d-1}}$$
$$\leqslant 1 - \left[1 - \frac{\mathbf{E}[(\log(1 + \max_{\zeta} c(z, \zeta)))^K]}{(mr)^K} \right]^{2d(2(l-l_0+r)+1)^{d-1}} \tag{80}$$

When $K > d + 1$, we can use the Borel–Cantelli lemma and condition (41) to conclude. □

To prove our last result, we will use the following lemma. Let,

$$c_x = \max_{\zeta} c(x, \zeta), \qquad \Gamma_{xy} = \max_{\zeta} \big| c(x, \zeta) - c(x, \zeta^y) \big|. \tag{81}$$

LEMMA 1. – *The basic coupling between the RIPS P^t and $\overline{P}^t$ (44) obeys*

$$\operatorname{Prob}\big[\sigma_t(x) \neq \sigma_t'(x) | \zeta, \zeta\big]$$
$$\leqslant \frac{t^2}{2} \Big[\sum_{y \sim x} \Gamma_{xy} \Big(c_y + \sum_{z \sim y} \Gamma_{yz}(1 + c_z) + c_y^2 \Big) + c_x^2 \Big]. \tag{82}$$

The proof of Lemma 1 is postponed to the end of the paper.

Proof of Theorem 5. – The proof is an extension of a result in [14] for non random IPS. Take $t = n\delta + s$, $0 \leqslant s < \delta$,

$$\big\| P^t f - P_\delta^{\lfloor t/\delta \rfloor} f \big\| \leqslant \big\| P^t f - P^{n\delta} f \big\| + \big\| P^{n\delta} f - P_\delta^n f \big\|. \tag{83}$$

The first term of the right hand side is bounded by

$$s\|Lf\| \leqslant s\sum_{z}\Delta_z f \max_{\zeta} c(z,\zeta). \tag{84}$$

Because,

$$P^{n\delta} - P_\delta^n = \sum_{r=1}^{n} P_\delta^{n-r}\left(P^\delta - P_\delta\right)P^{(r-1)\delta}, \tag{85}$$

the second term of (83) is bounded by

$$\sum_{r=1}^{n}\sum_{z\in\mathbb{Z}^d}\Delta_z\left(P^{(r-1)\delta}f\right)\mathrm{Prob}\left[\sigma_\delta(z)\neq\sigma'_\delta(z)|\zeta,\zeta\right]. \tag{86}$$

Prob is the basic coupling between the probability measures induced by the original process P^t and the auxiliary dynamics $\overline{P}^t$ (as defined in (44)) respectively. Using (59) to bound $\Delta_z(P^{(r-1)\delta}f)$, we can show that (86) is smaller than

$$\sum_{r=1}^{\infty}\sum_{y,z\in\mathbb{Z}^d}\Delta_y f G^{(r-1)\delta}(y,z)\mathrm{Prob}\left[\sigma_\delta(z)\neq\sigma'_\delta(z)|\zeta,\zeta\right]. \tag{87}$$

Under the conditions of Theorem 5, there exists, for some $v > 1$, $m > 0$, **Q**-a.s. a finite constant $B_x = B_x(x,\{\lambda_{xy}\},\{\delta_x\})$ such that (see Proposition 2)

$$\sum_{r=1}^{\infty}G^{\delta(r-1)}(x,y) \leqslant B_x\exp(-md(x,y))\sum_{r=1}^{\infty}\exp\left(-m\left(\log\left(1+\delta(r-1)\right)\right)^v\right). \tag{88}$$

Moreover,

$$\sum_{r=1}^{\infty}\exp\left(-m\left(\log\left(1+\delta(r-1)\right)\right)^v\right) \leqslant \left\lceil\frac{1}{\delta}\right\rceil\sum_{r=1}^{\infty}\exp\left(-m(\log r)^v\right). \tag{89}$$

Applying Lemma 1, gives, together with (87)–(89) the following upper bound for (87)

$$\frac{\delta}{2}\tilde{C} \sup_{y\in\operatorname{supp} f} \sum_{z\in\mathbb{Z}^d} \mathrm{e}^{-md(y,z)}\left(\sum_{u\sim z}\Gamma_{zu}\left(c_u+\sum_{v\sim u}\Gamma_{uv}(1+c_v)+c_u^2\right)+c_z^2\right), \tag{90}$$

with $\tilde{C}=\tilde{C}(f,d,m,\{\lambda_{xy}\},\{\delta_x\})<\infty$ $\mathbf{Q}$-a.s.

Theorem 5 is proven if we can show that for every $y\in\mathbb{Z}^d$ the sum in (90) is finite. Therefore we apply the Borel–Cantelli lemma to the probabilities

$$\mathbf{Q}\left\{\sum_{z:\ d(y,z)=L}\left(\sum_{u\sim z}\Gamma_{zu}\left(c_u+\sum_{v\sim u}\Gamma_{uv}(1+c_v)+c_u^2\right)+c_z^2\right)>\mathrm{e}^{mL}\right\}, \tag{91}$$

which are summable over L.

Indeed, introduce $\Gamma(z)$ as

$$\Gamma(z)=\sum_{u\sim z}\Gamma_{zu}\left(c_u+\sum_{v\sim u}\Gamma_{uv}(1+c_v)+c_u^2\right)+c_z^2, \tag{92}$$

then, (91) is not larger than

$$\begin{aligned}&\mathbf{Q}\left\{\exists z:\ d(y,z)=L,\ \Gamma(z)>\frac{\mathrm{e}^{mL}}{2d(2L+1)^{d-1}}\right\}\\ &\quad\leqslant 2d(2L+1)^{d-1}\mathbf{Q}\left\{\Gamma(z)>\frac{\mathrm{e}^{mL}}{2d(2L+1)^{d-1}}\right\},\end{aligned} \tag{93}$$

by subadditivity.

Repeating this argument for all the sums and using that $\Gamma_{xy}<c_x$, we easily see that all the terms that appear, are, for large L, dominated by

$$C_1L^{d-1}\mathbf{Q}\left\{\left(\log(1+c_x)\right)^{d+1}>C_2L^{d+1/2}\right\}\leqslant C_1L^{d-1}\frac{\mathbf{E}[(\log(1+c_x))^{d+1}]}{L^{d+1/2}} \tag{94}$$

for some constants C_1, C_2. Condition (41) implies that the sum over L converges. □

Proof of Lemma 1. – Following [14] we construct for both IPS a random PCA with transition probabilities parametrized by $\delta>0$:

$$p_x^{(\delta)}(\sigma(x)|\zeta)=\begin{cases}\mathrm{e}^{-\delta c(x,\zeta)} & \text{if } \sigma(x)=\zeta(x),\\ 1-\mathrm{e}^{-\delta c(x,\zeta)} & \text{if } \sigma(x)\neq\zeta(x),\end{cases} \tag{95}$$

and

$$q_x^{(\delta)}(\sigma(x)|\zeta) = \begin{cases} \mathrm{e}^{-\delta d(x,\zeta)} & \text{if } \sigma(x) = \zeta(x), \\ 1 - \mathrm{e}^{-\delta d(x,\zeta)} & \text{if } \sigma(x) \neq \zeta(x). \end{cases} \tag{96}$$

The Trotter–Kurtz Theorem implies that

$$\lim_{\delta \downarrow 0} P_\delta^{\lfloor t/\delta \rfloor} = P^t f(\zeta), \qquad \lim_{\delta \downarrow 0} \overline{P}_\delta^{\lfloor t/\delta \rfloor} = \overline{P}^t f(\zeta). \tag{97}$$

In the same way the basic coupling between the two RIPS P_t and $\overline{P}_t$ is equal to the limit towards infinite small timesteps $\delta \downarrow 0$ of the basic couplings between the associated random PCA. $\mathbf{E}^{\zeta,\tilde{\zeta}}$ is the corresponding expectation value in the coupled process with as initial configurations ζ and $\tilde{\zeta}$.

Define, for the configurations σ and σ', the indicator function $I\{\sigma(x) \neq \sigma'(x)\}$ that gives one when $\sigma(x) \neq \sigma'(x)$ and zero otherwise.

$$\frac{d}{dt}\mathrm{Prob}\left[\sigma_t(x) \neq \sigma_t'(x)|\zeta,\zeta\right]$$
$$= \mathbf{E}^{\zeta,\zeta}\left[\lim_{\delta\downarrow 0}\frac{1}{\delta}\left(\mathrm{var}\left(p_x^{(\delta)}(\cdot|\sigma_t), q_x^{(\delta)}(\cdot|\sigma_t')\right) - I\left\{\sigma_t(x) \neq \sigma_t'(x)\right\}\right)\right]. \tag{98}$$

By the triangle inequality, this is smaller than

$$\mathbf{E}^{\zeta,\zeta}\left[\lim_{\delta\downarrow 0}\frac{1}{\delta}\left(\sum_{z\sim x}\max_{\zeta}\mathrm{var}\left(p_x^{(\delta)}(\cdot|\zeta), p_x^{(\delta)}(\cdot|\zeta^z)\right)I\left\{\sigma_t(z) \neq \sigma_t'(z)\right\}\right.\right.$$
$$\left. + \max_{\zeta}\left(\mathrm{var}\left(p_x^{(\delta)}(\cdot|\zeta), p_x^{(\delta)}(\cdot|\zeta^x)\right) - 1\right)\right)I\left\{\sigma_t(x) \neq \sigma_t'(x)\right\}$$
$$\left. + \mathrm{var}\left(p_x^{(\delta)}(\cdot|\sigma_t'), q_x^{(\delta)}(\cdot|\sigma_t')\right)\right]. \tag{99}$$

In the limit $\delta \downarrow 0$ this yields

$$\frac{d}{dt}\mathrm{Prob}\left[\sigma_t(x) \neq \sigma_t'(x)|\zeta,\zeta\right] \leqslant \sum_{z\sim x}\Gamma_{xz}\mathrm{Prob}\left[\sigma_t(z) \neq \sigma_t'(z)|\zeta,\zeta\right]$$
$$- \min_{\zeta}\left|c(x,\zeta) + c(x,\zeta^x)\right|\mathrm{Prob}\left[\sigma_t(x) \neq \sigma_t'(x)|\zeta,\zeta\right]$$
$$+ \mathbf{E}^{\zeta,\zeta}\left[\left|c(x,\sigma_t') - d(x,\sigma_t')\right|\right]. \tag{100}$$

The last term is bounded by

$$\mathbf{E}^{\zeta,\zeta}\left[\left|c(x,\sigma_t') - d(x,\sigma_t')\right|I\left\{\sigma_t'(x) = \zeta(x)\right\}\right.$$
$$\left. + \left|c(x,\sigma_t') - d(x,\sigma_t')\right|I\left\{\sigma_t'(x) \neq \zeta(x)\right\}\right]$$

$$\leqslant \sum_{z\sim x} \Gamma_{xz}\mathrm{Prob}\big[\sigma_t'(z) \neq \zeta(z)\big] + c_x\mathrm{Prob}\big[\sigma_t'(x) \neq \zeta(x)\big]. \quad (101)$$

Consider for any $x \in \mathbb{Z}^d$

$$\begin{aligned}\frac{d}{dt}\mathrm{Prob}\big[\sigma_t(x) \neq \zeta(x)|\zeta,\zeta\big] &= \frac{d}{dt}\mathbf{E}^{\zeta,\zeta}\big[I\{\sigma_t'(x) \neq \zeta(x)\}\big]\\ &= \mathbf{E}^{\zeta,\zeta}\big[d(x,\sigma_t)\big(I\{\sigma_t'(x) = \zeta(x)\} - I\{\sigma_t'(x) \neq \zeta(x)\}\big)\big]\\ &\leqslant c_x. \end{aligned} \quad (102)$$

Hence,

$$\mathrm{Prob}\big[\sigma_t(x) \neq \zeta(x)|\zeta,\zeta\big] \leqslant c_x t. \quad (103)$$

When we substitute this in (100) and integrate both sides between 0 and δ, we get

$$\begin{aligned}\mathrm{Prob}\big[\sigma_\delta(x) \neq \sigma_\delta'(x)|\zeta,\zeta\big] &\leqslant \sum_{z\sim x}\Gamma_{xz}\int_0^\delta ds\,\mathrm{Prob}\big[\sigma_s(z) \neq \sigma_s'(z)|\zeta,\zeta\big]\\ &\quad + \frac{\delta^2}{2}\Big(\sum_{z\sim x}\Gamma_{xz}c_z + c_x^2\Big). \end{aligned} \quad (104)$$

Expanding the integral term in a Taylor series for small δ gives

$$\begin{aligned}&\int_0^\delta ds\,\mathrm{Prob}\big[\sigma_s(z) \neq \sigma_s'(z)|\zeta,\zeta\big]\\ &\quad = \frac{\delta^2}{2}\frac{d}{dt}\mathrm{Prob}\big[\sigma_t(z) \neq \sigma_t'(z)|\zeta,\zeta\big]\big|_{t=\theta\delta}, \end{aligned} \quad (105)$$

for some $0 < \theta < 1$.

So, we can use (100) for the second time to conclude. □

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