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Spontaneously broken symmetries

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ABSTRACT. – The possibility of broken symmetry can be decided in the fixpoint algebra of the symmetry group by solving an appropriate eigenvalue problem. The method is applied for the fermi algebra and spin systems. © Elsevier, Paris

Key words: Symmetry, automorphisms, extremal invariant states, crossed product, fixed point algebra.

RÉSUMÉ. – La possibilité de symétries spontanément brisées peut être décidée par l'existence d'une valeur propre 1 d'un certain opérateur. La méthode est illustrée par des exemples comme les algèbres de spin et CAR. © Elsevier, Paris

1. INTRODUCTION

Spontaneous symmetry breaking is one of the main themes of physics in the past decades and crucial effects in elementary particle physics and condensed matter physics are attributed to it [GHK]. Since this phenomenon was mainly studied within the context of quantum field theory it became considered as typical for infinite quantum systems. We shall study it in this paper in the context of general dynamical systems and we will realize that the mechanism of symmetry breaking can be found also in some finite systems. For finite systems one is used to have a unique Hamiltonian determining the dynamics, and the degeneracy of its pure point spectrum is responsible for spontaneous symmetry breaking. If the system allows inequivalent representations (which is always the case for classical systems and for infinite quantum systems) then the symmetry breaking is related to the unitary implementability of automorphisms. But this implementability has different consequences if one is in a pure state or in a mixed state. We shall consider in detail only the simplest case of a discrete symmetry group and there it is possible to decide whether spontaneous symmetry breaking can occur by examining whether an appropriate operator has eigenvalue one.

We will use this strategy to show e.g. that translation invariant states on the fermi fields have to be even and that symmetry breaking in the X-Y model is only possible for appropriate parameters in the ground state but not in faithful states like temperature states. In particular relativistic QFT with a Poincaré-invariant vacuum cannot give a nonvanishing vacuum expectation value of a fermi (i.e. anticommuting) field irrespective of whether the vacuum is degenerated or not as long as it is translation invariant. This is surprising in supersymmetric theories where bose and fermi fields can be mixed and bose fields can certainly have nonvanishing vacuum expectation values.

In Section 2 we shall give the general definitions and recapitulate in various examples where symmetry breaking does occur or not. We shall see that the interesting examples have the form of a crossed product and in Section 3 we shall derive the main theorem which states when for such a structure spontaneous symmetry breaking can occur. Our results are not limited to vacuum or equilibrium states but we need some properties of the time evolution of the algebra invariant under the symmetry. This will shed some light on the question why the infinite spin chain and the fermionic chain behave so differently in this respect though for finite chains they are isomorphic. Finally we shall illustrate the power of this theorem by applying it to the rotation algebra.

2. DEFINITIONS AND EXAMPLES

A dynamical system $(\mathcal{A}, \tau)$ consists of a C*-algebra $\mathcal{A}$ (the "observables") and a one parameter group τ^t of automorphisms of $\mathcal{A}$, $t \in R$ or Z (the "time evolution"). A state $\omega : \mathcal{A}^+ \rightarrow R^+$ is called invariant if $\omega \circ \tau = \omega$ and extremal invariant if it is not a convex combination of different invariant states. A symmetry $\gamma \in \operatorname{Aut} \mathcal{A}$ is said to be dynamically broken if $[\tau, \gamma] \neq 0$

and spontaneously broken by an invariant state ω if $[\tau, \gamma] = 0$ and $\omega \circ \gamma \neq \omega$. We are interested only in the latter case.

For the question whether a symmetry can be broken spontaneously we can restrict ourselves to study extremal invariant states because if they respect a symmetry so does a mixture of them. Since R and Z are amenable groups there always exists at least one invariant state $\bar{\omega}$ by taking the mean over t of $\omega \circ \tau^t$, ω any state, and one invariant state with respect to τ and γ by averaging $\bar{\omega} \circ \gamma^n$ over n. Thus the only remaining question is whether there are extremal τ-invariant states which break γ.

EXAMPLES 2.1

1. Classical mechanics in one dimension.

   $\mathcal{A} = \{f(x,p)\}$, τ given by the flow of the Hamiltonian $H = p^2 + V(x)$, $\gamma : (x,p) \to (-x,-p)$ (parity) is dynamically unbroken if $V(x) = V(-x)$. Invariant states are given by probability distributions $\rho(H)$.

   (a) $V = x^2$. The states $c \cdot \delta(H - E)$, $E > 0$ are extremal invariant and γ is never broken spontaneously.

   (b) $V = x^4 - x^2$. The states $c \cdot \delta(H - E)$, $E > 0$ are extremal invariant and do not break γ spontaneously. For $-3/16 < E < 0$ $\delta(H - E)\Theta(\pm x)$ is extremal invariant and γ is broken spontaneously. Of course, generally if ω is invariant so is $\frac{1}{2}(\omega + \omega \circ \gamma)$ and does not break γ but is not necessarily extremal τ-invariant.

   (c) $V = x^2 + 1/|x|$. Here all invariant states $c \cdot \delta(H - E)$ are not extremal invariant and its components $c \cdot \delta(H - E)\Theta(\pm x)$ break γ.

   Thus the extremal invariant states may (a) never, (b) sometimes, (c) always break γ but there are always invariant states where γ remains unbroken.

2. Finite quantum systems where the Hamiltonian H has pure point spectrum.

   $\mathcal{A} = \mathcal{B}(\mathcal{H})$, $\tau^t(A) = e^{iHt} A e^{-iHt}$, $A \in \mathcal{A}$. A symmetry γ is given by any unitary $V \in \mathcal{A}$, $\gamma(A) = V \mathcal{A} V^{-1}$, $[V, H] = 0$. The extremal invariant states are $\omega_i(A) = \langle i|A|i\rangle$, $H|i\rangle = E_i|i\rangle$. We see that if $E_i \neq E_j$ $\forall\ i \neq j$ all symmetries of the dynamical system $(\mathcal{A}, \tau)$ are of the form $V(H)$ and cannot be broken spontaneously. If for some $i \neq j$ we have $E_i = E_j$ there is some $V \neq 1$ in the degeneracy space $[V, H] = 0$ and V is not a function of H. The corresponding symmetry is broken spontaneously by ω_i but not by $\frac{1}{2}(\omega_i+\omega_j)$. Thus there is spontaneous symmetry breaking iff there is degeneracy.

3. Infinite quantum systems.

Here new features appear since τ may not be implemented by some e^{iHt} and if it is in some representations H may have continuous spectrum and may not belong to $\mathcal{A}$. Then $V(H) = e^{if(H)}$ will generally not generate an automorphism of $\mathcal{A}$. This new situation arises already for

(a) Free fermions

The CAR algebra $\mathcal{A}$ is generated by the $a(f)$, $f \in L^2$, $[a(f), a(g)]_+ = 0$, $[a(f), a^*(g)]_+ = \langle f|g\rangle$, $\tau^t(a(f)) = a(e^{-iht}f)$, $h = h^* \in \mathcal{L}(L^2)$, $\gamma(a(f)) = -a(f)$.

THEOREM 2.2. – *If h has purely absolutely continuous spectrum γ cannot be broken spontaneously.*

REMARK. – The theorem appears absurd since the following seems an easy counterexample. Take a translation ($= \sigma_x$) invariant Hamiltonian H and add to it $\varepsilon \int dx(a(x) + a^*(x))$. Assume this creates another time evolution $\tau_\varepsilon \neq \gamma^{-1}\tau_\varepsilon\gamma$, $[\tau_\varepsilon, \sigma_x] = 0$. This τ_ε will have a KMS state ω which we may assume to be σ-invariant, if it is not we can average $\omega \circ \sigma_x$ over x. Then $\omega \circ \gamma$ is KMS with respect to $\gamma^{-1}\tau_\varepsilon\gamma$ and thus $\omega \neq \omega \circ \gamma$ since a state cannot be KMS with respect to the two different automorphisms τ_ε and $\gamma^{-1}\tau_\varepsilon\gamma$. Thus ω is σ-translation invariant but not γ invariant and we have a contradiction to the theorem in its general form since the generator of translations has a purely absolutely continuous spectrum. The solution to this puzzle is that τ_ε does not exist since the formal expression H_ε does not generate an automorphism of $\mathcal{A}$.

Because the theorem is surprising we shall give its easy proof.

Proof. – We have to show that $\omega \circ \tau = \omega$ implies $\omega(A) = 0$ whenever A is an odd polynomial in the $a(f)$ and $a^*(f)$. Since h has absolutely continuous spectrum $\forall\, f, g, \varepsilon > 0\ \exists\, T$ such that

$$\|[a(f), \tau^t a^*(g)]_+\| = |\langle f|e^{iht}g\rangle| < \varepsilon \qquad \forall\, t \geq T.$$

Therefore also for any odd polynomial $A\ \exists\ T$ such that $\|[A, \tau^t A]_+\| < \varepsilon\ \forall\, t > T$. Thus since $|\omega(A)|^2 \leq \frac{1}{2}\omega([A^*, A]_+)$

we see

$$\begin{aligned}|\omega(A)|^2 &= \left|\frac{1}{N}\sum_{k=1}^{N}\omega(\tau^{Tk}A)\right|^2 \\ &\leq \frac{1}{2N^2}\omega\left(\sum_{j,k=1}^{N}[\tau^{Tk}A^*,\tau^{Tj}(A)]_+\right) \\ &= \frac{1}{2N^2}\omega\left(\sum_{j,k=1}^{N}[A^*,\tau^{(j-k)T}A]_+\right) \\ &\leq \frac{1}{2N}+\frac{N(N-1)}{2N^2}\varepsilon \quad \text{if } \|A\|\leq 1.\end{aligned}$$

Since this holds for all $\varepsilon > 0$ and N we conclude $\omega(A) = 0$.

REMARKS

1. We see that the proof is more general, τ need not to be quasifree but only asymptotically anticommutative.
2. The result does not extend to the bigger gauge group $\gamma(a) = ia$. By a translation invariant Bogoliubov transformation one can construct a translation invariant state $\bar{\omega}$ with $\bar{\omega}(a(f)a(g)) \neq 0$.

(b) The spin chain (X-Y model)
$\mathcal{A} = \bigotimes_{j\in Z} M_2^{(j)}$ generated by $\{\vec{\sigma}_i\}$. Let τ be determined by the (formal) Hamiltonian

$$H = -J\sum_j[(1+c)\sigma_j^x\sigma_{j+1}^x + (1-c)\sigma_j^y\sigma_{j+1}^y + 2\lambda\sigma_j^z],$$

$$c \in (0,1),\ \lambda \in R$$

and

$$\gamma\sigma_j^{x,y} = -\sigma_j^{x,y}.$$

Here we have the

THEOREM 2.3 [AM]

(a) $|\lambda| \geq 1 : \exists!$ *ground state. For each* β $\exists!$ *KMS state.*

(b) $|\lambda| < 1 : \exists$ 2 *ground states. For each* β $\exists!$ *KMS state.*

COROLLARY. – *Since* $\omega \circ \gamma$ *has the same KMS or ground state properties as* ω, *no symmetry is broken in KMS states but* γ *is* *broken in the ground states iff* $|\lambda| < 1$ *since they are not* γ *invariant.*

4. The quantum CAT (= quantized torus).

Here $\mathcal{A}_\Theta$ is the subalgebra of the Weyl algebra linearly generated by $W(\vec{n}) = e^{i(n_1 x + n_2 p)}$ when $\vec{n}$ is restricted to the lattice Z^2. It is characterized by the relations

$$W(\vec{n})W(\vec{m}) = e^{i\pi\Theta\sigma(\vec{n},\vec{m})}W(\vec{n}+\vec{m})$$

where $\vec{n}, \vec{m} \in Z^2$. $\sigma(\vec{n},\vec{m})$ is the symplectic form $n_1 m_2 - n_2 m_1$ and $\Theta \in (-1,1]$ plays the rôle of Planck's constant. Linear symplectic transformations of $\vec{n}$ give automorphisms of the Weyl algebra and they leave $\mathcal{A}_\Theta$ invariant if they have integer coefficients. Thus we consider a discrete time evolution $\tau_T(W(\vec{n})) = W(T\vec{n})$ with $T \in \mathrm{Sp}\,(2, Z) = SL(2, Z)$ or explicitly

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a, b, c, d \in Z \quad \text{and} \quad ad - bc = 1.$$

The shifts $\gamma_{\vec{k}} W(\vec{n}) = e^{i\langle \vec{k}|\vec{n}\rangle} W(\vec{n})$ are symmetries of $(\mathcal{A}, \tau_T)$ iff $\langle \vec{k} - T^*\vec{k}|\vec{n}\rangle = 2\pi g$, $g \in Z \ \forall\, \vec{n}$, which holds if $(1 - T^*) \cdot \vec{k} \in 2\pi Z^2$. Nevertheless the spontaneous symmetry breaking in this seemingly finite quantum system is severely restricted by the

THEOREM 2.4 [NT1, N1]. – *For any* $T \in SL(2, Z) \setminus \{1\}$ *with probability 1 in* Θ *there is only one invariant state of* $(\mathcal{A}_\Theta, \tau_T)$*, namely the trace* $\omega_0(W(\vec{n})) = \delta_{\vec{n},0}$.

COROLLARY. – *With probability one in* Θ *there is no spontaneous breaking of any symmetry* γ *of* $(\mathcal{A}_\Theta, \tau_T)$.

Proof. – $\omega_0 \circ \gamma$ is also τ_T invariant, $\omega_0 \circ \gamma \circ \tau_T = \omega_0 \circ \tau_T \circ \gamma = \omega_0 \circ \gamma$ and the latter must equal ω_0 since it is also tracial and the trace is unique.

REMARKS 2.5

1. Probability is with respect to the Lebesgue measure of $(-1, 1)$. The statement does not hold for $\Theta =$ rational and a countable subset of irrationals but it holds nevertheless for almost all Θ.
2. In physics it is quite unusual that there is only one invariant state over a local C*-algebra, one expects at least KMS states for all temperatures. T can be extended to a continuous dilation automorphism over the whole Weyl algebra but this automorphism does not have KMS states [NT2].

3. For Θ rational it is shown in [N1] that spontaneous symmetry breaking is possible by some states not normal to the trace-representation. In the classical case $\Theta = 0$ τ_T has some periodic orbits and taking the average of the functions on such orbits gives other invariant states which may break a symmetry. For $\Theta = 1/2$ $\mathcal{A}_{1/2}$ is generated by 2 anticommuting unitaries u, v, $uv = -vu$ and it is easy to find other invariant states, for instance, if

$$T = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \Rightarrow \tau_T(u) = uv^2, \qquad \tau_T(v) = v,$$

$$\omega(v^n u^m) = \frac{1 + (-)^m}{2}$$

is τ_T invariant. τ_T commutes with the symmetries $\gamma_1(u, v) = (-u, v)$, $\gamma_2(u, v) = (u, -v)$ and ω does not break γ_1 but it does γ_2. In the next section we shall develop a systematic way of constructing such states.

Our findings so far are actually more general since the symmetry breaking properties of τ's which are conjugated with automorphisms respecting γ are the same

LEMMA 2.6. – *Let $\beta \in$ Aut $\mathcal{A}$ commute with γ and $\tilde{\tau} = \beta^{-1} \circ \tau \circ \beta$. Then $(\mathcal{A}, \tilde{\tau})$ can break γ spontaneously if and only if $(\mathcal{A}, \tau)$ does.*

Proof. – Let $\tilde{\omega} \circ \tilde{\tau} = \tilde{\omega} \neq \tilde{\omega} \circ \gamma$. Then $\tilde{\omega} \circ \beta^{-1}$ is τ-invariant, $\tilde{\omega} \circ \beta^{-1} \circ \tau = \tilde{\omega} \circ \tilde{\tau} \circ \beta^{-1} = \tilde{\omega} \circ \beta^{-1}$ but breaks γ: $\tilde{\omega} \circ \beta^{-1} \circ \gamma = \tilde{\omega} \circ \gamma \circ \beta^{-1} \neq \tilde{\omega} \circ \beta^{-1}$. Since β is invertible the conclusion goes in both directions.

REMARKS 2.7

1. For finite quantum systems where all automorphisms are unitarily implementable, $\beta = \text{ad } e^{iB} := e^{iB} \cdot e^{-iB}$, $\gamma = \text{ad } G$, $[G, B] = 0$, and H has a pure point spectrum then the Lemma simply says that H and $\beta(H)$ have the same spectrum such that the degeneracy criterium works the same way in both cases.
2. Let $a(f) \rightarrow a(Mf) + a^*(Nf)$ be a Bogoliubov transformation in example 3(a). It respects γ and thus the Lemma says that if a Hamiltonian can be diagonalized by a Bogoliubov transformation and then exhibits an absolutely continuous spectrum γ cannot be broken spontaneously.
3. If $\tau, \tilde{\tau} \in$ Aut $\mathcal{A}$ both respect γ and approach each other asymptotically such that the Møller automorphism $\lim_{t \to \infty} \tau^{-1} \tilde{\tau}^t = \Omega \in$ Aut $\mathcal{A}$,

$\lim_{t\to\infty} \tilde{\tau}^{-t}\tau^t = \Omega^{-1}$, exists then the spontaneous symmetry breaking properties of $(\mathcal{A}, \tau)$ and $(\mathcal{A}, \tilde{\tau})$ are the same.

3. CROSSED PRODUCTS

There is a striking difference between the fermionic and the spin chain, though for finite chains they are isomorphic: translation invariant states for fermions are even whereas for the spin chain any constant magnetic field not in the z-axis can be used to construct as equilibrium states states which are translation invariant and break the symmetry $\gamma\sigma^{x,y} = -\sigma^{x,y}$. Yet the two systems have many things in common:

3.1. Relation of $\mathcal{A}_f$ and $\mathcal{A}_s$

The algebra $\mathcal{A}_e$ of fixpoints of γ of the two algebras $\mathcal{A}_f$ and $\mathcal{A}_s$ are isomorphic

$\mathcal{A}_s \supset \mathcal{A}_e = \{\text{polynomials in } \vec{\sigma}_k \text{ where } \#\sigma^x + \#\sigma^y \text{ is even}\}$

$\mathcal{A}_f \supset \mathcal{A}_e = \{\text{polynomials in } a_k, a_\ell^* \text{ where } \#a_k + \#a_\ell^* \text{ is even}\}$.

They are identified via

$$\sigma_j^z = a_j^* a_j - a_j a_j^*$$

$$\sigma_j^x \sigma_k^x = (a_j + a_j^*) \prod_{\ell=j}^{k-1} (a_\ell^* a_\ell - a_\ell a_\ell^*)(a_k + a_k^*).$$

REMARK. – For finite chains also $\mathcal{A}_f$ and $\mathcal{A}_s$ can be identified via the above Klein transformation. But for the infinite chain $\prod_{\ell=-\infty}^{k}(a_\ell^* a_\ell - a_\ell a_\ell^*)$ is ill defined, whereas for $\mathcal{A}_e$ it reduces to a well defined finite product.

3.2. $\mathcal{A}_f$ and $\mathcal{A}_s$ as crossed products

It is possible to pass from $\mathcal{A}_e$ to $\mathcal{A}_f$ resp. to $\mathcal{A}_s$ by adding a missing element, e.g. $(a_0 + a_0^*)$ for $\mathcal{A}_f$ and σ_0^x for $\mathcal{A}_s$. These elements are unitary and implement an automorphism α resp. $\bar{\alpha}$ on $\mathcal{A}_e$, satisfying $\alpha^2 = \bar{\alpha}^2 = id.$ Thus

$$\mathcal{A}_f = \mathcal{A}_e \overset{\alpha}{\times} Z^{(2)}, \qquad \mathcal{A}_s = \mathcal{A}_e \overset{\bar{\alpha}}{\times} Z^{(2)}, \qquad \alpha^2 = \bar{\alpha}^2 = \text{id}.$$

$Z^{(2)}$ is the group with two elements. $\mathcal{A}_e$ is embedded in $\mathcal{A}_f$ resp. $\mathcal{A}_e$ as fixedpoint algebra of the gauge group, $\gamma A = A$, $A \in \mathcal{A}_e$. Gauge invariant automorphisms $\bar{\tau}$, $\bar{\tau}\gamma = \gamma\bar{\tau}$ on $\mathcal{A}_f$ resp. $\mathcal{A}_s$ are also automorphisms τ on

$\mathcal{A}_e$ and can be regained from τ in a unique way. States on $\mathcal{A}_e$ with $\omega \circ \tau = \omega$ can also be enlarged to states on $\mathcal{A}_f$ resp. $\mathcal{A}_s$ and whether this extension is unique (no symmetry breaking) or not can be decided on the level of $\mathcal{A}_e$.

In the following we will demonstrate these statements.

PROPOSITION 3.3. – *Let $\mathcal{A}_e$ be a C*-algebra, $\alpha \in \text{Aut } \mathcal{A}_e$, $\alpha \neq \text{ad } V$, $V \in \mathcal{A}_e$, $\alpha^2 = \text{ad } W$, $\alpha W = W$, $W \in \mathcal{A}_e$. Let $\widehat{\mathcal{A}} = \mathcal{A}_e \overset{\alpha}{\times} Z^{(2)}$. Then*

(i) $\widehat{\mathcal{A}} = \begin{pmatrix} A & B \\ \alpha(B)W & \alpha(A) \end{pmatrix}$ *is a C*-subalgebra of $\mathcal{A}_e \otimes M_2$.*

(ii) *α can be extended to $\widehat{\mathcal{A}}$ and there it is inner, $\widehat{\alpha} = \text{ad} \begin{pmatrix} 0 & 1 \\ W & 0 \end{pmatrix}$.*

(iii) *There is a grading on $\widehat{\mathcal{A}}$, i.e. an automorphism γ, $\gamma^2 = \text{id}$, $\gamma = \text{ad}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \text{Aut } \widehat{\mathcal{A}}$. γ need not be inner, $\widehat{\alpha} \circ \gamma = \gamma \circ \widehat{\alpha}$ and $\mathcal{A}_e$ is the fixpoint algebra of $\widehat{\mathcal{A}}$ with respect to γ.*

(iv) *Automorphisms that are equivalent to α modulo inner automorphisms, i.e. $\alpha_x = ad\ x \circ \alpha$, $x \in \mathcal{A}_e$, lead with $W_x = x\ \alpha(x)W$ to algebras $\widehat{\mathcal{A}}_x \subset \mathcal{A}_e \otimes M_2$, which are unitarily related*

$$\widehat{\mathcal{A}}_x = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \widehat{\mathcal{A}} \begin{pmatrix} 1 & 0 \\ 0 & x^* \end{pmatrix}$$

such that γ is respected.

The various claims are verified by straightforward calculation and do not warrant an ingenious proof.

REMARKS 3.4

1. If $\alpha^2 = 1$, $W = 1$, $\widehat{\mathcal{A}}$ is isomorphic to the crossed product $\mathcal{A}_e\, \alpha \times Z^{(2)}$ [BR,T]. Thus, if W can be written as $W = \alpha(x^*)x^*$ then $\widehat{\mathcal{A}}$ is unitarily equivalent to a crossed product.
2. Conversely, given an algebra $\widehat{\mathcal{A}}$ with a grading $\gamma \in \text{Aut } \widehat{\mathcal{A}}$, $\gamma^2 = 1$, and given an odd unitary element $u = -\gamma u \in \widehat{\mathcal{A}}$ then $\widehat{\mathcal{A}}$ can be written as sum of its even and odd part

$$C = A + Bu, \qquad \gamma A = A, \qquad \gamma B = B, \qquad A = \frac{1}{2}(C + \gamma C).$$

With $\alpha = \text{ad } u$, $W = u^2$ the multiplication law is the same as the one of the matrices in (3.3.i). Subsequently we shall concentrate on the special case $\alpha^2 = 1$. The general case only makes the notation a little heavier. It is considered e.g. in [N1].

3. If $\mathcal{A}_e$ has trivial center, $\alpha^2 = \text{ad } W$ implies already $\alpha W = \pm W$.

   Proof. – $\alpha^3 = \text{ad } W \circ \alpha = \alpha \circ \text{ad } W = \text{ad } \alpha W \circ \alpha$ and therefore $W^* \alpha W = c\mathbf{1}$ together with $\alpha^2 W = W$ this implies $c = \pm 1$.

PROPOSITION 3.5. – *Let τ be an automorphism of $\mathcal{A}_e$. It can be extended to $\hat{\tau} \in Aut\, \hat{\mathcal{A}}$, if and only if*

$$\tau\, \alpha^1 \tau^{-1} \alpha^{-1} = ad\, V, \qquad V \in \mathcal{A}_e$$

and $V^{-1} = V^* = \alpha(V)$. For algebras with trivial center $\hat{\tau}$ is unique up to $\hat{\tau} \to \hat{\tau} \circ \gamma$. If τ is implemented by U, then $\hat{\tau}$ is implemented by $\begin{pmatrix} U & 0 \\ 0 & V^* U \end{pmatrix}$.

Proof. – $\hat{\mathcal{A}}$ is generated algebraically by $\mathcal{A}_e$ which is isomorphic to the subalgebra of elements of the form

$$\begin{pmatrix} A & 0 \\ 0 & \alpha A \end{pmatrix}, \qquad A \in \mathcal{A}_e, \qquad \text{and} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus $\hat{\tau}$ is determined by its action on $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Since it maps even elements into even elements as bijection it has to map odd elements into odd elements. Therefore there must be a $V \in \mathcal{A}_e$ such that

$$\hat{\tau}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & V \\ \alpha V & 0 \end{pmatrix}.$$

Unitarity of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ demands $\alpha V = V^*$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & V \\ \alpha V & 0 \end{pmatrix}^2$ implies

$$V \alpha V = V V^* = V^* V = \mathbf{1}.$$

Since $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ implements $\hat{\alpha}$ we have

$$\begin{aligned} &\hat{\tau}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & \alpha A \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \tau\alpha A & 0 \\ 0 & \alpha\tau\alpha A \end{pmatrix} = \begin{pmatrix} V\alpha\tau A V^* & 0 \\ 0 & \alpha(V\alpha\tau A V^*) \end{pmatrix} \end{aligned}$$

which proves $\tau\alpha\tau^{-1}\alpha^{-1} = \text{ad } V$. V is fixed up to unitaries z of the center $\mathcal{Z}$. For $\mathcal{Z} = C\mathbf{1}$ with $\alpha(V) = V^*$ only $z = \pm 1$ is allowed, which corresponds to the freedom $\hat{\tau}$ or $\hat{\tau} \circ \gamma$.

Conversely, if $\tau\alpha\tau^{-1}\alpha^{-1} = \text{ad } V$, then $\alpha\tau\alpha\tau^{-1} = \text{ad } \alpha V = \text{ad } V^*$ and V can be used for the extension $\hat{\tau}$. If $\hat{\tau}$ exists, $[\hat{\tau}, \gamma] = 0$ and $\alpha = \text{ad } U$, then (3.5) holds with $V = \hat{\tau}(U)$, $U \in \mathcal{A}_e$.

PROPOSITION 3.6. – *If ω is a unique τ invariant state on $\mathcal{A}_e$ there exists a $\hat{\tau}$ invariant state $\hat{\omega}$ with $\hat{\omega} \circ \gamma = \hat{\omega}$, $\hat{\omega}|_{\mathcal{A}_e} = \omega$. Furthermore any extension $\hat{\omega}_1$ of ω is dominated by $2\hat{\omega}$.*

Proof. – In the GNS representation π_ω of ω we have $\omega(A) = \langle \Omega | \pi_\omega(A) | \Omega \rangle$. We define

$$\hat{\omega}(A + Bu) = \left\langle \begin{matrix} \Omega \\ 0 \end{matrix} \middle| \begin{matrix} \pi_\omega(A) & \pi_\omega(B) \\ \pi_\omega(\alpha B) & \pi_\omega(\alpha A) \end{matrix} \middle| \begin{matrix} \Omega \\ 0 \end{matrix} \right\rangle.$$

This state satisfies obviously all requirements. The second claim follows from $\hat{\omega} = \frac{1}{2}(\hat{\omega}_1 + \hat{\omega}_1 \circ \gamma) \geq \frac{1}{2}\hat{\omega}_1$.

PROPOSITION 3.7. – $\left| \begin{matrix} \Omega \\ \psi \end{matrix} \right\rangle$, $0 \neq \psi \in \mathcal{H}_\omega$ *extends ω iff α can be unitarily implemented in the representation, i.e. $\alpha(A) = WAW^*$, W = unitary $\in \mathcal{B}(\mathcal{H}_\omega)$ for $\psi = W\Omega$ (we write A instead of $\pi_\omega(A)$).*

Proof. – $\left| \begin{matrix} \Omega \\ \psi \end{matrix} \right\rangle$ gives extension of $\omega \Leftrightarrow \langle \psi | \alpha(A) | \psi \rangle = \langle \Omega | A | \Omega \rangle$. If $\alpha(A) = WAW^*$ take $|\psi\rangle = W|\Omega\rangle$. Conversely, if $\langle \psi | \alpha(A) | \psi \rangle = \langle \Omega | A | \Omega \rangle$, define $WA|\Omega\rangle = \alpha(A)|\psi\rangle \ \forall \ A \in \mathcal{A}$. W is densely defined, unitary, and $\langle \Omega | B^* W^* AWC | \Omega \rangle = \langle \psi | \alpha(B^* \alpha^{-1}(A)C) | \psi \rangle = \langle \Omega | B^* \alpha^{-1}(A) C | \Omega \rangle \Rightarrow$ W implements α. W is not uniquely fixed, $B'W$ with B' a unitary from $\pi_\omega(\mathcal{A}_e)'$ is just as good.

If Ω is cyclic and separating we shall normalize W by requiring $W = JWJ =: W'$ where J generates the canonical conjugation $\pi_\omega(\mathcal{A}_e)' = J\pi_\omega(\mathcal{A}_e)J$ [BR]. If $\pi_\omega(\mathcal{A}_e)$ is a factor together with $\pi_\omega(\mathcal{A}_e)'$ it generates all of $\mathcal{B}(\mathcal{H}_\omega)$. Since W^2 generates the identity transformation on both in this case we have $W^2 = \mathbf{1}$ or $W = W^{-1} = W^*$.

However, this extension is not necessarily $\hat{\tau}$-invariant. There remains the problem whether $\hat{\omega}$ is the only $\hat{\tau}$ invariant extension of ω and if not, whether the other extensions break the γ symmetry. As we have seen it is necessary for symmetry breaking that $\hat{\omega}$ can be decomposed into other $\hat{\tau}$ invariant states. We use the following well established fact [BR]

PROPOSITION 3.8. – *If $\hat{\omega}_1$ is dominated by a multiple of $\hat{\omega}$, then there exists a unique positive operator $T \in \pi_{\hat{\omega}}(\hat{\mathcal{A}})'$ such that*

$$\hat{\omega}_1(A) = \langle \Omega_{\hat{\omega}} | \pi_{\hat{\omega}}(A) T | \Omega_{\hat{\omega}} \rangle.$$

$\widehat{\omega}_1$ is $\widehat{\tau}$ invariant, iff $\widehat{\tau}T := \widehat{U}T\widehat{U}^* = T$, where $\widehat{U}$ implements $\widehat{\tau}$ and $\widehat{U}|\Omega_{\widehat{\omega}}\rangle = |\Omega_{\widehat{\omega}}\rangle$.

One finds easily that the commutant $\widehat{\pi}_\omega(\widehat{\mathcal{A}})'$ are operators of the form $\begin{pmatrix} A' & W \\ W & A' \end{pmatrix}$, where $A' \in \pi_\omega(\mathcal{A}_e)'$ and $W\pi_\omega(A) = \pi_\omega(\alpha A)W \ \forall\ A \in \mathcal{A}_e$. ¿From $\alpha^2 = 1$, $\alpha(A^*) = \alpha(A)^*$ we infer that $W^2, W^*W, WW^* \in \pi_\omega(\mathcal{A}_e)'$. Such a W will not exist if the representations π_ω and $\pi_\omega \circ \alpha$ are disjoint. In this case there are sequences $A_n \rightharpoonup 0$, $\alpha(A_n) \not\rightharpoonup 0$ which contradicts $WA_n = \alpha(A_n)W$ for $W \neq 0$.

PROPOSITION 3.9. – *Assume that π_ω and $\pi_\omega \circ \alpha$ are disjoint. Then $\widehat{\omega}$ cannot be decomposed into gauge breaking invariant states.*

Proof. – In this case

$$\pi_{\widehat{\omega}}(\widehat{\mathcal{A}})' = \left\{ \begin{pmatrix} A' & 0 \\ 0 & A' \end{pmatrix}, \qquad A' \in \pi_\omega(\mathcal{A}_e)' \right\}.$$

$\widehat{\omega}$ can only be decomposed by elements with $\tau A' = A'$, which just correspond to decompositions of ω and do not mix between the even and odd part.

COROLLARY 3.10. – *If ω is τ extremal invariant for $\mathcal{A}$ and π_ω and $\pi_\omega \circ \alpha$ are disjoint, then $\widehat{\omega}$ is $\widehat{\tau}$ extremal invariant for $\widehat{\mathcal{A}}$ and γ cannot be broken by $\widehat{\omega}$.*

THEOREM 3.11. – *Let ω denote a τ-invariant state over $\mathcal{A}_e$ faithful over $\pi_\omega(\mathcal{A}_e)''$ and $\{U\}' \cap \pi_\omega(\mathcal{A}_e)' = \lambda \mathbf{1}$ where τ is implemented by U and extends to $\mathcal{B}(\mathcal{H}_\omega)$. U is chosen in the standard way $U = JUJ$ where J is the modular conjugation of ω. Let α be unitarily implemented. Then ω has a γ breaking extension iff one of the four following equivalent conditions hold.*

(i) *There exists $\bar{W}$ implementing α such that $T = \begin{pmatrix} 1 & \bar{W} \\ \bar{W} & 1 \end{pmatrix} \in \pi_{\widehat{\omega}}(\widehat{\mathcal{A}})'$ is τ-invariant, $\bar{W} = \bar{W}^*$,*

(ii) *$V = \tau(\bar{W})\bar{W}^* = \tau(\bar{W}^*)\bar{W}$ (V from (3.5) and the second equality is implied by $\alpha(V) = V^*$ from the first),*

(iii) *$\bar{W}$ can be written as $B''W_0$, $B'' \in \pi_\omega(\mathcal{A}_e)''$, unitary, $W_0 \in \mathcal{B}(\mathcal{H}_\omega)$, $[W_0, U] = 0$, $V = \tau(B'')B''^* = W_0^*\tau(B''^*)B''W_0$,*

(iv) *$\exists\ |\psi\rangle \in \mathcal{H}_\omega$, $U|\psi\rangle = V|\psi\rangle$*

Proof. – γ breaking $\Leftrightarrow$ (i): A' in (3.8) has to be a c number, the choice $c = \mathbf{1}$ makes T positive.

(i) $\Leftrightarrow$ (ii):

$$\begin{pmatrix} U & 0 \\ 0 & V^*U \end{pmatrix}\begin{pmatrix} 0 & \bar{W} \\ \bar{W} & 0 \end{pmatrix}\begin{pmatrix} U^* & 0 \\ 0 & U^*V \end{pmatrix} = \begin{pmatrix} 0 & U\bar{W}U^*V \\ V^*U\bar{W}U^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & \bar{W} \\ \bar{W} & 0 \end{pmatrix}$$

is equivalent to (ii).

(iii) $\Rightarrow$ (ii):

$$\tau(\bar{W})\bar{W}^* = UB''W_0U^*W_0^*B''^* = \tau(B'')B''^* = V$$
$$\tau(\bar{W}^*)\bar{W} = UW_0^*B''^*UB''W_0 = W_0^*\tau(B''^*)B''W_0 = V.$$

(ii) $\Rightarrow$ (iii): With the standard $W = W^* = JWJ$ (3.7) we have $UWU^*W = VV'$, $V' = JVJ \in \pi_\omega(\mathcal{A}_e')$ and we can write $\bar{W} = B'^*W$, $B' \in \pi_\omega(\mathcal{A}_e)'$. Then

$$V = \tau(\bar{W})\bar{W}^* = UB'^*WU^*WB'$$
$$= \tau(B'^*)VV'B' \Rightarrow V' = \tau(B')B'^* \Rightarrow V = \tau(B'')B''^*$$

with $B'' = JB'J$ since J commutes with τ. Now define $W_0 = B''^*\bar{W}$, then

$$W_0^*\tau(W_0) = \bar{W}^*B''\tau(B''^*)\tau(\bar{W}) = \bar{W}^*V^*V\bar{W} = 1 \Rightarrow \tau(W_0) = W_0.$$

Finally

$$V = \tau(\bar{W}^*)\bar{W} = \tau(W_0^*B''^*)B''W_0 = W_0^*\tau(B''^*)B''W_0.$$

(ii) $\Rightarrow$ (iv): $|\psi\rangle = \bar{W}|\Omega\rangle$.

(iv) $\Rightarrow$ (iii): Since ω is faithful for $\pi_\omega(\mathcal{A}_e)''$ and

$$AW|\Omega\rangle = 0 \Leftrightarrow \langle\Omega|\alpha(A^*A)|\Omega\rangle = 0 \Leftrightarrow A = 0$$

we see that $W|\Omega\rangle$ is separating for $\pi_\omega(\mathcal{A}_e)''$ and thus cyclic for $\pi_\omega(\mathcal{A}_e)'$. Since we assumed that ω is the only τ invariant state in the folium of ω we have $\langle\psi|\alpha A|\psi\rangle = \omega(A)\ \forall\ A \in (\mathcal{A}_e)''$. Hence we may write $|\psi\rangle = B'^*W|\Omega\rangle$, $B' \in \pi_\omega(\mathcal{A}_e)'$, $B'B'^* = 1$. If $B'^*W|\Omega\rangle = V^*UB'^*WU^*|\Omega\rangle$ we have

$$|\Omega\rangle = WB'V^*\underbrace{UB'^*U^*}_{\tau(B'^*)}\underbrace{UWU^*W}_{VV'}W|\Omega\rangle = \alpha(B'\tau(B'^*)V')|\Omega\rangle$$

by extending α to $\pi_\omega(\mathcal{A}_e)'$. Since $|\Omega\rangle$ is separating for $\pi_\omega(\mathcal{A}_e)'$ we conclude

$$V' = \tau(B')B'^* \Rightarrow V = \tau(B'')B''^*.$$

At this point we can identify B'^*W with $\bar{W}$ from (ii) and proceed as in (ii) $\Rightarrow$ (iii).

REMARK 3.12. – If ω is a pure state (for instance, a ground state) then the symmetry can only be broken if α is inner in $\pi_\omega(\mathcal{A}_e)''$ since it is all of $\mathcal{B}(\mathcal{H}_\omega)$. If $W \in \pi_\omega(\mathcal{A}_e)''$ the algebra generated by $A + BW \in \pi_\omega(\mathcal{A}_e)''$ with $A, B \in \pi_\omega(\mathcal{A}_e)$ is isomorphic to $\pi_\omega(\mathcal{A}_e)''$. Thus in this case the "fermionic operators" (odd elements) can be obtained as strong limits of "bosonic operators" (even elements). On the other hand, if α is inner in $\pi_\omega(\mathcal{A}_e)''$, (iii) is satisfied with an appropriate choice for the sign of V which then determines $\hat{\tau}$. That the γ symmetry is actually broken follows since $\langle\Omega|\pi_\omega(B)|\psi\rangle = 0 \; \forall \; B \in \mathcal{A}_e$ implies $\psi = 0$ since $|\Omega\rangle$ is cyclic for $\mathcal{A}_e$.

3.3. Cluster properties

The uniqueness of the τ invariant state ω as assumed in (3.11) is guaranteed if $(\mathcal{A}_e, \tau, \omega)$ is weakly asymptotically abelian in its folium and ω is extremally invariant. The dynamical system $(\mathcal{A}_e, \tau, \omega)$ is clustering which is equivalent to saying that U has one eigenvalue 1 (eigenvector $|\Omega\rangle$) and otherwise a continuous spectrum. We shall now investigate whether these properties carry over to $(\widehat{\mathcal{A}}, \hat{\tau}, \hat{\omega})$.

a) No spontaneous symmetry breaking

We have to establish the spectrum of $\widehat{U} = \begin{pmatrix} U & 0 \\ 0 & V^*U \end{pmatrix}$. By assumption V^*U does not have an eigenvalue 1 and we shall first show that it has no other proper eigenvalues. From $V|\psi\rangle = e^{i\lambda}U|\psi\rangle$ we would conclude as in the proof of (3.11), (iv) $\Rightarrow$ (iii) that $V = e^{i\lambda}\tau(B'')B''^*$. We shall show that for $\lambda \neq 0, \pi$ this is incompatible with $\alpha(V) = V^* : e^{i\lambda}\alpha(V^*) = e^{2i\lambda}\tau(B'')B''^* = WB''\tau(B''^*)W$ where we have again $W = B'B''W_0$. Thus the equation continues $= W_0^*B'^*\tau(B''^*)B'B''W_0 = \tau(W_0^*B''^*W_0)W_0^*B''W_0$ or $\tau(B''^*W_0^*B''^*W_0) = e^{2i\lambda}B''W_0^*B''^*W_0$. But our assumptions on $(\mathcal{A}_e, \tau, \omega)$ imply that $\pi_\omega(\mathcal{A}_e)''$ does not have an element $\neq c\mathbf{1}$ with periodic time dependence thus $e^{2i\lambda} = 1$, $B''W_0^*B''^*W_0 = c\mathbf{1}$. Thus $\widehat{U}$ has only the eigenvector $\left|\begin{matrix}\Omega \\ 0\end{matrix}\right\rangle$ if there is no γ-breaking. ($e^{i\lambda} = -1$ corresponds to our previous ambiguity $V \to -V$.) For a good criterium to exclude the singular continuous spectrum further knowledge on the spectrum of U and the relation between τ and α would be necessary.

PROPOSITION 3.13. – *If there is no spontaneous γ-breaking then V^*U has only continuous spectrum and $(\widehat{\mathcal{A}}, \widehat{\tau}, \widehat{\omega})$ is η-abelian (i.e. invariant means of operators are c numbers in π_ω). If the spectrum of V^*U is absolutely continuous the system is weakly asymptotically abelian and extremal invariant states are clustering.*

b) Spontaneous γ-breaking

Since we can expect clustering only for extremally invariant states we first have to decompose $\widehat{\omega}$ with τ-invariant projections from $\pi_{\widehat{\omega}}(\widehat{\mathcal{A}})'$. This is easily done since we will learn from a) that $\bar{W} = B''W_0$ is actually hermitian. We know that W is hermitian, $W = B'B''W_0 = W^* = W_0^*B''^*B'^*$ and from a) we get with an appropriate choice of the phase $B''^* = W_0^*B''W_0$. Combining the two we see $B'B''W_0 = W_0^*B'^*W_0^*B''W_0$. Thus with $J \cdot J$ of the first relation we have $B' = W_0^*B'^*W_0 = W_0^*B'^*W_0^* \Rightarrow W_0^* = W_0$ and $\bar{W} = B''W_0 = W_0B''^* = (B''W_0)^* = \bar{W}^*$.

Thus $P_\pm = \frac{1}{2}\begin{pmatrix} 1 & \pm\bar{W} \\ \pm\bar{W} & 1 \end{pmatrix}$ are τ-invariant projectors $\in \pi_{\widehat{\omega}}(\widehat{\mathcal{A}})'$ and the states $\widehat{\omega}_\pm(\cdot) = \widehat{\omega}(\cdot P_\pm)$ are extremally invariant. They correspond to the vectors

$$|\widehat{\Omega}_\pm\rangle = P_\pm|\widehat{\Omega}\rangle/\|P_\pm|\widehat{\Omega}\rangle\| = \frac{1}{\sqrt{2}}\left|\begin{matrix} \Omega \\ \pm\bar{W}\Omega \end{matrix}\right\rangle.$$

For them we have indeed clustering and weak asymptotic abelianness since $\widehat{U}^n$ converges weakly to the projector onto the $\widehat{U}$-invariant subspace. With $V^*U = \bar{W}U\bar{W}$ we have

$$\begin{aligned}\widehat{U}^n &= \begin{pmatrix} U^n & 0 \\ 0 & (\bar{W}U\bar{W})^n \end{pmatrix} \rightharpoonup \begin{pmatrix} |\Omega\rangle\langle\Omega| & 0 \\ 0 & \bar{W}|\Omega\rangle\langle\Omega|\bar{W} \end{pmatrix} \\ &= |\widehat{\Omega}_+\rangle\langle\widehat{\Omega}_+| + |\widehat{\Omega}_-\rangle\langle\widehat{\Omega}_-|.\end{aligned}$$

Further

$$\text{w-}\lim P_\pm \widehat{U}^n P_\pm = \begin{pmatrix} |\Omega\rangle\langle\Omega| & \bar{W}|\Omega\rangle\langle\Omega| \\ |\Omega\rangle\langle\Omega|\bar{W} & \bar{W}|\Omega\rangle\langle\Omega|\bar{W} \end{pmatrix}$$

is a one-dimensional projection.

PROPOSITION 3.14. – *For spontaneous symmetry breaking weak asymptotic abelianness carries over to the extremal invariant states over $\widehat{\mathcal{A}}$ without further assumptions.*

4. EXAMPLES

4.1. A baby model

To illustrate that it is not so much the infinity of the system that opens the possibility of symmetry breaking but on the contrary the existence of inequivalent representations which can hinder symmetry breaking we consider the 1 spin system:

$$\widehat{\mathcal{A}} = \{1, \vec{\sigma}\}, \qquad \gamma\sigma^{x,y} = -\sigma^{x,y}, \qquad \mathcal{A}_e = \{1, \sigma^z\},$$
$$\alpha = \text{ad } \sigma^x, \qquad \alpha\sigma^z = -\sigma^z, \qquad \tau = \text{id}.$$

Therefore

$$\widehat{\pi}_\omega(\sigma^z) = \begin{pmatrix} \pi_\omega(\sigma^z) & 0 \\ 0 & -\pi_\omega(\sigma^z) \end{pmatrix}, \qquad \widehat{\pi}_\omega(\sigma^x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(i) If π_ω is pure, i.e. $\omega_\pm(\sigma^z) = \pm 1 = \pi_\pm(\sigma^z)$, then $\pi_\pm$ are inequivalent representations and $\widehat{\pi}_\omega$ is irreducible. The extension $\widehat{\omega}(\sigma^z) = \pm 1$ is by (3.9) even, generally $\omega(\vec{\sigma}) = \vec{s}$, $\|\vec{s}\| \leq 1$ and $|\omega(\sigma^z)| = 1$ implies $\omega(\sigma^{x,y}) = 0$.

(ii) If π_ω is mixed, i.e. $\omega(\sigma^z) = \cos^2\alpha - \sin^2\alpha$, then $\mathcal{H}_\omega$ is two dimensional and we span $\mathcal{B}(\mathcal{H}_\omega)$ by Pauli matrices $\{1, \underline{\vec{\sigma}}\}$. It contains $\underline{\sigma}^x \notin \mathcal{A}_e$ that implements α. Since $\tau = \widehat{\tau} = \text{id}$, (3.11.iii) is trivially satisfied with $V = 1$. If we represent ω by the vector $\Omega = \left| \begin{matrix} \cos\alpha \\ \sin\alpha \end{matrix} \right\rangle$, then we take $|\psi\rangle = \underline{\sigma}^x|\Omega\rangle$ and

$$\widehat{\omega}(\sigma_x) = \frac{1}{2}\left\langle \begin{matrix} \Omega \\ \psi \end{matrix} \right| \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left| \begin{matrix} \Omega \\ \psi \end{matrix} \right\rangle = 2\cos\alpha\sin\alpha$$

breaks the symmetry and $\widehat{\mathcal{A}}$ is irreducibly represented.

(iii) If we take instead $\widehat{\tau} = \text{ad } (\sigma^z)^t$ (notice that $\mathcal{A}_e$ is commutative, therefore the extension is not unique) and if we write again $\pi_\omega(\sigma^z) = \underline{\sigma}^z$, $\widehat{U}_t = (\sigma^z)^t \otimes (\underline{\sigma}^z)^t$ then (3.11.iii) with $W = \underline{\sigma}^x$ cannot be satisfied and the symmetry γ (which equals $\widehat{\tau}$) remains unbroken. Thus our intuition is correct that for a state invariant under rotations around the z-axis $\omega(\vec{\sigma})$ has to point in the z-direction.

This baby may be named SUSY since it is the simplest realization of $N = 2$ supersymmetry. $\widehat{\mathcal{A}}$ is generated by two fermionic charges $(Q_1, Q_2) = (\sigma^x, \sigma^y)$, $[Q_i, Q_j]_+ = H\delta_{ij}$, $H = 2$. The Witten parity $\frac{1}{iH}[Q_1, Q_2] = \sigma^z$ generates $\mathcal{A}_e$. Thus our crossed product realization coincides with the 2×2 matrix representation of supersymmetry which is nothing but the γ-symmetry. $H > 0$ says that it can be broeken by a ground state [J]. Our construction makes the two diagonal element of H isomorphic and thus does not cover the case when their zero eigenspaces have different dimensions.

4.2. The spin chain

Here $\widehat{\mathcal{A}}$ is the infinite tensor product $\otimes M_2$ of 2×2 matrices which we generate by the usual Pauli matrices. The grading is $\gamma\sigma_i^z = \sigma_i^z$, $\gamma\sigma_i^{x,y} = -\sigma_i^{x,y}$ $\forall$ i. $\mathcal{A}_e = \{$polynomials in $\vec{\sigma}, \#\sigma^x + \#\sigma^y$ even$\}$ and ad σ_0^x can be used to create $\widehat{\mathcal{A}}$ out of $\mathcal{A}_e$. For τ we consider the shift $\widehat{\tau}(\vec{\sigma}_i) = \vec{\sigma}_{i+1}$. It satisfies $\widehat{\tau}\alpha\widehat{\tau}^{-1}\alpha^{-1} = \text{ad}\ \sigma_1^x\sigma_0^x$ and $\sigma_1^x\sigma_0^x \in \mathcal{A}_e$, so that the construction of (3.5) works.

Assume $\widetilde{\omega}$ is an extremal translation invariant state and therefore clustering and $\widetilde{\omega}(\sigma_0^x) = a \neq 0$. Then

$$\lim_{N\to\infty} \langle\widehat{\Omega}| \left(\frac{1}{N} \sum_{n=1}^{N} \widehat{\pi}(\sigma_n^x\sigma_0^x) - a\widehat{\pi}(\sigma_0^x) \right)^2 |\widehat{\Omega}\rangle =$$

$$= \lim_{N\to\infty} \langle\widehat{\Omega}| \frac{1}{N^2} \sum_{n,k=1}^{N} \widehat{\pi}(\sigma_n^x\sigma_k^x) - \frac{2a}{N} \sum_{n=1}^{N} \widehat{\pi}(\sigma_n^x) + a^2|\widehat{\Omega}\rangle = 0$$

and therefore

$$\widehat{\pi}(\sigma_0^x) = \text{st-lim} \frac{1}{Na} \sum_{n=1}^{N} \sigma_n^x\sigma_0^x$$

where $\sigma_n^x\sigma_0^x \in \mathcal{A}_e$, so that $\widehat{\pi}(\sigma_0^x) = W \in \pi(\mathcal{A}_e)''$. Again (3.11.iii) is trivially satisfied and the γ-symmetry is broken by $\widetilde{\omega}$. We can pass from $\widetilde{\omega}$ to ω over $\mathcal{A}_e$ and the gauge invariant extension $\widehat{\omega}$ of ω can be decomposed into $\frac{1}{2}(\widetilde{\omega} + \widetilde{\omega} \circ \gamma)$ according to (3.11.i).

4.3. The fermion chain

We consider the CAR algebra for simplicity over ℓ^2, i.e. $\widehat{\mathcal{A}}$ is generated by a_i, a_k^*, $i, k \in Z$, $[a_i, a_k]_+ = 0$ and $[a_i, a_k^*]_+ = \delta_{ik}$. The grading $\gamma a_i = -a_i$ $\forall$ i defines the even algebra $\mathcal{A}_e$. For τ we take the shift, but any quasifree automorphism as in (2.3a) would work in the same way. We want to recover the result that any $\widehat{\tau}$ invariant state $\widehat{\omega}$ has to be even, $\widehat{\omega} \circ \gamma = \widehat{\omega}$.

For α we take ad $(a_0 + a_0^*)$ which evidently maps $\mathcal{A}_e$ into $\mathcal{A}_e$. Then

$$V = (a_1 + a_1^*)(a_0 + a_0^*) \tag{4.3.1}$$

or

$$\tau^n \alpha \tau^{-n} \alpha^{-1} = \text{ad } V_n = \text{ad } (a_n + a_n^*)(a_0 + a_0^*).$$

Notice that

$$\alpha V_n = -V_n \tag{4.3.2}$$

which corresponds to anticommutativity, but expressed in $\mathcal{A}_e$. In order to produce broken symmetry we have to find a solution to

$$V^* U|\psi\rangle = |\psi\rangle \quad \longrightarrow \quad U^n|\psi\rangle = V_n|\psi\rangle. \tag{4.3.3}$$

Now with $P = \text{w-}\lim_{n\to\pm\infty} U^n = |\Omega\rangle\langle\Omega|$

$$\lim_{n\to\infty} \langle\psi|U^n|\psi\rangle = \langle\psi|P|\psi\rangle = \lim_{n\to\infty} \langle\psi|V_n|\psi\rangle = \lim_{n\to\infty} \langle V_n^*\psi|\psi\rangle$$
$$\overset{(3.5)}{=} \lim_{n\to\infty} \langle\alpha V_n\psi|\psi\rangle = -\lim_{n\to\infty} \langle V_n\psi|\psi\rangle = -\lim_{n\to\infty} \langle\psi|U^{-n}|\psi\rangle = -\langle\psi|P|\psi\rangle.$$

Therefore $\langle\psi|P|\psi\rangle = 0$ hence $\langle\Omega|\psi\rangle = 0$ for any solution of (4.3.3). With such a ψ we calculate

$$\tilde{\omega}(\hat{\pi}_{\tilde{\omega}}(a_0 + a_0^*)) = \left\langle \begin{matrix}\Omega \\ \psi\end{matrix} \middle| \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \middle| \begin{matrix}\Omega \\ \psi\end{matrix} \right\rangle = 2\langle\Omega|\psi\rangle = 0.$$

The same argument works if we recall (3.3.iv) and replace α by $\alpha \circ \text{ad } B$, $B \in \mathcal{A}_e$, such that now $\text{st-}\lim_{n\to\infty}(\alpha V_n + V_n) = 0$,

$$\tilde{\omega}(\hat{\pi}_{\tilde{\omega}} B(a_0 + a_0^*)) = 0,$$

and therefore $\tilde{\omega}$ does not break the γ symmetry.

To show explicitly that (3.11.iii) cannot be met take an even polynomial $g_0(a_{-k}^{\#} \dots a_k^{\#})$ and $n > 2k + 1$. Now multiply the relation $W^* U^n W = V_n^* U^n = (a_0 + a_0^*)(a_n + a_n^*)U^n$ with $g_0^* \cdot g_0$. If $\text{w-}\lim_{n\to\infty} U^n = |\Omega\rangle\langle\Omega|$ we see with $g_n = U^n g_0 U^{-n}$

$$\begin{aligned}
0 &\le |\langle\Omega|W g_0|\Omega\rangle|^2 \leftarrow \langle\Omega|g_0^* W^* U^n W g_0|\Omega\rangle \\
&= \langle\Omega|g_0^*(a_0 + a_0^*)(a_n + a_n^*)g_n|\Omega\rangle = -\langle\Omega|(a_n + a_n^*)g_0^* g_n(a_0 + a_0^*)|\Omega\rangle \\
&= -\langle\Omega|(a_n + a_n^*)g_n g_0^*(a_0 + a_0^*)|\Omega\rangle \\
&= -\langle\Omega|(\alpha(g_0))_n(a_n + a_n^*)(a_0 + a_0^*)\alpha(g_0)^*|\Omega\rangle \\
&= -\langle\Omega|\alpha(g_0)^*(a_0 + a_0^*)U^n(a_0 + a_0^*)\alpha(g_0)|\Omega\rangle \\
&= -\langle\Omega|\alpha(g_0)^* W^* U^n W \alpha(g_0)|\Omega\rangle \to -|\langle\Omega|W\alpha(g_0)|\Omega\rangle|^2 \le 0 \\
&\Rightarrow |\langle\Omega|W g_0|\Omega\rangle|^2 = 0.
\end{aligned}$$

Since k was arbitrary the $g_0|\Omega\rangle$ are dense in $\mathcal{H}_\omega$ we conclude $W|\Omega\rangle = 0$ which contradicts the unitarity of W.

4.4. The rotation algebra

So far (3.11) has not given more than (2.2). But next we will apply (3.11) to systems which are not anticommutative. We return to the quantized torus (2.4) and choose the rotation parameter $\Theta = 1/2$. With the grading

$$\gamma(W(n_1, n_2)) = e^{i\pi n_1} W(n_1, n_2)$$

the even algebra $\mathcal{A}_e$ built by $W(2n_1, n_2)$ is abelian. It is isomorphic to the functions on the torus T^2 which we shall represent by $W(2,0) = u^2 = e^{2\pi i x}$ and $W(0,1) = v = e^{2\pi i y}$, $x, y \in [0, 1)$. We write $\mathcal{A}_{1/2}$ as the crossed product of $\mathcal{A}_e$ with the automorphism $\alpha = \text{ad } W(1,0) = \text{ad } u$, $\alpha u^2 = u^2$, $\alpha v = -v$. Thus we represent $\mathcal{A}_{1/2}$

$$\text{if } m = \text{even } \pi(v^n u^m) = \begin{pmatrix} v^n u^m & 0 \\ 0 & (-)^n v^n u^m \end{pmatrix},$$

$$\text{if } m = \text{odd } \pi(v^n u^m) = \begin{pmatrix} 0 & v^n u^{m-1} \\ (-)^n v^n u^{m+1} & 0 \end{pmatrix}.$$

As automorphism we consider τ_T with $T = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$. Extremal states on $\mathcal{A}_e$ are point measures on T^2. Among them the fixed points of τ give also invariant states. They have to satisfy

$$\tau(x, y) = (x + g_1, y + g_1), \qquad (g_1, g_2) \in Z^2,$$

i.e., a) $x = y = 0$ and b) $x = 0$, $y = 1/4$), c) $x = 0$, $y = 3/4$ are solutions.

To enlarge the corresponding states on $\mathcal{A}_{1/2}$ we calculate

$$\alpha\tau\alpha^{-1}\tau^{-1} \rightarrow u^2 v = V.$$

Since the representation on the abelian $\mathcal{A}_e$ is pure and concentrated on a fixed point, $U = 1$. Though the faithfulness assumed in (3.11) is not satisfied nevertheless a solution of (iv) allows to construct symmetry breaking invariant states and we have to solve $V\psi = \psi$

(a) $V = \mathbf{1}$. We have symmetry breaking. The vectors $\left|\begin{matrix} 1 \\ 1 \end{matrix}\right\rangle$ and $\left|\begin{matrix} 1 \\ -1 \end{matrix}\right\rangle$ define invariant states.

(b), (c) $V = \pm i$. The unique extension is even.

One can also consider periodic orbits, that is to say, states that are given by fixed points of τ^N, $N \in Z^+$. The U is then an $N \times N$ matrix and so is V. Again UV has eigenvalue 1 in some (but not all) examples and symmetry breaking is possible.

4.5. The X-Y model

We have already stated the relation of the spin chain and the fermion chain (3.1). The passage from $\mathcal{A}_e$ to $\mathcal{A}_f$ resp. $\mathcal{A}_s$ uses $\alpha_f = \text{ad}\,(a_0 + a_0^*)$ or $\alpha_s = \text{ad}\,\sigma_0^x$. The two are related via

$$\alpha_s \circ \alpha_f = \Theta_-,$$

$$\Theta_-(a_n) = \varepsilon_n a_n, \qquad \Theta_-(\sigma_n^x) = \varepsilon_n \sigma_n^x,$$

$$\Theta_-(\sigma_n^z) = \sigma_n^z, \qquad \varepsilon_n = \begin{matrix} 1 \text{ for } n \geq 0 \\ -1 \text{ for } n < 0. \end{matrix}$$

This relation is the main input in the analysis of the X-Y model in [AM] of (2.2). Whereas space translations are sufficiently explicit for an analysis (norm asymptotic abelian) we have to extract the structure of time invariant states with respect to H in 2.b) from our knowledge of the time evolution on $\mathcal{A}_e$ where it reduces to a quasifree time evolution with absolutely continuous one particle spectrum after an appropriate Bogoliubov transformation. In order to be in a Fock space the Bogoliubov transformation has to be adjusted such that the one particle spectrum is positive. Now α_f is a Bogoliubov transformation that turns exactly one particle into an antiparticle, which means that it has index **1** (or odd). In (2.3) the index of the Bogoliubov transformation Θ_- is evaluated for the different parameters in the Hamiltonian, e.g. for $|\lambda| < 1$, $\gamma \neq 0$, therefore for a sufficiently weak magnetic field in the z-direction and a sufficiently strong attraction of the spins if they point in the x-direction, it is odd. Combined with α_f we get a Bogoliubov transformation of even index together with a scattering transformation. This scattering transformation is inner in the Fock representation, and so is the even Bogoliubov transformation. Therefore in the Fock representation we satisfy the demands of (3.11.i) and there is symmetry breaking. But the scattering transformation is not inner in $\mathcal{A}_e$ and in general, especially in faithful quasifree representations it is either not implementable or at least not inner in $\pi_\omega(\mathcal{A}_e)''$ [N2]. This corresponds to the results (2.3) that for finite temperatures there is no symmetry breaking.

If, on the other hand, the Bogoliubov transformation is even (as it happens for sufficiently strong magnetic fields) then the odd Bogoliubov transformation α_f hinders broken symmetry as for the fermi system.

Finally we have to consider the spectral properties of V^*U. We consider quasifree states. There U is a quasifree evolution of the CCR algebra built by creation and annihilation operators of the algebra and its commutant whose spectrum is absolutely continuous. V corresponds to a perturbation on the one particle level by an operator of finite rank which thus can only change the pure point spectrum and not the absolutely continuous spectrum. This guarantees that for all extremally invariant states the system is clustering and otherwise the evolution is weakly asymptotically abelian.

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