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# A calculus of observables on a Dirac particle

by

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ABSTRACT. – We construct a calculus of observables suitable for a description of measurements associated with a particle satisfying the Dirac wave equation. The calculus, built on a four-component phase space, is fully covariant with respect to all the usual symmetries of the Dirac equation, including the discrete ones. Some simple classical observables correspond to operators arising from representation-theoretic considerations or from some Clifford analysis on the mass-shell; a discussion of position operators is included. © Elsevier, Paris

*Key words:* Symbolic calculus; Dirac equation; quantization; Poincaré group; position operator

RÉSUMÉ. – On construit un calcul symbolique des observables associé à une particule satisfaisant à l'équation de Dirac libre : ce calcul est l'analogue, pour ce qui concerne cette équation, de ce que sont le calcul de Weyl ou celui de Klein-Gordon pour une particule satisfaisant à l'équation de Schrödinger libre ou à celle de Klein-Gordon. Les observables classiques (les « symboles » pour ce calcul) sont à valeurs vectorielles, et le calcul est covariant à l'égard de toutes les symétries classiques de l'équation de Dirac (celles qui proviennent du groupe de Poincaré restreint tout autant que les symétries discrètes). On calcule les symboles, dans ce calcul, des opérateurs infinitésimaux de la représentation du groupe de Poincaré; on revient également au très classique problème de l'opérateur de position, pour lequel on est conduit par ce calcul symbolique à un point de vue nouveau. © Elsevier, Paris

*Mots clés :* Calcul symbolique ; équation de Dirac ; quantification ; groupe de Poincaré ; opérateur de position

## 1. INTRODUCTION

Whatever their true nature, quantum phenomena concern us only insofar as they interact with our classical world. That they can produce appreciable effects, and conversely that classical fields act on quantum systems is of course crucial to the whole of physics. A more historical, if foundational, role was played in quantum mechanics by the so-called measurement process. It is true, too, that overemphasis on this scheme sometimes led to depressing philosophical developments. What it did help bring to light, however, was a wealth of new mathematical methods or domains (the theory of operators, unitary representation theory, the Weyl calculus) which, besides their fundamental mathematical interest, found their place as essential tools in mathematical physics, though probably not at the exact location they had been meant to fill.

In the measurement process, the central role is ascribed to a (generally unknown) *quantization rule*, that associates an operator on some Hilbert space with every suitable classical observable. The basic demands one may raise about such a rule regard the compatibility between a geometric structure on some phase space and a quantum analogue which can best be put, up to some point, in representation-theoretic terms. It is therefore no accident that the availability of such a construction should seem to depend on the existence of sufficient symmetries in the (classical or quantum) problem involved. A most interesting case occurs in connection with species of elementary particles specified by some free wave equation. For instance, the Schrödinger equation for a free non-relativistic particle yields the well-known *Weyl calculus* of operators, as shown in [13], section 3, it being granted that this calculus also arises from a variety of other ways. In the same monograph, we developed, under the name of *Klein-Gordon calculus*, a quantization procedure in connection with the square-root Klein-Gordon equation. We here build a calculus associated with the Dirac equation: the constraint that it should be covariant with respect to all symmetries of the Dirac equation, including the discrete symmetries, is fully satisfied. Our present *Dirac calculus* should set itself aside from the numerous papers devoted to such classical issues as that of a position operator for the electron in that it is concerned with a description of *all* operators acting on the space of solutions of the given wave equation.

It is our belief that, generally speaking, calculi of observables should be given some status in the bag of mathematical tools considered in elementary particle theory: indeed, their construction should make considerable use of the symmetries characterizing the species of particle under consideration, as is the case here with the symmetries of spacetime, while internal symmetries should play a role too for stranger particles. Also, even though it may be a little too early to tell, we believe that such calculi might contribute to some extent to a proper understanding of the fields that interact with the particle. We are fully aware, on the other hand, that in its present state this work bears no obvious link with the domain where the most interesting physics take place, namely that where collisions do occur. But, to (mis)quote the humorous preface of D. Kastler's book [7], mathematicians have a right, after all, to take delight in trying to take part in the permanent refurbishing of the first floor of the grandiose edifice of Physics. If nothing else, this may lead to some non-trivial new mathematics (*cf.* e.g. [14], [15] for applications of the Klein-Gordon calculus to the theory of Mathieu functions, or to a generalization of the hypergeometric equation).

At this point, it may be useful to tell the reader that, even though the Klein-Gordon calculus is sometimes alluded to, mostly in the present introduction, no knowledge of it is a prerequisite towards reading this paper. In graduating from the (square-root) Klein-Gordon calculus to the present Dirac calculus, there are several major difficulties or novelties, some of which it may be of some interest to report.

First, there is the obvious fact that Dirac wave functions are no longer scalar, but vector-valued; as an acknowledgement of this fact, one has to use as classical observables (mathematicians would call these functions symbols), so to speak, matrix-valued functions on the phase space. It is essential, however, to identify these with functions having some physical interpretation: what we get here is functions whose range of values is the same as that of the electromagnetic vector potential. Up to the choice of some electromagnetic unit, this is of course just the same as functions valued in Minkowski's space, but the electromagnetic vector potential also interacts (through the minimal coupling) with Dirac particles in an already known way. This coincidence is necessary if we wish to consider the case of particles in some external field, a question we hope to return to in some other work, and which we certainly do not view as a *generalization* of the free case, rather as a situation calling (in the present frame in which observables are considered) for some fundamental new ideas: indeed, when *general* external fields are present, symmetries are lost. Let us insist, on the other hand, on the fact that one should not confuse two issues: the discussion of some wave equation (which, indeed, is trivial in the free case), and that of measurements which do not perturb the evolutionary process associated with the given wave equation. Even though it deals with observables acting on free Dirac particles only, the whole construction described in the present paper is just as relevant as the one which yields the Weyl symbolic calculus from the consideration of a free non-relativistic particle. That we get general symbolic calculi from free wave equations is due in both cases to the fact that the wave equation only serves as a means of extending a function to the whole spacetime from its restriction to some appropriate hypersurface, thus giving functions on the hypersurface more elbow-room for the action of symmetries: spacetime, not space, is where the action lies!

Next, there is the important question, which has been met by all the people who tried to define, in the Dirac case, such operators as the position operator [10] (whatever this may mean): namely, should one consider only those operators which preserve the sign of the energy, thus avoiding the phenomenon known as *Zitterbewegung*? Our answer is yes, not only from such considerations, but from other ones too, which shall be explained in due time.

Finally, the greatest difficulty arose from the following circumstance. There is a concept of restricted observers, the set of which constitute a 7-dimensional manifold; it is elementary, but quite deep in some sense, that it should coincide with the phase-space (spacetime $\times$ energy-momentum: time is included) corresponding to a free relativistic particle of mass 1, with positive frequency. Moreover, some natural equivalence relation (which cuts down the dimension from 7 to 6) yields the set of straight worldlines as a quotient set: classical observables, in the Klein-Gordon calculus, were just functions living on this quotient set. One major difficulty, in the Dirac calculus, comes from the fact that there is no genuine concept of what a classical analogue of a Dirac particle should be, thus preventing the second interpretation of the phase space given above to enter the picture; however, there is a notion of observer under the extended Poincaré group, which is just what we need. Making a choice between four components in view of the four types of observers, or two in view of the two possible signs of the energy was finally settled in favour of the observers.

Some common spirit guides the present work and that of Cordes [4], in which an algebra of observables invariant under the time evolution associated with a Dirac equation is constructed: the concern there lies with the study of hyperbolic systems, not the quantization problem. The Dirac equation itself is treated in many physics textbooks: we relied on several ones [2], [6], [11] but, above all, on Thaller's book [12], where one can find a very lucid discussion of some of the difficulties associated with the position operator and related observables.

In order to keep the present paper within a reasonable size, we stopped short of developing the Dirac calculus as a full pseudodifferential analysis, along the lines of what has been done for the Klein-Gordon analysis in [13]. For this would require extensive work (part of it could be saved by relying on the Klein-Gordon analysis itself), even though the results might not appear as the most novel feature of the present calculus. Let us also admit that we do have in mind some possibly more urgent, or more exciting, developments. Besides constructing the calculus, we have been satisfied, in the present paper, with listing a few important operators together with their symbols. Considering first the infinitesimal generators of the Poincaré representation $\pi$, we show that the symbols of the operators $d\pi(X)$, as $X$ describes the set of spacetime translations (resp. the Lie algebra of the Lorentz group) are canonically associated, in the most natural way, with elements of the Minkowski space $\mathbb{M}$ (resp. with elements of $\Lambda^2\mathbb{M}$). Next, we discuss the issue of position operators. Finally, we note in the last section that from the consideration of position operators or, rather, of position symbols, there arises in a very natural way a certain operator which, when Dirac wave functions are viewed as sections of some linear bundle on the mass-shell, turns out to be just the Dirac operator associated with this spinor bundle. Here, the word "Dirac operator" should be taken in the sense ascribed to it by Riemannian geometers, or "Clifford analysts" (*cf.* [1], [3], [5], [8]).

One final remark: in our presentation of the Dirac equation, which is just the usual chiral representation under some slight disguise, we found it much better, for several reasons, to rely on abstract data (for instance, the space of spinors is just a 2-dimensional complex vector space, not $\mathbb{C}^2$) than on column or row vectors and matrices. The major reason is that the concept of observer our whole construction is made of is just some additional structure put on Minkowski's space $\mathbb{M}$: now, this is better understood if $\mathbb{M}$ is not encumbered with a *fixed* isomorphism with $\mathbb{R}^4$. On the other hand, we have made it our policy to denote the relevant coordinate-free concepts by the same letters ($\gamma, \sigma \ldots$) they are usually denoted by in their matrix-realizations. Last, we have carefully avoided all kinds of pedantry (like defining the Minkowski space as an affine rather than as a vector space) our mathematical taste might have led us to.

## 2. SPINORS AND SPACETIME

Sections 2 to 4 of the present paper are a reminder of well-known facts and notions: however, it has been found necessary to define the various concepts in a coordinate-free version, equivalent to the usual one as soon as assorted bases have been introduced. We take spinors, rather than space-time, as the prime notion.

AXIOMS 2.1. – *We assume that* $V$ *and* $W$ *are two given complex vector spaces of dimension two; besides, there is a given non-degenerate sesquilinear form on* $V \times W$ *(antilinear with respect to the first variable), denoted as* $(v, w)$, $v \in V$, $w \in W$. *Finally, one has given the phase-class* $|\eta|$ *(*i.e. *the class up to multiplication by a complex constant of modulus one) of some non-zero complex two-form* $\eta$ *on* $V \times V$.

*On the other hand, there is a given 4-dimensional real vector space* $\mathbb{M}$, *together with an* $\mathbb{R}$*-linear isomorphism* $\sigma$ *from* $\mathbb{M}$ *onto the space* $\mathrm{Herm}(W)$ *of all hermitian forms on* $W$.

*Remark.* – The spaces $V$ and $W$ stand for the spaces of spinors with undotted (resp.dotted) indices; $\mathbb{M}$ of course stands for Minkowski's space. So far as physical dimensions are concerned, we assume that $c = 1$, thus assigning the dimension $L$ to elements of $\mathbb{M}$: then (so as to have a dimensionless $\sigma$), we assume that the elements of $W$ have dimension $L^{-\frac{1}{2}}$, consequently that those of $V$ have dimension $L^{\frac{1}{2}}$. A two-form on $V$ thus has dimension $L^{-1}$, so that $|\eta|^{-1}$ is nothing but a length-unit: we shall assume that it is precisely the Compton wavelength $\frac{h}{mc}$, where $m$ is the mass of the electron.

To allow a comparison with a more traditional setting, let us introduce, in a coherent way, bases for the linear spaces $V, W$ and $\mathbb{M}$: we shall refer to the various bases involved as to *assorted bases*. To start with, the pair $\{\varepsilon_1, \varepsilon_2\}$ shall be any basis of $V$ subject to the condition $|\eta(\varepsilon_1, \varepsilon_2)| = 1$; then, $\{\varepsilon_1^*, \varepsilon_2^*\}$ shall be the dual basis of $W$, i.e. the one characterized by $(\varepsilon_j, \varepsilon_k^*) = \delta_{jk}$. Since the given sesquilinear form ( , ) on $V \times W$ permits to identify $W$ to the space of antilinear forms on $V$, hermitian forms $h[\ ,\ ]$ on $W$ can, and will, be canonically identified with hermitian maps from $W$ to $V$ under the rule

$$h[w_1, w_2] = (hw_1, w_2) = \overline{(hw_2, w_1)}. \tag{2.1}$$

Given the assorted bases, defined above, of $V$ and $W$, we define the assorted basis $\{e_0, e_1, e_2, e_3\}$ of $\mathbb{M}$, and the corresponding set of linear coordinates $\{x^0, x^1, x^2, x^3\}$, in such a way that, for any $x \in \mathbb{M}$, the matrix representing $\sigma(x) \in \mathrm{Herm}(W) \subset \mathcal{L}(W, V)$ should be

$$\sigma(x) = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}; \tag{2.2}$$

in other words $\sigma(x) = \sum x^\mu \sigma_\mu$ where $\sigma_0$ is the identity matrix and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.3}$$

Let us now take Planck's constant as a unit of action: then the space $\mathbb{M}'$, linear dual of $\mathbb{M}$, should be interpreted as the space of energy-momentum covectors. On the other hand, there is an intrinsic duality between $\mathrm{Herm}(W) \subset \mathcal{L}(W,V)$ and $\mathrm{Herm}(V) \subset \mathcal{L}(V,W)$, namely that defined by the pairing

$$< h, g >_{\mathrm{Herm}(W)\times\mathrm{Herm}(V)} = \frac{1}{2} Tr(h \circ g). \tag{2.4}$$

We may then define $\tilde{\sigma} : \mathbb{M}' \to \mathrm{Herm}(V)$ as the contragredient

$$\tilde{\sigma} = \sigma'^{-1} \tag{2.5}$$

of $\sigma$: in matrix-form, $\tilde{\sigma}(p)$ is the same as $\sigma(x)$, substituting the components of $p$ for those of $x$.

The canonical Minkowski form $Q$ on $\mathbb{M}$ is defined, in assorted coordinates, as

$$Q(x) = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2, \tag{2.6}$$

which is just the determinant of the matrix (2.2) representing $\sigma(x)$. This is an intrinsic notion, whose polarized form $Q(x,y)$ permits to define the linear isomorphism $\theta$ from $\mathbb{M}'$ onto $\mathbb{M}$ through the equation

$$Q(x,y) = < x, \theta^{-1} y >_{\mathbb{M}\times\mathbb{M}'} . \tag{2.7}$$

The following formulas, whose verification is immediate, are useful:

$$\tilde{\sigma}(q)\sigma(\theta p) + \tilde{\sigma}(p)\sigma(\theta q) = 2 < \theta q, p > Id_W , \tag{2.8}$$
$$\sigma(\theta p)\tilde{\sigma}(q) + \sigma(\theta q)\tilde{\sigma}(p) = 2 < \theta q, p > Id_V , \quad q, p \in \mathbb{M}' \tag{2.9}$$

and (a particular case since $< p, \theta p >= Q(\theta p)$)

$$\tilde{\sigma}(p)\sigma(\theta p) = Q(\theta p)\, Id_W , \quad \sigma(\theta p)\tilde{\sigma}(p) = Q(\theta p)\, Id_V , \quad p \in \mathbb{M}'. \tag{2.10}$$

Fix a two-form $\eta$ in the given phase-class $|\eta|$. Then there is a unique antilinear map $\kappa : V \to W$ such that the identity

$$\overline{\eta(v_1, v_2)} = (v_2, \kappa v_1) \tag{2.11}$$

holds for every pair $(v_1, v_2)$ of points of $V$. If the basis $\{\varepsilon_1, \varepsilon_2\}$ of $V$ is chosen so that $\eta(\varepsilon_1, \varepsilon_2) = 1$, one then has

$$\kappa\varepsilon_1 = \varepsilon_2^*, \qquad \kappa\varepsilon_2 = -\varepsilon_1^*. \tag{2.12}$$

Of course, if only $|\eta|$, not $\eta$, is defined, which is our genuine assumption, then only the phase-class of $\kappa$ will be defined: this is just what is needed, since $\kappa$ will be essential principally in the construction of the antilinear maps (for instance the charge conjugation) that occur in the theory of the Dirac equation, and these should be defined only up to some constant phase. The formula

$$\tilde{\sigma}(p) = -\kappa\,\sigma(\theta p)\,\kappa, \qquad p \in \mathbb{M}', \tag{2.13}$$

will be useful later, and may be proved as a consequence of (2.12) and (2.2) together with the fact that, in assorted coordinates, one has, for every $p \in \mathbb{M}'$,

$$\theta p = x \quad \text{with} \quad x^0 = p_0,\, x^j = -p_j, \quad (j = 1, 2, 3). \tag{2.14}$$

The definition of the standard covering homomorphism

$$\Lambda : SL(V) \to \mathcal{L}_+^\uparrow, \tag{2.15}$$

where $SL(V)$ stands for the group of linear automorphisms of $V$ which preserve any non-degenerate complex two-form on $V$ (it does not depend on which two-form we choose), and $\mathcal{L}_+^\uparrow$ stands for the restricted Lorentz group of $\mathbb{M}$, i.e. the connected component of the identity in the Lorentz group $\mathcal{L}$ consisting of all linear transformations of $\mathbb{M}$ which preserve the Minkowski form $Q$, is well-known: namely, define $s^* \in SL(W)$ through

$$(v, s^*w) = (sv, w), \qquad v \in V,\, w \in W \tag{2.16}$$

for any $s \in SL(V)$: then, given $x \in \mathbb{M}$ and $s \in SL(V)$, the point $\Lambda(s)x \in \mathbb{M}$ is defined by the equation

$$\sigma(\Lambda(s)x)[w_1, w_2] = \sigma(x)[s^*w_1, s^*w_2], \qquad w_1, w_2 \in W, \tag{2.17}$$

in other words

$$\sigma(\Lambda(s)x) = s\sigma(x)s^*. \tag{2.18}$$

Let us list the formulas

$$\theta\,\Lambda(s)' = \Lambda(s)^{-1}\theta, \tag{2.19}$$

where $\Lambda(s)'$ denotes the transpose of $\Lambda(s)$, and

$$\tilde{\sigma}(\Lambda(s)'p) = s^*\tilde{\sigma}(p)s, \qquad p \in \mathbb{M}', \tag{2.20}$$

or

$$\tilde{\sigma}(\Lambda(s)'p)(v_1, v_2) = \tilde{\sigma}(p)(sv_1, sv_2), \qquad p \in \mathbb{M}',\, v_1, v_2 \in V, \tag{2.21}$$

which will all be useful later.

## 3. OBSERVERS

The following concept, adapted to our needs, is not exactly the usual one. In connection with Minkowski's space $\mathbb{M}$, we define an unrestricted observer $\omega$ as a set

$$\omega = (x, T_\omega, S_\omega, \updownarrow, \varepsilon) \tag{3.1}$$

where the data are the following: $x$ is a point of $\mathbb{M}$; $T_\omega$ is a one-dimensional linear subspace of $\mathbb{M}$ such that $Q(y, y) > 0$ for all $y \in T_\omega$; the three-dimensional subspace $S_\omega$ of $\mathbb{M}$ is the orthogonal of $T_\omega$ with respect to the Minkowski form $Q$. Finally, $\updownarrow$ and $\varepsilon$ together amount to an orientation of $T_\omega$ and an orientation of $S_\omega$ according to the following rules: as $\mathbb{M}$ is provided with a given causality, it has a well-defined cone of the future, and the variable $\updownarrow$ shall be set to the value $\uparrow$ or $\downarrow$ according to whether a vector in $T_\omega$, positive with respect to the given orientation of $T_\omega$, does point in the direction of the future or not. Last, the variable $\varepsilon$ stands for a global orientation of $\mathbb{M}$ : to simplify notations, we set its value to 1 or -1 according to whether it is compatible or not with the orientation of $\mathbb{M}$ canonically associated with the isomorphism $\sigma : \mathbb{M} \to \mathrm{Herm}(W)$. We denote as $\Omega$ the set of all observers: it splits into four connected components as

$$\Omega = \Omega_+^\uparrow \cup \Omega_-^\uparrow \cup \Omega_+^\downarrow \cup \Omega_-^\downarrow. \tag{3.2}$$

We may refer to $S_\omega$ and $T_\omega$ as to the space and time as viewed from the point of view of the observer $\omega$. On the space of all observers, there is a natural equivalence relation, which identifies two observers $(x, \ldots)$ and $(y, \ldots)$ if they share the same splitting $\mathbb{M} = T \oplus S$ of spacetime, the same concepts of orientation and causality, finally if $x - y$ lies in $T$: from the point of view of the observer $(x, \ldots)$, this means that, just sitting and getting older, he would eventually reach the point $y$ in spacetime, unless it is after one has exchanged $x$ and $y$ that this situation should prevail.

We shall denote as $\mathcal{M}$ the *mass-hyperboloid*, i.e. the subset of $\mathbb{M}'$ characterized as

$$\mathcal{M} = \{p \in \mathbb{M}' : Q(\theta p) = 1\} = \mathcal{M}^\uparrow \cup \mathcal{M}^\downarrow, \tag{3.3}$$

where $\theta(\mathcal{M}^\uparrow)$ and $\theta(\mathcal{M}^\downarrow)$ are the two components of $\{x \in \mathbb{M} : Q(x) = 1\}$ which lie within the cone of the future, or that of the past respectively. As is well known, $\mathcal{M}^\uparrow$ is a Riemannian space with a $ds^2$ written, in assorted coordinates, as

$$ds^2 = -dp_0^2 + dp_1^2 + dp_2^2 + dp_3^2. \tag{3.4}$$

For each $p \in \mathcal{M}^{\uparrow}$, let $S_p$ denote the linear map: $\mathbb{M}' \to \mathbb{M}'$ defined as

$$S_p q = -q + 2 < \theta p, q > p, \qquad q \in \mathbb{M}'. \tag{3.5}$$

As is well-known too, $\mathcal{M}^{\uparrow}$ is actually a Riemannian *symmetric* space, and the restriction of the map $S_p$ to $\mathcal{M}^{\uparrow}$ is nothing but the geodesic symmetry of $\mathcal{M}^{\uparrow}$ around $p$.

The same holds with $\mathcal{M}^{\downarrow}$ in place of $\mathcal{M}^{\uparrow}$: observe that, for $p \in \mathcal{M}^{\uparrow}$, $S_p$ also acts within $\mathcal{M}^{\downarrow}$, but we shall never consider this, as $S_p$ is just the same as $S_{-p}$. We shall use at some point the easily proven identity

$$\tilde{\sigma}(S_p q)\sigma(\theta p) = \tilde{\sigma}(p)\sigma(\theta q), \qquad p \in \mathcal{M}^{\uparrow}, \, q \in \mathbb{M}'. \tag{3.6}$$

Given an oriented one-dimensional space $T_\omega$ entering the definition of some observer (3.1), let $u_\omega$ denote the vector in $T_\omega$, normalized by the condition $Q(u_\omega) = 1$ and positive with respect to the given orientation; we also set

$$p_\omega = \theta^{-1} u_\omega, \tag{3.7}$$

which is a vector that lies in $\mathcal{M}$. From the pair $(x, p_\omega)$ one can recover all the data that enter the definition (3.1) of the observer $\omega$, except for the global orientation $\varepsilon$ (or, what amounts just to the same, the orientation of $S_\omega$). Thus any of the two spaces of observers

$$\Omega_+^{\uparrow} \cup \Omega_+^{\downarrow} \qquad \text{or} \qquad \Omega_-^{\uparrow} \cup \Omega_-^{\downarrow} \tag{3.8}$$

can be identified, through the map $\omega \mapsto (x, p_\omega)$, with the space $\mathbb{M} \times \mathcal{M}$; under this identification, the two shells of $\mathcal{M}$ correspond to the two possible causalities associated with $\omega$ (we get the $\uparrow$ or $\downarrow$ sign according to whether this causality is compatible or not with the one canonically inherited from the isomorphism $\sigma$ introduced in the Axioms 2.1). Observe that the component $\mathbb{M} \times \mathcal{M}^{\uparrow}$ can be thought of as the classical phase space (including time: the standard phrase is *extended phase space* but we do not want to risk any confusion with the extended Poincaré group) associated with a free relativistic particle of mass 1: $p$ would stand for the energy-momentum of the particle, and $x$ for its location in spacetime.

Any set of assorted coordinates defines an observer $\omega_0$ that may serve as an observer of reference, namely that for which $x = 0$ and $T_\omega$ and $S_\omega$ are generated by the bases $\{e_0\}$ and $\{e_1, e_2, e_3\}$ respectively, and these two bases are compatible with the orientation and causality on $\mathbb{M}$ as viewed by $\omega_0$. Now, if $\omega$ is any other observer with the same concept of causality as

$\omega_0$, and $u_\omega = \lambda e_0 + w_\omega$ for some $\lambda > 0$ and $w_\omega \in \mathrm{span}(e_1, e_2, e_3)$, one may think of the vector $v_\omega = \lambda^{-1} w_\omega$, which is purely spatial from the point of view of the observer of reference and satisfies $\|v_\omega\|_0 := -Q(v_\omega) < 1$, as the velocity of the observer $\omega$ with respect to the observer of reference; then, $\lambda = (1 - \|v_\omega\|_0^2)^{-\frac{1}{2}}$ and, in assorted coordinates,

$$p_\omega = \theta u_\omega = \lambda\theta(e_0 + v_\omega) \tag{3.9}$$

may be written as $(1 - \|v_\omega\|_0^2)^{-\frac{1}{2}}(1, -v_1, -v_2, -v_3)$ if $(v_1, v_2, v_3)$ is the set of (assorted) components of $v_\omega$; then $p_\omega$ is nothing but the energy-momentum of a classical relativistic particle of mass 1 that would move with the velocity $v_\omega$ with respect to the observer of reference. Thus, under the identification just described of $\Omega_+^\uparrow$ with $\mathbb{M} \times \mathcal{M}^\uparrow$, an equivalence class of observers transforms to a worldline times the point $p$ in $\mathcal{M}^\uparrow$ whose associated velocity $v = -p_0^{-1}\mathbf{p}$ is such that the vector $(1, v) \in \mathbb{M}$ is parallel to the given worldline: it may thus be identified with the worldline itself.

The coincidence just explained, to wit that the space of observers $\Omega_+^\uparrow$ (we shall also refer to this space as to the space of *restricted observers*) can be identified with the classical phase space corresponding to a relativistic particle of mass 1, in a way which transforms an equivalence class of observers into a straight worldline, was basic in our construction of the Klein-Gordon calculus. Actually, in that case, symbols (classical observables) were just scalar functions living on these sets of equivalence classes. This cannot be carried over in the Dirac case to a full extent. Indeed, there is no genuine classical analogue of a Dirac particle. Even if we agreed, to account for the two possible signs of the frequency, to say that in that case $\mathcal{M}^\uparrow$ should be replaced by $\mathcal{M}$, this would only double the number of components of the extended phase space for the given particle, whereas the space of observers gets four components. As will be seen, if we demand that the Dirac calculus be covariant under all symmetries of the Dirac wave equation, including the discrete symmetries, we have to give observers the precedence.

## 4. THE DIRAC EQUATION

This section contains no novel features: only we must fix our notations. Complex-linear endomorphisms of the space of *bispinors* $V \oplus W$ may be represented as block-matrices relative to this decomposition. In particular, given $p \in \mathbb{M}'$, set

$$\gamma(p) = \begin{pmatrix} 0 & \sigma(\theta p) \\ \tilde{\sigma}(p) & 0 \end{pmatrix} \quad : \tag{4.1}$$

in assorted bases, this is the same as $\gamma(p) = \sum \gamma^\mu p_\mu$ with

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \gamma^j = \begin{pmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{pmatrix} \ (j = 1, 2, 3) , \tag{4.2}$$

where the $\sigma_j$'s were defined in (2.3).

Functions, or rather tempered distributions $\Psi$ on $\mathbb{M}$ valued in $V \oplus W$ have Fourier transforms, defined componentwise as

$$(\mathcal{F}\Psi)(p) := \widehat{\Psi}(p) = \int_{\mathbb{M}} \Psi(x)\, e^{-2i\pi<x,p>}\, dx, \qquad p \in \mathbb{M}', \tag{4.3}$$

where the measure $dx$ is the standard Lebesgue measure on $\mathbb{R}^4$ in assorted coordinates. The Dirac equation is the equation

$$\gamma(p)\, \widehat{\Psi}(p) = \widehat{\Psi}(p) \tag{4.4}$$

or, equivalently (in assorted coordinates)

$$(2i\pi)^{-1} \sum \gamma^\mu \frac{\partial}{\partial x^\mu}\, \Psi = \Psi, \tag{4.5}$$

the usual Dirac wave equation since the Compton wavelength $\frac{h}{mc}$ has been chosen as a length unit.

As a consequence of (2.10), one has

$$(\gamma(p))^2 = Q(\theta p) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{4.6}$$

so that, for any solution of (4.5), $\mathcal{F}\Psi$, as a $(V \oplus W)$-valued distribution on $\mathbb{M}'$, has its support contained in $\mathcal{M}$. The full Lorentz group $\mathcal{L}$ acts on $\mathbb{M}'$ through the contragredient representation

$$(\Lambda, p) \mapsto \Lambda'^{-1} p, \qquad \Lambda \in \mathcal{L},\, p \in \mathbb{M}', \tag{4.7}$$

thus preserving the mass hyperboloid $\mathcal{M}$. Under this action of $\mathcal{L}$ on $\mathcal{M}$, the positive measure $\delta(Q(\theta p) - 1)$ defined on $\mathcal{M}$, in assorted coordinates, as

$$\delta(Q(\theta p) - 1) :=< \mathbf{p} >^{-1}\, d\mathbf{p} \tag{4.8}$$

with

$$p = (p_0, \mathbf{p}), \qquad < \mathbf{p} >= (1 + |\mathbf{p}|^2)^{\frac{1}{2}} = |p_0|, \tag{4.9}$$

is invariant, as is well-known. We shall prefer the more obvious, though less intrinsic, notation $< \mathbf{p} >^{-1} d\mathbf{p}$ to denote the measure (4.8) on $\mathcal{M}$, and the notation $\delta(Q(\theta p)-1)$ to denote the corresponding measure on $\mathbb{M}'$ supported on $\mathcal{M}$, a slightly distinct object.

Now a $p$-dependent Hilbert space structure on $V \oplus W$ can be defined ($p \in \mathcal{M}^{\uparrow}$) as

$$\|\begin{pmatrix} v \\ w \end{pmatrix}\|_p^2 = \tilde{\sigma}(p)[v,v] + \sigma(\theta p)[w,w]. \tag{4.10}$$

The Hilbert space $\mathcal{H}$ associated with the free Dirac equation (4.5) consists of all solutions of (4.5) which satisfy the additional property that

$$\widehat{\Psi}(p) = (\mathcal{G}\Psi)(p)\,\delta(Q(\theta p)-1) \tag{4.11}$$

for some $(V \oplus W)$-valued function $\mathcal{G}\Psi$ on $\mathcal{M}$ such that the function $p \mapsto \|(\mathcal{G}\Psi)(p)\|_{p\ \mathrm{sign}(p_0)}$ lies in $L^2(\mathcal{M}) := L^2(\mathcal{M}; <\mathbf{p}>^{-1} d\mathbf{p})$; one then sets

$$\|\Psi\|_{\mathcal{H}}^2 = \int_{\mathcal{M}} \|(\mathcal{G}\Psi)(p)\|_{p\ \mathrm{sign}(p_0)}^2 <\mathbf{p}>^{-1} d\mathbf{p} \tag{4.12}$$

where, of course, $p\ \mathrm{sign}(p_0) = \pm p$ according to whether $p \in \mathcal{M}^{\uparrow}$ or $\mathcal{M}^{\downarrow}$. The Dirac equation (4.4) can then be rephrased as

$$\gamma(p)(\mathcal{G}\Psi)(p) = (\mathcal{G}\Psi)(p). \tag{4.13}$$

Also, one may denote as $L^2(\mathcal{M}, \mathrm{Ker}(\gamma-1))$ the Hilbert space which consists of all functions $\mathcal{G}\Psi$, $\Psi \in \mathcal{H}$.

Defining the canonical hermitian form $(\!( \ , \ )\!)$ on $V \oplus W$ as

$$(\!(\xi_1, \xi_2)\!) = (v_1, w_2) + \overline{(v_2, w_1)} \qquad \text{if } \xi_1 = \begin{pmatrix} v_1 \\ w_1 \end{pmatrix},\ \xi_2 = \begin{pmatrix} v_2 \\ w_2 \end{pmatrix} \tag{4.14}$$

and noting that, for every $p \in \mathcal{M}^{\uparrow}$, one has

$$(\!(\xi, \xi)\!) = \|\xi\|_p^2 \qquad \text{if } \xi \in \mathrm{Ker}(\gamma(p)-1) \tag{4.15}$$

and

$$(\!(\xi, \xi)\!) = -\|\xi\|_p^2 \qquad \text{if } \xi \in \mathrm{Ker}(\gamma(-p)-1), \tag{4.16}$$

one sees that if $\Psi$ is a solution of the Dirac equation (4.5), then

$$\|\Psi\|_{\mathcal{H}}^2 = \int_{\mathcal{M}} (\!(\,(\mathcal{G}\Psi)(p), (\mathcal{G}\Psi)(p)\,)\!)\, p_0^{-1} d\mathbf{p} : \tag{4.17}$$

observe that $p_0^{-1} d\mathbf{p} = - <\mathbf{p}>^{-1} d\mathbf{p}$ on $\mathcal{M}^{\downarrow}$, a non-positive measure.

We now turn to a discussion of the Poincaré group symmetries of $\mathcal{H}$. The Poincaré group $\mathcal{P}$, a semi-direct product of $\mathcal{L}$ and $\mathbb{M}$, is the group of affine transformations of $\mathbb{M}$, denoted as $(\Lambda, a)$, with $\Lambda \in \mathcal{L}$ and $a \in \mathbb{M}$, and

$$(\Lambda, a).x = \Lambda x + a, \qquad x \in \mathbb{M}. \tag{4.18}$$

The restricted (resp.orthochronous) group $\mathcal{P}_+^\uparrow$ (resp. $\mathcal{P}^\uparrow$) is of course obtained when $\Lambda$ varies through $\mathcal{L}_+^\uparrow$ (resp. $\mathcal{L}^\uparrow$).

We also set

$$SL_{\text{in}} = \{(s, a) : \, s \in SL(V), a \in \mathbb{M}\}, \tag{4.19}$$

where the subscript stands for *inhomogeneous*: this is a group if the multiplication is defined as

$$(s_1, a_1)(s_2, a) = (s_1 s_2, \, a_1 + \Lambda(s_1)a_2) : \tag{4.20}$$

note that

$$(s, a)^{-1} = (s^{-1}, -\Lambda(s)^{-1}a), \tag{4.21}$$

and that the map $(s, a) \mapsto (\Lambda(s), a)$ is a homomorphism from $SL_{\text{in}}$ onto $\mathcal{P}_+^\uparrow$. Given $s \in SL(V)$, one also sets

$$\Xi(s) = \begin{pmatrix} s & 0 \\ 0 & s^{*-1} \end{pmatrix}, \tag{4.22}$$

an automorphism of the space of bispinors $V \oplus W$. The representation $\pi$ of the group $SL_{\text{in}}$ in $\mathcal{H}$ is then defined as

$$(\pi(s, a)\Psi)(x) = \Xi(s) \, \Psi(\Lambda(s)^{-1}(x - a)) : \tag{4.23}$$

indeed, this leads to

$$\mathcal{F}(\pi(s, a)\Psi)(p) = e^{-2i\pi <a,p>} \, \Xi(s) \, \widehat{\Psi}(\Lambda(s)'p) \tag{4.24}$$

and $\pi(s, a)\Psi$ is again a solution of Dirac's equation since, as a consequence of (2.18) and (2.19) on one hand, of (2.20) on the other, one has

$$\gamma(p)\Xi(s) = \Xi(s) \, \gamma(\Lambda(s)'p). \tag{4.25}$$

As $\Lambda(s)'$ preserves $\mathcal{M}$ together with its measure $< \mathbf{p} >^{-1} d\mathbf{p}$, one can also write

$$(\mathcal{G} \, \pi(s, a)\Psi)(p) = e^{-2i\pi <a,p>} \, \Xi(s) \, (\mathcal{G}\Psi)(\Lambda(s)'p). \tag{4.26}$$

It immediately follows from (4.14) and (2.16) that

$$((\Xi(s)\xi_1, \Xi(s)\xi_2)) = ((\xi_1, \xi_2)), \qquad s \in SL(V),\ \xi_1,\ \xi_2 \in V \oplus W. \quad (4.27)$$

As is well-known, $\pi$ is a unitary representation of the group $SL_{\text{in}}$, a two-fold covering of $\mathcal{P}_+^\uparrow$.

Let us recall now the so-called discrete symmetries of the full Poincaré group $\mathcal{P}$, traditionally denoted as $C, P$ and $T$, though, what amounts to the same, we shall consider $C, P$ and $CPT$ instead. One should remember important differences among these three symmetries. First, contrary to the other two ones, the charge conjugation $C$ is antilinear rather than complex-linear and is not associated with any geometric transformation of $\mathbb{M}$. Next, the decomposition of the Hilbert space $L^2(\mathcal{M}, \operatorname{Ker}(\gamma - 1))$ as the direct sum of its two subspaces consisting of sections supported in $\mathcal{M}^\uparrow$ or $\mathcal{M}^\downarrow$ respectively yields, under $\mathcal{G}$, a corresponding Hilbert space decomposition

$$\mathcal{H} = \mathcal{H}^\uparrow \oplus \mathcal{H}^\downarrow \quad (4.28)$$

which is preserved under the representation of the group $SL_{\text{in}}$. This decomposition is also preserved under $P$, while the other two discrete symmetries $C$ and $CPT$ switch the two summands. Finally, as will be seen from the definitions, the charge conjugation, or rather its phase-class, is an intrinsic (i.e. not observer-dependent) symmetry; the transformation $CPT$, on the other hand, is related to the choice of an origin in Minkowski's space $\mathbb{M}$, and that such a choice appears as canonical is only due to the fact that, to avoid mathematical pedantry, we have decided to make $\mathbb{M}$ a vector space, not an affine space. The map $P$, which is related to the geometric transformation which expresses itself, in assorted coordinates, as $(x^0, x^1, x^2, x^3) \mapsto (x^0, -x^1, -x^2, -x^3)$, really depends on the choice of a class of restricted observers: actually, two restricted observers have the same notion of parity $P$ if and only if they are equivalent under the equivalence discussed right after (3.2). To comply with tradition, we have decided in the present paper to deal with the parity transform $P$ under the tacit assumption that an observer of reference has been chosen; but it would be just as well to define and study parity transformations $P_\omega$: this was the point of view adopted in our development of the Klein-Gordon calculus, where the basic definition of the calculus was actually given in terms of the operators $P_\omega$. The definitions of $C, P$ and $CPT$ are as follows.

Introduce the block-matrices (with respect to the direct sum $V \oplus W$)

$$K = \begin{pmatrix} 0 & \kappa^{-1} \\ \kappa & 0 \end{pmatrix}, \qquad \gamma^0 = \gamma((e^*)^0), \qquad \gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad (4.29)$$

in which the entries of $K$ are antilinear. Then define

$$\begin{aligned}(C\Psi)(x) &= K\,\Psi(x)\\ (P\Psi)(x) &= \gamma^0\,\Psi(Jx)\\ (CPT\Psi)(x) &= \gamma^5\,\Psi(-x), \qquad x \in \mathbb{M}\,,\end{aligned} \tag{4.30}$$

where, in the definition of the (observer-dependent) parity transform $P$, the linear transformation $J$ of $\mathbb{M}$ is defined, in the coordinates chosen, as $J(x^0, \mathbf{x}) = (x^0, -\mathbf{x})$; observe that $J'^{-1}(p_0, \mathbf{p}) = (p_0, -\mathbf{p})$ for every $p \in \mathbb{M}'$, so that (from (3.5)), $J'^{\pm 1}$ coincides with the map $S_{(e^*)^0}$ associated with the base-point $(e^*)^0$ of $\mathcal{M}^\uparrow$, the first basis vector in the assorted basis of $\mathbb{M}'$ considered. Then

$$\begin{aligned}(\mathcal{G}C\Psi)(p) &= K\,(\mathcal{G}\Psi)(-p)\\ (\mathcal{G}\,P\Psi)(p) &= \gamma^0(\mathcal{G}\Psi)(J'^{-1}p)\\ (\mathcal{G}\,CPT\,\Psi)(p) &= \gamma^5\,(\mathcal{G}\Psi)(-p), \qquad p \in \mathcal{M}.\end{aligned} \tag{4.31}$$

It is immediate (and well-known) that

$$\begin{aligned}\gamma(p)K &= K\gamma(p)\\ \gamma(p)\gamma^0 &= \gamma^0\gamma(J'^{-1}p)\\ \gamma(p)\gamma^5 &= -\gamma^5\gamma(p), \qquad p \in \mathbb{M}',\end{aligned} \tag{4.32}$$

from which it follows that $C$, $P$ and $CPT$ are isometries (antilinear in the first case, linear in the other two ones) of $\mathcal{H}$: only the second one preserves the decomposition (4.28), and one has $C^2 = P^2 = (CPT)^2 = I$.

## 5. THE DIRAC SYMBOLIC CALCULUS

As a consequence of the Dirac equation (4.13), the image under $\mathcal{G}$ of the Hilbert space $\mathcal{H}$ appears as a space of sections of the complex-linear bundle $p \mapsto \mathrm{Ker}(\gamma(p) - 1)$ above $\mathcal{M}$: this is a subbundle of rank two of the trivial bispinor bundle $\mathcal{M} \times (V \oplus W)$. It is thus natural that operators acting on $\mathcal{H}$ should appear first as associated with appropriate sections of some matrix-bundle above $\mathcal{M}$. This will give rise to our first symbolic calculus, in which the map *symbols* $\mapsto$ *operators* shall be denoted as $\widetilde{Op}$. However, this will be only the first step in a two-step construction: the final version of the calculus will be denoted as $Op$. In the first step, we shall deal with symbols that will be functions (or, rather, sections of an appropriate bundle) on $\mathbb{M} \times \mathcal{M}$, a two-component space; on the other hand, after the second step has been completed, symbols shall become functions living on the four-component space $\Omega$ (*cf.* (3.2)). *We shall postpone all non-formal considerations to the next section and shall behave, in the present one, as if all integrals did converge.*

Let $\updownarrow$ be a variable set to the value $\uparrow$ or $\downarrow$: we want to associate with appropriate symbols $(x,p) \mapsto f(x,p)$ living on $\mathbb{M} \times \mathcal{M}^{\updownarrow}$ operators from the space $\mathcal{H}^{\updownarrow}$ to itself; by convention, such operators will be zero on the orthogonal of $\mathcal{H}^{\updownarrow}$ in $\mathcal{H}$ (*cf.* (4.28)). For simplicity of notation, it is just as well in what follows to assume that $\updownarrow$ has been set to the value $\uparrow$ since the modifications connected with the other choice would be obvious.

Denote as $\Delta^{\uparrow}$ the set of worldlines which is by definition the quotient of the space $\mathbb{M} \times \mathcal{M}^{\uparrow}$ by the equivalence relation which, in accordance with the discussion at the end of section 3, identifies two equivalent restricted observers (in the same way, one might identify the quotient of the two-component space $\mathbb{M} \times \mathcal{M}$ by the same equivalence relation with the space of *oriented* worldlines). One can define a measure $dm$ on $\Delta^{\uparrow}$ by means of the choice of some set of representatives in $\mathbb{M} \times \mathcal{M}^{\uparrow}$. Two such convenient possible choices are

$$\mathbb{M}_0 \times \mathcal{M}^{\uparrow} = \{(x,p) : \; p \in \mathcal{M}^{\uparrow}\, ; \, x = (0, \mathbf{x})\} \tag{5.1}$$

or, for any smooth real function $k$ of $p$ alone,

$$\mathbb{E}_k^{\uparrow} = \{(x,p) \in \mathbb{M} \times \mathcal{M}^{\uparrow} : < x, p >= k(p)\}. \tag{5.2}$$

Observe that the first choice actually depends on that of a set of (assorted) coordinates on $\mathbb{M}$, not the second one. We then set

$$dm(x,p) = d\mathbf{x}\, d\mathbf{p} \qquad \text{in the first realization} \tag{5.3}$$

and

$$dm(x,p) = \frac{d\mathbf{x}\, d\mathbf{p}}{p_0^2} \tag{5.4}$$

in the second one. That these two measures agree is due to the fact that, given $(x,p) \in \mathbb{E}_k^{\uparrow}$, the unique point $(0, \mathbf{y}, p)$ equivalent to $(x,p)$ is given by

$$\mathbf{y} = \mathbf{x} + \frac{x_0}{p_0}\mathbf{p} = \mathbf{x} - \frac{< \mathbf{x}, \mathbf{p} >}{p_0^2}\,\mathbf{p} + \frac{k(p)}{p_0^2}\mathbf{p}, \tag{5.5}$$

and it is an easy matter (*cf.* [13], appendice, p.208) to check that $\frac{d\mathbf{y}}{d\mathbf{x}} = p_0^{-2}$.

The restricted Poincaré group $\mathcal{P}_+^\uparrow$ acts on the space $\mathbb{M} \times \mathcal{M}^\uparrow$ according to the rule

$$(\Lambda, a).(x, p) = (\Lambda x + a, {\Lambda'}^{-1} p) \tag{5.6}$$

which extends (4.18). This action is compatible with the equivalence which identifies two points on the same worldline since, obviously, if $x - y$ is a multiple of $\theta p$ for some $p \in \mathcal{M}^\uparrow$ (*cf.* (3.7)), then $\Lambda x - \Lambda y$ is a multiple of $= \Lambda\,\theta p = \theta({\Lambda'}^{-1} p)$ (*cf.* (2.19)). It thus defines an action on the quotient set $\Delta^\uparrow$, and it is easy to see that this action preserves the measure $dm$. The easiest thing to do is to check the invariance of $dm$ under restricted Lorentz transformations in the realization $\mathbb{E}_0^\uparrow$ (*cf.* (5.2)) since $\mathbb{E}_0^\uparrow$ itself is preserved under $\mathcal{L}_+^\uparrow$, and the invariance of $dm$ under purely spatial translations in the realization $\mathbb{M}_0 \times \mathcal{M}^\uparrow$, as these transformations together generate $\mathcal{P}_+^\uparrow$; or one can look at [13], p. 31.

A scalar function on $\mathbb{M} \times \mathcal{M}^\uparrow$ which is invariant under the equivalence relation above (i.e. which is associated with some function on $\Delta^\uparrow$) shall be called *admissible*. The same notion makes sense for sections of any linear bundle over $\mathbb{M} \times \mathcal{M}^\uparrow$ which is actually the pullback of some bundle over $\mathcal{M}^\uparrow$.

In order to emphasize the fact that our present discussion of the quantizing map $\widetilde{Op}$ only plays a temporary role, we shall call *presymbol* on $\mathbb{M} \times \mathcal{M}^\uparrow$ any *admissible* function

$$f : (x, p) \mapsto f(x, p) \in \mathrm{End}(V \oplus W) \tag{5.7}$$

where $\mathrm{End}(V \oplus W)$ stands for the space of complex-linear endomorphisms of the bispinor space $V \oplus W$, satisfying the property that, for every $(x, p) \in \mathbb{M} \times \mathcal{M}^\uparrow$, $f(x, p)$ commutes with $\gamma(p)$.

DEFINITION 5.1. – *Given a presymbol $f$ on $\mathbb{M} \times \mathcal{M}^\uparrow$, we define the operator $\widetilde{Op}^\uparrow(f)$ as follows: for every $\Psi \in \mathcal{H}^\uparrow$, the function $\widetilde{Op}^\uparrow(f)\,\Psi$ is defined so that*

$$(\mathcal{G}\,\widetilde{Op}^\uparrow(f)\,\Psi)(q) = 8 \int_{\Delta^\uparrow} \gamma(\mathrm{mid}\,(p, q))\, f(x, p)\, \gamma(\mathrm{mid}\,(p, S_p q)) \\ (\mathcal{G}\Psi)(S_p q)\, e^{2i\pi <x, S_p q - q>}\, dm(x, p) \tag{5.8}$$

*for all $q \in \mathcal{M}^\uparrow$, where* $\mathrm{mid}(p, q)$ *stands for the geodesic middle of $p$ and $q$ on $\mathcal{M}^\uparrow$. Also, by convention, the operator is zero on $\mathcal{H}^\downarrow$.*

Recall from (5.3) or (5.4) that $dm(x, p)$ is indeed a measure on $\Delta^\uparrow$. Although we have stressed the point that all questions of convergence would be eluded in the present section, what we still have to do, at least formally, is to show that the function $\widetilde{\mathcal{G}Op}^{\uparrow}(f)\,\Psi$ so defined satisfies Dirac's equation (4.13), i.e. that, as a function of $q$, the right-hand side lies in $\mathrm{Ker}(\gamma(q)-1)$. This is a consequence of (iii) in the following lemma.

LEMMA 5.2. – *Let $s \in SL(V)$, $p, q \in \mathcal{M}^{\uparrow}$. Then:*
(i) $\Xi(s)$ : $\mathrm{Ker}(\gamma(\Lambda(s)'q)-1) \longrightarrow \mathrm{Ker}(\gamma(q)-1)$;
(ii) $\gamma(p)$ : $\mathrm{Ker}(\gamma(S_p q)-1) \longrightarrow \mathrm{Ker}(\gamma(q)-1)$;
(iii) $\gamma(\mathrm{mid}\,(p,q))$ : $\mathrm{Ker}(\gamma(q)-1) \longrightarrow \mathrm{Ker}(\gamma(p)-1)$.

*Moreover, all these maps are isometries if, for each $p \in \mathcal{M}^{\uparrow}$,* $\mathrm{Ker}(\gamma(p)-1)$ *is given the Hilbert space structure defined in (4.11).*

*Proof.* – The point (i) is a consequence of (4.25). Since $p$ is the geodesic middle of $q$ and $S_p q$, (ii) and (iii) are just the same. Then (ii) would be a consequence of

$$\gamma(p)\gamma(S_p q) = \gamma(q)\gamma(p), \tag{5.9}$$

or, making everything explicit,

$$\begin{aligned}\sigma(\theta p)\tilde{\sigma}(S_p q) &= \sigma(\theta q)\tilde{\sigma}(p)\\ \tilde{\sigma}(p)\sigma(\theta\, S_p q) &= \tilde{\sigma}(q)\sigma(\theta p).\end{aligned} \tag{5.10}$$

Now the last two equalities can both be derived from (3.6): for the first one, multiply (3.6) on the left by $\sigma(\theta p)$, on the right by $\tilde{\sigma}(p)$ and use (2.10); to get the second one, substitute $S_p q$ for $q$ in (3.6).

Denoting, for every $p \in \mathcal{M}^{\uparrow}$, as $(V \oplus W)_p$ the whole space $V \oplus W$ endowed with the norm $\|\ \|_p$ defined in (4.10), we show that $\Xi(s)$ is an isometry from $(V \oplus W)_{\Lambda(s)'q}$ onto $(V \oplus W)_q$ and that $\gamma(p)$ is an isometry from $(V \oplus W)_{S_p q}$ onto $(V \oplus W)_q$. To show that

$$\left\|\Xi(s)\begin{pmatrix} v \\ w \end{pmatrix}\right\|_q^2 = \left\|\begin{pmatrix} v \\ w \end{pmatrix}\right\|_{\Lambda(s)'q}^2 \tag{5.11}$$

amounts to showing that

$$\tilde{\sigma}(q)[sv, sv] = \tilde{\sigma}(\Lambda(s)'q)[v, v]$$

and

$$\sigma(\theta q)[s^{*-1}w, s^{*-1}w] = \sigma(\theta\Lambda(s)'q)[w, w],$$

a consequence of (2.21) for the first equation, of (2.17) and (2.19) for the second one. The last assertion follows in the same way from the equalities (5.10). □

We now prove the *semi-covariance* of the calculus $Op^{\uparrow}$ under the action of $\mathcal{P}_+^{\uparrow}$, a word that we shall explain soon. Using (4.26) together with (4.21), we first get

$$(\mathcal{G}\pi(s,a)^{-1}\,\Psi)(q) = e^{2i\pi<\Lambda(s)^{-1}a,q>}\,\Xi(s^{-1})\,(\mathcal{G}\Psi)(\Lambda(s)'^{-1}q), \qquad (5.12)$$

from which it follows, after a straightforward computation (starting from (5.8)) that

$$\begin{aligned}&(\mathcal{G}\pi(s,a)\widetilde{Op}^{\uparrow}(f)\pi(s,a)^{-1}\,\Psi)(q)\\ &= 8\int_{\Delta^{\uparrow}} \Xi(s)\;\gamma(\text{mid}\,(p,\Lambda(s)'q))\;f(x,p)\;\gamma(\text{mid}\,(p,S_p(\Lambda(s)'q))\;\Xi(s^{-1})\\ &e^{2i\pi<x+\Lambda(s)^{-1}a,S_p(\Lambda(s)'q)-\Lambda(s)'q>}(\mathcal{G}\Psi)(\Lambda(s)'^{-1}S_p(\Lambda(s)'q))\;dm(x,p).\end{aligned} \qquad (5.13)$$

Let us now perform the change of variables

$$x \mapsto \Lambda(s)^{-1}(x-a), \qquad p \mapsto \Lambda(s)'p, \qquad (5.14)$$

which, in the notation (5.2), transforms, say, the 6-dimensional surface $\mathbb{E}_0^{\uparrow}$ into $\mathbb{E}_{<a,p>}^{\uparrow}$: it preserves the measure (5.4) according to what has been said right after (5.6). Setting

$$g(x,p) = f(\Lambda(s)^{-1}(x-a),\Lambda(s)'p) \qquad (5.15)$$

and observing that

$$\text{mid}\,(\Lambda(s)'p,\Lambda(s)'q) = \Lambda(s)'\text{mid}\,(p,q) \qquad (5.16)$$

since the Riemannian structure on $\mathcal{M}^{\uparrow}$ is Lorentz-invariant, also that

$$S_{\Lambda(s)'p}\,(\Lambda(s)'q) = \Lambda(s)'\,S_pq, \qquad (5.17)$$

we get

$$\begin{aligned}&(\mathcal{G}\,\pi(s,a)\,\widetilde{Op}^{\uparrow}(f)\,\pi(s,a)^{-1}\,\Psi)(q)\\ &= 8\int_{\Delta^{\uparrow}} \Xi(s)\;\gamma(\Lambda(s)'\text{mid}\,(p,q))\;g(x,p)\;\gamma(\Lambda(s)'\text{mid}\,(p,S_pq))\;\Xi(s^{-1})\\ &e^{2i\pi<x,S_pq-q>}\;(\mathcal{G}\Psi)(S_pq)\;dm(x,p).\end{aligned} \qquad (5.18)$$

Using (4.25) to the effect that

$$\Xi(s)\,\gamma(\Lambda(s)'\mathrm{mid}\,(p,q)) = \gamma(\mathrm{mid}\,(p,q))\,\Xi(s), \tag{5.19}$$

we finally get the semi-covariance formula

$$\pi(s,a)\,\widetilde{Op}^{\uparrow}(f)\,\pi(s,a)^{-1} = \widetilde{Op}^{\uparrow}(f_{s,a}), \tag{5.20}$$

with

$$f_{s,a}(x,p) = \Xi(s)\,f(\Lambda(s)^{-1}(x-a), \Lambda(s)'p)\,\Xi(s)^{-1}. \tag{5.21}$$

One still has to show, of course, that $f_{s,a}$ is admissible, which follows from (2.19), also that $f_{s,a}(x,p)$ commutes with $\gamma(p)$, a fact that follows from two applications (one on the left, one on the right) of (4.25).

We consider (5.21) as a *semi*-covariance property only because $\Xi(s)$ is still there, on the right-hand side: now $\Xi(s)$ is a mathematical object related to bispinors, not a classical one. This slightly unsatisfactory behaviour will disappear in the final quantization rule $Op$, in which conjugation under $\pi(s,a)$ will transform symbols in a purely geometric way (i.e. meaningful in a classical, non quantum, sense). Meanwhile, still using $\widetilde{Op}$, we study how the discrete symmetries $C, P$ and $CPT$ act on the calculus: again, some "matrices", actually objects related to bispinors, will appear in the transformation rules, only to disappear with the use of $Op$ instead.

The formulas for the action on presymbols of discrete symmetries are the following:

$$C\,\widetilde{Op}^{\uparrow}(f)\,C = \widetilde{Op}^{\downarrow}(f_C) \quad \text{with} \quad f_C(x,p) = K\,f(x,-p)\,K, \tag{5.22}$$

$$P\,\widetilde{Op}^{\uparrow}(f)\,P = \widetilde{Op}^{\uparrow}(f_P) \quad \text{with} \quad f_P(x,p) = \gamma^0\,f(Jx, J'^{-1}p)\,\gamma^0 \tag{5.23}$$

and

$$\begin{aligned} &(CPT)\,\widetilde{Op}^{\uparrow}(f)\,(CPT) = \widetilde{Op}^{\downarrow}(f_{CPT}) \\ &\text{with} \quad f_{CPT}(x,p) = \gamma^5\,f(-x,-p)\,\gamma^5, \end{aligned} \tag{5.24}$$

in which, in assorted coordinates, $J(x_0, \mathbf{x}) = (x_0, -\mathbf{x})$, and where $K, \gamma^0$ and $\gamma^5$ have been defined in (4.29). The proofs are rather straightforward: one starts from (5.8) and from the expressions (4.31) for $\mathcal{G}C\Psi$, $\mathcal{G}P\Psi$ and $\mathcal{G}(CPT)\Psi$ respectively, using (4.32) to commute $\gamma(p)$ with $K, \gamma^0$ or $\gamma^5$.

LEMMA 5.3. – *Extend* $\sigma : \mathbb{M} \to \mathrm{Herm}(W) \subset \mathcal{L}(W,V)$ *and* $\tilde{\sigma} : \mathbb{M}' \to \mathrm{Herm}(V) \subset \mathcal{L}(V,W)$ *in a complex-linear way into maps still denoted as* $\sigma$

*and* $\tilde{\sigma}$ *from* $\mathbb{M} \otimes \mathbb{C}$ *into* $\mathcal{L}(W, V)$ *and* $\mathbb{M}' \otimes \mathbb{C}$ *into* $\mathcal{L}(V, W)$ *respectively. Then, for every* $A \in \mathbb{M} \otimes \mathbb{C}$ *and every* $p \in \mathcal{M}$, *the block-matrix*

$$\begin{pmatrix} 0 & \sigma(A) \\ \tilde{\sigma}(p)\sigma(A)\tilde{\sigma}(p) & 0 \end{pmatrix} \in \operatorname{End}(V \oplus W)$$

*commutes with* $\gamma(p)$, *thus defines an endomorphism* $\Theta_p(A)$ *of the two-dimensional complex vector space* $\operatorname{Ker}(\gamma(p) - 1)$. *The map* $\Theta_p : \mathbb{M} \otimes \mathbb{C} \to \operatorname{End}(\operatorname{Ker}(\gamma(p) - 1))$ *so defined is a linear isomorphism, and the image of* $\mathbb{M}$ *under* $\Theta_p$ *coincides with the set of endomorphisms of* $\operatorname{Ker}(\gamma(p) - 1)$ *which are self-adjoint for the Hilbert space structure on* $\operatorname{Ker}(\gamma(p) - 1)$ *defined by the norm* $\| \ \|_{p\,\mathrm{sign}(p_0)}$ *introduced in (4.10). In the same way, the map*

$$A^\dagger \in \mathbb{M}' \otimes \mathbb{C} \longmapsto \begin{pmatrix} 0 & \sigma(\theta p)\tilde{\sigma}(A^\dagger)\sigma(\theta p) \\ \tilde{\sigma}(A^\dagger) & 0 \end{pmatrix} \in \operatorname{End}(V \oplus W)$$

*identifies* $\mathbb{M}' \otimes \mathbb{C}$ *with* $\operatorname{End}(\operatorname{Ker}(\gamma(p) - 1))$, *and the image of* $\mathbb{M}'$ *under this map is the same as the one before.*

*Proof.* – That the block-matrix associated with $A \in \mathbb{M} \otimes \mathbb{C}$ commutes with $\gamma(p)$ is an immediate consequence of (2.10). Since a pair $\binom{v}{w} \in V \oplus W$ lies in $\operatorname{Ker}(\gamma(p) - 1)$ if and only if $v = \sigma(\theta p)w$, one sees that $\Theta_p(A)$ is zero if and only if $\sigma(A)w = 0$ for every $w \in W$, so that $\Theta_p$ is one-to-one. We now show that, if $A \in \mathbb{M}$, one has

$$((\Theta_p(A)\xi_1\,,\,\xi_2)) = ((\xi_1\,,\,\Theta_p(A)\xi_2)) \tag{5.25}$$

for every pair $\xi_1 = \binom{v_1}{w_1}$, $\xi_2 = \binom{v_2}{w_2}$ of points of $V \oplus W$: this will imply that $\Theta_p(A)$ is self-adjoint in view of (4.15) and (4.16). If one uses (4.14), the verification amounts to checking that

$$\begin{aligned}(\sigma(A)w_1, w_2) &+ \overline{(v_2, \tilde{\sigma}(p)\sigma(A)\tilde{\sigma}(p)v_1)} \\ &= (v_1, \tilde{\sigma}(p)\sigma(A)\tilde{\sigma}(p)v_2) + \overline{(\sigma(A)w_2, w_1)},\end{aligned} \tag{5.26}$$

which is a consequence of (2.1) since $\sigma(A)$ is hermitian in this case. The part of the lemma relative to $\mathbb{M}'$ instead of $\mathbb{M}$ can be proved in the same way as the first part, or derived from it. □

DEFINITION 5.4. – *Given* $\updownarrow = \uparrow$ *or* $\downarrow$ *and* $\varepsilon = \pm 1$, *so that* $\Omega_\varepsilon^\updownarrow$ *is one of the four components of the set of observers* $\Omega$ *occurring in the decomposition (3.2), a symbol on* $\Omega_\varepsilon^\updownarrow$ *shall be a function on* $\mathbb{M} \times \mathcal{M}^\updownarrow$ *valued in* $\mathbb{M} \otimes \mathbb{C}$ *(resp.* $\mathbb{M}' \otimes \mathbb{C}$*) when* $\varepsilon = 1$ *(resp.* $-1$*), and admissible in the sense that it should take the same value at* $(x, p)$ *and* $(y, p)$ *if* $x - y$ *is a multiple of* $\theta p$ *(cf. (3.7)).*

*We shall usually denote a symbol as $A = A(x,p)$ in the case when $\varepsilon = 1$, and as $A^\dagger = A^\dagger(x,p)$ in the case when $\varepsilon = -1$. The operators $Op_+^{\uparrow\downarrow}(A)$ (resp. $Op_-^{\uparrow\downarrow}(A^\dagger)$) associated with the symbol $A$ (or $A^\dagger$) shall then be defined as follows:*

$$Op_+^{\uparrow\downarrow}(A) = \widetilde{Op}^{\uparrow\downarrow}(f) \quad \text{with } f(x,p) = \begin{pmatrix} 0 & \sigma(A(x,p)) \\ \tilde{\sigma}(p)\sigma(A(x,p))\tilde{\sigma}(p) & 0 \end{pmatrix}$$

*and*

$$Op_-^{\uparrow\downarrow}(A^\dagger) = \widetilde{Op}^{\uparrow\downarrow}(f)$$

*with*

$$f(x,p) = \begin{pmatrix} 0 & \sigma(\theta p)\tilde{\sigma}(A^\dagger(x,p))\sigma(\theta p) \\ \tilde{\sigma}(A^\dagger(x,p)) & 0 \end{pmatrix}.$$

Under the symmetries of the extended Poincaré group, the symbolic calculus just defined is covariant in a way which can be described by the following set of formulas:

THEOREM 5.5. – *Under the action of (the two-fold covering of) the group $\mathcal{P}_+^\uparrow$, $\mathbb{M}$-valued symbols transform like vectors, and $\mathbb{M}'$-valued symbols transform like covectors. More precisely*

$$\begin{aligned}&\pi(s,a)\, Op_+^{\uparrow\downarrow}(A)\, \pi(s,a)^{-1} \\ &\quad = Op_+^{\uparrow\downarrow}\left((x,p) \mapsto \Lambda(s)\, A(\Lambda(s)^{-1}(x-a), \Lambda(s)'p)\right)\end{aligned}$$

*and*

$$\begin{aligned}&\pi(s,a)\, Op_-^{\uparrow\downarrow}(A^\dagger)\, \pi(s,a)^{-1} \\ &\quad = Op_-^{\uparrow\downarrow}\left((x,p) \mapsto \Lambda(s)'^{-1}\, A^\dagger(\Lambda(s)^{-1}(x-a), \Lambda(s)'p)\right).\end{aligned}$$

*The action of discrete symmetries is given by the formulas*

$$\begin{aligned} C\, Op_+^\uparrow(A)\, C &= Op_-^\downarrow(A^\dagger) \quad \text{with } A^\dagger(x,p) = -\theta^{-1}A(x,-p), \\ P\, Op_+^\uparrow(A)\, P &= Op_-^\uparrow(A^\dagger) \quad \text{with } A^\dagger(x,p) = \theta^{-1}JA(Jx, J'^{-1}p) \end{aligned}$$

*and*

$$(CPT)\, Op_+^\uparrow(A)\, (CPT) = Op_+^\downarrow((x,p) \mapsto -A(-x,-p)),$$

*and the same set of formulas remains valid after all symbols $\uparrow$ and $\downarrow$ have been switched with one another; recall that $J(x^0, \mathbf{x}) = (x^0, -\mathbf{x})$.*

*Proof.* – To prove the first two formulas, starting from (5.20) and definition 5.4, it suffices to check that

$$\begin{aligned}\Xi(s)&\begin{pmatrix}0 & \sigma(A)\\ \tilde{\sigma}(\Lambda(s)'p)\sigma(A)\tilde{\sigma}(\Lambda(s)'p) & 0\end{pmatrix}\Xi(s^{-1})\\ &=\begin{pmatrix}0 & \sigma(\Lambda(s)A)\\ \tilde{\sigma}(p)\sigma(\Lambda(s)A)\tilde{\sigma}(p) & 0\end{pmatrix}\end{aligned}$$

and that

$$\begin{aligned}\Xi(s)&\begin{pmatrix}0 & \sigma(\theta\Lambda(s)'p)\tilde{\sigma}(A^\dagger)\sigma(\theta\Lambda(s)'p)\\ \tilde{\sigma}(A^\dagger) & 0\end{pmatrix}\Xi(s^{-1})\\ &=\begin{pmatrix}0 & \sigma(\theta p)\tilde{\sigma}(\Lambda(s)'^{-1}A^\dagger)\sigma(\theta p)\\ \tilde{\sigma}(\Lambda(s)'^{-1}A^\dagger) & 0\end{pmatrix},\end{aligned}$$

a straightforward task if one uses (2.18) and (2.20) to get the first one, using also (2.19) to get the second one. The action of discrete symmetries is studied in the same way, substituting $K, \gamma^0$ or $\gamma^5$ for $\Xi(s)$. Using (2.13) in the first case, (3.6) together with $J'^{-1}\theta^{-1} = (\theta J')^{-1} = (J^{-1}\theta)^{-1} = \theta^{-1}J$ in the second, we are done; the last case is even simpler. □

REMARK 5.6. – The isomorphism $\theta^{-1} : \mathbb{M} \to \mathbb{M}'$ defined in (2.7) is intrinsic. On the other hand, for every $p \in \mathcal{M}$, there is a euclidean norm $|\ |_p$ on $\mathbb{M}$, depending only on $\pm p$, defined by

$$|x|_p^2 = Q(x_{T_p}) - Q(x_{T_p^\perp}), \tag{5.27}$$

where $x_{T_p}$ and $x_{T_p^\perp}$ are the two projections of $x$ on the subspace $T_p$ of $\mathbb{M}$ generated by $\theta p$ (*cf.* (3.7)) and on its Minkowski-orthogonal $T_p^\perp$ (in accordance with (3.1), we might have denoted this latter space as $S_p$, but this notation has been preempted by that (*cf.* (3.5)) for geodesic symmetries on the mass-shell). This $p$-dependent euclidean structure on $\mathbb{M}$ gives rise to another isomorphism $\iota_p : \mathbb{M} \to \mathbb{M}'$ which, as a consequence of (3.5) and (2.10), can be defined by the formula $\iota_p = S_p\,\theta^{-1}$; or, using (3.6), one can also write, for every $y \in \mathbb{M}$,

$$\tilde{\sigma}(\iota_p y) = \tilde{\sigma}(p)\sigma(y)\tilde{\sigma}(p). \tag{5.28}$$

Extending $\iota_p$ in a $\mathbb{C}$-linear way as a map from $\mathbb{M}\otimes\mathbb{C}$ to $\mathbb{M}'\otimes\mathbb{C}$, one can then write, following Definition 5.4,

$$Op_+^{\uparrow\downarrow}(A) = Op_-^{\uparrow\downarrow}(A^\dagger) \qquad \text{if} \quad A^\dagger(x,p) = \iota_p\, A(x,p) \tag{5.29}$$

for all $(x,p) \in \mathbb{M} \times \mathcal{M}$.

Thus, if it were only for the sake of getting a *complete* calculus, we could dispense with the $Op_-^{\uparrow\downarrow}$-part in Definition 5.4. However, one has to introduce the 4-component version of the calculus to get the simplest possible covariance formulas (*cf.* Theorem 5.5) for the conjugation under $C$ and $P$. These are just what one should expect, namely $C$ reverses orientation and causality, $P$ orientation only, $CPT$ causality only.

REMARK 5.7. – Let us say that a symbol $A$ is purely scalar if $A(x,p) = \phi(x,p)\,\theta p$ for some scalar function $\phi(x,p)$ (i.e. if $A(x,p)$ is carried by the one-dimensional space $T_p$ with the notation in Remark 5.6). In this case, one has for instance

$$Op_+^{\uparrow}(A) = \widetilde{Op}^{\uparrow}(f) \qquad \text{with} \qquad f(x,p) = \phi(x,p)\gamma(p) \tag{5.30}$$

so that, using Definition 5.1, $Op_+^{\uparrow}(A)$ acts on $\mathcal{H}^{\uparrow}$ as expressed by the formula

$$Op_+^{\uparrow}(A) = 8\int_{\Delta^{\uparrow}} \phi(x,p)\; P_{(x,p)}\; dm(x,p), \tag{5.31}$$

with

$$\begin{aligned}(\mathcal{G}\,P_{(x,p)}\Psi)(q) \\ = \gamma(\mathrm{mid}\,(p,q))\,\gamma(p)\,\gamma(\mathrm{mid}\,(p,S_pq))\,(\mathcal{G}\,\Psi)(S_pq)\,e^{2i\pi<x,S_pq-q>}.\end{aligned} \tag{5.32}$$

Now, given $p$ and $q \in \mathcal{M}^{\uparrow}$, it follows from (3.6) and (2.10) (expanding the $\gamma$-operators according to (4.1)) that

$$\gamma(q)\gamma(p)\gamma(S_pq) = \gamma(p). \tag{5.33}$$

Since $S_p(\mathrm{mid}\,(p,q)) = \mathrm{mid}\,(p,S_pq)$ (only think of the geodesic line through $p$ and $q$), one also has

$$\gamma(\mathrm{mid}\,(p,q))\gamma(p)\gamma(\mathrm{mid}\,(p,S_pq)) = \gamma(p), \tag{5.34}$$

thus

$$(\mathcal{G}\,P_{(x,p)}\Psi)(q) = \gamma(p)\,(\mathcal{G}\,\Psi)(S_pq)\,e^{2i\pi<x,S_pq-q>}. \tag{5.35}$$

In particular, $P_{(0,(e^*)^0)}$ reduces to $P$ according to (4.35). In view of the covariance of the calculus, one may consider $P_{(x,p)}$ as the parity operator associated with the equivalence class of observers containing the point $(x,p)$. This gives the true meaning of formula (5.31).

Let us recall at this point that the (square-root) Klein-Gordon analogue of (5.31) was used in [13] as a *definition* of the Klein-Gordon calculus. In the

Dirac case, the present discussion shows that such a definition would yield only the operators whose symbols, in the Dirac calculus, are purely scalar.

REMARK 5.8. – The standard Schrödinger equation of non-relativistic quantum mechanics for a particle of mass 1 is

$$4i\pi\,\frac{\partial\Psi}{\partial t}=(-\Delta+V)\,\Psi, \tag{5.36}$$

where the potential $V$, which accounts for some external field, is a function of $\mathbf{x}$ alone: but general symbols are functions of $(\mathbf{x},p)$ (*cf.* [13], section 3, for an interpretation of the Weyl calculus in connection to the free non-relativistic Schrödinger equation). In just the same way, the electromagnetic vector potential $A(x)=(A^\mu(x))$ yields (through the minimal coupling) the perturbation term which one classically adds to the free Dirac equation to account for some external field: but our present symbols are functions $A=A(x,p)$, which could be identified with their time-zero restrictions in view of the admissibility condition. We hope to come back at some other time to the very difficult problem of coupling the quantization problem with the presence of external fields.

## 6. A BASIC ESTIMATE

All definitions in the last section were formal, waiting for a proof of the convergence of the integrals involved. It will be possible to develop the Dirac calculus as a full pseudodifferential analysis, as was done in [13] for the Klein-Gordon calculus: however, all this, in the Klein-Gordon case, required considerable work (actually, the bulk of the monograph just quoted was devoted to just that), and we shall only give hints in this direction at the end of the present section. We shall be satisfied, in the present paper, with proving the following minimal fact: that symbols (i.e. classical observables) which are square-integrable functions in some natural sense, yield Hilbert-Schmidt operators in the $Op$-calculus.

LEMMA 6.1. – *Given $p\in\mathcal{M}^\uparrow$, recall the euclidean norm $|\ |_p$ on $\mathbb{M}$ defined in (5.27) and extend it as the Hilbert space norm $|\ |_p$ on $\mathbb{M}\otimes\mathbb{C}$ for which $\mathbb{M}$ and $i\mathbb{M}$ are orthogonal. Then:*

(i) *for each $x\in\mathbb{M}\otimes\mathbb{C}$, $p\in\mathcal{M}^\uparrow$, $s\in SL(V)$, one has $|x|_{\Lambda(s)'p}=|\Lambda(s)x|_p$;*

(ii) *for each* $x \in \mathbb{M} \otimes \mathbb{C}$, *the Hilbert-Schmidt norm of the operator* $\begin{pmatrix} 0 & \sigma(x) \\ \tilde{\sigma}(p)\sigma(x)\tilde{\sigma}(p) & 0 \end{pmatrix}$ *as an endomorphism of the space* $\mathrm{Ker}(\gamma(p)-1)$ *(cf.* Lemma 5.2*) is* $2^{\frac{1}{2}}|x|_p$.

*Proof.* – For the first point, one may assume that $x$ lies in $\mathbb{M}$. By (2.19), $\Lambda(s)$ sends $\theta\Lambda(s)'p$ to $\theta p$, thus the space $T_{\Lambda(s)'p}$ to $T_p$: since, also, $\Lambda(s)$ is a Lorentz transformation, the first point is proven. Then, in assorted coordinates, set $p = \Lambda(s)'(e^*)^0$ with $s \in SL(V)$. As a consequence of Lemma 5.2, the Hilbert-Schmidt norm we wish to compute is the same as that of the operator

$$\begin{aligned}
&\Xi(s)\begin{pmatrix} 0 & \sigma(x) \\ \tilde{\sigma}(p)\sigma(x)\tilde{\sigma}(p) & 0 \end{pmatrix}\Xi(s^{-1}) \\
&= \begin{pmatrix} 0 & s\sigma(x)s^* \\ s^{*-1}\tilde{\sigma}(p)\sigma(x)\tilde{\sigma}(p)s^{-1} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \sigma(\Lambda(s)x) \\ \tilde{\sigma}((e^*)^0)\sigma(\Lambda(s)x)\tilde{\sigma}((e^*)^0) & 0 \end{pmatrix}
\end{aligned}$$

as en endomorphism of the space $\mathrm{Ker}(\gamma^0 - 1)$ (recall that $\gamma^0 = \gamma((e^*)^0)$). Now (in assorted coordinates, identifying bispinors with vectors in $\mathbb{C}^4$), an orthonormal basis of the kernel of $\gamma^0 - 1 = \begin{pmatrix} -I & I \\ I & -I \end{pmatrix}$ consists of the two vectors

$$\begin{pmatrix} v_1 \\ w_1 \end{pmatrix} = 2^{-\frac{1}{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v_2 \\ w_2 \end{pmatrix} = 2^{-\frac{1}{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \tag{6.1}$$

so that, making $\sigma(x)$ explicit as in (2.2), one sees that the square of the Hilbert-Schmidt norm of the *matrix* $\begin{pmatrix} 0 & \sigma(x) \\ \sigma(x) & 0 \end{pmatrix}$ is the sum of the squared norms of the vectors

$$2^{-\frac{1}{2}}\begin{pmatrix} x^0 + x^3 \\ x^1 + ix^2 \\ x^0 + x^3 \\ x^1 + ix^2 \end{pmatrix} \quad \text{and} \quad 2^{-\frac{1}{2}}\begin{pmatrix} x^1 - ix^2 \\ x^0 - x^3 \\ x^1 - ix^2 \\ x^0 - x^3 \end{pmatrix},$$

i.e. $2|x|^2$, where $|x| = |x|_{(e^*)^0}$ stands for the canonical norm of $x \in \mathbb{C}^4$. □

So far as estimates are concerned, it suffices to deal with the $Op_+^\uparrow$-part of the calculus, as defined in Definition 5.4. Let us introduce the (intrinsically defined) d'Alembert operator $\Box$, acting on functions on $\mathbb{M}$ or on $\mathbb{M} \times \mathcal{M}^{\uparrow}$, defined in assorted coordinates as

$$\Box = \left(\frac{\partial}{\partial x^0}\right)^2 - \left(\frac{\partial}{\partial x^1}\right)^2 - \left(\frac{\partial}{\partial x^2}\right)^2 - \left(\frac{\partial}{\partial x^3}\right)^2. \tag{6.2}$$

On (say, scalar) admissible symbols $f = f(x,p)$, $\Box$ acts as a positive elliptic operator since, under the admissibility condition, one may write in assorted coordinates

$$\Box f = p_0^{-2} \sum p_j p_k \frac{\partial^2 f}{\partial x^j \partial x^k} - \sum \frac{\partial^2 f}{\partial x^j \partial x^j} \tag{6.3}$$

and observe that $p_0^{-2} < \mathbf{p}, \boldsymbol{\xi} >^2 - |\boldsymbol{\xi}|^2 \leq -p_0^{-2} |\boldsymbol{\xi}|^2$, an inequality in which $\boldsymbol{\xi}$ stands as a dual Fourier variable of $\mathbf{x}$. One should therefore not find it surprising that one can define, for every real number $\lambda$, the operator

$$\nabla^\lambda = \left(1 + \frac{\Box}{16\,\pi^2}\right)^{\lambda/2}. \tag{6.4}$$

As done in [13], p. 33, one can use the Fourier transform $\mathcal{F}_1$, which, when acting on functions of $(x,p)$, stands for the Fourier transformation with respect to the $x$-variables, setting (with $\xi = (\xi_0, \boldsymbol{\xi})$)

$$\mathcal{F}_1(\nabla^\lambda f)(\xi, p) = \left(1 + \frac{1}{4}(|\boldsymbol{\xi}|^2 - \xi_0^2)\right)^{\lambda/2} (\mathcal{F}_1 f)(\xi, p). \tag{6.5}$$

Now it is useful, in the computation of (partial) Fourier transforms of admissible symbols, to represent an admissible (scalar, as yet) function $f$ as a genuine function of 6 variables, setting

$$f_{\mathbb{E}_0^\uparrow}(\mathbf{x}, p) = f(-p_0^{-1} < \mathbf{x}, \mathbf{p} >, \mathbf{x}\,; p), \tag{6.6}$$

which is nothing else than the restriction of the function $f$ to the space $\mathbb{E}_0^\uparrow$ (*cf.* (5.2)), when expressed in terms of the coordinates $\mathbf{x}$. The function $f_{\mathbb{E}_0^\uparrow}$, which (in assorted coordinates) lives on $\mathbb{R}^3 \times \mathcal{M}^\uparrow$, has an $\mathbf{x}$-Fourier transform, still denoted as $\mathcal{F}_1 f_{\mathbb{E}_0^\uparrow}$, and the operator $\nabla^\lambda$ can then be characterized by the identity (*cf.* [13], p. 35)

$$(\mathcal{F}_1 \nabla^\lambda f_{\mathbb{E}_0^\uparrow})(\boldsymbol{\eta}, p) = \left(1 + \frac{1}{4}(|\boldsymbol{\eta}|^2 + < \boldsymbol{\eta}, \mathbf{p} >^2)\right)^{\lambda/2} (\mathcal{F}_1 f_{\mathbb{E}_0^\uparrow})(\boldsymbol{\eta}, p). \tag{6.7}$$

In particular, for $\lambda \leq 0$, the operator $\nabla^\lambda$ acts as a bounded operator, with norm $\leq 1$, in the space of scalar (admissible) symbols characterized by the condition

$$\int_{\Delta^\uparrow} |f(x,p)|^2 \, dm(x,p) \ < \ \infty. \tag{6.8}$$

The definition of the operator $\nabla^\lambda$ extends to vector-valued admissible symbols like $A = A(x,p)$, letting the operator act componentwise.

THEOREM 6.2. – *Let $A$ be an $\mathbb{M}$-valued admissible symbol, defined on $\mathbb{M} \times \mathcal{M}^\uparrow$, satisfying the condition that*

$$\int_{\Delta^\uparrow} |A(x,p)|_p^2 \, dm(x,p) \ < \ \infty.$$

*Then, under* Definition 5.4, *$Op_+^\uparrow(A)$ is well-defined as a Hilbert-Schmidt endomorphism of the space $\mathcal{H}^\uparrow$. Moreover, the square of the Hilbert-Schmidt norm of $Op_+^\uparrow(A)$ is given by the formula*

$$\|Op_+^\uparrow(A)\|_{\text{H.S.}}^2 = 2 \int_{\Delta^\uparrow} |(\nabla^{-3/2} A)(x,p)|_p^2 \, dm(x,p). \tag{6.9}$$

*Proof.* – Set $\mathcal{A} = \mathcal{G}\, Op_+^\uparrow(A)\, \mathcal{G}^{-1}$, an operator considered as a would-be endomorphism of $\mathcal{G}\mathcal{H}^\uparrow$, a Hilbert space which was also denoted as $L^2(\mathcal{M}^\uparrow, \text{Ker}(\gamma - 1))$ right after (4.13). One has to give a meaning to the defining formula

$$(\mathcal{A}\Phi)(q) = 8 \int_{\mathbb{E}_0^\uparrow} \gamma(\text{mid}\,(p,q)) \begin{pmatrix} 0 & \sigma(A(x,p)) \\ \tilde{\sigma}(p)\sigma(A(x,p))\tilde{\sigma}(p) & 0 \end{pmatrix}$$
$$\gamma(\text{mid}\,(p, S_p q))\, \Phi(S_p q)\, e^{2i\pi < x, S_p q - q >}\, p_0^{-2}\, d\mathbf{x}\, d\mathbf{p}. \tag{6.10}$$

Defining $A_{\mathbb{E}_0^\uparrow}$, componentwise, as in (6.6), and noting that

$$< x,\, q - S_p q >= -\frac{2\, q_0}{p_0} < \mathbf{x}, \mathbf{p} > +2 < \mathbf{x}, \mathbf{q} > \tag{6.11}$$

when $< x, p >= 0$, one gets

$$(\mathcal{A}\Phi)(q) = 8 \int_{\mathcal{M}^\uparrow} \gamma(\text{mid}\,(p,q)) \begin{pmatrix} 0 & \sigma(B(q,p)) \\ \tilde{\sigma}(p)\sigma(B(q,p))\tilde{\sigma}(p) & 0 \end{pmatrix}$$
$$\gamma(\text{mid}\,(p, S_p q))\, \Phi(S_p q)\, p_0^{-2}\, d\mathbf{p} \tag{6.12}$$

with

$$B(q,p) := \int A(x,p)\, e^{2i\pi <x, S_p q - q>}\, d\mathbf{x} = (\mathcal{F}_1 A_{\mathbb{E}_0^\uparrow}) \left( 2\left(\mathbf{q} - \frac{q_0}{p_0}\mathbf{p}\right), p \right). \tag{6.13}$$

We eliminate the variable $p$ to the benefit of the variable $r = S_p q$, so that, as shown in ([13], (7.6) and (7.7)), one has

$$\frac{d\mathbf{r}}{d\mathbf{p}} = 8(< \theta p, q >)^2\, \frac{r_0}{p_0} \ : \tag{6.14}$$

it follows that the integral kernel $k(q,r)$ of the operator $\mathcal{A}$ with respect to the measure $r_0^{-1} d\mathbf{r}$ is given by

$$k(q,r) = p_0^{-1} (< \theta p, q >)^{-2}\, \gamma(\mathrm{mid}\,(q,p)) \begin{pmatrix} 0 & \sigma(B(q,p)) \\ \tilde{\sigma}(p)\sigma(B(q,p))\tilde{\sigma}(p) & 0 \end{pmatrix} \gamma(\mathrm{mid}\,(p,r)), \tag{6.15}$$

where $p = \mathrm{mid}\,(q,r)$. Now, the square of the Hilbert-Schmidt norm of $\mathcal{A}$ as an endomorphism of $L^2(\mathcal{M}^\uparrow, \mathrm{Ker}(\gamma - 1))$ is given by

$$\|\mathcal{A}\|^2_{\mathrm{H.S.}} = \int \|k(q,r)\|^2_{q,r}\, \frac{d\mathbf{r}}{r_0}\, \frac{d\mathbf{q}}{q_0}, \tag{6.16}$$

where $\|k(q,r)\|_{q,r}$ stands for the Hilbert-Schmidt norm of $k(q,r)$ regarded as a map from $\mathrm{Ker}(\gamma(r) - 1)$ to $\mathrm{Ker}(\gamma(p) - 1)$. As a consequence of Lemma 5.2(iii) and Lemma 6.1, we thus get

$$\|Op_+^\uparrow(A)\|^2_{\mathrm{H.S.}} = 2 \int_{\mathcal{M}^\uparrow \times \mathcal{M}^\uparrow} |B(q,p)|^2_p\ p_0^{-2}\, (< \theta p, q >)^{-4}\, \frac{d\mathbf{q}}{q_0}\, \frac{d\mathbf{r}}{r_0} \tag{6.17}$$

with $p = \mathrm{mid}\,(q,r)$ and $B(q,p)$ as defined in (6.12). The rest of the proof goes exactly like the corresponding fact in the Klein-Gordon calculus ([13], p. 35-36). One reverts to the variables $q$ and $p$ rather than $q$ and $r$, thus doing the same change of variables as above, in the reverse direction; then one lets the variable $q$ down to the advantage of the new variable $\boldsymbol{\eta} = 2(\mathbf{q} - \frac{q_0}{p_0}\mathbf{p})$, so that

$$\frac{d\boldsymbol{\eta}}{d\mathbf{q}} = 8(p_0 q_0)^{-1} < \theta p, q >, \tag{6.18}$$

after which one gets the formula

$$\|Op_+^{\uparrow}(A)\|^2_{\mathrm{H.S.}}$$
$$=2\int_{\mathcal{M}^{\uparrow}\times\mathbb{R}^3} |(\mathcal{F}_1 A_{\mathbb{E}_0^{\uparrow}})(\boldsymbol{\eta},p)|_p^2 \left(1+\frac{1}{4}(|\boldsymbol{\eta}|^2+<\mathbf{p},\boldsymbol{\eta}>^2)\right)^{-3/2} p_0^{-2}\, d\mathbf{p}\, d\boldsymbol{\eta}, \tag{6.19}$$

which concludes the proof of theorem 6.2. □

PROPOSITION 6.3. – *Let $A$ be an $\mathbb{M}$-valued admissible symbol, defined on $\mathbb{M}\times\mathcal{M}^{\uparrow}$, satisfying the square-integrability condition of* Theorem 6.2. *Then the adjoint of the operator $Op_+^{\uparrow}(A)$ as a continuous endomorphism of $\mathcal{H}^{\uparrow}$ is $Op_+^{\uparrow}(\bar{A})$. In other words, real classical observables yield hermitian operators under the Dirac quantization rule.*

*Proof.* – Starting from the definition of $Op_+^{\uparrow}(A)$ and from the polarized form of the expression (4.17) for the squared norm in $\mathcal{H}^{\uparrow}$, we get, for $\Psi_1$ and $\Psi_2$ in $\mathcal{H}^{\uparrow}$, the following formula:

$$(Op_+^{\uparrow}(A)^*\Psi_1,\Psi_2)=(\Psi_1,Op_+^{\uparrow}(A)\Psi_2)$$
$$=8\int_{\mathcal{M}^{\uparrow}} q_0^{-1}\, d\mathbf{q}\int_{\Delta^{\uparrow}} e^{2i\pi<x,S_pq-q>}\, dm(x,p)$$
$$\left(\!\!\left( (\mathcal{G}\Psi_1)(p)\,,\ \gamma(\mathrm{mid}\,(q,p))\begin{pmatrix} 0 & \sigma(A(x,p))\\ \tilde{\sigma}(p)\sigma(A(x,p))\tilde{\sigma}(p) & 0\end{pmatrix}\right.\right.$$
$$\left.\left.\gamma(\mathrm{mid}\,(p,S_pq))\,(\mathcal{G}\Psi_2)(p)\right)\!\!\right). \tag{6.20}$$

Now, observing that the exponential changes to its complex-conjugate under the change of variable $p\mapsto S_pq$, the proof boils down to showing that, given $\xi_1=\binom{v_1}{w_1}$ and $\xi_2=\binom{v_2}{w_2}$, one has

$$((\xi_1,\gamma(r)\xi_2))=((\gamma(r)\xi_1,\xi_2)) \tag{6.21}$$

for every $r\in\mathcal{M}^{\uparrow}$ and

$$\left(\!\!\left(\xi_1,\begin{pmatrix} 0 & \sigma(A)\\ \tilde{\sigma}(p)\sigma(A)\tilde{\sigma}(p) & 0\end{pmatrix}\xi_2\right)\!\!\right)$$
$$=\left(\!\!\left(\begin{pmatrix} 0 & \sigma(\bar{A})\\ \tilde{\sigma}(p)\sigma(\bar{A})\tilde{\sigma}(p) & 0\end{pmatrix}\xi_1,\xi_2\right)\!\!\right) \tag{6.22}$$

for every $A\in\mathbb{M}\otimes\mathbb{C}$. The second one is just (5.25) and, expanding (6.21) by means of the definition (4.14) of the hermitian form $((\ ,\ ))$, one sees that it follows from the hermitian property of $\tilde{\sigma}(r)$ or $\sigma(\theta r)$, as expressed by (2.1). □

REMARK 6.4. – Besides the *Zitterbewegung*, one of the other reasons why we have been satisfied with a calculus of operators all of which preserve the decomposition $\mathcal{H} = \mathcal{H}^{\uparrow} \oplus \mathcal{H}^{\downarrow}$ is that it would hardly be possible to interpret as a *hermitian* observable any operator that would violate this condition. It is our belief that, on *flat* Minkowski space, this was the correct choice. However, we do have in mind, as part of some future work, a mathematically non-trivial extension permitting the consideration of other operators too.

REMARK 6.5. – Here are a few hints towards the way the Dirac calculus could be developed as a pseudodifferential analysis: however, we shall give no proofs, as this would lengthen this paper in a considerable way, even though the following estimates are not the best one could get. One can define *classes of symbols*, meaning by this spaces of smooth symbols characterized by the validity of estimates bearing on their derivatives. In the $Op_+^{\uparrow}$-calculus, first decompose a symbol $A$ as

$$A = A_{el} + A_{mag}, \tag{6.23}$$

where the two terms in the decomposition of $A(x,p)$ are its electric and magnetic parts as viewed by any observer associated with the momentum $p$ (*cf.* (3.7)): in other words $A_{el}$ is purely scalar in the sense of Remark 5.7, and $< A(x,p), p >= 0$: thus $A_{el}(x,p) = \phi(x,p)\,\theta p$ with $\phi(x,p) =< A(x,p), p >$. Then define *symbols of weight 1* to be those symbols $A$ which satisfy the following property: any of the scalar functions $\phi$ and $A^{\mu}_{mag}$, when restricted to $\mathbb{R}^3 \times \mathcal{M}^{\uparrow}$ and expressed in the $(\mathbf{x}, \mathbf{p})$-coordinates, is smooth, bounded, and remains so after one has applied to it any differential operator in the algebra generated by the operators $\frac{\partial}{\partial x^j}, \sum p_j \frac{\partial}{\partial x^j}, \frac{\partial}{\partial p_j}, \sum p_j \frac{\partial}{\partial p_j}$. Then the definition of the quantizing map $Op_+^{\uparrow}$ can be uniquely extended in a (suitably defined) continuous way so that operators $Op_+^{\uparrow}$ whose symbols are symbols of weight 1 make sense as bounded operators in the Hilbert space $\mathcal{H}^{\uparrow}$; they constitute an algebra.

One can also, of course, define more general weights $m$ (exactly as in [13], def. 8.10) and associate with these classes of symbols of weight $m$ (as in [13], def. 8.13).

## 7. SYMBOLS OF THE GENERATORS OF THE POINCARÉ REPRESENTATION

In this section, we compute the symbols of the infinitesimal generators of the representation $\pi$ of $SL_{\text{in}}$ in $\mathcal{H}^{\uparrow}$. These will be polynomials in $x$, of degree 0 or 1, with coefficients depending on $p$, thus will not satisfy the square-integrability condition which would make an application of Theorem 6.2 possible. However, it is a lengthy if feasible thing to develop the Dirac pseudodifferential analysis to a point where slowly increasing smooth symbols can be considered (*cf.* the Klein-Gordon analysis): this is why we shall not make too much fuss, in the present section, about extending the analysis up to a point lacking complete justification (this will be repaired at some latter opportunity).

The Lie algebra $\mathfrak{g}$ of $SL_{\mathrm{in}}$ is the same as that of the Poincaré group $\mathcal{P}_+^\uparrow$, and its elements $X$ are best described through the action defined in (4.18) of $e^{tX}$ on $\mathbb{M}$. We get (in assorted coordinates) the four following types, corresponding to time-translation, space-translation in the direction $\frac{\partial}{\partial x^j}$, boost in the same direction, finally rotation in the plane generated by $\{\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\}$, with $j < k$:

Type 1: $e^{tX}.x = (x^0 + t, \mathbf{x})$,

Type 2: $e^{tX}.x = (x^0, \ldots x^j + t, \ldots)$,

Type 3: $e^{tX}.x = (x^0 \cosh t + x^j \sinh t, \ldots, x^0 \sinh t + x^j \cosh t, \ldots)$,

Type 4: $e^{tX}.x = (x^0, \ldots, x^j \cos t - x^k \sin t, \ldots, x^j \sin t + x^k \cos t, \ldots)$.

Given $X \in \mathfrak{g}$, interpreted this time as an element in the Lie algebra of $SL_{\mathrm{in}}$, we set

$$d\pi(X) = -\frac{1}{2i\pi} \frac{d}{dt}\bigg|_{t=0} \pi(e^{tX}), \tag{7.1}$$

thus normalizing the definition of the infinitesimal representation $d\pi$ so as to get self-adjoint operators. Let us call $D_0, D_j, B_j$ and $R_{jk}$ the operators $d\pi(X)$ associated with the four types of $X$'s defined above. Here, of course, the "$B$" (resp."$R$") stands for *boost*, resp. *rotation*; also, even though $R_{jk}$ makes sense, in connection with the $X$ of the fourth type defined above, only when $j < k$, it is of course natural to define $R_{kj} = -R_{jk}$. The computation of $d\pi(X)$ has probably been done thousands of times: however, since what we are really interested in is the operator $\mathcal{G} d\pi(X) \mathcal{G}^{-1}$, it is much faster do redo the computation, starting from the expression (4.26) of $\mathcal{G}\pi(s,a)\mathcal{G}^{-1}$. We immediately get, of course, whether $\mu = 0$ or $j$ $(j = 1, 2, 3)$,

$$(\mathcal{G}\, D_\mu \Psi)(p) = p_\mu\, (\mathcal{G}\, \Psi)(p). \tag{7.2}$$

To compute $B_j$ or $R_{jk}$, we first lift the element $e^{tX}$ of $\mathcal{P}_+^\uparrow$ (actually, even $\mathcal{L}_+^\uparrow$) to be considered to an element $s$ of $SL(V)$, i.e. we solve $\Lambda(s) = e^{tX}$ with $s$ near the identity for small $|t|$. As is well-known (*cf.* [12], (2.136) and

(2.138)), the answer is best given with the help of Pauli's matrices. We get

$$s = \sigma_0 \cosh \frac{t}{2} + \sigma_j \sinh \frac{t}{2} \tag{7.3}$$

in the first case, and

$$s = \sigma_0 \cos \frac{t}{2} - i\sigma_l \sin \frac{t}{2} \tag{7.4}$$

in the second one, with $\{l, j, k\}$ an even permutation of $\{1, 2, 3\}$. Recalling the definition (4.22) of $\Xi(s)$, we see that

$$\frac{d}{dt}\bigg|_{t=0} \Xi(s) = \begin{pmatrix} \frac{1}{2}\sigma_j & 0 \\ 0 & -\frac{1}{2}\sigma_j \end{pmatrix}, \qquad \text{resp.} \begin{pmatrix} -\frac{i}{2}\sigma_l & 0 \\ 0 & -\frac{i}{2}\sigma_l \end{pmatrix} \tag{7.5}$$

in the two cases under study. Thus, as a consequence of (4.26),

$$(\mathcal{G}B_j\,\Psi)(p) = -\frac{1}{2i\pi}\left[p_0\frac{\partial}{\partial p_1} + \frac{1}{2}\begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix}\right](\mathcal{G}\Psi)(p) \tag{7.6}$$

and

$$(\mathcal{G}\,R_{jk}\Psi)(p) = -\frac{1}{2i\pi}\left[p_k\frac{\partial}{\partial p_j} - p_j\frac{\partial}{\partial p_k} - \frac{i}{2}\begin{pmatrix} \sigma_l & 0 \\ 0 & \sigma_l \end{pmatrix}\right](\mathcal{G}\Psi)(p). \tag{7.7}$$

In these two formulas it is understood that $\{p_1, p_2, p_3\}$ is taken as the set of coordinates on $\mathcal{M}^\uparrow$.

Obviously, in view of the covariance of the full calculus with respect to the discrete symmetries, it suffices to compute the symbols of the generators of the Poincaré representation in the $Op_+^\uparrow$ calculus.

LEMMA 7.1. – *Let $L$ be an antisymmetric linear operator from $\mathbb{M}'$ to $\mathbb{M}$ and set $A(x, p) = Lp$. The operator with symbol $A$ is given by*

$$(\mathcal{G}\,Op_+^\uparrow(A)\,\Psi)(p) = \begin{pmatrix} 0 & \sigma(Lp) \\ -\tilde{\sigma}(\theta^{-1}Lp) & 0 \end{pmatrix}(\mathcal{G}\Psi)(p).$$

*Proof.* – Starting from (5.8) and using $\mathbb{M}_0 \times \mathcal{M}^\uparrow$ (*cf.* (5.1)) as a set of representatives of $\Delta^\uparrow$, we first perform the change of variables which substitutes the variable $r = S_p q$ for $p$; as shown in ([13], p. 65), one has

$$\frac{d\mathbf{r}}{d\mathbf{p}} = 8(<\theta p, q>)^2\,\frac{r_0}{p_0}. \tag{7.8}$$

In our case, following Definition 5.4,

$$f(x,p) = \begin{pmatrix} 0 & \sigma(Lp) \\ \tilde{\sigma}(p)\sigma(Lp)\tilde{\sigma}(p) & 0 \end{pmatrix} \tag{7.9}$$

depends only on $p$ so that (with a full recognition that what we claim here should actually wait for an extension of the calculus permitting to handle symbols which are slowly varying, e.g. constant, as functions of $x$), we get

$$\begin{aligned}(\mathcal{G}\, Op_+^{\uparrow}(A)\,\Psi)(p) &= \gamma(q)f(q)\gamma(q)(\mathcal{G}\Psi)(q) \\ &= \begin{pmatrix} 0 & \sigma(Lq) \\ \tilde{\sigma}(q)\sigma(Lq)\tilde{\sigma}(q) & 0 \end{pmatrix}(\mathcal{G}\Psi)(q). \end{aligned} \tag{7.10}$$

Now, all that has to be shown to complete the proof of Lemma 7.1 is that

$$\tilde{\sigma}(p)\sigma(Lp)\tilde{\sigma}(p) = -\tilde{\sigma}(\theta^{-1}Lp), \qquad p \in \mathcal{M}^{\uparrow}, \tag{7.11}$$

under the assumption that $L$ is antisymmetric. Now one can give a matrix proof, choosing assorted bases so that $p$ coincides with $(e^*)^0$, the first basis vector of $\mathbb{M}'$. Then the matrix for $\tilde{\sigma}(p)$ is just the identity, so that, with $L = (L^{\mu\nu})$, we have $L^{00} = 0$ and, *in matrices*, $\sigma(L(e^*)^0) = \sigma(\sum L^{\mu 0}e_\mu) = \sum L^{j0}\sigma_j$ (since $L$ is antisymmetric, and with the usual convention that $0 \leq \mu \leq 3$ but $1 \leq j \leq 3$); on the other hand, $-\tilde{\sigma}(\theta^{-1}L(e^*)^0) = -\tilde{\sigma}(\theta^{-1}\sum L^{\mu 0}e_\mu) = \tilde{\sigma}(\sum L^{j0}e_j) = \sum L^{j0}\sigma_j$. □

LEMMA 7.2. – *If $A(x,p) = (p_0x^1 + p_1x^0)\,\theta p$, the operator $Op_+^{\uparrow}(A)$ is given by the equation*

$$-2i\pi\,(\mathcal{G}\,Op_+^{\uparrow}(A)\Psi)(p) = \left[p_0\frac{\partial}{\partial p_1} - \frac{1}{2}(p_0\gamma^1 + p_1\gamma^0)\right](\mathcal{G}\Psi)(p)$$

*and, if $A(x,p) = (p_3x^2 - p_2x^3)\,\theta p$, one has*

$$-2i\pi\,(\mathcal{G}\,Op_+^{\uparrow}(A)\Psi)(p) = \left[p_3\frac{\partial}{\partial p_2} - p_2\frac{\partial}{\partial p_3} - \frac{1}{2}(p_3\gamma^2 - p_2\gamma^3)\right](\mathcal{G}\Psi)(p).$$

*Proof.* – First observe that, in both cases, $A(x,p)$ is indeed an admissible symbol. Starting this time from (5.31) (since we are dealing here with purely scalar symbols in the sense of Remark 5.7) and (5.35), we use the same change of variable $\mathbf{p} \mapsto \mathbf{r}$ as in the proof of Lemma 7.1, thus getting in both cases

$$(\mathcal{G}\,Op_+^{\uparrow}(A)\Psi)(q) = \int \phi(0,\mathbf{x};p)\,(<\theta p, q>)^{-2}\,\frac{p_0}{r_0}\,\gamma(p)\,(\mathcal{G}\Psi)(r)\,e^{2i\pi<\mathbf{x},\mathbf{r}-\mathbf{q}>}\,d\mathbf{x}\,d\mathbf{r}, \tag{7.12}$$

with the understanding that $p = \mathrm{mid}\,(q, r)$, and that

$$\phi(0, \mathbf{x}, p) = p_0 x^1, \qquad \text{resp. } p_3 x^2 - p_2 x^3 \tag{7.13}$$

in the two cases under consideration. Setting for the duration of the proof of the present lemma

$$\partial_j = \left.\frac{\partial}{\partial r_j}\right|_{r=q}, \tag{7.14}$$

and using the well-known formula (which expresses that, considering $\mathcal{M}^\uparrow$ as embedded in $\mathbb{M}'$, the geodesic middle on $\mathcal{M}^\uparrow$ is proportional to the affine middle in $\mathbb{M}'$)

$$p = \mathrm{mid}\,(q, r) = 2^{-\frac{1}{2}}(< \theta q, r > +1)^{-\frac{1}{2}}\,(q + r), \tag{7.15}$$

we get the following set of formulas (not forgetting, either, that $q_0$ is not an independent variable, but a function of the $q_j$'s):

$$\partial_j < \theta q, r > = \partial_j (q_0 r_0 - q_j r_j) = \left.(q_0 \frac{r_j}{r_0} - q_j)\right|_{r=q} = 0, \tag{7.16}$$

$$\partial_j p_k = \frac{1}{2}\delta_{jk}, \qquad \partial_j p_0 = \frac{1}{2}\frac{q_j}{q_0}, \qquad \partial_j < \theta q, p > = 0 \tag{7.17}$$

and

$$\begin{aligned} \partial_j\left(\frac{p_0}{r_0}(< \theta p, q >)^{-2}\gamma(p)\right) &= \frac{1}{2}\frac{q_j}{q_0^2}\gamma(q) - q_0^{-2} q_j \gamma(q) + \frac{1}{2}\gamma^j + \frac{1}{2}\frac{q_j}{q_0}\gamma^0 \\ &= -\frac{1}{2}\frac{q_j}{q_0^2}\gamma(q) + \frac{1}{2}(\gamma^j + \frac{q_j}{q_0}\gamma^0). \end{aligned} \tag{7.18}$$

Starting from (7.12), an integration by parts yields, in the first case,

$$(\mathcal{G}\, Op_+^\uparrow(A)\Psi)(q) = -\frac{1}{2i\pi}\,\partial_1 \left[p_0\,(\frac{p_0}{r_0}(< \theta p, q >)^{-2}\,\gamma(p))\,(\mathcal{G}\Psi)(r)\right], \tag{7.19}$$

from which the first formula asserted in Lemma 7.2 is proven, if one uses also Dirac's equation (4.13) to get

$$\begin{aligned} q_0\gamma(q)\frac{\partial}{\partial q_1}\,(\mathcal{G}\Psi)(q) &= q_0 \frac{\partial}{\partial q_1}\,(\gamma(q)\mathcal{G}\Psi(q)) - q_0 \frac{\partial \gamma(q)}{\partial q_1}\,(\mathcal{G}\Psi)(q) \\ &= \left[q_0 \frac{\partial}{\partial q_1} - (q_0\gamma^1 + q_1\gamma^0)\right](\mathcal{G}\Psi)(q). \end{aligned} \tag{7.20}$$

The second formula is proven in just the same way. □

LEMMA 7.3. – *In assorted coordinates, the matrices*

$$\begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix} + p_0\,\gamma^1 + p_1\,\gamma^0 + i\left[p_2\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} - p_3\begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}\right]$$

*and*

$$\begin{pmatrix} -i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix} + p_3\,\gamma^2 - p_2\,\gamma^3 + i\left[p_1\begin{pmatrix} 0 & -\sigma_0 \\ \sigma_0 & 0 \end{pmatrix} + p_0\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}\right]$$

*vanish as linear operators on the space* $\mathrm{Ker}(\gamma(p) - 1)$.

*Proof.* – Writing the $\gamma$-matrices as block-matrices as in (4.2), using also (2.10), finally recalling that $\mathrm{Ker}(\gamma(p) - 1)$ consists of all vectors of the kind $\binom{v}{\tilde{\sigma}(p)v}$ with $v \in V$, one sees that one has to check the two equalities

$$\sigma_1\sigma(\theta p) + p_1\sigma_0 - p_0\sigma_1 + i(p_2\sigma_3 - p_3\sigma_2) = 0 \qquad (7.21)$$

and

$$-i\sigma_1\sigma(\theta p) + p_2\sigma_3 - p_3\sigma_2 + i(p_0\sigma_1 - p_1\sigma_0) = 0, \qquad (7.22)$$

which can be done by brute force. □

PROPOSITION 7.4. – *The symbol of the identity operator (on $\mathcal{H}^\uparrow$) is the function $A(x,p) = \theta p$, the unit vector pointing towards the future for the observer associated with p. The symbol of the generators of the Poincaré representation are as follows. The symbol of $D_\mu$ is the function $p_\mu\,\theta p$. The symbols of the boosts and rotation operators are given by*

$$B_j = Op_+^\uparrow\left[\,(p_0x^j + p_jx^0)\,\theta p + \frac{1}{4\pi}\,(p_ke_l - p_le_k)\right],$$

*and*

$$R_{jk} = Op_+^\uparrow\left[\,(p_kx^j - p_jx^k)\,\theta p + \frac{1}{4\pi}\,(p_0e_l - p_le_0)\right],$$

*where $\{j,k,l\}$ is an even permutation of $\{1,2,3\}$.*

*Proof.* – The first two points follow from (7.2) and (7.12). According to Lemma 7.1, one has

$$(\mathcal{G}\;Op_+^\uparrow(p_2e_3 - p_3e_2)\,\Psi)(p) = \left[p_2\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} - p_3\begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}\right](\mathcal{G}\Psi)(p) \qquad (7.23)$$

and

$$(\mathcal{G}\, Op_+^{\uparrow}(p_0e_1-p_1e_0)\,\Psi)(p) = \left[p_1\begin{pmatrix} 0 & -\sigma_0 \\ \sigma_0 & 0 \end{pmatrix} + p_0\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}\right] (\mathcal{G}\Psi)(p). \tag{7.24}$$

Then, (7.6) (resp. (7.7)), Lemma 7.2 and the last two equations yield

$$\begin{aligned}&(\mathcal{G}\, B_1\Psi)(p) - (\mathcal{G}\, Op_+^{\uparrow}\left[(p_0x^1 + p_1x^0)\,\theta p + \frac{1}{4\pi}(p_2e_3 - p_3e_2)\right]\Psi)(p)\\ &= -\frac{1}{4i\pi}\left[\begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix} + p_0\gamma^1\right.\\ &\left.+p_1\gamma^0 + ip_2\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} - ip_3\begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}\right] (\mathcal{G}\Psi)(p)\end{aligned} \tag{7.25}$$

and

$$\begin{aligned}&(\mathcal{G}\, R_{23}\Psi)(p) - (\mathcal{G}\, Op_+^{\uparrow}\left[(p_3x^2 - p_2x^3)\,\theta p + \frac{1}{4\pi}(p_0e_1 - p_1e_0)\right]\Psi)(p)\\ &= -\frac{1}{4i\pi}\left[\begin{pmatrix} -i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix} + p_3\gamma^2 - p_2\gamma^3\right.\\ &\left.+ip_0\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} + ip_1\begin{pmatrix} 0 & -\sigma_0 \\ \sigma_0 & 0 \end{pmatrix}\right](\mathcal{G}\Psi)(p).\end{aligned} \tag{7.26}$$

One concludes thanks to Lemma 7.3, using also the covariance of the calculus under rotations. □

Introducing the completely antisymmetric Levi-Civita tensor $(\varepsilon_{\mu\nu\alpha\beta})$ characterized by $\varepsilon_{0123} = 1$, it is customary (*cf.* [9], p. 87) to set $\varepsilon^{\mu\nu\alpha\beta} = -\varepsilon_{\mu\nu\alpha\beta}$ and to associate with each 2-form $F = (F_{\mu\nu})$ on Minkowski's space the 2-vector $*F$ such that

$$(*F)^{\alpha\beta} = \frac{1}{2}\sum_{\mu\nu} \varepsilon^{\mu\nu\alpha\beta}\, F_{\mu\nu}. \tag{7.27}$$

The 2-vector $*F$ can be interpreted as a bilinear antisymmetric form $(p,q) \mapsto (*F)(p,q)$ on $\mathbb{M}'$, but also as a linear map: $\mathbb{M}' \to \mathbb{M}$ through the equation $< (*F)(p), q >_{\mathbb{M}\times\mathbb{M}'} = (*F)(p,q)$.

THEOREM 7.5. – *The set of symbols, in the $Op_+^{\uparrow}$-calculus, of all the infinitesimal generators of the Poincaré representation restricted to the Lorentz subgroup $\mathcal{L}_+^{\uparrow}$ of $\mathcal{P}_+^{\uparrow}$ coincides with the set of functions*

$$(x,p) \longmapsto F(x,\theta p)\,\theta p + \frac{1}{4\pi}(*F)(p)$$

*as $F$ describes the set of all real 2-forms on $\mathbb{M}$.*

*Proof.* – When $F(x,y) = -x^0y^1+x^1y^0$, one has $F(x,\theta p) = p_0x^1+p_1x^0$ and $(*F)(p,q) = p_2q_3 - p_3q_2$, i.e. $(*F)(p) = p_2e_3 - p_3e_2$; when $F(x,y) = -x^2y^3+x^3y^2$, one has $F(x,\theta p) = p_3x^2-p_2x^3$ and $(*F)(p,q) = p_0q_1 - p_1q_0$, i.e. $(*F)(p) = p_0e_1 - p_1e_0$, so that Theorem 7.5 is a corollary of Proposition 7.4. □

REMARK 7.6. – The two terms in the preceding decomposition of the symbol under consideration are none other than its electrical and magnetic parts in the sense of Remark 6.5.

## 8. POSITION OPERATORS

In this section, we study, or discard, operators which have been set forth, classically, as possible candidates for *position operators*. In the next one, we shall suggest our own, with a full recognition of the fact that it does not answer the same problem. However, in some sense, the *a priori* search for a position operator is as much an emotional problem as a problem in physics or mathematics. Indeed, one might not take for granted that, given any classical notion, or observable, there should correspond to it an analogous quantum notion: or, if one does, one is, consciously or not, referring to some rule that would establish such a correspondence. But, as this has been precisely the whole point of the present paper, our (admittedly personal) point of view is that the question was as yet not general enough. Whatever the case may be, there is only one operator, among the classical ones very thoroughly discussed in Thaller's book [12], that we can retain in the present frame: then, we shall show that its symbol, in the Dirac calculus, is as simple as one might wish.

Classical position operators are defined with respect to some observer of reference, since the time-zero restriction of Dirac wave functions is essential to their definition. The first idea, which yields the *standard* position operator denoted as $\mathbf{x}$ in [12] (it is really a triple $(\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3)$) is to consider the operator which multiplies the time-zero restriction $\Psi(0,\mathbf{x})$ of $\Psi$ by $x^j$, using the Dirac equation to extend the result as a wave living on spacetime again. Then, as explained in [12], one gets the so-called *Zitterbewegung* as an unwanted effect: this is traced to the operator $\mathbf{x}^j$ mixing the wave functions with positive or negative energy, a situation which we have refrained from in our calculus. The second operator, denoted as $\tilde{\mathbf{x}}$ in [12], is the standard position operator cured from this defect in the most obvious way: namely, if, say, $\Psi$ lies in $\mathcal{H}^\uparrow$ or, rather, in some appropriate dense subspace of $\mathcal{H}^\uparrow$, one defines the image of $\Psi$ under the operator $\tilde{\mathbf{x}}^j$ as the image of $\Psi$ under the preceding operator followed by the orthogonal projection from $\mathcal{H}$ to $\mathcal{H}^\uparrow$. Let us mention at once that this is the operator, hereafter called $\mathrm{Pos}_j$, that fits well with our Dirac calculus: one may regret that the three operators $\tilde{\mathbf{x}}^j = \mathrm{Pos}_j$ do not commute with one another, but there is nothing one can do about it. The third possibility is the so-called Newton-Wigner position operator (*cf.* again [12] for a discussion, as well as [10]): as it is based on the Foldy-Wouthuysen transformation, it is incompatible with Einstein's causality. Let us mention at this point that it would have been an easy, but very disappointing task, to base a symbolic calculus on the Foldy-Wouthuysen transformation. The simple trick would have been to use the transformation to reduce an operator in this calculus to a $4 \times 4$-matrix of operators associated with the square-root Klein-Gordon equation, for which the Klein-Gordon calculus would have been available. A comparison between our present Dirac calculus and this kind of machine may still prove useful, technically, since the Klein-Gordon analysis has been developed to some further extent in [13]. However, one of the major points, in going from the Klein-Gordon equation to the Dirac equation, is to regain causality, which partly explains why nothing really interesting could be gotten from such a calculus. Let us take this opportunity to mention that a major part of the work done in preparation for the present paper has been discarding a number of other possibilities that may have suggested themselves, before reaching the present calculus which, from our point of view at least, is completely satisfactory.

After this discussion, we turn to a computation of the symbol of the operator which we have been led to retaining. This necessitates, as a preliminary, an expression of the positive and negative-energy parts of a wave $\Psi$ in terms of the time-zero restriction of $\Psi$; in all this section, we assume that assorted bases have been chosen.

LEMMA 8.1. – *Let $\Psi \in \mathcal{H}$. With $\Psi_0(\mathbf{x}) = \Psi(0, \mathbf{x})$, the decomposition of $\Psi$ into its positive and negative-energy parts is given by*

$$\Psi = \mathrm{pr}^\uparrow \Psi + \mathrm{pr}^\downarrow \Psi,$$

*where*

$$(\mathcal{G}\,\mathrm{pr}^{\uparrow\downarrow}\Psi)(p) = \frac{\dot{\varepsilon}}{2}\begin{pmatrix} \sigma(e_0)\tilde{\sigma}(J'^{-1}p) & \sigma(e_0) \\ \tilde{\sigma}((e^*)^0) & \tilde{\sigma}((e^*)^0)\sigma(\theta J'^{-1}p) \end{pmatrix}$$
$$\hat{\Psi}_0(\mathbf{p}) \times \mathrm{char}(p \in \mathcal{M}^{\uparrow\downarrow}) :$$

*here* $\varepsilon = \pm 1$ *according to whether the arrow points upwards or downwards, and* $J(x^0, \mathbf{x}) = (x^0, -\mathbf{x})$.

*Proof.* – One first checks that, for every $p \in \mathcal{M}$, the image of the matrix

$$\begin{pmatrix} \sigma(e_0)\tilde{\sigma}(J'^{-1}p) & \sigma(e_0) \\ \tilde{\sigma}((e^*)^0) & \tilde{\sigma}((e^*)^0)\sigma(\theta J'^{-1}p) \end{pmatrix}$$

is a subset of Ker($\gamma(p) - 1$): expanding everything, this is a straightforward consequence of the equations (3.6), with $(e^*)^0$ substituted for $p$, and $p$ or $J'^{-1}p$ substituted for $q$. Thus, the right-hand side of the equation in Lemma 8.1 is, indeed, the image under $\mathcal{G}$ of some distribution $\mathrm{pr}^{\uparrow\downarrow} \in \mathcal{H}^{\uparrow\downarrow}$.

Now, as a consequence of (4.12), one has

$$\Psi(0, \mathbf{x}) = \int_{\mathcal{M}} e^{2i\pi <\mathbf{x}, \mathbf{p}>} (\mathcal{G}\Psi)(p) \frac{d\mathbf{p}}{<\mathbf{p}>}, \qquad \mathbf{x} \in \mathbb{R}^3 \tag{8.1}$$

for every $\Psi \in \mathcal{H}$: applying this with $\mathrm{pr}^{\uparrow\downarrow}\Psi$ instead of $\Psi$, and observing that if $p_0 > 0$, $(-p_0, \mathbf{p}) = -J'^{-1}p$ and

$$\begin{aligned} &\frac{1}{2}\begin{pmatrix} \sigma(e_0)\tilde{\sigma}(J'^{-1}p) & \sigma(e_0) \\ \tilde{\sigma}((e^*)^0) & \tilde{\sigma}((e^*)^0)\sigma(\theta J'^{-1}p) \end{pmatrix} \\ &- \frac{1}{2}\begin{pmatrix} -\sigma(e_0)\tilde{\sigma}(p) & \sigma(e_0) \\ \tilde{\sigma}((e^*)^0) & -\tilde{\sigma}((e^*)^0)\sigma(\theta p) \end{pmatrix} \\ &= p_0 \, I_{V\oplus W}, \end{aligned} \tag{8.2}$$

we get

$$(\mathrm{pr}^{\uparrow}\Psi)(0, \mathbf{x}) + (\mathrm{pr}^{\downarrow}\Psi)(0, \mathbf{x}) = \int e^{2i\pi <\mathbf{x}, \mathbf{p}>} \hat{\Psi}_0(\mathbf{p}) \, d\mathbf{p} = \Psi_0(\mathbf{x}). \tag{8.3}$$

□

PROPOSITION 8.2. – *Consider the position operator* $\mathrm{Pos}_j$ *on* $\mathcal{H}^{\uparrow}$ *defined as the product* $\mathrm{pr}^{\uparrow}\, \mathbf{x}^j$, *where* $\mathbf{x}^j$ *stands for the operator characterized by* $(\mathbf{x}^j\, \Psi)(0, \mathbf{x}) = x^j \Psi(0, \mathbf{x})$: *in other words, as a triple of operators on* $\mathcal{H}^{\uparrow}$, $(\mathrm{Pos}_j)_{j=1,2,3}$ *is the operator denoted as* $\tilde{\mathbf{x}}$ *in* ([12], p. 24). *The* $Op_+^{\uparrow}$*-symbol* $\mathrm{pos}_j$ *of* $\mathrm{Pos}_j$ *and the* $Op_+^{\uparrow}$-symbol $b_j$ *of* $B_j$ *are related by*

$$b_j(x, p) = p_0 \; \mathrm{pos}_j(x, p).$$

*Proof.* – We first compute $Op_+^{\uparrow}(p_0^{-1} b_j)$, following what was done in Lemmas 7.1 and 7.2, and Proposition 7.4. Apart from multiplying all results by $p_o^{-1}$, the sole difference comes from the analogue of (7.19), where the term arising from differentiating $p_0$ itself on the right-hand side does not occur. Thus

$$\begin{aligned}(\mathcal{G}Op_+^{\uparrow}(p_0^{-1}b_j)\Psi)(q) &= q_0^{-1}(\mathcal{G}B_j\Psi)(q) + \frac{1}{4i\pi}\frac{q_j}{q_0}(\mathcal{G}\Psi)(q)\\ &= -\frac{1}{2i\pi}\left[\frac{\partial}{\partial q_j} + \frac{1}{2q_0}\begin{pmatrix}\sigma_j & 0\\ 0 & -\sigma_j\end{pmatrix} - \frac{1}{2}\frac{q_j}{q_0^2}\right](\mathcal{G}\Psi)(q).\end{aligned} \tag{8.4}$$

On the other hand, with $\Psi \in \mathcal{H}^{\uparrow}$,

$$(\mathcal{G}\mathrm{Pos}_j^{\uparrow})(p) = \frac{1}{2}\begin{pmatrix}\sigma(e_0)\tilde{\sigma}(J'^{-1}p) & \sigma(e_0)\\ \tilde{\sigma}((e^*)^0) & \tilde{\sigma}((e^*)^0)\sigma(\theta J'^{-1}p)\end{pmatrix} \left(-\frac{1}{2i\pi}\frac{\partial}{\partial p_j}\right)(\hat{\Psi}_0(p)). \tag{8.5}$$

Since $\Psi \in \mathcal{H}^{\uparrow}$, Lemma 8.1 applied again yields

$$\begin{aligned}(\mathcal{G}\mathrm{Pos}_j^{\uparrow}\Psi)(p) &= -\frac{1}{2i\pi}\frac{\partial}{\partial p_j}(\mathcal{G}\Psi)(p)\\ &+ \frac{1}{4i\pi}\left[\frac{\partial}{\partial p_j}\begin{pmatrix}\sigma(e_0)\tilde{\sigma}(J'^{-1}p) & \sigma(e_0)\\ \tilde{\sigma}((e^*)^0) & \tilde{\sigma}((e^*)^0)\sigma(\theta J'^{-1}p)\end{pmatrix}\right] p_0^{-1}(\mathcal{G}\Psi)(p)\\ &= -\frac{1}{2i\pi}\left[\frac{\partial}{\partial p_j} - \frac{1}{2}\begin{pmatrix}-\sigma_j + \frac{p_j}{p_0} & 0\\ 0 & \sigma_j + \frac{p_j}{p_0}\end{pmatrix}p_0^{-1}\right](\mathcal{G}\Psi)(p),\end{aligned} \tag{8.6}$$

which should only be compared to (8.4). □

REMARK 8.3. – As $B_j$ is the symmetrized product of $\mathrm{Pos}_j$ with the energy operator (*cf.* [12], (1.39)), the result of Proposition 8.2 is as simple as one could expect.

## 9. THE DIRAC OPERATOR ON THE MASS-SHELL AS A SCALAR POSITION OPERATOR

Given $(x,p) \in \mathbb{M} \times \mathcal{M}^{\uparrow}$, the vector

$$A(x,p) = x - < x,p > \theta p \tag{9.1}$$

is the projection of $x$ on $T_p^{\perp}$ relative to the Minkowski-orthogonal decomposition $\mathbb{M} = T_p \oplus T_p^{\perp}$, where $T_p$ is the one-dimensional subspace of $\mathbb{M}$ generated by $\theta p$ (*cf.* (3.7)), i.e. the time-line from the point of view of any observer associated with $p$. One may thus consider $A(x,p)$ as the *position* part of the vector $x$ relative to $p$. As $A(x,p)$ is admissible in the sense of Definition 5.4, it is tempting to look at the associated operator $Op_+^{\uparrow}(A)$, which is the purpose of the present section.

One could hardly compare the operator thus introduced to the position operators discussed in the preceding section: for, even though the classical observable $A(x,p)$ is vector-valued (as, indeed, are all symbols in the present Dirac calculus), we only get one operator, not a triple. In other words, the associated hermitian observable yields a scalar, not a vector. However, $Op_+^{\uparrow}(A)$ is a very interesting operator to discuss, since it is Lorentz-invariant (i.e. it commutes with all operators $\pi(s,0)$, $s \in SL(V)$), as it follows from the covariance of the calculus together with the observation that

$$\Lambda(s)\, A(\Lambda(s)^{-1}x, \Lambda(s)'p) = A(x,p) \tag{9.2}$$

for every $s \in SL(V)$. Under the $\mathcal{G}$-transformation, $Op_+^{\uparrow}(A)$ may be viewed as an operator acting on a space of sections of the linear bundle $p \mapsto \mathrm{Ker}(\gamma(p) - 1)$ above $\mathcal{M}^{\uparrow}$. What we are going to show presently is that $\mathcal{G}Op_+^{\uparrow}(A)\mathcal{G}^{-1}$ is nothing else than the *Dirac operator* associated with some natural Clifford module structure and connection on this bundle. Here, the phrase "Dirac operator" has to be taken in the sense of Riemannian geometers (*cf.* [1], [3], [5], [8]), not in that of physicists (or *pseudo*-riemannian geometers).

Recall that there is a canonical Clifford bundle $Cl(\mathcal{M}^{\uparrow})$ over $\mathcal{M}^{\uparrow}$, the fiber of which above $p \in \mathcal{M}^{\uparrow}$ is the associative algebra with unit generated by $\mathcal{M}_p$, the space tangent to $\mathcal{M}^{\uparrow}$ at $p$, under the defining relation

$$X.Y + Y.X = -2\,(X,Y)_p\,, \qquad X, Y \in \mathcal{M}_p, \tag{9.3}$$

where $(X,Y)_p$ denotes the scalar product of $X$ and $Y$ with respect to the Euclidean structure on $\mathcal{M}_p$ associated with the Riemannian structure on $\mathcal{M}^{\uparrow}$: recall that the latter one was defined in (3.4).

We shall always, relying on the canonical embedding $\mathcal{M}^{\uparrow} \subset \mathbb{M}'$, consider elements $X$ of $\mathcal{M}_p$ as vectors in $\mathbb{M}'$, namely vectors satisfying $< \theta p, X >= 0$; in other words, we shall always make the identification

$$\sum a_j \frac{\partial}{\partial p_j} = X = (p_0^{-1} < \mathbf{a}, \mathbf{p} >, \mathbf{a}) \in \mathbb{M}'. \tag{9.4}$$

Then, if $X$ and $Y$ lie in $\mathcal{M}_p$, one has

$$(X,Y)_p = - < \theta X, Y >_{\mathsf{M} \times \mathsf{M}'} \tag{9.5}$$

since this is obvious when $p = (e^*)^0$ and can be carried through to any other point if one uses the transitive group $\mathcal{L}_+^\uparrow$ of isometries of $\mathcal{M}^\uparrow$.

Next, the complex linear bundle $p \mapsto \mathrm{Ker}(\gamma(p) - 1)$ over $\mathcal{M}^\uparrow$ can be given the structure of a bundle of left modules over $Cl(\mathcal{M}^\uparrow)$, letting $X \in \mathcal{M}_p$ act on $\mathrm{Ker}(\gamma(p) - 1)$ as

$$X.\xi = i \begin{pmatrix} 0 & -\sigma(\theta X) \\ \tilde{\sigma}(X) & 0 \end{pmatrix} \xi \,, \qquad \xi \in \mathrm{Ker}(\gamma(p) - 1). \tag{9.6}$$

Indeed, recall from (5.28) that $\tilde{\sigma}(p)\sigma(\theta X)\tilde{\sigma}(p) = \tilde{\sigma}(\iota_p \theta X)$ with $\iota_p \theta X = S_p X = -X$ since $< \theta p, X >= 0$, so that

$$X.\xi = -i\, \Theta_p(\theta X)\, \xi \tag{9.7}$$

in the sense of Lemma 5.3: thus $X.\xi$ belongs to $\mathrm{Ker}(\gamma(p) - 1)$ if $\xi$ does. If $Y$, too, lies in $\mathcal{M}_p$, one has

$$\begin{aligned} &X.(Y.\xi) + Y.(X.\xi) \\ &= \begin{pmatrix} \sigma(\theta X)\tilde{\sigma}(Y) + \sigma(\theta Y)\tilde{\sigma}(X) & 0 \\ 0 & \tilde{\sigma}(X)\sigma(\theta Y) + \tilde{\sigma}(Y)\sigma(\theta X) \end{pmatrix} \xi \\ &= 2 < \theta X, Y > \xi \end{aligned} \tag{9.8}$$

according to (2.8) and (2.9). In view of (9.3) and (9.5), this action of $\mathcal{M}_p$ on $\mathrm{Ker}(\gamma(p) - 1)$ thus extends to an action of the Clifford algebra $Cl(\mathcal{M}^\uparrow)$.

Now the bundle $\mathrm{Ker}(\gamma - 1)$ is endowed with a Riemannian metric if one sets, in accordance with (4.15), $\|\xi\|_p^2 = ((\xi, \xi))$ for all $\xi \in \mathrm{Ker}(\gamma(p) - 1)$. Then, if $X \in \mathcal{M}_p$ satifies $(X, X)_p = 1$, i.e. $< \theta X, X >= -1$, one has

$$\|X.\xi\|_p^2 = ((\, \Theta_p(\theta X)\, \xi \,,\, \Theta_p(\theta X)\, \xi\, )) \tag{9.9}$$

for all $\xi \in \mathrm{Ker}(\gamma(p) - 1)$: in view of (5.25), then (9.8) and (9.5), this yields

$$\|X.\xi\|_p^2 = ((\, \xi \,,\, (\Theta_p(\theta X))^2\, \xi\, )) = - < \theta X, X > ((\xi, \xi)) = \|\xi\|_p^2, \tag{9.10}$$

which means that unit vectors in $\mathcal{M}_p$ act isometrically on $\mathrm{Ker}(\gamma(p) - 1)$.

The Clifford bundle $Cl(\mathcal{M}^\uparrow)$ is endowed with a canonical connection inherited from the Riemannian structure of $\mathcal{M}^\uparrow$: as such, it is $\mathcal{L}_+^\uparrow$-invariant, $\mathcal{L}_+^\uparrow$ acting as a group of isometries of $\mathcal{M}^\uparrow$. To provide the bundle $\mathrm{Ker}(\gamma - 1)$

with a connection, we first identify any global section $\Phi$ of this bundle with a map $\Phi^{\check{}}$: $\mathcal{M}^{\uparrow} \to V \oplus W$ through the family of canonical injections: $\mathrm{Ker}(\gamma(p) - 1) \to V \oplus W$. Given $p \in \mathcal{M}^{\uparrow}$ and $X \in \mathcal{M}_p$, $\Phi^{\check{}}$ can be differentiated at $p$, along the vector $X$, in a straight way: let us denote as $X.\Phi^{\check{}}$ the result of this operation. We then define

$$\nabla_X \Phi = X.\Phi^{\check{}} - \frac{1}{2} \begin{pmatrix} \sigma(\theta X)\tilde{\sigma}(p) & 0 \\ 0 & \tilde{\sigma}(X)\sigma(\theta p) \end{pmatrix} \Phi^{\check{}} \tag{9.11}$$

and observe that, indeed, $\nabla_X \Phi$ lies in $\mathrm{Ker}(\gamma(p) - 1)$. With $\Phi^{\check{}}(p) = \left(\begin{smallmatrix} v(p) \\ \tilde{\sigma}(p)v(p) \end{smallmatrix}\right)$, this amounts to proving that

$$\begin{pmatrix} 0 \\ (X.\tilde{\sigma}(p))\, v(p) \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \sigma(\theta X)\tilde{\sigma}(p)v(p) \\ \tilde{\sigma}(X)v(p) \end{pmatrix} \in \mathrm{Ker}(\gamma(p) - 1), \tag{9.12}$$

i.e. that

$$-\tilde{\sigma}(p)\sigma(\theta X)\tilde{\sigma}(p) = 2\, X.\tilde{\sigma}(p) - \tilde{\sigma}(X). \tag{9.13}$$

With $X = \sum a_j \frac{\partial}{\partial p_j}$, one has

$$X.\tilde{\sigma}(p) = \sum a_j \sigma_j + p_0^{-1} < \mathbf{a}, \mathbf{p} > \sigma_0 = \tilde{\sigma}(X), \tag{9.14}$$

where the second equation is a consequence of (9.4). On the other hand, we already noted (right before (9.7)), that $\tilde{\sigma}(p)\sigma(\theta X)\tilde{\sigma}(p) = -\tilde{\sigma}(X)$, so we are done.

Some easy, but lengthy, details are to be checked now: that this is a Riemannian connection, that it is compatible with the canonical connection on $Cl(\mathcal{M}^{\uparrow})$, finally that it is $SL(V)$-invariant. The first point, namely that

$$(\!( \nabla_X \Phi_1 \, , \, \Phi_2 )\!) + (\!( \Phi_1 \, , \, \nabla_X \Phi_2 )\!) = X.(\!( \Phi_1 \, , \, \Phi_2 )\!) \tag{9.15}$$

for any pair $(\Phi_1, \Phi_2)$ of sections of the bundle $\mathrm{Ker}(\gamma - 1)$, reduces after expansion to the already proven identity $\tilde{\sigma}(X)\sigma(\theta p) = -\tilde{\sigma}(p)\sigma(\theta X)$, together with the fact that all four operators involved in this identity are hermitian. Next, we show that the connection is $SL(V)$-invariant, letting $s \in SL(V)$ act on sections of $\mathrm{Ker}(\gamma - 1)$ through

$$(s.\Phi)(p) = \Xi(s)\, \Phi(\Lambda(s)'p) \tag{9.16}$$

(*cf.* (4.26)). We want to check that the identity

$$\nabla_X(s.\Phi) \;=\; \Xi(s)\; \nabla_{\Lambda(s)'X}\, \Phi \tag{9.17}$$

holds for every $s \in SL(V)$) and every section $\Phi$ of the bundle under consideration; the verification amounts to

$$\begin{pmatrix} \sigma(\theta X)\tilde{\sigma}(p) & 0 \\ 0 & \tilde{\sigma}(X)\sigma(\theta p) \end{pmatrix} \Xi(s)$$
$$= \Xi(s) \begin{pmatrix} \sigma(\theta\Lambda(s)'X)\tilde{\sigma}(\Lambda(s)'p) & 0 \\ 0 & \tilde{\sigma}(\Lambda(s)'X)\sigma(\theta\Lambda(s)'p) \end{pmatrix}. \quad (9.18)$$

Expanding everything, one sees that (9.18) is a consequence of the definition (4.22) of $\Xi(s)$ together with (2.18), (2.20) and (2.19). In view of the $SL(V)$-invariance of the two connections (that on $Cl(\mathcal{M}^\uparrow)$ and that on $\mathrm{Ker}(\gamma - 1)$), it suffices, in view of proving the compatibility of the two connections, to check the identity

$$\nabla_X(Y.\Phi) = (\nabla_X Y).\Phi + Y.(\nabla_X \Phi), \quad (9.19)$$

where $X \in \mathcal{M}_p$ and $Y$ is a smooth section of the tangent bundle of $\mathcal{M}^\uparrow$, in the case when $p = (e^*)^0$: at $(e^*)^0$, all Christoffel symbols vanish in the coordinates $\{p_1, p_2, p_3\}$. We may also assume that

$$X = \frac{\partial}{\partial p_j} = \frac{p_j}{p_0}(e^*)^0 + (e^*)^j \quad (9.20)$$

and

$$Y = \frac{\partial}{\partial p_k} = \frac{p_k}{p_0}(e^*)^0 + (e^*)^k. \quad (9.21)$$

Then

$$Y.\Phi = i \begin{pmatrix} 0 & -\sigma(\theta Y) \\ \tilde{\sigma}(Y) & 0 \end{pmatrix} \Phi^{\sim} = i \begin{pmatrix} 0 & -\frac{p_k}{p_0}\sigma_0 + \sigma_k \\ \frac{p_k}{p_0}\sigma_0 + \sigma_k & 0 \end{pmatrix} \Phi^{\sim}. \quad (9.22)$$

One may then check, at $p = (e^*)^0$, the identity

$$\nabla_{\frac{\partial}{\partial p_j}} \left[ \begin{pmatrix} 0 & -\frac{p_k}{p_0}\sigma_0 + \sigma_k \\ \frac{p_k}{p_o}\sigma_0 + \sigma_k & 0 \end{pmatrix} \Phi^{\sim} \right] = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \nabla_{\frac{\partial}{\partial p_j}} \Phi^{\sim}, \quad (9.23)$$

using the definition (9.11) of the connection on $\mathrm{Ker}(\gamma - 1)$: if one expands everything, the verification is an immediate consequence of the identities $\sigma_j\sigma_k + \sigma_k\sigma_j = 2\,\delta_{jk}$.

We can now (*cf.* [8], p. 113) define the Dirac operator $D = \mathrm{Dir}(\mathcal{M}^\uparrow, \mathrm{Ker}(\gamma - 1))$ as

$$D\Phi = \sum_j f^j.\nabla_{f_j} \Phi, \quad (9.24)$$

where $\{f^j\}$ (usually called $\{e_j\}$ in Clifford geometry, but this usage has been preempted) is an orthonormal basis of $\mathcal{M}_p$.

Let us compute $D$ in an explicit way, at any point $p \in \mathcal{M}^\uparrow$. First set $p = \Lambda'(e^*)^0$, with $\Lambda' = (\Lambda_\mu^{\ \nu})$ a Lorentz matrix; one may then take $f^j = \Lambda'(e^*)^j = \sum_\mu \Lambda_\mu^{\ j}(e^*)^\mu$. Then one has

$$(D\Phi)(p) = i \sum_j \begin{pmatrix} 0 & -\sigma(\theta f^j) \\ \tilde{\sigma}(f^j) & 0 \end{pmatrix} (\nabla_{f^j} \Phi)(p). \tag{9.25}$$

Let us set $\{\mu\} = 1$ if $\mu = 0$, $-1$ if $\mu = 1, 2, 3$; also recall that latin indices $j, k, \ldots$ only vary from 1 to 3, greek indices from 0 to 3. Then

$$(\nabla_{f^j} \Phi)(p) = \sum_k \Lambda_k^{\ j} \frac{\partial \Phi^{\check{}}}{\partial p_k} - \frac{1}{2} \sum_\mu \Lambda_\mu^{\ j} \begin{pmatrix} \{\mu\}\sigma_\mu \tilde{\sigma}(p) & 0 \\ 0 & \sigma_\mu \sigma(\theta p) \end{pmatrix} \Phi^{\check{}}. \tag{9.26}$$

Also

$$\begin{pmatrix} 0 & -\sigma(\theta f^j) \\ \tilde{\sigma}(f^j) & 0 \end{pmatrix} = \sum_\nu \Lambda_\nu^{\ j} \begin{pmatrix} 0 & -\{\nu\}\sigma_\nu \\ \sigma_\nu & 0 \end{pmatrix}, \tag{9.27}$$

thus

$$\begin{aligned}(D\Phi)(p) = i \sum_{j,\nu} \Lambda_\nu^{\ j} \begin{pmatrix} 0 & -\{\nu\}\sigma_\nu \\ \sigma_\nu & 0 \end{pmatrix} \\ \Big[\sum_k \Lambda_k^{\ j} \frac{\partial \Phi^{\check{}}}{\partial p_k} - \frac{1}{2} \sum_\mu \Lambda_\mu^{\ j} \begin{pmatrix} \{\mu\}\sigma_\mu \tilde{\sigma}(p) & 0 \\ 0 & \sigma_\mu \sigma(\theta p) \end{pmatrix} \Phi^{\check{}}\Big].\end{aligned} \tag{9.28}$$

Using the condition which expresses that $\Lambda$ is Lorentz, to wit,

$$\sum_\nu \{\nu\}\, \Lambda_\mu^{\ \nu} \Lambda_\rho^{\ \nu} = \{\mu\}\, \delta_{\mu\rho}, \tag{9.29}$$

or

$$\sum_j \Lambda_\mu^{\ j} \Lambda_\rho^{\ j} = p_\mu p_\rho - \{\mu\}\delta_{\mu\rho}, \tag{9.30}$$

we get after a straightforward computation the final expression

$$\begin{aligned}\frac{1}{i}\,(D\Phi)(p) = \begin{pmatrix} 0 & -\sigma(\theta p) \\ \tilde{\sigma}(p) & 0 \end{pmatrix} \sum_k p_k \frac{\partial \Phi^{\check{}}}{\partial p_k} \\ + \sum_k \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \frac{\partial \Phi^{\check{}}}{\partial p_k} + \frac{3}{2} \begin{pmatrix} 0 & -\sigma(\theta p) \\ \tilde{\sigma}(p) & 0 \end{pmatrix} \Phi^{\check{}}.\end{aligned} \tag{9.31}$$

THEOREM 9.1. – *Let* $A(x,p) = x - < x,p > \theta p$. *The operator* $\mathcal{G}\, Op_+^{\uparrow}(A)\, \mathcal{G}^{-1}$ *on* $\mathcal{H}^{\uparrow}$ *and the Dirac operator on* $C^{\infty}(\mathcal{M}^{\uparrow}, Ker(\gamma - 1))$ *associated with the connection (9.11) are related, on their common domain, by*

$$\mathcal{G}\, Op_+^{\uparrow}(A)\, \mathcal{G}^{-1} = \frac{4}{\pi}\ \mathrm{Dir}(\mathcal{M}^{\uparrow}, \mathrm{Ker}(\gamma - 1)).$$

*Proof.* – Let us use the formula (5.8) for the definition of $Op_+^{\uparrow}(A)$, realizing the space of straight worldlines $\Delta^{\uparrow}$ as $\mathbb{E}_0^{\uparrow}$ (*cf.* (5.2)): then $A(x,p) = x$ and $f(x,p)$, as introduced in Definition 5.4, reduces to

$$f(x,p) = \begin{pmatrix} 0 & \sigma(x) \\ \tilde{\sigma}(p)\sigma(x)\tilde{\sigma}(p) & 0 \end{pmatrix}. \tag{9.32}$$

As a consequence of (5.28), one has

$$\tilde{\sigma}(p)\sigma(x)\tilde{\sigma}(p) = \tilde{\sigma}(\iota_p x) = \tilde{\sigma}(S_p \theta^{-1} x) = \tilde{\sigma}(-\theta^{-1} x), \tag{9.33}$$

where the last equality comes from (3.5) and, again, from the equation $< x,p >= 0$. Thus

$$\begin{aligned} &(\mathcal{G}\, Op_+^{\uparrow}(A)\, \Psi)(q) \\ &= \int_{<x,p>=0} \gamma(\mathrm{mid}\,(p,q)) \begin{pmatrix} 0 & \sigma(x) \\ -\tilde{\sigma}(\theta^{-1}x) & 0 \end{pmatrix} \gamma(\mathrm{mid}\,(p, S_p q)) \\ &(\mathcal{G}\,\Psi)(S_p q)\ e^{2i\pi <x, S_p q - q>}\ \frac{d\mathbf{x}\, d\mathbf{p}}{p_0^2}. \end{aligned} \tag{9.34}$$

In view of the fact that both operators the identity of which is the object of Theorem 9.1 commute with all operators $\pi(s_1)$, $s_1 \in SL(V)$, it suffices to compute $(\mathcal{G}\, Op_+^{\uparrow}(A)\, \Psi)((e^*)^0)$. We set $p = \Lambda(s)'^{-1}(e^*)^0$, $\Lambda(s)$ chosen as a boost ($\Lambda(s) = \Lambda(s)'$ in matrices): in an explicit way, with $|\mathbf{p}| = \sinh t$, we may take (*cf.* [12], (2.136))

$$s = \sigma_0 \cosh\frac{t}{2} - \sum_j \frac{p_j}{|\mathbf{p}|}\,\sigma_j \sinh\frac{t}{2}. \tag{9.35}$$

Let us then perform the change of variables $\mathbf{p} \mapsto \mathbf{r}$, with $r = \Lambda(s)'(e^*)^0 = S_{(e^*)^0} p = J'^{-1} p$ where, in the usual way, $J'^{-1}(p_0, \mathbf{p}) = (p_0, -\mathbf{p})$: then $\frac{d\mathbf{p}}{p_0} = \frac{d\mathbf{r}}{r_0}$. At the same time, we set $x = \Lambda(s) y$ with $y = \binom{0}{\mathbf{y}}$ so that $d\mathbf{x} = p_0\, d\mathbf{y} = r_0\, d\mathbf{y}$. Then

$$\begin{pmatrix} 0 & \sigma(x) \\ \tilde{\sigma}(\theta^{-1}x) & 0 \end{pmatrix} = \Xi(s) \begin{pmatrix} 0 & \sigma(y) \\ \tilde{\sigma}(\theta^{-1}y) & 0 \end{pmatrix} \Xi(s^{-1}) \tag{9.36}$$

as a consequence of the definition (4.22) of $\Xi(s)$, of (2.18), (2.20) and (2.19). We also use (5.19) to write

$$\begin{aligned}\gamma(\text{mid}\,(p,(e^*)^0))\,\Xi(s) &= \Xi(s)\,\gamma(\Lambda(s)'\text{mid}\,(p,(e^*)^0)) \\ &= \Xi(s)\,\gamma(\text{mid}\,((e^*)^0,r))\end{aligned} \tag{9.37}$$

and

$$\begin{aligned}\Xi(s)^{-1}\,\gamma(\text{mid}\,(p,S_p(e^*)^0) &= \gamma(\text{mid}\,((e^*)^0,\Lambda(s)'S_p(e^*)^0)\ \Xi(s^{-1}) \\ &= \gamma(\text{mid}\,((e^*)^0,{J'}^{-1}r)\ \Xi(s^{-1}).\end{aligned} \tag{9.38}$$

Next, we note that

$$S_p((e^*)^0) = (2\,r_0^2 - 1, -2\,r_0\,\mathbf{r}) \tag{9.39}$$

and

$$\begin{aligned}< x, S_p((e^*)^0) - (e^*)^0 > &=< y, \Lambda(s)'(S_p((e^*)^0) - (e^*)^0) > \\ &=< y, {J'}^{-1}r - r > \\ &= -2\ < \mathbf{y},\mathbf{r} > .\end{aligned} \tag{9.40}$$

Thus

$$\begin{aligned}(\mathcal{G}\,Op_+^\uparrow(A)\,\Psi)((e^*)^0) = 8\sum_j \int_{\mathbb{R}^3\times\mathcal{M}^\uparrow} \Xi(s)\,\gamma(\text{mid}\,((e^*)^0,r)\,y^j \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0\end{pmatrix} \\ \gamma(\text{mid}\,((e^*)^0,{J'}^{-1}r))\,\Xi(s^{-1})\ (\mathcal{G}\,\Psi)(2\,r_0^2 - 1, -2\,r_0\,\mathbf{r})\,e^{-4i\pi<\mathbf{y},\mathbf{r}>}\,d\mathbf{y}\,\frac{d\mathbf{r}}{r_0},\end{aligned} \tag{9.41}$$

with

$$s = \sigma_0\,\cosh\frac{t}{2} + \sum_j \frac{r_j}{|\mathbf{r}|}\,\sigma_j\,\sinh\frac{t}{2}, \qquad |\mathbf{r}| = \sinh t. \tag{9.42}$$

Setting

$$\partial_j = \frac{\partial}{\partial r_j}\bigg|_{\mathbf{r}=0} \tag{9.43}$$

and integrating by parts, we get

$$(\mathcal{G}\, Op_+^{\uparrow}(A)\, \Psi)((e^*)^0) = \frac{2}{i\pi} \sum_j \partial_j \left[\Xi(s)\, \gamma(\mathrm{mid}\,((e^*)^0, r)) \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}\right.$$
$$\left. \gamma(\mathrm{mid}\,((e^*)^0, {J'}^{-1} r))\, \Xi(s^{-1})\, (\mathcal{G}\, \Psi)(\cosh 2t, -2\,\mathbf{r} \cosh t)\, r_0^{-1}\right] \qquad (9.44)$$

with $t$ and $s$ as in (9.42). Now

$$\partial_j s = \frac{1}{2}\sigma_j, \quad \partial_j \Xi(s) = \frac{1}{2}\begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix}, \quad \partial_j(\Xi(s^{-1})) = -\frac{1}{2}\begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix},$$
$$\partial_j(\gamma(\mathrm{mid}\,((e^*)^0, r))) = \frac{1}{2}\gamma^j = \frac{1}{2}\begin{pmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{pmatrix},$$
$$\partial_j(\gamma(\mathrm{mid}\,((e^*)^0, {J'}^{-1} r))) = -\frac{1}{2}\begin{pmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{pmatrix} \qquad (9.45)$$

and, using again $\mathbf{p}$ as a set of coordinates on $\mathcal{M}^{\uparrow}$,

$$\partial_j(\mathcal{G}\Psi)(\cosh 2t, -2\,\mathbf{r} \cosh t) = -2 \left.\frac{\partial}{\partial p_j}\right|_{p=(e^*)^0} (\mathcal{G}\Psi)(p). \qquad (9.46)$$

Thus

$$(\mathcal{G}\, Op_+^{\uparrow}(A)\, \Psi)((e^*)^0) = -\frac{4}{i\pi} \sum_j \left[\gamma^0 \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \gamma^0 \left.\frac{\partial}{\partial p_j}\right|_{p=(e^*)^0} (\mathcal{G}\Psi)(p)\right.$$
$$\left. - \frac{1}{4}\,\mathrm{Mat}_j (\mathcal{G}\Psi)((e^*)^0)\right], \qquad (9.47)$$

where $\mathrm{Mat}_j \in \mathrm{End}(V \oplus W)$ is given by

$$\mathrm{Mat}_j = \begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix} \gamma^0 \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \gamma^0 + \begin{pmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \gamma^0$$
$$- \gamma^0 \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{pmatrix} - \gamma^0 \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \gamma^0 \begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix} \qquad (9.48)$$
$$= \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}. \qquad (9.49)$$

As a consequence

$$(\mathcal{G}\, Op_+^{\uparrow}(A)\, \Psi)((e^*)^0) = -\frac{4}{i\pi} \left[\sum_j \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \left.\frac{\partial}{\partial p_j}\right|_{p=(e^*)^0} (\mathcal{G}\Psi)(p)\right.$$
$$\left. - \frac{3}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (\mathcal{G}\Psi)((e^*)^0)\right], \qquad (9.50)$$

an expression which needs only be compared to (9.31), the latter one being evaluated at $p = (e^*)^0$. □

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