ANNALES DE L'I. H. P., SECTION A

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Harmonic maps of the hyperbolic space and development of singularities in wave maps and Yang-Mills fields

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by

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ABSTRACT. – In this article we explore some of the connections between the theories of Yang-Mills fields, wave maps, and harmonic maps. It has been shown that the search for similarity solutions of wave maps leads to harmonic maps of the hyperbolic space. On the other hand, Glassey and Strauss have shown that the equations for an $SO(3)$-equivariant Yang-Mills connection on the Minkowski space $\mathbb{R}^{3,1}$ with gauge group $SU(2)$ reduce to a certain nonlinear wave equation, which we can now identify as a wave map on $\mathbb{R}^{1,1}$. More generally, we will here show the reduction under *equivariance* of a Yang-Mills system on the Minkowski space $\mathbb{R}^{n,1}$ to a wave map system on $\mathbb{R}^{n-2,1}$ in the specific case of $SO(n)$ bundles with $SO(n)$ symmetry. We then prove for odd n the existence of equivariant

harmonic maps from the hyperbolic space $\mathbb{H}^n$ that are smooth at the ideal boundary of $\mathbb{H}^n$, thus establishing the existence of similarity solutions for equivariant wave maps and Yang-Mills fields. As a consequence we show that for $n \geq 7$, it is possible to have a wave map into a negatively curved target manifold that develops from smooth initial data and blows up in finite time, in sharp contrast to the elliptic case of harmonic maps. Finally we show how these singular solutions can be lifted to one dimension higher to produce singular travelling waves. © Elsevier, Paris

RÉSUMÉ. – Dans cet article, nous explorons certaines relations entre les théories de champs de Yang-Mills, des applications d'onde, et des applications harmoniques. On sait que la recherche de solutions similaires d'applications d'onde conduit à des applications harmoniques de l'espace hyperbolique. Par ailleurs, Glassey et Strauss ont montré que les équations pour une connexion de Yang-Mills $SO(3)$-équivariante sur l'espace de Minkowski $\mathbb{R}^{3,1}$ muni du groupe de jauge $SU(2)$ se réduisent à une équation des ondes non-linéaire, que l'on peut en fait identifier à une application d'onde sur $\mathbb{R}^{1,1}$. Plus généralement, nous établissons ici la réduction *équivariante* d'un système de Yang-Mills sur l'espace de Minkowski $\mathbb{R}^{n,1}$ à un système d'applications d'onde sur $\mathbb{R}^{n-2,1}$ dans le cas spécifique des fibrés $SO(n)$ avec symétrie $SO(n)$. Nous montrons ensuite pour n pair l'existence d'applications harmoniques équivariantes de l'espace hyperbolique $\mathbb{H}^n$ qui sont régulières à la frontière idéale de $\mathbb{H}^n$, établissant ainsi l'existence de solutions similaires pour les applications d'onde équivariantes et les champs de Yang-Mills. Il en résulte que pour $n \geq 7$, il est possible de construire une application d'onde dans une variété cible à courbure négative, qui possède une donnée initiale régulière et explose en temps fini, ce qui diffère du cas elliptique des applications harmoniques. Enfin, à l'aide d'un relèvement, nous en déduisons l'existence d'ondes progressives singulières dans la dimension immédiatement supérieure. © Elsevier, Paris

1. INTRODUCTION

In this article we explore some of the connections between the theories of Yang-Mills fields, wave maps, and harmonic maps. It has been shown that the search for similarity solutions of wave maps leads to harmonic maps of the hyperbolic space [13]. On the other hand, Glassey and Strauss [6] have

shown that the equations for an $SO(3)$-equivariant Yang-Mills connection on the Minkowski space $\mathbb{R}^{3,1}$ with gauge group $SU(2)$ reduce to a certain nonlinear wave equation, which we can now identify as a wave map on $\mathbb{R}^{1,1}$. More generally, we will here show the reduction under *equivariance* of a Yang-Mills system on the Minkowski space $\mathbb{R}^{n,1}$ to a wave map system on $\mathbb{R}^{n-2,1}$ in the specific case of $SO(n)$ bundles with $SO(n)$ symmetry. We then prove for odd n the existence of equivariant harmonic maps from the hyperbolic space $\mathbb{H}^n$ that are smooth at the ideal boundary of $\mathbb{H}^n$, thus establishing the existence of similarity solutions for equivariant wave maps and Yang-Mills fields. As a consequence we show that for $n \geq 7$, it is possible to have a wave map into a negatively curved target manifold that develops from smooth initial data and blows up in finite time, in sharp contrast to the elliptic case of harmonic maps. Finally we show how these singular solutions can be lifted to one dimension higher to produce singular travelling waves.

This Section is organized as follows: in §1.1-1.3 we review the known results on wave maps and give the Yang-Mills reduction under equivariance. In §1.4 we state our main results, which are then proved in Sections 2 and 3.

1.1. Wave Maps

Let M be the Minkowski space $\mathbb{R}^{n,1}$, with coordinates $x = (x^0, x^1, \dots, x^n) = (t, \bar{x})$ and let N be a smooth, complete, rotationally symmetric d-dimensional Riemannian manifold without boundary. This means (cf. [14]) that N can be identified with a ball of radius $R \in \mathbb{R}^+ \cup \{\infty\}$ in $\mathbb{R}^d$ equipped with a metric of the form

$$ds^2 = du^2 + g^2(u)\gamma_{AB}d\theta^A d\theta^B, \tag{1.1}$$

where (u, θ) are polar coordinates on $\mathbb{R}^d$, γ is the standard metric on the sphere $\mathbb{S}^{d-1}$, and g is as follows:

$$g \in C^\infty(\mathbb{R}, \mathbb{R}), \quad g(-x) = -g(x), \quad xg(x) \geq 0, \quad g'(0) = 1. \tag{1.2}$$

In addition, if $R = \infty$, we assume that there exists a constant $C > 0$ such that

$$g(x)g'(x) \geq -C, \qquad \forall x > 0. \tag{1.3}$$

A *wave map* $U : M \to N$ is a stationary point (with respect to compactly supported variations) of the functional

$$\mathcal{A}[U] = \frac{1}{2}\int_M \langle \partial_\mu U, \partial^\mu U \rangle = \frac{1}{2}\int_M \partial_\mu u \partial^\mu u + g^2(u)\gamma_{AB}\partial_\mu\theta^A \partial^\mu\theta^B.$$

At a stationary point, the vector $U = u \cdot \theta \in \mathbb{R}^d$ satisfies

$$\begin{cases} \partial_\mu \partial^\mu u + g(u)g'(u)\gamma_{AB}\partial_\mu\theta^A\partial^\mu\theta^B = 0 \\ \partial^\mu(g^2(u)\gamma_{AB}\partial_\mu\theta^B) = 0. \end{cases} \tag{1.4}$$

Let $(t, \rho, \omega) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{S}^{n-1}$ be spatial polar coordinates on M, in which the line element of M can be expressed as

$$ds^2 = -dt^2 + d\rho^2 + \rho^2\gamma_{AB}d\omega^A d\omega^B.$$

One Ansatz of equivariance which is compatible with the wave map structure (by no means the most general one) is to require

$$u = u(t, \rho) \qquad \text{and} \qquad \theta = \theta(\omega), \tag{1.5}$$

This is referred to as the *corotational* Ansatz. It follows (cf. [8]) that $\theta : \mathbb{S}^{n-1} \to \mathbb{S}^{d-1}$ has to be an *eigenmap*, i.e., a harmonic map of constant energy density

$$k = |\nabla_\omega\theta|^2.$$

The wave map system (1.4) for U then reduces to the following single scalar wave equation for the spatially radial function $u : M \to \mathbb{R}$:

$$u_{tt} - u_{\rho\rho} - \frac{n-1}{\rho}u_\rho + \frac{k}{\rho^2}g(u)g'(u) = 0. \tag{1.6}$$

Remark 1. – If $\theta : \mathbb{S}^{n-1} \to \mathbb{S}^{d-1}$ is a harmonic map, then the unit vector $\theta(x) \in \mathbb{R}^d$ satisfies $\Delta_{\mathbb{S}^{n-1}}\theta + |\nabla_\omega\theta|^2\theta = 0$. Hence if the energy density $|\nabla_\omega\theta|^2$ is constant, then each component of θ is an eigenfunction for $\Delta_{\mathbb{S}^{n-1}}$, the spectrum of which is $\{\lambda_\ell = \ell(\ell + n - 2) \mid \ell \in \mathbb{N}\}$. The corresponding eigenspace to λ_ℓ is the vector space $\mathcal{H}_\ell$ of ℓ-homogeneous harmonic polynomials in $\mathbb{R}^n$, restricted to the unit sphere, and

$$d(n, \ell) := \dim \mathcal{H}_\ell = \frac{(2\ell + n - 2)(\ell + n - 3)!}{\ell!(n-2)!}.$$

Let $\{\Phi^1, \ldots, \Phi^{d(n,\ell)}\}$ be an orthonormal basis for $\mathcal{H}_\ell$. Then the map $\theta : \mathbb{S}^{n-1} \to \mathbb{S}^{d(n,\ell)-1}$ defined by $\theta^j = \sqrt{\frac{\lambda_\ell}{n-1}}\Phi^j\big|_{\mathbb{S}^{n-1}}$ for $j = 1, \ldots, d(n, \ell)$ is a λ_ℓ-eigenmap, i.e., $\Delta\theta + \lambda_\ell\theta = 0$ and $\sum_{i=1}^{d(n,\ell)}(\theta^j(x))^2 = 1$ for all $x \in \mathbb{S}^{n-1}$ (See [5 Chap. 8] and references cited therein).

We also note that since θ is a homogeneous polynomial of degree ℓ in its variables, for $U(t, \rho, \omega) = u(t, \rho) \cdot \theta(\omega)$ to be a smooth map, we must have that $u(t, \rho)/\rho^\ell$ is smooth near $\rho = 0$.

The following results regarding existence, uniqueness, regularity and break-down of solutions to (1.6) were obtained in [13]:

1. There exists $T > 0$ such that the Cauchy problem for (1.6) with initial data $(u_0, u_1) \in \dot{H}^{n/2} \times \dot{H}^{n/2-1}$ has a unique solution u such that

$$(u, u_t) \in C([0, T); \dot{H}^{n/2} \times \dot{H}^{n/2-1}) \cap L^q([0, T); \dot{B}_\sigma^{q,2} \times \dot{B}_{\sigma-1}^{q,2}),$$

where $q = 2(n+3)/(n+1)$, $\sigma = (n-1)/2$, and $\dot{H}^s$ and $\dot{B}_s^{q,p}$ are homogeneous Sobolev and Besov spaces of functions on $\mathbb{R}^n$.

2. For $n = 2$, the Cauchy problem for (1.6) with smooth initial data of small total *energy*

$$E(u_0, u_1) = \|u_0\|_{H^1(\mathbb{R}^2)} + \|u_1\|_{L^2(\mathbb{R}^2)}$$

has a unique smooth solution defined for all time. Moreover, if the initial data is smooth and if g satisfies

$$g(x) + xg'(x) > 0 \text{ for } x > 0, \tag{1.7}$$

then the energy of the solution cannot concentrate at a point, and therefore the solution remains smooth for all time.

3. Let $n = 3$, and suppose there exists $a > 0$ such that $g'(a) = 0$ and $g''(a) < 0$. Choose ℓ large enough such that $-\ell(\ell+1)g(a)g''(a) > 1/4$. Then (1.6) admits solutions which are *self-similar* (see below) inside a null cone. These solutions develop from Cauchy data which are smooth and of compact support, and they suffer a gradient blowup in finite time. Moreover, there exist data in H^s_{loc} with $s < n/2$ such that the corresponding Cauchy problem has two distinct H^s_{loc} solutions.

1.2. Yang-Mills Connections

Let P be a principal fiber bundle with base manifold B of dimension n, structure group G and canonical projection π. Let $\mathfrak{g}$ be the Lie algebra of G. A *connection* on this bundle can be thought of locally as a $\mathfrak{g}$-valued 1-form A defined on the coordinate patches U_α of the base manifold, $A = A_\mu(x)dx^\mu$, $A_\mu : U_\alpha \to \mathfrak{g}$. Thus for $v \in T_x B$ a tangent vector, $A(x, v) = A_\mu(x)v^\mu$. Under a gauge transformation $g : U_\alpha \cap U_\beta \to G$ that changes coordinates on fibers in the overlap of the two trivializations $\pi^{-1}(U_\alpha)$ and $\pi^{-1}(U_\beta)$, the form A transforms according to

$$A_\mu(x) \mapsto g^{-1}(x)A_\mu(x)g(x) + g^{-1}(x)\partial_\mu g(x).$$

The curvature of the connection A is defined to be the 2-form $F = F_{\mu\nu}dx^\mu dx^\nu$, with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

The Yang-Mills functional

$$\mathcal{L}(A) = \int_B F_{\mu\nu} F^{\mu\nu}$$

is invariant under gauge transformations, and its critical points are called Yang-Mills connections. These are solutions of the Euler-Lagrange equations

$$D^\mu F_{\mu\nu} = 0, \qquad (1.8)$$

where $D_\mu := \partial_\mu + [A_\mu, \cdot]$ is the covariant derivative associated with the connection A. If B is a globally hyperbolic Lorentzian manifold with time function x^0 and Cauchy hypersurface $\Sigma = \{x^0 = 0\}$, then equations (1.8), modulo the gauge invariance, form a hyperbolic PDE system for which a Cauchy problem is well-posed. A Cauchy data set for (1.8) is then a pair $(\bar{A}, \bar{E})$ of $\mathfrak{g}$-valued 1-forms on Σ satisfying the constraint div $\bar{E} + [\bar{A}, \bar{E}] = 0$. The *Cauchy problem in the temporal gauge* for (1.8) consists of finding a connection A with $A_0 \equiv 0$ which satisfies (1.8) and such that $A = \bar{A}$ and $\partial_0 A = \bar{E}$ on Σ.

Assume that a symmetry group S acts on the base manifold B, $\alpha : S \times B \to B$ with $\alpha(s, x) = sx$, and that this action has a lift to an action on the bundle P, $\bar{\alpha} : S \times P \to P$ such that (i) $\alpha \circ \pi = \pi \circ \bar{\alpha}$ and (ii) $\bar{\alpha}$ is a bundle map, *i.e.* it commutes with the right action of G on P. This implies (cf. [7]) that there is a homomorphism $\lambda : S \to G$ such that on a trivialization $\pi^{-1}(U)$ we have $\bar{\alpha}(s, (x, h)) = (sx, \lambda(s)h)$ for all $x \in U$, $s \in S$ and $h \in G$. A connection A on P is *equivariant* with respect to the S-action if the transformation induced on the 1-forms A_μ by this action of S amounts to a (global) gauge transformation. That is to say, for a fixed $s \in S$,

$$A(x, v) = \lambda(s)^* A(sx, s_* v),$$

for all $x \in B$ and $v \in T_x B$.

We now consider a special case of the above, when B is the Minkowski space $\mathbb{R}^{N,1}$, with global coordinates $x = (x^0, x^1, \ldots, x^N) = (t, \bar{x})$, $G = S = SO(N)$ the group of orthogonal transformations on $\mathbb{R}^N$ and

$\lambda = \mathrm{id}_G$. Thus $A_\mu(x)$ is an $N \times N$ skew-symmetric matrix A_μ^{ij}. The appropriate Ansatz of equivariance in this situation is (cf. [4])

$$A_\mu^{ij}(x) = (\delta_\mu^i x^j - \delta_\mu^j x^i) h(t, r),$$

where $h : B \to \mathbb{R}$ is a spatially radial function and $r = |\bar{x}|$. The Ansatz also implies that we are in the temporal gauge $A_0 \equiv 0$. Substituting the above in the equations for a Yang-Mills connection and setting $u := r^2 h$ and $n := N - 2$, we obtain the following semilinear wave equation for u:

$$u_{tt} - u_{rr} - \frac{n-1}{r} u_r + \frac{2n}{r^2} u(1-u)\left(1 - \frac{1}{2}u\right) = 0, \qquad (1.9)$$

which is the Euler-Lagrange equation for the critical points of the corresponding reduced functional

$$\int_B F_{\mu\nu} F^{\mu\nu} = c_n \int_{-\infty}^{\infty} \int_0^{\infty} -u_t^2 + u_r^2 + \frac{2n}{r^2} u^2 \left(1 - \frac{1}{2}u\right)^2 r^{n-1} dr dt.$$

Remark 2. – We observe that (1.9) is the same as equation (1.6) obtained before for a corotational wave map, with $\ell = 2$ and $g(u) = u(1 - \frac{1}{2}u)$. The only difference is that this g is not odd, which means that the nonlinearity in (1.9), unlike the one in (1.6), has a quadratic term.

For $n = 2$, u is a solution to (1.9) iff the function $h = u/r^2$ satisfies a wave equation in 6 space dimensions:

$$h_{tt} - h_{rr} - \frac{5}{r} h_r = 6h^2 - 2r^2 h^3,$$

2 is the critical power in 6 dimensions, and this is why, despite the presence of the quadratic term in the nonlinearity in (1.9), the arguments in [13] apply and we have local and global existence results for (1.9) which are analogous to those mentioned in §1.1 for wave maps. Note that $u \in H^{n/2}(\mathbb{R}^n)$ iff $A \in H^{(N-2)/2}(\mathbb{R}^N)$. The singularity result mentioned in §1.1 holds for (1.9) as well, because the conditions stated are satisfied for $a = 1$ and $\ell = 2$.

1.3. Similarity Solutions and Development of Singularities

A similarity solution (or self-similar solution) of the Cauchy problem for a wave map U from the Minkowski space $M = \mathbb{R}^{n,1}$ into a Riemannian manifold N is a solution that is only a function of x/t, where $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ are coordinates on M. Such solutions are constant along rays emanating from the origin of the space-time, and thus, if nontrivial, have a gradient

singularity at the origin. They can be regarded as maps defined on the hyperboloid $t^2 - |x|^2 = 1$, which, as a spacelike hypersurface of M, inherits a Riemannian metric from it that has constant curvature -1 and thus is a model for the hyperbolic space $\mathbb{H}^n$. Therefore the restriction of a self-similar wave map U to the interior of the light cone at the origin can be regarded as a harmonic map from $\mathbb{H}^n$ into N. Suppose $\Phi : \mathbb{H}^n \to N$ is a smooth, nontrivial harmonic map. We take the ball model for $\mathbb{H}^n$, *i.e.*, the unit ball B^n in $\mathbb{R}^n$ with the following metric

$$ds^2 = \frac{1}{(1 - r^2)^2} dr^2 + \frac{r^2}{1 - r^2} \gamma_{AB} d\omega^A d\omega^B,$$

where $r = |x|$, $\omega = x/|x|$, $x \in \mathbb{R}^n$. Let $\tilde{\Phi} : \mathbb{R}^n \to N$ be a smooth map such that

$$\tilde{\Phi}(x) = \begin{cases} \Phi(x) |x| \leq 1 \\ \Phi(0) |x| > 2 \end{cases}$$

Then inside the light cone $t^2 - |x|^2 = 0$ the unique solution to the Cauchy problem for a wave map $U : M \to N$ with the following smooth, finite energy initial data prescribed at $t = -1$:

$$U(x, -1) = \tilde{\Phi}(x), \qquad U_t(x, -1) = x^i \partial_i \tilde{\Phi}(x),$$

is nothing but

$$U(x, t) = \Phi\left(\frac{x}{-t}\right),$$

which means that U will undergo a gradient blowup at $t = 0$. However, we see that in order for this to be considered a true formation of singularity we need $\tilde{\Phi}$ and therefore Φ to be smooth at the boundary of the unit ball, which corresponds to the ideal boundary of the hyperbolic space $\mathbb{H}^n$. Although the Dirichlet problem at infinity for harmonic maps of the hyperbolic space has been solved (see [3]), boundary regularity is, to the best of our knowledge, an open problem. Following Shatah's work [12], in [13] the authors used equivariance to reduce this elliptic PDE problem to the study of an ODE, and were able to prove the existence of smooth nontrivial solutions in the dimension $n = 3$. We will take the same approach here in treating the problem in higher dimensions.

1.4. Main Results

The following theorem extends the results stated in § 1.1 regarding singular solutions of the Cauchy problem for wave maps:

THEOREM 1.1. – *Let M be the Minkowski space $\mathbb{R}^{n,1}$ with $n \geq 4$ and let N be a rotationally symmetric Riemannian manifold of dimension d, with a metric of the form (1.1), where g, in addition to (1.2)-(1.3) satisfies one of the following: Either*

$$g'(x) > 0 \; \forall x > 0 \text{ and } f := gg' \text{ satisfies (2.19)}, \qquad (1.10)$$

or else,

$$\exists a > 0 \text{ s.t. } g'(a) = 0 \text{ and } g''(a) < 0. \qquad (1.11)$$

Consider the Cauchy problem for a wave map $U = u \cdot \theta : M \to N$. If the dimension d is large enough, then

1. *There exist smooth, compactly supported initial data such that the corresponding unique smooth solution to (1.4) develops a singularity in finite time.*

2. *If g satisfies (1.11), then there exist Cauchy data (U_0, U_1) which belong to $H^s_{\text{loc}} \times H^{s-1}_{\text{loc}}$ for all $s < n/2$ and such that the corresponding Cauchy problem for (1.4) has locally nonunique solutions in H^s_{loc}.*

3. *If n is even, then there exists a sequence of smooth compactly supported initial data (U_0^j, U_1^j) for (1.4), with*

$$\|U_0^j\|_{H^s} + \|U_1^j\|_{H^{s-1}} \to 0 \text{ as } j \to \infty,$$

for all $s < n/2$, such that at the same time $T_j \to 0$, where T_j is the lifespan of the corresponding smooth solution to (1.4).

We will show that the following is a consequence of the above:

THEOREM 1.2. – *There exist smooth Riemannian manifolds N with all sectional curvatures everywhere negative, such that the Cauchy problem for a wave map from $\mathbb{R}^{7,1}$ into N admits a self- similar solution which develops from smooth Cauchy data of compact support and has a gradient blow-up in finite time.*

We also have the analogous results for the Cauchy problem of Yang-Mills fields:

THEOREM 1.3. – *Consider the Cauchy problem in the temporal gauge for a connection A on the principal bundle $\mathbb{R}^{N,1} \times SO(N)$ satisfying (1.8).*

1. *For $5 \leq N \leq 8$, there exist a family of smooth, compactly supported Cauchy data such that the corresponding unique smooth solution develops a singularity in finite time.*

2. *For $5 \le N \le 8$, there exist Cauchy data $(\bar{A}, \bar{E}) \in H_{\text{loc}}^s \times H_{\text{loc}}^{s-1}$ for all $s < (N-2)/2$ such that the corresponding Cauchy problem has locally nonunique solutions in H_{loc}^s for all time*

3. *For $N = 6, 8$ there exists a sequence of smooth, compactly supported initial data $(\bar{A}^j, \bar{E}^j)$, with*

$$\|\bar{A}^j\|_{H^s} + \|\bar{E}^j\|_{H^{s-1}} \to 0 \text{ as } j \to \infty,$$

for all $s < (N-2)/2$, such that at the same time $T_j \to 0$ where T_j is the lifespan of the corresponding smooth solution.

The proof of Theorem 1.1 consists of constructing self-similar solutions for the corotational wave map problem in the case n is odd (Theorems 2.2 and 2.4), and then lifting those to create singular travelling waves in one dimension higher, to cover the case when n is even (Theorem 3.1). Theorem 1.3 is essentially a consequence of Theorem 1.1 because of the reduction of the Yang-Mills equations to wave maps as seen in the above.

2. SINGULAR SOLUTIONS IN ODD DIMENSIONS

It is easily seen that equations (1.6) and (1.9) will have a similarity solution $u(\rho, t) = \phi(\rho/t)$ if ϕ satisfies the following ordinary differential equation in $r = \rho/t$:

$$\phi'' + \left(\frac{n-1}{r} + \frac{(n-3)r}{1-r^2}\right)\phi' = \frac{k}{r^2(1-r^2)}f(\phi), \qquad (2.1)$$

where $f := gg'$ and $k := \ell(\ell + n - 2)$, $\ell \in \mathbb{N}$. In the interval $[0, 1]$ the only possible singularities for a solution to (2.1) are clearly $r = 0$ and $r = 1$.

We assume that f satisfies:

(H1) $f \in C^\infty(\mathbb{R}, \mathbb{R})$, $f(0) = 0$, $f'(0) = 1$.

(H2) $F(x) \le Cx^2 \qquad \forall x \in \mathbb{R}$, where $F(x) \stackrel{\text{def}}{=} \int_0^x f(s)ds$.

We note that (H1) holds for the nonlinearity in the corotational wave map equation (1.6) as well as the equivariant Yang-Mills equation (1.9). Moreover, once the solution to (2.1) is proved to be bounded, (H2) can be satisfied by modifying f for large x. The following, third hypothesis is however not satisfied by the nonlinearity in (1.9), and it only holds for the one in (1.6) if the target N is *geodesically convex*:

(H3) $xf(x) > 0 \quad \forall x \ne 0$.

We prove the existence of smooth solutions to (2.1) in the following way: If (H3) holds, we use a shooting argument to construct a solution that satisfies the appropriate boundary conditions at $r = 1$. If (H3) does not hold, we take a variational approach, proving existence of a solution by minimizing a suitably renormalized functional. This plan is carried out in §2.3 and §2.4. But first we need to study the behavior of the solutions of (2.1) near the two endpoints.

2.1. The Behavior at $r = 0$

LEMMA 2.1. – *Let ϕ be a solution to (2.1) in (0,1) which is bounded near $r = 0$. Then we must have $f(\phi(0)) = 0$.*

Proof. – Suppose this is not true, then ϕ satisfies the equation

$$\phi'' + \frac{n - 1 - 2r^2}{r(1 - r^2)}\phi' = \frac{b(r)}{r^2}, \qquad (2.2)$$

where $b(r) := kf(\phi(r))/(1 - r^2)$, so that $b_0 := b(0) \neq 0$. Let μ be the integrating factor for this equation. Thus

$$\frac{\mu'}{\mu} = \frac{n - 1 - 2r^2}{r(1 - r^2)},$$

so that $\mu = \frac{r^{n-1}}{(1-r^2)^{(n-3)/2}} = r^{n-1} + \cdots$ for r small. Thus we have

$$(\mu\phi')' = \frac{b(r)}{r^2}\mu(r),$$

which upon integration on (a, r) yields

$$\mu(r)\phi'(r) = \mu(a)\phi'(a) + \int_a^r \mu(s)\frac{b(s)}{s^2}ds.$$

Integrating one more time we obtain

$$\phi(r) = \phi(a) + \mu(a)\phi'(a)\int_a^r \frac{ds}{\mu(s)} + \int_a^r \frac{\mu(s)}{s^2}b(s)\int_s^r \frac{dt}{\mu(t)}.$$

Multiplying by r^{n-2} and letting $r \to 0$ we obtain

$$\mu(a)\phi'(a) = \frac{b_0}{n - 2}a^{n-2} + \cdots,$$

so that $(n - 2)\phi'(a) = b_0/a + \cdots$ and thus

$$\phi(y) - \phi(x) = \frac{b_0}{n - 2} \ln \frac{y}{x} + \cdots,$$

which is a contradiction since ϕ is assumed to be bounded near the origin. ∎

LEMMA 2.2. – *Let ϕ be a bounded solution to (2.1) with $\phi(0) = 0$ and $\phi(r) \not\equiv 0$ for small $r > 0$. Then $\phi(r)/r^\ell$ is smooth in a neighborhood of $r = 0$.*

Proof. – We multiply (2.1) by ϕ and rearrange terms to obtain

$$\left(\phi\phi' + \frac{n - 1}{2r}\phi^2\right)' = \frac{k}{r^2(1 - r^2)}\phi f(\phi) - \frac{n - 1}{2r^2}\phi^2 + \phi'^2 - \frac{(n - 3)r}{1 - r^2}\phi\phi'.$$

By completing the square in the last two terms in the above we have

$$\phi'^2 - \frac{(n - 3)r}{1 - r^2}\phi\phi' = \left(\phi' - \frac{n - 3}{2(1 - r^2)}r\phi\right)^2 - \frac{(n - 3)^2\phi^2}{4(1 - r^2)^2}r^2 \geq -Cr^2.$$

By (H1) we know that $f(\phi) \geq 3\phi/4$ for ϕ small, *i.e.* for small r. Since $k \geq n - 1$ we see that

$$\left(\phi\phi' + \frac{n - 1}{2r}\phi^2\right)' \geq c\frac{\phi^2}{r^2},$$

for $0 < r \leq b$ with b small enough. Integrating the above on $[a, b]$ we obtain

$$\frac{1}{2}(\phi^2)'(b) + \frac{n - 1}{2b}\phi^2(b) - \frac{1}{2}(\phi^2)'(a) - \frac{n - 1}{2a}\phi^2(a) \geq c\int_a^b \frac{\phi^2}{r^2}dr.$$

Letting $a \to 0$ and using the fact that ϕ^2 has to be increasing close to $r = 0$, we obtain

$$\int_0^b \frac{\phi^2}{r^2}dr \leq C(b), \tag{2.3}$$

for $b > 0$ small enough.

Now let $\phi = r^\ell w$. Then w satisfies

$$w'' + \frac{n + 2\ell - 1}{r}w' = \frac{k}{r^{\ell+2}}(f(\phi) - \phi) + \frac{\ell(\ell + 1)w - (n - 3)rw'}{1 - r^2}.$$

We view this as

$$\Delta w = R(r, w),$$

where Δ is the Laplacian on $\mathbb{R}^{n+2\ell}$. By assumption, ϕ is small for small r, and by (H1), $f(\phi) - \phi = O(\phi^2)$ for small ϕ, so that we have

$$|R(r, w)| \leq C(1 + |w|^{1+\frac{2}{\ell}}).$$

Let $p := 2 + \frac{n+1}{\ell}$ and let B denote the ball of radius b centered at the origin in $\mathbb{R}^{n+2\ell}$. Then by (2.3),

$$\|w\|_{L^p(B)}^p = \int_0^b |w|^{2+\frac{n+1}{\ell}} r^{n+2\ell-1} dr = \int_0^b \frac{\phi^{2+\frac{n+1}{\ell}}}{r^2} dr \leq C(b)$$

so that by elliptic regularity theory, $w \in W^{2,q}(B)$ with $q = p/(1 + \frac{2}{\ell})$ and by the Sobolev imbedding theorem, $w \in L^s(B)$ with $\frac{1}{s} = \frac{1}{q} - \frac{2}{n+2\ell}$, which implies that $s > p$, $i.e.$ the regularity of w can be improved and by repeating the argument we can eventually show that w is smooth in a neighborhood of zero. ∎

2.2. The Behavior at $r = 1$

If $n = 3$, then equation (2.1) dictates that for a solution ϕ to be smooth at $r = 1$ we must have $f(\phi(1)) = 0$. For $n \geq 4$ however, the situation is different. If we multiply (2.1) by $1 - r^2$ and set $r = 1$, we do not obtain a value for $\phi(1)$ but only a condition that has to be satisfied at the boundary if ϕ is to be smooth there:

$$(n - 3)\phi'(1) = kf(\phi(1)).$$

More generally, let $\phi_j \stackrel{\text{def}}{=} \dfrac{d^j \phi}{dr^j}$. Equation (2.1) can be written as

$$p\phi_0'' + q\phi_0' = kf(\phi_0),$$

with $p \stackrel{\text{def}}{=} r^2 - r^4$ and $q \stackrel{\text{def}}{=} (n-1)r - 2r^3$. Thus the equation satisfied by ϕ_j is

$$p\phi_j'' + q_1^j \phi_j' + q_2^j \phi_j + q_3^j \phi_{j-1} + q_4^j \phi_{j-2} = k \frac{d^j}{dr^j} f(\phi(r)), \qquad (2.4)$$

where

$$q_i^j \stackrel{\text{def}}{=} \binom{j}{i} \frac{d^i p}{dr^i} + \binom{j}{i-1} \frac{d^{i-1} q}{dr^{i-1}}, \qquad 1 \leq i \leq 4.$$

Let

$$a_j \overset{\text{def}}{=} \phi_j(1), \qquad f_j \overset{\text{def}}{=} \frac{d^j}{d\phi^j}\bigg|_{\phi=a_0} f(\phi).$$

By setting $r = 1$ in (2.4) we see that a necessary condition for the smoothness of ϕ at $r = 1$ is that

$$\mathfrak{B}_j = 0, \qquad j = 0, 1, 2, \dots$$

where

$$\mathfrak{B}_j := \sum_{i=1}^{4} c_i^j a_{j-i+2} - \ell \mathfrak{F}_j(a_1, \dots, a_j), \qquad (2.5)$$

with $c_i^j := q_i^j(1)$ and in particular $c_1^j = n - 3 - 2j$. Moreover,

$$F_0 := f_0, \qquad F_j := \frac{d^j}{dr^j}\bigg|_{r=1} f(\phi(r)) = \sum_{i=1}^{j} f_i p_i^j, \quad j \geq 1,$$

where p_i^j is a polynomial of order i in $a_1, \dots, a_j$ which can be obtained from the following recursion relation:

$$p_1^j := a_j, \qquad p_i^j(a_1, \dots, a_j) := a_1 p_{i-1}^{j-1} + \sum_{l=1}^{j-1} a_{l+1} \frac{\partial p_i^{j-1}}{\partial a_l}$$

In particular, $F_j = f_1 a_j + \cdots + f_j(a_1)^j$. Let

$$m := \frac{n-3}{2}.$$

The following proposition is immediate:

PROPOSITION 1. – *If n is odd, the set of boundary conditions $\mathfrak{B}_j = 0$ for $j = 0, \dots, m$ will form a closed algebraic system for $a_0, \dots, a_m$. The linear part of this system is lower-triangular, and thus by forward substitution we can reduce the system to a single nonlinear equation for $a_0 = \phi(1)$ of the following form: Let $\alpha = (\alpha_0, \dots, \alpha_m)$ denote a multi-index and let f^α be the function $f^{\alpha_0}(f')^{\alpha_1} \dots (f^{(m)})^{\alpha_m}$. We can find numbers $b_{j,\alpha}$ such that the equation for a_0 is*

$$p_m(a_0, k) := f_0 \sum_{j=0}^{m} C_j(a_0) k^j = 0,$$

where

$$C_j(s) = \sum_{|\alpha| \le j} b_{j,\alpha} f^\alpha(s), \qquad 0 \le j \le m. \tag{2.6}$$

Remark 3. – The first nontrivial case is $n = 5$, where we see that the two boundary conditions (2.5) for $j = 0, 1$ imply

$$(kf_1 + 2)f_0 = 0.$$

As a further example, if $n = 7$, then $m = 2$ and the boundary conditions are

$$4a_1 = kf_0$$
$$2a_2 = kf_1 a_1$$
$$-10a_2 - 12a_1 = kf_1 a_2 + kf_2 a_1^2,$$

which again reduce to

$$((f_2 f_0 + 2f_1^2)k^2 + 20f_1 k + 48)f_0 = 0. \tag{2.7}$$

LEMMA 2.3. – *If ϕ is a solution of (2.1) with $\phi \in C^{\frac{1}{2}}((0,1])$, then $\phi \in C^{m,\frac{1}{2}}((0,1])$ and $\mathfrak{B}_j = 0$ for $j = 0, \dots, m - 1$.*

Proof. – We proceed by induction. Fix $j \in \{0, \dots, m-1\}$ and assume $\phi \in C^{j,\frac{1}{2}}$. Then from (2.4) we have

$$\phi_j'' + \frac{q_1^j}{p} \phi_j' = \frac{1}{p} K_j(1) + O\left(\frac{1}{\sqrt{1-r}}\right),$$

where $K_j(r) \overset{\text{def}}{=} - q_2^j \phi_j - q_3^j \phi_{j-1} - q_4^j \phi_{j-2} + k d^j f(\phi(r))/dr^j$, and thus $K_j \in C^{\frac{1}{2}}$. Let μ be the integrating factor for this equation. Thus

$$\frac{\mu'}{\mu} = \frac{q_1^j}{p} = \frac{j - m}{1 - r} + \cdots,$$

near $r = 1$, so that

$$\mu = (1 - r)^{j-m} + \cdots,$$

and

$$(\mu \phi_j')' = \frac{\mu}{p} K_j(1) + O((1 - r)^{j-m-\frac{1}{2}}).$$

Integrating this on (a, r) and dividing by μ we obtain

$$\phi_j'(r) - \frac{(1-r)^{m-j}}{(1-a)^{m-j}} \phi_j'(a) = \frac{1}{2(j-m)} K_j(1) + O(\sqrt{1-r}),$$

which implies that $\phi'_j(1) = \frac{1}{2(j-m)}K_j(1)$, i.e. $\mathfrak{B}_j = 0$, and that $\phi'_j(r) - \phi'_j(1) = O(\sqrt{1-r})$, i.e., $\phi_{j+1} \in C^{\frac{1}{2}}((0,1])$. ∎

Remark 4. – If n is even, a similar argument shows that a solution which is at least $C^{\frac{1}{2}}$ near $r = 1$ is actually in $C^{\frac{n}{2}-1,\alpha}$, for all $\alpha < \frac{1}{2}$.

LEMMA 2.4. – *If ϕ is as in Lemma 2.3 with $\mathfrak{B}_m = 0$, then $\phi \in C^{m+l}((0,1])$ for all $l \geq 1$.*

Proof. – First we show that if $\mathfrak{B}_m = 0$ holds, then $\phi \in C^{m+1,\frac{1}{2}}((0,1])$: Equation (2.4) for $j = m$ can be written as

$$p\phi''_m + Q\phi'_m + R\phi_m = S,$$

with $Q \overset{\text{def}}{=} q_1^m = (2n-4)r(1-r^2)$, $R \overset{\text{def}}{=} q_2^m(r) - kf'(\phi(r))$ and S containing the remaining terms, which only depend on $\phi_0, \ldots, \phi_{m-1}$. Thus by the previous Lemma, $S \in C^{1,\frac{1}{2}}$. In this notation, the boundary condition $\mathfrak{B}_m = 0$ reads

$$R(1)\phi_m(1) = S(1).$$

Thus subtracting the two lines above we obtain

$$\begin{aligned}
r^2\phi''_m + (2n-4)r\phi'_m &= \frac{S(r)-S(1)}{1-r^2} + \frac{R(r)-R(1)}{1-r^2}\phi_m(r) \\
&\quad + \frac{\phi_m(r)-\phi_m(1)}{1-r^2}R(1) \\
&:= I + II + III.
\end{aligned} \tag{2.8}$$

It is easy to see that $I, II \in C^{\frac{1}{2}}$. Let us set $y \overset{\text{def}}{=} \phi_m(r) - \phi_m(1)$. Thus $y(1) = 0$ and by Lemma 2.3, $|y| \leq c\sqrt{1-r}$. From (2.8) we have

$$(r^{2n-4}y')' = O\left(\frac{1}{\sqrt{1-r}}\right),$$

which upon integration yields $y'(r) - y'(1) = O(\sqrt{1-r})$, i.e., $y \in C^{1,\frac{1}{2}}$ and thus $\phi \in C^{m+1,\frac{1}{2}}$.

We now proceed by induction on l. Suppose $\phi \in C^{m+l-1}((0,1])$ for $l \geq 2$. Going back to (2.8) we then have $I, II \in C^{l-1}$ and $III \in C^{l-2}$. Thus the right hand side of the equation for ϕ_m is in C^{l-2} and therefore $\phi_m \in C^l$. ∎

2.3. The Convex Case

LEMMA 2.5. – *Let f satisfy* (H1) *and* (H2). *Then for every $c \in \mathbb{R}$ there exists a unique solution $\phi \in C^\infty([0,1)) \cap C^{1/2}([0,1])$ of* (2.1) *such that*

$$\lim_{r \to 0} \frac{\phi(r)}{r^\ell} = c, \qquad \lim_{r \to 0} \frac{\phi'(r)}{r^{\ell-1}} = \ell c. \tag{2.9}$$

If, in addition, f satisfies (H3), *then $\phi > 0$ and $\phi' > 0$ on* $(0,1)$.

Proof. – Set $\phi = r^\ell w$. Equation (2.1) is then equivalent to

$$w'' + \left(\frac{2(\ell+1)(1-r^2) + n - 3}{r(1-r^2)} \right) w' - \frac{\ell(\ell+1)}{1-r^2} w$$
$$= \frac{k}{r^{\ell+2}(1-r^2)} (f(r^\ell w) - r^\ell w). \tag{2.10}$$

Let

$$h(x) := \frac{f(x) - x}{x^2} = \int_0^1 \int_0^1 s f''(s x \sigma) \, d\sigma ds. \tag{2.11}$$

It follows that $h \in C^\infty(\mathbb{R}, \mathbb{R})$ and equation (2.10) becomes

$$w'' + \left(\frac{2(\ell+1)(1-r^2) + n - 3}{r(1-r^2)} \right) w' - \frac{\ell(\ell+1)}{1-r^2} w = k \frac{r^{\ell-2}}{1-r^2} w^2 h(r^\ell w). \tag{2.12}$$

Note that

$$\left(\frac{r^{2(\ell+1)+n-3}}{(1-r^2)^{\frac{n-3}{2}}} w' \right)'$$
$$= \frac{r^{2(\ell+1)+n-3}}{(1-r^2)^{\frac{n-3}{2}}} \left(w'' + \left(\frac{2(\ell+1)(1-r^2) + n - 3}{r(1-r^2)} \right) w' \right);$$

and thus w solves equation (2.12) with initial conditions

$$w(0) = c, \qquad w'(0) = 0,$$

if and only if w solves the integral equation

$$w(r) = c + r^2 \int_0^1 \int_0^1 s \sigma^{2(\ell+1)+n-3} \left(\frac{1 - r^2 s^2}{1 - r^2 s^2 \sigma^2} \right)^{\frac{n-3}{2}}$$
$$\times H(rs\sigma, w(rs\sigma)) \, d\sigma ds, \tag{2.13}$$

where

$$H(x, w) \overset{\text{def}}{=} \frac{\ell(\ell+1)}{1-x^2} w + k \frac{x^{\ell-2}}{1-x^2} w^2 h(x^\ell w).$$

Equation (2.13) is solved locally by a contraction mapping argument. It follows that there exists a unique, maximal solution $w \in C^2([0, R^*))$ (with $0 < R^* \leq 1$) of (2.12) such that $w(0) = c$ and $w'(0) = 0$. The solution w is maximal in the sense that if $R^* < 1$, then $w(r) \to \infty$ as $r \nearrow R^*$. Furthermore, since $h \in C^\infty(\mathbb{R}, \mathbb{R})$ it follows easily from (2.13) that $w \in C^\infty([0, R^*))$. Therefore, there exists a unique, maximal solution $\phi \in C^\infty([0, R^*))$ (with $0 < R^* \leq 1$) of (2.1) such that (2.9) is satisfied. ϕ is maximal in the sense that if $R^* < 1$, then $\phi(r) \to \infty$ as $r \nearrow R^*$. Multiplying equation (2.1) by $r^2(1-r^2)\phi'$, we obtain

$$\frac{d}{dr}\left(\frac{r^2(1-r^2)}{2}\phi'^2 - kF(\phi)\right) + (n-2)r\phi'^2 = 0. \qquad (2.14)$$

Integrating the above identity on $(0, r)$, with $0 < r \leq R^*$, we obtain

$$\frac{r^2(1-r^2)}{2}\phi'(r)^2 + (n-2)\int_0^r s\phi'(s)^2 ds = kF(\phi(r)). \qquad (2.15)$$

It follows from (2.15) and (H2) that

$$|\phi'| \leq K \frac{|\phi|}{\sqrt{1-r}}, \qquad (2.16)$$

for $R^*/2 \leq r < R^*$, with $K^2 = \frac{16kC}{R^{*2}(2+R^*)}$. Since $(1-r)^{-1/2}$ is integrable on $(0, 1)$, this implies that ϕ is bounded on $[0, R^*)$; and so, $R^* = 1$ and ϕ is bounded on $[0, 1)$. Estimate (2.16) now implies that there exists K' such that

$$|\phi'| \leq K'(1-r)^{-1/2},$$

which implies that $\phi \in C^{1/2}([0, 1])$.

It also follows from (2.9) that $\phi' > 0$ for r small. Suppose that ϕ' has a first zero $r_0 \in (0, 1)$. It follows from (2.1) and (H3) that $\phi''(r_0) > 0$, which is absurd. ∎

Remark 5. – If we now let $r \to 1$ in (2.15) we obtain

$$(n-2)\int_0^1 s\phi'(s)^2 ds = kF(\phi(1)).$$

Thus we see that if $\phi(1) = 0$, then for $n \neq 2$, ϕ must be the constant solution $\phi \equiv 0$. Therefore for $n \neq 2$ there are no nontrivial solutions to (2.1) with $\phi(0) = \phi(1) = 0$.

LEMMA 2.6. – *Suppose that f satisfies* (H1-H3) *and that there exists $s_0 > 0$ such that $C_m(s_0) < 0$, with C_m as in* (2.6). *Then for $k > 0$ large enough there exists $s_* \in (0, s_0)$ such that if there exists a solution ϕ to* (2.1) *with $\phi(1) = s_*$, then $\mathfrak{B}_m = 0$ and hence ϕ is in $C^\infty([0, 1])$.*

Proof. – We need to show that the equation $p_m(s, k) = 0$ has a solution in s for k fixed and large enough. Notice that by (H3), $f_0 \neq 0$ unless $a_0 = 0$, which by virtue of Remark 5 leads to the trivial solution $\phi \equiv 0$. A long but straightforward calculation shows that C_m, the coefficient of the highest degree in p_m, has the following form

$$C_m(s) = \sum_{\alpha \in A} c_\alpha f^\alpha,$$

where the index set A is as follows:

$$A \overset{\text{def}}{=} \left\{ \alpha \mid \sum_{i=0}^{m} \alpha_i = m, \text{ and } \sum_{i=0}^{m} i\alpha_i = m \right\}.$$

We observe that for $\alpha \in A$ if $\alpha_0 = 0$, then $\alpha_2 = \alpha_3 = \cdots = \alpha_m = 0$ as well, and thus $\alpha_1 = m$. Now because of this and (H1) we have that $C_m(0) > 0$. Hence for large enough k, $p_m(0, k) > 0$.

Now suppose that there exists $s_0 > 0$ such that $C_m(s_0) < 0$. Then for large enough k, $p_m(s_0, k) < 0$. Since p_m is obviously continuous in s it thus follows that for each large enough k there exists $s_* \in (0, s_0)$ such that $p_m(s_*, k) = 0$, and thus the boundary condition $\mathfrak{B}_m = 0$ is satisfied, so that by Lemma 2.4 the corresponding solution is in $C^\infty([0, 1])$. ∎

We now need to show that there exists a $c > 0$ such that the solution to (2.1)-(2.9) achieves $\phi(1) = s_*$. This is done in the course of the following four Lemmas. For notational convenience, we denote by ϕ_c the solution of (2.1)-(2.9).

LEMMA 2.7. – *The mapping $c \mapsto \phi_c$ is continuous as a map of $[0, \infty) \to C([0, 1])$.*

Proof. – We proceed in three steps:

Step 1. – For every $0 < \epsilon < 1$, the mapping $c \mapsto \phi_c$ is continuous from $[0, \infty) \to C([0, 1 - \epsilon])$. This follows from a classical continuous dependence argument applied to equation (2.10).

Step 2. – If $c_n \to c$, then $\phi_{c_n}(1) \to \phi_c(1)$: Let $\epsilon > 0$. Since ϕ_{c_n} is an increasing function of r, we have $\phi_{c_n}(1) > \phi_{c_n}(1 - \epsilon)$, and so $\liminf_{n \to \infty} \phi_{c_n}(1) > \phi_c(1 - \epsilon)$, by Step 1. Since ϵ was arbitrary, we

obtain $\liminf_{n\to\infty}\phi_{c_n}(1) \geq \phi_c(1)$. Therefore it remains to show that $\limsup_{n\to\infty}\phi_{c_n}(1) \leq \phi_c(1)$. To see this we argue by contradiction: Assume that there exists $\delta > 0$ and a subsequence, which we still denote by $(c_n)_{n\geq 0}$ such that $\phi_{c_n}(1) \geq \phi_c(1) + \delta$. Since $(c_n)_{n\geq 0}$ is bounded, it follows easily from (2.15) that there exists K independent of n such that $\phi_{c_n} \leq K$ and $\sqrt{1-r}\,\phi'_{c_n}(r) \leq K$ for $\frac{1}{2} \leq r < 1$. Therefore, there exists $\mu > 0$ independent of n such that $\phi_{c_n}(r) \geq \phi_{c_n}(1) - \mu\sqrt{1-r}$, for all $\frac{1}{2} \leq r < 1$. It follows that

$$\begin{aligned}
\phi_{c_n}(r) - \phi_c(r) &\geq \phi_{c_n}(1) - \phi_c(1) + \phi_c(1) - \phi_c(r) - \mu\sqrt{1-r} \\
&\geq \delta + \phi_c(1) - \phi_c(r) - \mu\sqrt{1-r} \\
&\geq \frac{\delta}{2} - \mu\sqrt{1-r} \geq \frac{\delta}{4},
\end{aligned}$$

if $1 - r$ is small enough. This contradicts Step 1.

Step 3. – Conclusion. We argue by contradiction. Suppose that there exists a $\delta > 0$ and a subsequence, still denoted by $(c_n)_{n\geq 0}$, and a sequence $(r_n)_{n\geq 0} \in [0,1]$ such that $|\phi_{c_n}(r_n) - \phi_c(r_n)| \geq \delta$. It follows from Step 1 that $r_n \to 1$ as $n \to \infty$, and so $\phi_c(r_n) \to \phi_c(1)$. Therefore, for n large enough, we have $|\phi_{c_n}(r_n) - \phi_c(1)| \geq \delta/2$. Since $\phi_{c_n}(r_n) \leq \phi_{c_n}(1)$, it follows from Step 2 that $\limsup_{n\to\infty}\phi_{c_n}(r_n) \leq \phi_c(1)$; and so, we have $\phi_{c_n}(r_n) \leq \phi_c(1) - \delta/2$ for n large enough. Let now $0 < \epsilon < 1$ and let n_0 be large enough so that $r_n \geq 1 - \epsilon$ for $n \geq n_0$. It follows that $\phi_{c_n}(1 - \epsilon) \leq \phi_{c_n}(r_n)$; and so, by Step 1, $\phi_c(1 - \epsilon) \leq \phi_c(1) - \delta/4$ for n large enough. But this is absurd since $\phi_c(1 - \epsilon) \to \phi_c(1)$ as $\epsilon \nearrow 1$. ∎

Let $\tilde{f} \in C^\infty(\mathbb{R}, \mathbb{R})$ verify assumption (H1-H3), and further assume that $\tilde{f}$ is increasing on $[0, \infty)$ and that $\tilde{f} \leq f$ on $[0, \infty)$. Given $c > 0$, let us denote by ϕ_c and $\tilde{\phi}_c$ the solutions to (2.1)-(2.9) corresponding to f and $\tilde{f}$ respectively.

LEMMA 2.8. – *For every $c \geq 0$, we have $\tilde{\phi}_c \leq \phi_c$.*

Proof. – Since $\tilde{f} \leq f$ it follows that $\tilde{h} \leq h$ where h is as in (2.11). Therefore, since $w = r^{-\ell}\phi_c$ and $\tilde{w} = r^{-\ell}\tilde{\phi}_c$ are constructed by a contraction mapping argument applied to equation (2.13), it follows that there exists $r_0 > 0$ such that $\tilde{\phi}_c \leq \phi_c$ and $\tilde{\phi}'_c \leq \phi'_c$ on $[0, r_0]$. Set now $z = \phi_c - \tilde{\phi}_c$. We have $z \geq 0$ and $z' \geq 0$ on $[0, r_0]$. Furthermore,

$$\begin{aligned}
z'' + \frac{n - 1 - 2r^2}{r(1-r^2)}z' &= \frac{k}{r^2(1-r^2)}\Big(f(\phi_c) - \tilde{f}(\phi_c) + \tilde{f}(\phi_c) - \tilde{f}(\tilde{\phi}_c)\Big) \\
&\geq \frac{k}{r^2(1-r^2)}\left(\frac{\tilde{f}(\phi_c) - \tilde{f}(\tilde{\phi}_c)}{\phi_c - \tilde{\phi}_c}\right)z.
\end{aligned}$$

Setting

$$A(r) = \int_{r_0}^{r} \frac{n - 1 - 2s^2}{s(1 - s^2)} ds,$$

we obtain

$$(e^A z')' \geq \frac{ke^A}{r^2(1 - r^2)} \left(\frac{\tilde{f}(\phi_c) - \tilde{f}(\tilde{\phi}_c)}{\phi_c - \tilde{\phi}_c} \right) z,$$

and so, since $z(r_0) \geq 0$ and $z'(r_0) \geq 0$,

$$e^A z' \geq \int_{r_0}^{r} \frac{ke^A}{s^2(1 - s^2)} \left(\frac{\tilde{f}(\phi_c) - \tilde{f}(\tilde{\phi}_c)}{\phi_c - \tilde{\phi}_c} \right) z \, ds,$$

for $r \geq r_0$. This implies easily that $z' \geq 0$. Hence $z \geq 0$ on $[r_0, 1]$ as well. ∎

LEMMA 2.9. – *The mappings $c \mapsto \tilde{\phi}_c$ and $c \mapsto \tilde{\phi}'_c$ are increasing as maps of $[0, \infty) \to C([0, 1])$.*

Proof. – Let $c' > c$ and set $z = \tilde{\phi}_{c'} - \tilde{\phi}_c$. We have $z > 0$ and $z' > 0$ for small r, since $z \sim (c' - c)r^\ell$ and $z' \sim \ell(c' - c)r^{\ell-1}$ near $r = 0$. On the other hand, we have

$$r^2(1 - r^2)z'' + ((n - 1)r - 2r^3)z' = k(\tilde{f}(\tilde{\phi}_{c'}) - \tilde{f}(\tilde{\phi}_c)).$$

If there exists r such that $z'(r) \leq 0$, then there exits r' such that $z(r') > 0$, $z'(r') = 0$, and $z''(r') \leq 0$, which is ruled out by the above equation. It follows that $z' > 0$ on $(0, 1]$ and thus $z > 0$ on $(0, 1]$. ∎

Finally, we have

LEMMA 2.10.

$$\lim_{c \to \infty} \tilde{\phi}_c(1) = +\infty. \tag{2.17}$$

Proof. – We argue by contradiction: Assume (2.17) is false. Let $\alpha(r) \overset{\text{def}}{=} \tilde{\phi}_\infty(r)$. Let $\delta > 0$ be fixed and small. From (2.16) we know $|\tilde{\phi}'_c(r)| \leq K|\alpha|/\sqrt{1 - r}$ for $r \geq \delta$, so that $|\tilde{\phi}_c(r) - \tilde{\phi}_c(1)| \leq A\sqrt{1 - r}$ for $r \geq \delta$ with A independent of c. Also by Lemma 2.3, since $\tilde{\phi}'_c(1) = k\tilde{f}(\tilde{\phi}_c(1))/(n - 3)$, we have that $|\tilde{\phi}'_c(r)| \leq B$ for $r \geq \delta$, with B independent of c. These uniform bounds imply that $\alpha \in C^1((0, 1])$, and that α solves the equation (2.1) in $(0, 1)$. Since $\alpha' \geq 0$, α is monotone and

thus can be extended as a bounded continuous function to $[0, 1]$. Hence by Lemma 2.1, $\tilde{f}(\alpha(0)) = 0$, which implies that

$$\alpha(0) = 0$$

since we are assuming that $\tilde{f}$ satisfies (H3). Now because of this and (2.9) we have that

$$\alpha'(0) = \infty. \tag{2.18}$$

On the other hand, by Lemma 2.2, α is C^1 near $r = 0$. Thus we obtain a contradiction and the Lemma is be proved. ∎

Let P_m be the homogeneous polynomial of degree m in $m + 1$ variables defined as follows:

$$P_m(x) \overset{\text{def}}{=} \sum_{|\alpha| \leq m} b_{m,\alpha} x^\alpha,$$

where $b_{m,\alpha}$ are as in Proposition 1.

THEOREM 2.1. – *Let f satisfy (H1–H3). If*

$$P_m(f(s), f'(s), \ldots, f^{(m)}(s)) < 0, \quad \text{for some } s > 0, \tag{2.19}$$

then for ℓ positive and large enough equation (2.1) with $k = \ell(\ell + n - 2)$ will have a solution $\phi \in C^\infty([0, 1])$ with

$$\phi(0) = 0, \qquad \phi(1) \leq s. \tag{2.20}$$

Furthermore, ϕ will be monotone increasing on $[0, 1]$.

Proof. – The hypothesis of the theorem implies that $C_m(s) < 0$, where C_m is as in (2.6). By Lemma 2.6 for ℓ large enough there exists a $s_* \in (0, s)$ such that a solution ϕ with $\phi(1) = s_*$ will be smooth in $[0,1]$.

By (H2) we can assume that $f(t) \geq \delta t$ for all $t \geq 0$ and some $\delta > 0$. Then, there exists a function $\tilde{f} \in C^\infty(\mathbb{R}, \mathbb{R})$ satisfying (H1-H3) such that $\tilde{f}$ is increasing and $\tilde{f} \leq f$ on $[0, \infty)$. By Lemma 2.10 there exists a c' such that $\tilde{\phi}_{c'}(1) > s_*$, and thus by Lemma 2.8, $\phi_{c'}(1) > s_*$, so that by Lemma 2.7, there exists a $0 < c < c'$ such that $\phi_c(1) = s_*$. ∎

We can now apply this result to wave maps:

THEOREM 2.2. – *Let $n \geq 5$ be an odd integer. If d is large enough, then the equation (1.4) for a wave map from $\mathbb{R}^{n,1}$ into N^d admits solutions which are self-similar inside a null cone, provided the function $f = gg'$ satisfies (2.19).*

These solutions develop from smooth Cauchy data of compact support, and suffer a gradient blow-up in finite time.

Proof. – The function $f = gg'$ satisfies (H1) and (H2) by assumption, and if (2.19) holds for $s = s_0$, then we can change f for $s > s_0$ such that (H3) is also satisfied. By Theorem 2.1 the resulting solution ϕ to (2.1) is monotone and stays below s_0, so that it will not be affected the modification.

With ℓ as large as required by Lemma 2.6, let $d = d(n, \ell)$, and let $\theta_0 : \mathbb{S}^{n-1} \to \mathbb{S}^{d-1}$ be the eigenmap constructed in Remark 1. Let Φ be the harmonic map of the hyperbolic space $\mathbb{H}^n$ into N^d defined by $\Phi(r, \omega) = \phi(r)\theta_0(\omega)$. We then use this Φ to construct a self-similar wave map as in §1.3. ∎

Notice that f is the derivative of a metric coefficient for the target manifold N. It thus follows that in seven or more dimensions, the condition on N for the existence of smooth solutions to (2.1) involves metric derivatives of order 3 or higher, and thus is not a curvature condition. Thus, in seven dimensions, we can find examples of *negatively* curved targets that admit self-similar wave maps:

Proof of Theorem 1.2. – Let $g(u) = u + \frac{10b^2}{3}u^3 - u^5$, with b to be determined later, and let N be the rotationally symmetric Riemannian manifold whose metric is given by (1.1). Thus $f(u) = g(u)g'(u) = u + 40b^2u^3/3 + \cdots + 5u^9$. If we set $h(u) := f(u)f''(u) + 2f'(u)^2$, then by Theorem 2.2 a sufficient condition for the existence of a self-similar wave map from $\mathbb{R}^{7,1}$ into N is that $h(s) < 0$ for some $s > 0$ and that $d = \dim N$ is large enough. Now $g''(u) = 20u(b^2 - u^2)$ so that $g''(b) = 0$ and we can easily check that

$$h(b) = 2(1 + 5b^4)\left(1 - 5b^4 - \frac{55}{3}b^8 + \frac{145}{9}b^{12}\right),$$

so that $h(b) < 0$ for $.61 \le b \le 1.075$. On the other hand, the metric on N is of the form (1.1). If we let $e_1 = \frac{\partial}{\partial u}$ and $e_j = \frac{\partial}{\partial \theta^j}$ for $j = 2, \ldots, d$ and denote by K_{ij} the sectional curvature of the two-plane spanned by e_i and e_j, then it is easy to compute that

$$K_{1i} = -\frac{g''}{g} \quad \text{and} \quad K_{ij} = \frac{1 - g'^2}{g^2} \quad \text{for} \quad 2 \le i < j \le d.$$

Hence N has all its sectional curvatures everywhere negative provided $g''(u) > 0$ and $|g'(u)| > 1$ for $u > 0$. Both these conditions are obviously satisfied for $u < b - \delta$, where δ is chosen sufficiently small such that $h(b - \delta)$ is still negative. For $u \ge b - \delta$ we can easily modify g so that they still hold. The self-similar solution constructed above will not be affected by this modification since it remains below $b - \delta$. ∎

2.4. The Nonconvex Case

Suppose now that the function f in (2.1) has a second zero, $f(\phi_*) = 0$ with $\phi_* > 0$, so that (H3) does not hold. For the Yang-Mills equation this is the case with $\phi_* = 1$, and for wave maps this implies that the target N is not geodesically convex. We present a variational argument similar to the one in [13], for the existence of a smooth solution to (2.1) with $\phi(1) = \phi_*$.

Let

$$E_\ell[\phi] := \frac{1}{2} \int_0^1 \{\phi'(r)^2 + \ell(\ell + n - 2)\frac{g^2(\phi) - g^2(\phi_*)}{r^2(1 - r^2)}\} \frac{r^{n-1}dr}{(1 - r^2)^{(n-3)/2}}. \tag{2.21}$$

It is easy to check that (2.1) is the Euler-Lagrange equation for a stationary point of this functional. Let

$$h(r) := \int_0^r \frac{s^{n-1}}{(1 - s^2)^{(n-3)/2}}ds,$$

so that the measure of integration in E_ℓ can be denoted by dh. Let the function space X_0 be defined as the closure of

$$\{u \in C^\infty([0,1]) \mid u(1) = 0, \|u'\|_{L^2(dh)} < \infty\},$$

with respect to the same norm. For $u \in X_0$ we have

$$|u(r)| = |\int_r^1 u'(s)ds| \le \left(\int_0^1 u'^2 dh(s)\right)^{1/2} \left(\int_r^1 \frac{(1 - s^2)^{(n-3)/2}}{s^{n-1}}ds\right)^{1/2}$$

$$\le C\|u'\|_{L^2(dh)}\frac{(1 - r)^{(n-1)/4}}{r^{(n-1)/2}}. \tag{2.22}$$

Now let

$$X_* := \{u : [0,1] \to \mathbb{R} \mid u - \phi_* \in X_0\}. \tag{2.23}$$

THEOREM 2.3. – *Let E_ℓ be as in (2.21), with $\ell \in \mathbb{N}$ and $g : \mathbb{R} \to \mathbb{R}$ a smooth function with the following properties:*

1. $g(0) = 0$, $g'(0) = 1$,
2. *There exists $\phi_* > 0$ such that $g'(\phi_*) = 0$ and $g'(s) > 0$ for $0 < s < \phi_*$,*
3. $g''(\phi_*) \neq 0$.

Let X_* be as in (2.23). Then for ℓ large enough such that $-4\ell(\ell + n - 2)g(\phi_*)g''(\phi_*) > (n - 2)^2$ the variational problem

$$\min_{u \in X_*} E_\ell[u] \tag{2.24}$$

has a smooth solution ϕ such that

$$\phi(r) - \phi_* \sim c(1-r)^{(n-1)/2} \text{ for } r \text{ near } 1, \tag{2.25}$$

$$0 \le \phi \le \phi_* \text{ and } \phi \text{ is monotone non-decreasing on } [0,1]. \tag{2.26}$$

$$\phi(r) \sim cr^\ell \text{ for } r \text{ near } 0. \tag{2.27}$$

Proof. – We begin by modifying g as follows: Let

$$\tilde{g}(s) := \begin{cases} g(s) & 0 \le s \le \phi_* \\ -g(-s) & -\phi_* \le s \le 0 \\ g(2\phi_* - s) & \phi_* \le s \le 2\phi_* \\ -g(2\phi_* + s) & -2\phi_* \le s \le -\phi_*. \end{cases}$$

We then extend $\tilde{g}$ outside the interval $[-2\phi_*, 2\phi_*]$ as a periodic function with period $4\phi_*$. Let $G(s) := \frac{1}{2}\tilde{g}^2(s)$ and let $G_* := G(\phi_*)$. We thus have that

1. $G \in C^2(\mathbb{R})$,
2. $0 \le G \le G_*$,
3. $G(s) = G(2\phi_* + s) = G(2\phi_* - s)$ for all $s \in \mathbb{R}$.

The theorem is proved with g replaced by $\tilde{g}$ in (2.21). We then prove that the minimizer is contained in $[0, \phi_*]$, and therefore is not affected by the modification.

We are going to show that E_ℓ is a C^1 functional over X_* which is bounded below.

Let

$$\mu(r) := \int_0^r \frac{s^{n-3}ds}{(1-s^2)^{(n-1)/2}}. \tag{2.28}$$

Now let $u, v \in X_*$. We have

$$|G(u) - G(v)| \le |G'(v)| \, |u - v| + K|u - v|^2. \tag{2.29}$$

We have $u - v \in X_0$ and $G'(v) \in X_0$ since $G'(\phi_*) = 0$. Therefore, by the above Lemma,

$$\int_0^1 \frac{|G(u) - G(v)|}{r^2(1-r^2)} dh \le \|G'(v)\|_{L^2(d\mu)}\|u-v\|_{L^2(d\mu)} + K\|u - v\|_{L^2(d\mu)}^2$$

$$\le C\|v'\|_{L^2(dh)}\|u' - v'\|_{L^2(dh)} + CK\|u' - v'\|_{L^2(dh)}^2,$$

which establishes the continuity of E_ℓ. Furthermore, for $\psi \in X_0$,

$$|\langle E_\ell'[u] - E_\ell'[v], \psi \rangle|$$

$$\le |\langle u' - v', \psi' \rangle_{L^2(dh)}| + c|\langle G'(u) - G'(v), \psi \rangle_{L^2(d\mu)}|$$

$$\le \|\psi'\|_{L^2(dh)}\|u' - v'\|_{L^2(dh)} + c\|\psi\|_{L^2(d\mu)}\|u - v\|_{L^2(d\mu)}$$

$$\le C\|\psi'\|_{L^2(dh)}\|u' - v'\|_{L^2(dh)},$$

establishing that E_ℓ is C^1.

To show that E_ℓ is bounded below, let $v \equiv \phi_*$ in (2.29). We then have that for all $u \in X_*$ and for $\epsilon > 0$ small enough,

$$\left| \int_{1-\epsilon}^1 (G(u) - G_*) d\mu \right| \leq c \int_{1-\epsilon}^1 |u - \phi_*|^2 d\mu$$
$$\leq c \|u'\|_{L^2(dh)}^2 \int_{1-\epsilon}^1 r^{-2} dr \leq \frac{1}{4} \|u'\|_{L^2(dh)}^2,$$

where we have used (2.22). On the other hand, we obviously have

$$\int_0^{1-\epsilon} (G(u) - G_*) d\mu \geq -C_\epsilon G_*,$$

so that putting all this together,

$$E_\ell[u] \geq \frac{1}{4} \|u'\|_{L^2(dh)}^2 - C_2, \tag{2.30}$$

which shows that E_ℓ is bounded below.

Let E_0 be the infimum of E_ℓ over X_*, and let u_n be a minimizing sequence. By (2.30) we have that u_n' is a bounded sequence in $L^2(dh)$, and thus we can extract a subsequence, also denoted by u_n such that

$$u_n \xrightarrow{\text{a.e.}} \phi \quad \text{and} \quad u_n' \xrightarrow{L^2} \phi',$$

for some $\phi \in X_*$. Let $\delta \in (0, \frac{1}{2})$ be fixed. If we write $E_\ell[u_n]$ in the following form:

$$E[u_n] = \frac{1}{2} \int_0^1 {u_n'}^2 dh + k \int_0^\delta G(u_n) d\mu + k \int_\delta^1 (G(u_n) - G_*) d\mu - k \int_0^\delta G_* d\mu,$$

we see that the first term is a convex functional and hence is lower semi-continuous with respect to weak convergence, while the second term is positive and hence lower semi-continuous by Fatou's lemma, and finally the third term is continuous by the dominated convergence theorem since for $\delta \leq r \leq 1$,

$$\left| \frac{G(u_n) - G_*}{r^2(1 - r^2)} \right| \leq C_\delta (1 - r)^{(n-3)/2},$$

which is integrable with respect to dh, and $G(u_n) \xrightarrow{\text{a.e.}} G(\phi)$. We thus have

$$E_0 = \liminf_{n \to \infty} E_\ell[u_n] \geq E_\ell[\phi]$$

which implies that $E_\ell[\phi] = E_0$, i.e. the infimum is achieved and the convergence is strong. ϕ solves (2.1) in $(0, 1)$, and by (2.22) is certainly $C^{\frac{1}{2}}$

near $r = 1$, so that by Lemma 2.3 it belongs to $C^{m,\frac{1}{2}}((0,1])$, $m = (n-3)/2$, and boundary conditions $\mathfrak{B}_j = 0$ are satisfied for $j = 0, \dots, m - 1$. This however, means that $\phi'(1) = \phi''(1) = \cdots = \phi^{(m)}(1) = 0$ since we have $f_0 = f(\phi_*) = 0$. It follows that $\mathfrak{B}_m = 0$ is automatically satisfied, and by Lemma 2.4 ϕ is C^∞ near $r = 1$. Moreover, (2.25) holds.

To prove (2.26), we use the following Lemma:

LEMMA 2.11. – *Let $\psi : [0, 1] \to \mathbb{R}$ be such that $E_\ell[\psi] < \infty$ and $\psi(1) = \phi_*$. Then one can always construct a continuous function $\tilde{\psi}$ that is monotone nondecreasing, $0 \leq \tilde{\psi} \leq \phi_*$, and $E_\ell[\tilde{\psi}] \leq E_\ell[\psi]$, with equality holding iff ψ is monotone.*

Proof. – We use the symmetries of the (modified) function g: Since by construction $G(s) = G(2j\phi \pm s)$ for all $j \in \mathbb{Z}$, by successive reflections of the graph of ψ across the lines $\psi = j\phi_*$ we can rearrange ψ to lie between 0 and ϕ_* without changing $E_\ell[\psi]$ at all. Now suppose ψ is not monotone. Let $s_{\max} \in [0, 1)$ be a point where ψ has a local maximum, and let $[a, b]$ be the largest interval contained in $[0, 1)$ and containing $s_{\max}$ such that $\psi(a) = \psi(b)$ and $\psi(s) \geq \psi(a)$ for $s \in [a, b]$. We define the rearrangement $\tilde{\psi}$ by setting $\tilde{\psi}(s) := \psi(a)$ for $s \in [a, b]$ and $\tilde{\psi}(s) := \psi(s)$ otherwise. Since g is increasing on $[0, \phi_*]$ it is easy to see that $E_\ell[\tilde{\psi}] < E_\ell[\psi]$. Repeating this procedure we can eliminate all local maxima of ψ so that the final rearrangement $\tilde{\psi}$ is monotone on $(0, 1)$. ∎

Finally, we need to address the behavior of ϕ near $r = 0$:

LEMMA 2.12. – *Let $\phi \in X_*$ be a minimizer of E_ℓ with $E_\ell[\phi] < 0$. Then $\phi(0) = 0$ and $\phi(r)/r^\ell$ is smooth near $r = 0$.*

Proof. – By the previous Lemma, $0 \leq \phi \leq \phi_*$, so that ϕ is bounded and thus by Lemma 2.1 we have $f(\phi(0)) = 0$, which means that either $\phi(0) = 0$ or else $\phi(0) = \phi_*$. But since ϕ is monotone and $\phi(1) = \phi_*$, the latter case implies that $\phi \equiv \phi_*$ on $[0, 1]$, which contradicts the assumption $E_\ell[\phi] < 0$. Hence $\phi(0) = 0$ and therefore by Lemma 2.2, $\phi \sim cr^\ell$ for r near 0. ∎

Since $E_\ell[\phi_*] = 0$, we prove that the minimizer ϕ has negative energy by taking a variation that decreases the energy of the constant map $\phi \equiv \phi_*$. First we need to show the following:

LEMMA 2.13.

$$\inf_{\eta \in X_0} \frac{\int_0^1 \eta'^2 dh}{\int_0^1 \eta^2 d\mu} = \left(\frac{n-2}{2} \right)^2.$$

Proof. – Let δ be the left hand side of the above. We first show that $\delta \geq (\frac{n-2}{2})^2$: For all $w \in X_0$ we have

$$\left(\int_0^1 \frac{w^2}{r^2(1-r^2)}\,dh\right)^{1/2} \leq \frac{2}{n-2}\left(\int_0^1 w'^2 dh\right)^{1/2}. \tag{2.31}$$

This is because

$$\int_0^1 w^2(r)\mu'(r)dr = -2\int_0^1 w(r)w'(r)\mu(r)dr := I.$$

The boundary term vanishes since $w^2(r) \sim (1-r)^{(n-1)/2}$ by (2.22), while $\mu(r) \sim (1-r)^{(n-3)/2}$. Moreover, by Hölder's inequality,

$$|I| \leq 2\|w'\|_{L^2(dh)}\|w\|_{L^2(d\mu)} \sup_{0 \leq r \leq 1}\left|\frac{\mu(r)}{\sqrt{\mu'(r)h'(r)}}\right|$$

$$= \frac{2}{n-2}\|w'\|_{L^2(dh)}\|w\|_{L^2(d\mu)},$$

which establishes (2.31). Now let

$$\eta_\epsilon(r) := \begin{cases} (r^{-\beta}-1)^\alpha & \epsilon \leq r \leq 1 \\ (\epsilon^{-\beta}-1)^\alpha & 0 \leq r \leq \epsilon \end{cases} \quad \text{where } \alpha := \frac{n+1}{4},\ \beta := 2\frac{n-2}{n+1}.$$

We can easily compute that

$$\delta \leq \frac{\int_0^1 \eta_\epsilon'^2 dh}{\int_0^1 \eta_\epsilon^2 d\mu} = \frac{(\frac{n-2}{2})^2(-\log\epsilon + O(1))}{-\log\epsilon + O(1)} \to (\frac{n-2}{2})^2 \text{ as } \epsilon \to 0,$$

which completes the proof. ∎

LEMMA 2.14. – *For ℓ large enough, there exists a function $\eta \in X_0$ such that*

$$E_\ell[\phi_* + \epsilon\eta] < 0,$$

for small enough $\epsilon > 0$.

Proof. – The constant function ϕ_* is a critical point of E_ℓ. We compute the second variation of E_ℓ at ϕ_*:

$$\frac{d^2}{d\epsilon^2}\bigg|_{\epsilon=0} E_\ell[\phi_* + \epsilon\eta] = \int_0^1 \eta'^2 dh - \nu\int_0^1 \eta^2 d\mu,$$

with $\nu := -kG''(\phi_*) = -\ell(\ell + n - 2)g(\phi_*)g''(\phi_*)$. Since by assumption, $g''(\phi_*) \neq 0$, we can take ℓ large enough so that $\nu > (\frac{n-2}{2})^2$. Then by the previous Lemma, there must exist an $\eta \in X_0$ such that

$$\int_0^1 {\eta'}^2 dh < \nu \int_0^1 \eta^2 d\mu,$$

i.e., the second variation is negative and hence E_ℓ can be made negative. ∎

Remark 6. – For the Yang-Mills equation (1.9) we have $\ell = 2$, so that $\nu = -(2n)(-1)(\frac{1}{2}) = n > (\frac{n-2}{2})^2$ only if $n \leq 7$. Indeed, the above argument shows that for $n \geq 8$, the second variation at ϕ_* is positive, so that the constant map $\phi \equiv \phi_*$ is a minimizer for E_ℓ.

This completes the proof of Theorem 2.3. ∎

We can now apply this result to wave maps and Yang-Mills fields:

THEOREM 2.4. – *Let g satisfy* (1.11). *For d large enough, equation* (1.4) *for a wave map $U : \mathbb{R}^{n,1} \to N^d$ admits solutions which are self-similar inside a null cone. Moreover, there exist Cauchy data which belong to H^s_{loc} for all $s < n/2$, such that the corresponding Cauchy problem has two distinct H^s_{loc} solutions.*

Proof. – Theorem 2.3 applies and produces a smooth function ϕ that solves (2.1) (with the original, unmodified f), for $0 \leq r \leq 1$. We can then proceed as in the proof of Theorem 2.2 to construct a self-similar wave map.

Now ϕ can be extended to $r > 1$ as a smooth solution of (2.1). Let

$$\lambda(r) := \frac{r^{n-1}}{(r^2 - 1)^{\frac{n-3}{2}}} = O(r^2) \text{ as } r \to \infty.$$

Then ϕ satisfies

$$(\lambda \phi')' = -\frac{r^{n-3}}{(r^2 - 1)^{\frac{n-1}{2}}} k f(\phi), \tag{2.32}$$

so that by (1.3) we have that for large r,

$$(\lambda \phi')' \leq \frac{C}{r^2}.$$

Integrating this on $(2, r)$ we get

$$\phi'(r) \leq \frac{C}{\lambda(r)} \leq \frac{C}{r^2},$$

so that integrating once more, we obtain $\phi(r) \leq C$, *i.e.*, ϕ remains bounded as $r \to \infty$. We can now use this information to rewrite (2.32) as:

$$(\lambda\phi')' = \frac{c}{r^2} + O\left(\frac{1}{r^3}\right),$$

so that upon integration we have

$$\phi'(r) = \frac{b}{r^2} + O\left(\frac{1}{r^3}\right),$$

and thus we obtain the following asymptotic behavior for ϕ as $r \to \infty$:

$$\phi(r) = a - \frac{b}{r} + O\left(\frac{1}{r^2}\right). \tag{2.33}$$

We prescribe the following Cauchy data for (1.6):

$$u(0,\rho) \equiv a, \qquad u_t(0,\rho) = \frac{-b}{\rho}. \tag{2.34}$$

Let Ω be the ball of radius R centered at the origin in $\mathbb{R}^n$ and let D_x denote the pseudo-differential operator $\sqrt{-\Delta_x}$, $x \in \mathbb{R}^n$. Thus

$$\|u_t(0,\rho)\|^2_{H^{s-1}(\Omega)} = |b| \|D_x^{s-1}\frac{1}{\rho}\|^2_{L^2(\Omega)} \leq C \int_0^R \rho^{n-1-2s} d\rho,$$

which shows that $u_t(0,\cdot) \in H^{s-1}_{\mathrm{loc}}(\mathbb{R}^n)$ for $s < n/2$. It is not hard to see (cf. [13]) that $u(t,\rho) = \phi(\rho/t)$ is a weak solution of the Cauchy problem (1.6-2.34). This solution is in fact smooth everywhere except at the origin of the spacetime and therefore is definitely in H^s_{loc} for all time.

One can also check that

$$\bar{u}(t,\rho) := \begin{cases} \phi(\rho/t) & \rho \geq t \\ \phi_* & \rho \leq t. \end{cases}$$

is another weak solution to the same Cauchy problem (cf. [13]). This solution is singular on the null cone $r = t$: Since $\phi^{(j)}(1) = 0$ for $j \leq m = \frac{n-3}{2}$, it follows that if ∂ denotes differentiation in a direction transverse to the null cone, then $\partial^{(n-1)/2}\bar{u}$ has a jump discontinuity across the cone, and thus $\bar{u} \in H^s_{\mathrm{loc}}$ for $s < n/2$. ∎

Remark 7. – Since $\bar{u}(t,0) = \phi_* \neq 0$, the map $\bar{U} : M \to N$ defined by $\bar{U}(t,\rho,\omega) = \bar{u}u(t,\rho)\theta_0(\omega)$ in addition to being singular on the null cone, is

also singular on the central line $\rho = 0$. It is easy to see that this singularity is once again in H^s for $s < n/2$.

And now the analogous result for Yang-Mills fields:

THEOREM 2.5. – *For $n = 3, 5, 7$, equation (1.9) admits solutions which are self-similar inside a null cone. These solutions develop in finite time from Cauchy data that are smooth and of compact support. Moreover, there exist Cauchy data in H^s_{loc} with $s < n/2$ such that the corresponding Cauchy problem for (1.9) has two distinct H^s_{loc} solutions.*

Proof. – Since here ℓ is fixed to be 2, in order for Theorem 2.3 to apply and give a smooth ϕ that solves (2.1) in $[0, 1]$, we need to restrict n, as in Remark 6. We then note that $g(u) = u(1 - u/2)$ satisfies (1.3), and the rest of the proof is the same as that of the previous theorem. ∎

3. SINGULAR SOLUTIONS IN EVEN DIMENSIONS

We now consider a wave map U from the Minkowski space $M = \mathbb{R}^{n+1,1}$ into a rotationally symmetric target N, where $n \geq 3$ is odd. We introduce cylindrical coordinates (t, r, z, ω) on M as follows: For $x \in M$, we let

$$t = x^0, \ \bar{x} = (x^1, \ldots, x^n), \ \rho = |\bar{x}|, \ \omega = \bar{x}/|\bar{x}|, \ z = x^{n+1}.$$

With (u, θ) coordinates on N as before, the corotational Ansatz reads: $u = u(t, \rho, z)$, $\theta = \theta(\omega)$, and u is a solution to

$$u_{tt} - u_{\rho\rho} - \frac{n-1}{\rho} u_\rho - u_{zz} + \frac{k}{\rho^2} f(u) = 0. \tag{3.1}$$

One obvious solution to the above which develops singularity in finite time is to take u to be self-similar in (ρ, t) and independent of z, *i.e.* $\bar{u}(t, \rho, z) = \phi(\rho/t)$ where ϕ is a solution to (2.1). This solution, however, at time $t = 0$ will develop a singularity along a whole line, and in particular will not be in $H^{(n+1)/2-\epsilon}$ at $t = 0$. This can be rectified by precomposing the map with a Lorentz boost: For $\beta > 1$ let L_β be the following Lorentz transformation: $L_\beta : M \to M$,

$$L_\beta(t, \rho, \omega, z) = \left(\frac{\beta t + z}{\sqrt{\beta^2 - 1}}, \rho, \omega, \frac{t + \beta z}{\sqrt{\beta^2 - 1}} \right).$$

We then let $u = \bar{u} \circ L_\beta$, *i.e.*,

$$u(t, \rho, z) = \phi\left(\frac{\sqrt{\beta^2 - 1}\rho}{\beta t + z} \right). \tag{3.2}$$

Note that L_β is an isometry of M, and thus wave maps are invariant under precomposition with L_β. Therefore (3.2) is a solution to (3.1). The effect of the boost is to tilt the line of singularity of $\bar{u}$ so that it is no longer contained in a time-slice, and thus (3.2) at each instant t is singular only at one point: $\rho = 0, z = -\beta t$, with the singularity developing at the speed $\beta > 1$. Such singular travelling waves have been used by Balabane [1], [2], Lindblad [9], and Lindblad-Sogge [10] to obtain counterexamples to global and local regularity results for nonlinear wave equations. In this section we prove the following theorem about singular solutions of (3.1). Similar results for the equivariant Yang-Mills equation follow easily.

THEOREM 3.1. – *Let $n \geq 3$ be an odd integer and let N be a rotationally symmetric Riemannian manifold of large enough dimension d and with a metric of the form (1.1) such that either (1.10-2.19) or (1.11) is true. Then the following hold:*

(1) *Equation (3.1) for a corotational wave map from $\mathbb{R}^{n+1,1}$ into N admits singular travelling wave solutions of the form (3.2).*

(2) *There exists a sequence (u_0^j, u_1^j) of smooth compactly supported Cauchy data for (3.1) such that*

$$\|u_0^j\|_{H^s} + \|u_1^j\|_{H^{s-1}} \to 0 \text{ as } j \to \infty,$$

for all $s < \frac{n+1}{2}$, while at the same time $T_j \to 0$, where T_j is the life-span of the corresponding smooth solution u^j.

(3) *If (1.11) holds, then there exists Cauchy data in H^s_{loc} with $s < \frac{n+1}{2}$ such that the corresponding Cauchy problem for (3.1) has two distinct H^s_{loc} solutions.*

Proof. – In order to have a Cauchy problem for (3.1) with initial data which are smooth and of compact support, we need to use a cut-off function. Let $a > 0$ and $\beta > 1$ be fixed. Let

$$T := \frac{a}{2\beta}, \qquad R := a\frac{\beta + 1}{2\beta}.$$

We prescribe Cauchy data at $t = 0$ which is supported in a ball centered at $x_0 := (0, \ldots, 0, -a) \in \mathbb{R}^{n+1}$. Let ϕ be a smooth solution to (2.1) provided by Theorem 2.1 or Theorem 2.3 and let

$$u(0, \rho, z) = u_0, \qquad u_t(0, \rho, z) = u_1,$$

where

$$u_0(\rho, z) := \phi\left(\frac{\sqrt{\beta^2 - 1}\rho}{z}\right)\chi(x),$$

$$u_1(\rho, z) := \frac{-\beta\sqrt{\beta^2-1}\rho}{z^2}\phi'\left(\frac{\sqrt{\beta^2-1}\rho}{z}\right)\chi(x),$$

(3.3)

with

$$\chi \in C_0^\infty(\mathbb{R}^{n+1}), \qquad \chi \equiv \begin{cases} 1 & \text{if } |x - x_0| \le R \\ 0 & \text{if } |x - x_0| \ge (a + R)/2 \end{cases} \qquad (3.4)$$

It is then clear that inside the following truncated past null cone

$$K_R(x_0) := \{(t, x) | 0 \le t \le R, 0 \le |x - x_0| \le R - t\}$$

the solution to the Cauchy problem for (3.1) with the above data is exactly (3.2). The first singularity of this solution takes place at time $t \le T$. This proves the first statement in the theorem (See Figure 1).
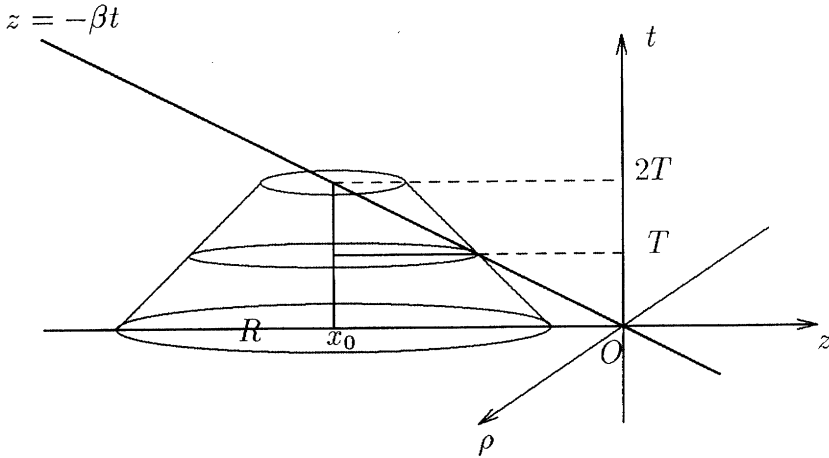

Figure 1.

Let $\psi(s) := \phi(1/s)$. Then by (2.27) and (2.33) we have

$$\psi(s) = \begin{cases} a - bs + \dots & s \text{ small} \\ cs^{-\ell} + \dots & s \text{ large} \end{cases}$$

so that $\psi \in H^s(\mathbb{R})$ for all $s > 0$. Let $\xi := \dfrac{\beta t + z}{\sqrt{\beta^2 - 1}\rho}$. Thus inside $K_R(x_0)$ the solution is $u(t, \rho, z) = \psi(\xi)$ and we have

$$\partial_\rho u = \psi'(\xi)\frac{-\xi}{\rho}, \qquad \partial_z u = \psi'(\xi)\frac{1}{\sqrt{\beta^2 - 1}\rho}.$$

Let $B_r(x_0)$ denote the ball of radius r centered at x_0 in $\mathbb{R}^{n+1}$. It is not hard to see that at time t, for β close to 1,

$$
\|D_x^s u\|_{L^2(B_{R-t}(x_0))}^2
$$
$$
\leq C \int_0^{R-t} \int_{-\sqrt{(R-t)^2 - \rho^2}}^{\sqrt{(R-t)^2 - \rho^2}} |D_\xi^s \psi|^2 \left(|\xi|^{2s} + \frac{1}{(\beta^2 - 1)^s} \right) dz \, \rho^{n-1-2s} \, d\rho
$$
$$
\leq \frac{C}{(\beta - 1)^{s-\frac{1}{2}}} \int_{-\infty}^{\infty} (|\xi|^{2s} + 1) |D_\xi^s \psi|^2 d\xi \int_0^{R-t} \rho^{n-2s} d\rho
$$
$$
\leq C \frac{(R - t)^{n+1-2s}}{(\beta - 1)^{s-\frac{1}{2}}}.
$$

This shows that the solution remains in H_{loc}^s for $s < \frac{n+1}{2}$. Now, given such an s, let $s' > s$ be such that $s' < \frac{n+1}{2}$. By the Sobolev inequality,

$$
\|u_0\|_{H^s} = \|D_x^s (\psi\chi)\|_{L^2} \leq \|D_x^{s'}(\psi\chi)\|_{L^q}, \qquad \frac{1}{q} = \frac{1}{2} + \frac{s' - s}{n + 1}.
$$

Let $p := \frac{n+1}{s'-s}$. By the "Leibnitz Rule" for fractional derivatives (cf. [11]), we then have

$$
\|D_x^{s'}(\psi\chi)\|_{L^q} \leq \|D_x^{s'}\psi\|_{L^2} \|\chi\|_{L^p} + \|D_x^{s'}\chi\|_{L^2} \|\psi\|_{L^p}.
$$

On the other hand,

$$
\|D_x^{s'}\chi\|_{L^2}^2 \leq C \int_R^{(a+R)/2} \frac{1}{(a - R)^{2s'}} r^n dr \leq C \frac{a^{n+1-2s'}}{(\beta - 1)^{2s'}},
$$

so that finally

$$
\|u_0\|_{H^s} \leq C \frac{a^{n+1-2s'}}{(\beta - 1)^{2s'}}.
$$

Let $\delta := n + 1 - 2s'$. If we now set $a = (\beta - 1)^{\frac{n+1}{\delta}}$ we then have $\|u_0\|_{H^s} \leq C(\beta - 1)^\delta \to 0$ as $\beta \to 1$. At the same time $T = \frac{a}{2\beta} \to 0$, proving the second statement in the theorem.

Finally, let χ be as in (3.4), and let us have the following Cauchy data for (3.1):

$$
u_0(\rho, z) = \phi_* \chi(\rho, z), \qquad u_1(\rho, z) = 0.
$$

Note that by Remark 7 the corresponding Cauchy data (U_0, U_1) for (1.4) is in H_{loc}^s for $s < \frac{n+1}{2}$. Now, inside the cone $K_R(x_0)$ an obvious solution

to the Cauchy problem for (3.1) with the above data is $u \equiv \phi_*$. For $t > T$ however, we also have another solution inside this cone, namely

$$\bar{u}(t,\rho,z) = \begin{cases} \phi_* & \beta t + z \leq \sqrt{\beta^2 - 1}\rho \\ \phi\left(\dfrac{\sqrt{\beta^2 - 1}\rho}{\beta t + z}\right) & \beta t + z \geq \sqrt{\beta^2 - 1}\rho, \end{cases}$$

where ϕ is the solution to (2.1) provided by Theorem 2.3. By previous remarks, $\bar{u}$ is also in H^s_{loc}, and this proves the third statement in the theorem. ∎

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