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Asymptotic behavior in time of solutions to the derivative nonlinear Schrödinger equation

by

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ABSTRACT. – We study the asymptotic behavior in time of solutions to the Cauchy problem for the derivative nonlinear Schrödinger equation

$$\left.\begin{array}{l} iu_t + u_{xx} + ia(|u|^2u)_x = 0, \quad (t,x) \in \mathbf{R}\times\mathbf{R}, \\ u(0,x) = u_0(x), \quad x \in \mathbf{R}, \end{array}\right\} \qquad \text{(DNLS)}$$

where $a \in \mathbf{R}$. We prove that if $\|u_0\|_{H^{1,2}} + \|u_0\|_{H^{3,0}}$ is sufficiently small, then the solution of (DNLS) satisfies the time decay estimate

$$\|u(t)\|_{L^\infty} \leq C(1+|t|)^{-1/2},$$

where $H^{m,s} = \{f \in \mathcal{S}'; \|f\|_{m,s} = \|(1+|x|^2)^{s/2}(1-\partial_x^2)^{m/2}f\|_{L^2} < \infty\}$, $m, s \in \mathbf{R}$. The above L^∞ time decay estimate is very important for the proof of existence of the modified scattering states to (DNLS). In order to derive the desired estimate we introduce a certain phase function. © Elsevier, Paris

Key words: Asymptotics for large time, modified scattering states, derivative NLS.

RÉSUMÉ. – Nous étudions le comportement asymptotique temporel des solutions du problème de Cauchy pour l'équation de Schrödinger non

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linéaire dérivée

$$\left.\begin{aligned} & iu_t + u_{xx} + ia(|u|^2u)_x = 0, \quad (t,x) \in \mathbf{R}\times\mathbf{R}, \\ & u(0,x) = u_0(x), \quad x \in \mathbf{R}, \end{aligned}\right\} \qquad \text{(DNLS)}$$

où $a \in \mathbf{R}$. Nous démontrons que si $\|u_0\|_{H^{1,2}} + \|u_0\|_{H^{3,0}}$ est suffisamment petit, alors la solution (DNLS) obéit à l'estimée temporelle

$$\|u(t)\|_{L^\infty} \le C(1+|t|)^{-1/2},$$

où $H^{m,s} = \{f \in \mathcal{S}'; \|f\|_{m,s} = \|(1+|x|^2)^{s/2}(1-\partial_x^2)^{m/2} f\|_{L^2} < \infty\}$, $m, s \in \mathbf{R}$. La borne L^∞ ci-dessus sur la décroissance temporelle est très importante pour la preuve d'existence d'états collisionels modifiés de (DNLS). Pour montrer la validité de cette borne, nous introduisons une fonction de phase appropriée.

1. INTRODUCTION

In this paper we study the asymptotic behavior in time of solutions to the Cauchy problem for the derivative nonlinear Schrödinger equation

$$\left.\begin{aligned} & iu_t + u_{xx} + ia(|u|^2u)_x = 0, \quad (t,x) \in \mathbf{R}\times\mathbf{R}, \\ & u(0,x) = u_0(x), \quad x \in \mathbf{R}, \end{aligned}\right\} \qquad \text{(DNLS)}$$

where the coefficient $a \in \mathbf{R}$. This equation was derived in [18], [19] to study the propagation of circular polarized Alfvén waves in plasma. The (DNLS) has an infinite family of conservation laws and can be solved exactly by the inverse scattering transform method [16]. The local existence of solutions to (DNLS) was proved in [23], [24] under the condition that $u_0 \in H^{s,0}$ $(s > 3/2)$ and the global existence of solutions to (DNLS) was also proved in [23], [24] for the initial data $u_0 \in H^{2,0}$ such that the norm $\|u_0\|_{H^{1,0}}$ is sufficiently small. These results were improved in [6],[7]. More precisely, the global existence of solutions to (DNLS) was shown in [6] if the initial data $u_0 \in H^{1,0}$ are sufficiently small in the norm $\|u_0\|_{L^2}$ and in [7] the smallness condition on $\|u_0\|_{L^2}$ was given explicitly in the form $\|u_0\|_{L^2}^2 \le 2\pi/|a|$ and also the smoothing effect of solutions was studied. The final state problem for (DNLS) was studied in [9] and the existence of modified wave operators and the non existence of the scattering states in

L^2 were shown. For the cubic nonlinear Schrödinger equation, the modified wave operators were constructed and the non existence of scattering states in L^2 were proved in paper [22] (*see* [4], [5] for the higher dimensional case). However the result in [9] does not say the asymptotics in large time of solutions to the Cauchy problem (DNLS). As far as we know there are no results concerning the time decay estimate of solutions to (DNLS). Our purpose in this paper is to prove L^∞ decay of solutions of (DNLS) with the same decay rate as that of solutions to the linear Schrödinger equation and to give the large time asymptotic formula for the solutions which shows the existence of the modified scattering states. Our proof of the results is based on the choice of the function space which was done in the recent paper [20], [21], the gauge transformation method used in [6], [7], [17] and the systematic use of the operator $J = x + 2it\partial_x$. The operator J was used previously to study (1) the scattering theory to nonlinear Schrödinger equations with power nonlinearities ([3], [11], [25]), (2) the time decay of solutions ([2], [10]) and (3) smoothing properties of solutions ([12], [13]) to some nonlinear Schrödinger equations. The main results in [3], [11], [25] are obtained through the following a priori estimate of solutions $\|Ju\|_{L^2} \leq C$. This estimate along with the Sobolev type inequality (Lemma 2.2) in the one dimensional case yields the required L^∞ time decay of solutions for the nonlinear Schrödinger equations with higher order power nonlinearity. However it seems to be difficult to get the same estimate $\|Ju\| \leq C$ in the case of cubic nonlinearity. To derive the desired a priori estimates of solutions in our function space taking L^∞ time decay estimates of solutions into acount we have to introduce a certain phase function since the previous methods ([3], [11], [25]) based solely on the a priori estimates of the value $(x + 2it\partial_x)u(t)$ without specifying any phase function does not work for (DNLS). The nonexistence of the usual L^2 scattering states shows that our result is sharp. Some phase functions were used in [4], [22] to prove the existence of the modified wave operators to the nonlinear Schrödinger equations with the critical power nonlinearity. Their results were shown through an integral equation corresponding to the original Cauchy problem and therefore the derivative loss in the equation can not be canceled via integration by parts. The method presented here is general enough, since it is also applicable to a wide class of nonlinear Schrödinger equations with nonlinearities containing derivatives of unknown function and for the generalized and modified Benjamin-Ono equations (*see* [14]), where derivatives are treated via integration by parts. We finally note that the cubic nonlinear Schrödinger equation of the form

$$iu_t + u_{xx} = (\partial_x |u|^2)(\alpha u + \beta u_x), \quad \alpha, \beta \in \mathbf{C}$$

was considered in [15] and the existence of scattering states in L^2 were shown by making use of a priori estimate of Ju for the solutions of this equation. However it is clear that their method does not work for (DNLS) because the non existence of scattering states in L^2 was shown in [9].

Let us explain the difference of our approach from the previous methods. The precise time decay estimate $\|u(t)\|_\infty \leq C(1+t)^{-1/2}$ of solutions u of (DNLS) will be obtained in Section 3 from the asymptotics of solutions u of (DNLS) for large time (Lemma 2.5). The main term of this asymptotics is determined by the Fourier transform of $U(-t)u(t)$ (where $U(t)$ is the free evolution group associated with the linear Schrödinger equation) and we also need a certain phase function to obtain the desired estimates concerning the main term (*see* Lemma 3.2). The remainder term of the asymptotics is estimated by the norm $\|Ju\|_{L^2}$ (*see* Lemma 2.5) so that the estimate $\|Ju\|_{L^2} \leq C(1+t)^\gamma$ (with some small positive γ) is sufficient for the precise L^∞ time decay estimates. Thus in our approach we can allow a little growth in time of the norm $\|Ju\|_{L^2}$. That is the main different point from the previous methods.

Before stating our results, we give

Notation and function spaces. – Let $\mathcal{F}f$ or $\hat{f}$ be the Fourier transform of f defined by

$$\mathcal{F}f(\xi) = \int e^{-ix\xi} f(x)dx.$$

The inverse Fourier transform $\mathcal{F}^{-1}$ is given by

$$\mathcal{F}^{-1}f(x) = \frac{1}{2\pi}\int e^{ix\xi} f(\xi)d\xi.$$

Let $U(t)$ be the free Schrödinger evolution group defined by

$$U(t)f = \frac{1}{\sqrt{4\pi it}}\int e^{-(x-y)^2/4ti} f(y)dy = \mathcal{F}^{-1}e^{-it\xi^2}\mathcal{F}f.$$

We introduce some function spaces. $L^p = \{f \in \mathcal{S}'; \|f\|_p < \infty\}$, where $\|f\|_p = (\int |f(x)|^p dx)^{1/p}$ if $1 \leq p < \infty$ and $\|f\|_\infty =$ ess.sup$\{|f(x)|; x \in \mathbf{R}\}$ if $p = \infty$. For simplicity we let $\|f\| = \|f\|_2$. Weighted Sobolev space $H_p^{m,s}$ defined by $H_p^{m,s} = \{f \in \mathcal{S}'; \|f\|_{m,s,p} = \|(1+x^2)^{s/2}(1-\partial_x^2)^{m/2}f\|_p < \infty\}$, $m, s \in \mathbf{R}$, $1 \leq p \leq \infty$; $H^{m,s} = H_2^{m,s}$, $\|\cdot\|_{m,s} = \|\cdot\|_{m,s,2}$. We let $(f,g) = \int f \cdot \overline{g} dx$. Different positive constants will be denoted by the same letter C.

In what follows we use the following commutation relation and identities freely.

$$J = x+2it\partial_x = S(t)2it\partial_x S(-t) = U(t)xU(-t), \quad [J, L] = JL-LJ = 0, \tag{C}$$

where $S(t) = e^{ix^2/4t}$ and $L = i\partial_t + \partial_x^2$.

Our main results are

THEOREM 1.1. – *We assume that $u_0 \in H^{3,0} \cap H^{1,2}$ and $\|u_0\|_{3,0} + \|u_0\|_{1,2} = \epsilon' \leq \epsilon$, where ϵ is sufficiently small. Then there exists a unique global solution u of (DNLS) such that*

$$u \in C(\mathbf{R}; H^{1,2} \cap H^{3,0}),$$

$$\|u(t)\|_\infty + \|u_x(t)\|_\infty \leq C(\|u_0\|_{3,0} + \|u_0\|_{1,2})(1+|t|)^{-1/2}.$$

THEOREM 1.2. – *Let u be the solution of (DNLS) obtained in Theorem 1.1. Then for any small initial data $u_0 \in H^{3,0} \cap H^{1,2}$ there exist unique functions $W, \Phi \in L^\infty$ such that*

$$\begin{aligned} &\left\|\mathcal{F}(U(-t)u)(t)\exp\left(\frac{ipa}{4\pi}\int_1^t |\hat{u}(\tau)|^2 \frac{d\tau}{\tau}\right) - W\right\|_\infty \\ &\leq C(\|u_0\|_{3,0} + \|u_0\|_{1,2})t^{-\beta+C\epsilon} \quad \textit{for} \quad t \geq 1, \end{aligned} \tag{1.1}$$

$$\begin{aligned} &\left\|\frac{-ipa}{4\pi}\int_1^t |\hat{u}(\tau)|^2 \frac{d\tau}{\tau} - \frac{-ipa}{4\pi}|\hat{u}(t)|^2 \log t - \Phi\right\|_\infty \\ &\leq C(\|u_0\|_{3,0} + \|u_0\|_{1,2})^2 t^{-\beta+C\epsilon} \quad \textit{for} \quad t \geq 1, \end{aligned} \tag{1.2}$$

where $C\epsilon < \beta < 1/4$. Furthermore we have the asymptotic formula for large time t

$$\begin{aligned} u(t,x) = &\frac{1}{\sqrt{4\pi i t}} W\left(\frac{x}{2t}\right)\exp\left(ix^2/4t - \frac{iax}{8\pi t}\left|W\left(\frac{x}{2t}\right)\right|^2 \log t + \Phi\left(\frac{x}{2t}\right)\right) \\ &+ O((\|u_0\|_{3,0} + \|u_0\|_{1,2})t^{-\frac{1}{2}-\beta+C\epsilon}) \end{aligned} \tag{1.3}$$

uniformly with respect to $x \in \mathbf{R}$.

The following corollary shows the existence of the modified scattering states.

COROLLARY 1.3. – *Let u be the solution of (DNLS) obtained in Theorem 1.1. Then for any small initial data $u_0 \in H^{3,0} \cap H^{1,2}$ there exist a unique function $W \in L^\infty$ such that*

$$\left\| \mathcal{F}(U(-t)u)(t) - W \exp\left(-\frac{iap}{4\pi} |W|^2 \right) \right\|_\infty \leq C(\|u_0\|_{3,0} + \|u_0\|_{1,2})^2 t^{-\beta + C\epsilon} \quad \textit{for } t \geq 1,$$

where $C\epsilon < \beta < 1/4$.

Remark 1.1. – The inequalities (1.1) and (1.2) show that W and Φ can be obtained from the initial function u_0 approximately.

Remark 1.2. – Our method can be applied to the following more general nonlinear Schrödinger equation.

$$\begin{cases} iu_t + u_{xx} + ia(|u|^2 u)_x + \gamma |u|^2 u + \lambda |u|^{p-1} u = 0 & (t,x) \in \mathbf{R} \times \mathbf{R}, \\ u(0,x) = u_0(x), \quad x \in \mathbf{R}, \end{cases} \tag{1.4}$$

where $\gamma, \lambda \in \mathbf{R}$ and $p > 3$ if $a = 0$, $p > 4$ if $a \neq 0$. The reason why we need the condition that $p > 4$ when $a \neq 0$ comes from the fact that our method requires some regularity of solutions to obtain time decay estimates of solutions. On the other hand, if $a = 0$, by using the method of this paper we can obtain the asymptotic formula and the time decay estimate in the $H^{1,0} \cap H^{0,1}$ space.

We organize our paper as follows. In Section 2 we give some preliminary results. Sobolev type inequalities are stated in Lemmas 2.1-2.2 and Lemma 2.3 is used to treat the nonlinear terms. In Lemma 2.4 some estimates of functions are shown which are needed when the gauge transformation technique is used. Lemma 2.5 says that the time decay of functions can be represented by using the free evolution group of the Schrödinger operator. In Section 3 we first give the local existence theorem (Theorem 3.1) without proof and we prove the main results of this paper by showing a priori estimates of local solutions to (DNLS) in Lemma 3.1 and Lemma 3.2.

2. PRELIMINARIES

LEMMA 2.1 (The Sobolev inequality). – *Let $1 \leq q, r \leq \infty$. Let $j, m \in \mathbf{N} \cup \{0\}$ satisfy $0 \leq j < m$. Let p and α satisfy $1/p = j + \alpha(1/r - m) + (1-\alpha)/q$; $j/m \leq \alpha < 1$ if $m - j - 1/r \in \mathbf{N} \cup \{0\}$, and $j/m \leq \alpha \leq 1$ otherwise. Then*

$$\|\partial_x^j \psi\|_p \leq C \|\partial_x^m \psi\|_r^\alpha \|\psi\|_q^{1-\alpha}, \tag{2.1}$$

provided that the right hand side is finite.

For Lemma 2.1, *see*, e.g., A. Friedman [1].

LEMMA 2.2. – *Let* $1 \le q, r \le \infty$. *Let* $j, m \in \mathbf{N} \cup \{0\}$ *satisfy* $0 \le j < m$. *Let* p *and* α *satisfy* $1/p = j + \alpha(1/r - m) + (1-\alpha)/q$; $j/m \le \alpha < 1$ *if* $m - j - 1/r \in \mathbf{N} \cup \{0\}$, *and* $j/m \le \alpha \le 1$ *otherwise. Then*

$$\|J^j f\|_p \le C|t|^{j-m\alpha}\|J^m f\|_r^\alpha \|f\|_q^{1-\alpha},$$

provided that the right hand side is finite.

Proof. – By applying Lemma 2.1 with $\psi = S(-t)f$, we obtain

$$\begin{aligned}|2t|^j\|\partial_x^j S(-t)f\|_p &\le C|2t|^j\|\partial_x^m S(-t)f\|_r^\alpha\|S(-t)f\|_q^{1-\alpha}\\ &\le C|t|^{j-m\alpha}\|Jf\|_r^\alpha\|f\|_q^{1-\alpha}.\end{aligned}$$

where we have used the identity (C). We again apply (C) to the above inequality to get the desired result.

LEMMA 2.3. – *We have*

$$\sum_{j=0}^{2}\|\partial_x^j(fg\bar{h})\| \le C(\|f\|_\infty^2 + \|g\|_\infty^2 + \|h\|_\infty^2)\sum_{j=0}^{2}(\|\partial_x^j f\| + \|\partial_x^j g\| + \|\partial_x^j h\|),$$

$$\sum_{j=0}^{2}\|J^j(fg\bar{h})\| \le C(\|f\|_\infty^2 + \|g\|_\infty^2 + \|h\|_\infty^2)\sum_{j=0}^{2}(\|J^j f\| + \|J^j g\| + \|J^j h\|),$$

provided the right hand sides are finite.

Proof. – We only prove the last inequality in the lemma since the first one is proved in the same way. From the identity

$$J(fg\bar{h}) = (Jf)g\bar{h} + f(Jg)\bar{h} - fg(\overline{Jh}),$$

we obtain

$$\begin{aligned}J^2(fg\bar{h}) = (J^2 f)g\bar{h} + f(J^2 g)\bar{h} + fg(\overline{J^2 h}) + 2(Jf)(Jg)\bar{h}\\ - 2(Jf)g(\overline{Jh}) - 2f(Jg)(\overline{Jh}).\end{aligned}$$

By applying Hölder's inequality and Lemma 2.2 to the above inequality we get

$$\begin{aligned}\|J^2(fg\bar{h})\| &\le \|g\|_\infty\|h\|_\infty\|J^2 f\| + \|f\|_\infty\|h\|_\infty\|J^2 g\| + \|f\|_\infty\|g\|_\infty\|J^2 h\|\\ &\quad + 2\|Jf\|_p\|Jg\|_p\|h\|_r + 2\|Jf\|_p\|g\|_r\|Jg\|_p + 2\|f\|_r\|Jg\|_p\|Jh\|_p\\ &\le \|g\|_\infty\|h\|_\infty\|J^2 f\| + \|f\|_\infty\|h\|_\infty\|J^2 g\| + \|f\|_\infty\|g\|_\infty\|J^2 h\|\\ &\quad + C|t|^{2(1-2\alpha_1-\alpha_2)}(\|f\|_\infty^{1-\alpha_1}\|g\|_\infty^{1-\alpha_1}\|h\|_\infty^{1-\alpha_2}\|J^2 f\|^{\alpha_1}\|J^2 g\|^{\alpha_1}\|J^2 h\|^{\alpha_2}\\ &\quad + \|f\|_\infty^{1-\alpha_1}\|g\|_\infty^{1-\alpha_2}\|h\|_\infty^{1-\alpha_1}\|J^2 f\|^{\alpha_1}\|J^2 g\|^{\alpha_2}\|J^2 h\|^{\alpha_1}\\ &\quad + \|f\|_\infty^{1-\alpha_2}\|g\|_\infty^{1-\alpha_1}\|h\|_\infty^{1-\alpha_1}\|J^2 f\|^{\alpha_2}\|J^2 g\|^{\alpha_1}\|J^2 h\|^{\alpha_1}),\end{aligned}$$

where

$$1/p = 1 + \alpha_1(1/2 - 2), \quad 1/r = \alpha_2(1/2 - 2), \quad 2/p + 1/r = 1/2.$$

A direct calculation shows that

$$2\alpha_1 + \alpha_2 = 1.$$

Hence Hölder's inequality gives the desired result.

LEMMA 2.4. – *We have*

$$\sum_{0\le j\le 2} \|J^j u\| \le \sum_{0\le j\le 2} \|J^j u^{(1)}\| + C|t|\|u\|_\infty^2 \sum_{0\le j\le 1} \|J^j u^{(1)}\| + Ct^2\|u\|_\infty^4\|u\|,$$

$$\begin{aligned}\sum_{0\le j\le 2} \|J^j u_x\| \le & \sum_{0\le j\le 2} \|J^j u^{(2)}\| \\ & + C\|u\|_\infty^2\Big((1 + |t|\|u\|_\infty^2)\big(\sum_{0\le j\le 2} \|J^j u^{(1)}\| \\ & \qquad + \|Ju^{(2)}\| + \|u_x\|\big) + t^2\|u\|_\infty^4\|u\|\Big),\end{aligned}$$

$$\sum_{0\le j\le 2} \|\partial_x^j u\| \le \sum_{0\le j\le 2} \|\partial_x^j u^{(1)}\| + C\|u\|_\infty^2 \sum_{0\le j\le 1} \|\partial_x^j u^{(1)}\| + C\|u\|_\infty^4\|u\|,$$

$$\begin{aligned}\sum_{0\le j\le 2} \|\partial_x^j u_x\| \le & \sum_{0\le j\le 2} \|\partial_x^j u^{(2)}\| \\ & + C\|u\|_\infty^2\Big((1 + \|u\|_\infty^2)\big(\sum_{0\le j\le 2} \|\partial_x^j u^{(1)}\| + \|\partial_x u^{(2)}\| + \|u_x\|\big) + \|u\|_\infty^4\|u\|\Big),\end{aligned}$$

provided that the right hand sides are finite, where

$$u^{(1)} = u \exp\left(ia \int_{-\infty}^x |u|^2 dx'\right),$$

$$\begin{aligned}u^{(2)} &= \exp\left(i\frac{a}{2}\int_{-\infty}^x |u|^2 dx'\right)\left(\partial_x\left(u \exp\left(i\frac{a}{2}\int_{-\infty}^x |u|^2 dx'\right)\right)\right) \\ &= (u_x + ia|u|^2 u)\exp\left(ia\int_{-\infty}^x |u|^2 dx'\right).\end{aligned}$$

Proof. – We only prove the first two inequalities since the other estimates are shown in the same way. A direct calculation shows

$$Ju^{(1)} = (Ju - 2ta|u|^2 u)\exp\left(ia\int_{-\infty}^x |u|^2 dx'\right),$$

$$
\begin{aligned}
J^2u^{(1)} &= (J^2u - 2taJ(|u|^2u) - 2ta|u|^2Ju + 4t^2a^2|u|^4u) \\
&\quad \times \exp\Big(ia\int_{-\infty}^{x}|u|^2dx'\Big), \\
Ju^{(2)} &= \big(Ju_x + iaJ(|u|^2u) - 2ta|u|^2u_x - 2ita^2|u|^4u)\big) \\
&\quad \times \exp\Big(ia\int_{-\infty}^{x}|u|^2dx'\Big), \\
J^2u^{(2)} &= \big(J^2u_x + iaJ^2(|u|^2u) - 2taJ(|u|^2u_x) - 2ita^2J(|u|^4u) \\
&\quad - (Ju_x + iaJ(|u|^2u) - 2ta|u|^2u_x - 2ita^2|u|^4u)2ta|u|^2\big) \\
&\quad \times \exp\Big(ia\int_{-\infty}^{x}|u|^2dx'\Big).
\end{aligned}
$$

By the Hölder's inequality we have

$$\|Ju\| \le \|Ju^{(1)}\| + C|t|\|u\|_\infty^2\|u\|, \tag{2.2}$$

$$\|J^2u\| \le \|J^2u^{(1)}\| + C|t|\|u\|_\infty^2\|Ju\| + Ct^2\|u\|_\infty^4\|u\|, \tag{2.3}$$

$$\|Ju_x\| \le \|Ju^{(2)}\| + C|t|\|u\|_\infty^2(\|Ju\| + \|u_x\|) + C|t|\|u\|_\infty^4\|u\|, \tag{2.4}$$

and

$$
\begin{aligned}
\|J^2u_x\| &\le \|J^2u^{(2)}\| + C\|u\|_\infty^2(\|J^2u\| + |t|\|Ju_x\|) \\
&\quad + C\|u\|_\infty^4(|t|\|Ju\| + t^2\|u_x\| + t^2\|u\|_\infty^2\|u\|) + C|t|\|u\|_\infty\|u_x\|_\infty\|Ju\|.
\end{aligned} \tag{2.5}
$$

We substitute (2.2) into (2.3) and (2.4) to get

$$\|J^2u\| \le \|J^2u^{(1)}\| + C|t|\|u\|_\infty^2\|Ju^{(1)}\| + Ct^2\|u\|_\infty^4\|u\| \tag{2.6}$$

and

$$\|Ju_x\| \le \|Ju^{(2)}\| + C|t|\|u\|_\infty^2(\|Ju^{(2)}\| + \|u_x\|) + C(|t| + t^2)\|u\|_\infty^4\|u\|. \tag{2.7}$$

We use (2.6) and (2.7) in the right hand side of (2.5) to obtain

$$
\begin{aligned}
\|J^2u_x\| &\le \|J^2u^{(2)}\| \\
&\quad + C\|u\|_\infty^2\big((1 + |t|\|u\|_\infty^2)(\|J^2u^{(1)}\| + |t|\|u_x\|) \\
&\qquad + |t|\|u\|_\infty^2\|Ju^{(2)}\| + t^2\|u\|_\infty^4\|u\|\big).
\end{aligned} \tag{2.8}
$$

The desired estimates follow from (2.2), (2.6), (2.7), (2.8) and the identities

$$\|u\| = \|u^{(1)}\|, \quad \|u_x + ia|u|^2u\| = \|u^{(2)}\|.$$

LEMMA 2.5. – *We let $u(t,x)$ be a smooth function. Then we have*

$$\sum_{0\le j\le 1} \|\partial_x^j u(t)\|_\infty \le C|t|^{-1/2}(\|\mathcal{F}(u)(t)\|_\infty + \|\mathcal{F}(u_x)(t)\|_\infty) + C|t|^{-1/2-\alpha/2}\|U(-t)u(t)\|_{1,1} \quad \textit{for} \quad |t|\ge 1,$$

where $\alpha \in [0,1/2)$.

Proof. – We have the identity

$$u(t,x) = U(t)U(-t)u(t,x) = \frac{1}{\sqrt{4\pi i t}}\int e^{i(x-y)^2/4t}U(-t)u(t,y)dy \tag{2.9}$$

The identity (2.9) can be written in the following way for $n = 0, 1$

$$\begin{aligned}\partial_x^n u(t,x) &= \frac{e^{ix^2/4t}}{\sqrt{4\pi i t}}\int e^{-ixy/2t}\partial_y^n U(-t)u(t,y)\{1+(e^{iy^2/4t}-1)\}dy \\ &= \frac{e^{ix^2/4t}}{\sqrt{4\pi i t}}\Big(i\frac{x}{2t}\Big)^n(\mathcal{F}U(-t)u(t))\Big(t,\frac{x}{2t}\Big) + R_n(t,x),\end{aligned} \tag{2.10}$$

where

$$R_n(t,x) = \frac{e^{ix^2/4t}}{\sqrt{4\pi i t}}\int e^{-ixy/2t}(e^{iy^2/4t}-1)\partial_y^n U(-t)u(t,y)dy.$$

We let α satisfy $0 < \alpha < 1/2$. Then we have the estimate

$$|e^{iy^2/4t} - 1| = 2\left|\sin\frac{y^2}{8t}\right| \le C\frac{|y|^\alpha}{|t|^{\alpha/2}}.$$

Hence we have by the Schwarz inequality

$$\begin{aligned}\|R_n(t)\|_\infty &\le C|t|^{-1/2-\alpha/2}\||y|^\alpha \partial_y^n U(-t)u(t,y)\|_1 \\ &\le C|t|^{-1/2-\alpha/2}\|U(-t)u(t)\|_{1,1} \quad \text{for} \quad |t|\ge 1\end{aligned} \tag{2.11}$$

since $\alpha < 1/2$. From (2.10) and (2.11) the lemma follows, since $\|\mathcal{F}(U(-t)\partial_x^j u)(t)\|_\infty = \|\mathcal{F}(\partial_x^j u)(t)\|_\infty$.

Proofs of Theorems. – We define the function space X_T as follows

$$\begin{aligned} X_T = \{f \in C([-T,T];\mathcal{S}');\ & |||f|||_{X_T} \\ &= \sup_{t\in[-T,T]}(1+|t|)^{-\tilde{\varepsilon}}\|f(t)\|_{3,0} \\ &+ \sup_{t\in[-T,T]}(1+|t|)^{-\tilde{\varepsilon}}\sum_{0\le j\le 2}\|J^j f(t)\|_{1,0} \\ &+ \sup_{t\in[-T,T]}(1+|t|)^{1/2}(\|f\|_\infty + \|f_x\|_\infty) < \infty\},\end{aligned}$$

where $\tilde{\epsilon}$ is a sufficiently small constant depending only on $\|u_0\|_{H^{3,0}} + \|u_0\|_{H^{1,2}}$.

To clarify the idea of the proof of the Theorem we only show a priori estimates of local solutions to (DNLS). For that purpose we assume that the following local existence theorem holds.

THEOREM 3.1. – *We assume that* $\|u_0\|_{3,0} + \|u_0\|_{1,2} = \epsilon' \leq \epsilon$ *and* ϵ *is sufficiently small. Then there exists a finite time interval* $[-T, T]$ *with* $T > 1$ *and a unique solution* u *of (DNLS) such that*

$$|||u|||_{X_T} \leq C\epsilon.$$

For the proof of Theorem 3.1, *see* [6]-[8].

In order to obtain the a priori estimates of solutions u to (DNLS) in X_T we translate the original equation into another system of equations. In the same way as in the derivation of [6, (2.3)] we find that $u^{(1)}$ and $u^{(2)}$ defined by Lemma 2.4 satisfy

$$\left.\begin{aligned} Lu^{(1)} &= ia(u^{(1)})^2\overline{u^{(2)}}, \\ Lu^{(2)} &= -ia(u^{(2)})^2\overline{u^{(1)}}, \end{aligned}\right\} \tag{3.1}$$

where $L = i\partial_t + \partial_x^2$.

LEMMA 3.1. – *Let* u *be the local solutions to (DNLS) stated in Theorem 3.1. Then we have for any* $t \in [-T, T]$

$$(1 + |t|)^{-C\epsilon}(\|u(t)\|_{3,0} + \sum_{1 \leq j \leq 2} \|J^j u(t)\|_{1,0}) \leq C(\|u_0\|_{3,0} + \|u_0\|_{1,2}),$$

where the positive constant C *does not depend on* $T > 1$.

Proof. – In what follows we only consider the positive time since the negative case can be treated similarly. By (C) we have by (3.1)

$$\left.\begin{aligned} LJ^2u^{(1)} &= iaJ^2\big((u^{(1)})^2\overline{u^{(2)}}\big), \\ LJ^2u^{(2)} &= -iaJ^2\big((u^{(2)})^2\overline{u^{(1)}}\big). \end{aligned}\right\} \tag{3.2}$$

The integral equations associated with (3.2) can be written as

$$\left.\begin{aligned} J^2u^{(1)}(t) &= U(t)x^2u^{(1)}(0) + ia\int_0^t U(t-s)J^2\big((u^{(1)})^2\overline{u^{(2)}}\big)(s)ds, \\ J^2u^{(2)}(t) &= U(t)x^2u^{(2)}(0) - ia\int_0^t U(t-s)J^2\big((u^{(2)})^2\overline{u^{(1)}}\big)(s). \end{aligned}\right\} \tag{3.3}$$

The operator $U(t)$ is a unitary operator in L^2. Therefore we easily obtain by (3.3) and Lemma 2.3

$$\begin{aligned}\|J^2u^{(1)}(t)\| + \|J^2u^{(2)}(t)\| &\leq \|x^2u^{(1)}(0)\| + \|x^2u^{(2)}(0)\| \\ &+ C\int_0^t (\|u^{(1)}(s)\|_\infty^2 + \|u^{(2)}(s)\|_\infty^2)(\|J^2u^{(1)}(s)\| + \|J^2u^{(2)}(s)\|)ds.\end{aligned} \tag{3.4}$$

Since

$$\|x^2u^{(1)}(0)\| = \|x^2u_0\|, \quad \|x^2u^{(2)}(0)\| \leq \|x^2u_{0x}\| + C\|u_0\|_\infty^2\|x^2u_0\|$$

and

$$\|u^{(1)}(t)\|_\infty \leq \|u(t)\|_\infty, \quad \|u^{(2)}(t)\|_\infty \leq \|u_x(t)\|_\infty + C\|u(t)\|_\infty^3,$$

we have from (3.4)

$$\begin{aligned}\|J^2u^{(1)}(t)\| + \|J^2u^{(2)}(t)\| &\leq 2\|x^2u_0\| + \|x^2u_{0x}\| \\ &+ C\int_0^t (\|u(s)\|_\infty^2 + \|u_x(s)\|_\infty^2)(\|J^2u^{(1)}(s)\| + \|J^2u^{(2)}(s)\|)ds,\end{aligned} \tag{3.5}$$

where we have used the condition that the initial data is sufficiently small and Theorem 3.1. We again use Theorem 3.1 to get

$$\begin{aligned}\|J^2u^{(1)}(t)\| + \|J^2u^{(2)}(t)\| &\leq 2\|x^2u_0\| + \|x^2u_{0x}\| \\ &+ C\epsilon\int_0^t (1+s)^{-1}(\|J^2u^{(1)}(s)\| + \|J^2u^{(2)}(s)\|)ds.\end{aligned} \tag{3.6}$$

Applying the Gronwall inequality to (3.6), we obtain

$$\|J^2u^{(1)}(t)\| + \|J^2u^{(2)}(t)\| \leq 2\|x^2u_0\| + \|x^2u_{0x}\|(1+t)^{C\epsilon}.$$

This implies

$$(1+t)^{-C\epsilon}(\|J^2u^{(1)}(t)\| + \|J^2u^{(2)}(t)\|) \leq 2\|x^2u_0\| + \|x^2u_{0x}\|. \tag{3.7}$$

In the same way as in the proof of (3.7) we have

$$\begin{aligned}(1+t)^{-C\epsilon}(\|u^{(1)}(t)\|_{2,0} + \|u^{(2)}(t)\|_{2,0} + \sum_{1\leq j\leq 2}(\|J^ju^{(1)}(t)\| + \|J^ju^{(2)}(t)\|)) \\ \leq C(\|u_0\|_{3,0} + \|u_0\|_{1,2}).\end{aligned} \tag{3.8}$$

The lemma follows by applying (3.8), the condition that the initial data is sufficiently small and Theorem 3.1 to Lemma 2.4.

LEMMA 3.2. – *Let u be the local solutions to (DNLS) stated in Theorem 3.1. Then we have for any $t \in [-T, T]$*

$$(1+|t|)^{1/2} \sum_{0 \leq j \leq 1} \|\partial_x^j u(t)\|_\infty \leq C(\|u_0\|_{3,0} + \|u_0\|_{1,2}),$$

where the positive constant C does not depend on $T > 1$.

Proof. – By Sobolev's inequality (Lemma 2.1) and Lemma 3.1 we have

$$(1+|t|)^{1/2} \sum_{0 \leq j \leq 1} \|\partial_x^j u(t)\|_\infty \leq C\|u(t)\|_{2,0} \leq C(\|u_0\|_{3,0}+\|u_0\|_{1,2}) \text{ for } t \leq 1. \tag{3.9}$$

We assume that $t \geq 1$. From Lemma 2.5 and Theorem 3.1 it follows that

$$\sum_{0 \leq j \leq 1} \|\partial_x^j u(t)\|_\infty \leq Ct^{-1/2-\alpha/2+C\epsilon}(\|u_0\|_{3,0} + \|u_0\|_{1,2})$$
$$+ Ct^{-1/2} \sum_{0 \leq j \leq 1} \|\mathcal{F}(\partial_x^j u)(t)\|_\infty. \tag{3.10}$$

We now consider the last term of the right hand side of (3.10). Multiplying both sides of (DNLS) by $U(-t)$, we find that

$$i(U(-t)u(t))_t + iaU(-t)(|u|^2 u(t))_x = 0.$$

We put $v(t) = U(-t)u(t)$. Then we have

$$iv_t(t) + iaU(-t)(|U(t)v|^2 U(t)v)_x = 0. \tag{3.11}$$

On the other hand we have

$$\begin{aligned} &U(-t)(|U(t)v|^2 U(t)v) = U(-t)((U(t)v)^2 U(-t)\bar{v}) \\ &= \lim_{A \to \infty} \frac{1}{(4\pi t)^2} \int_{-A}^{A} d\xi_1 \int\int\int d\xi_2 d\xi_3 d\xi_4 v(t,\xi_2)v(t,\xi_3)\bar{v}(t,\xi_4) \\ &\times \exp(-\frac{i}{4t}((x-\xi_1)^2 - (\xi_1-\xi_2)^2 - (\xi_1-\xi_3)^2 + (\xi_1-\xi_4)^2). \end{aligned}$$

A simple calculation gives

$$\begin{aligned} &(x-\xi_1)^2 - (\xi_1-\xi_2)^2 - (\xi_1-\xi_3)^2 + (\xi_1-\xi_4)^2 \\ &= (x^2 - \xi_2^2 - \xi_3^2 + \xi_4^2) + 2\xi_1(-x+\xi_2+\xi_3+\xi_4). \end{aligned}$$

Hence

$$U(-t)(|U(t)v|^2U(t)v) = \frac{1}{(4\pi t)^2}\int\int\int v(t,\xi_2)v(t,\xi_3)\bar{v}(t,\xi_4)$$
$$\times \exp\left(-\frac{i}{4t}(x^2-\xi_2^2-\xi_3^2+\xi_4^2)\right)d\xi_2 d\xi_3 d\xi_4$$
$$\times \left(\lim_{A\to\infty}\int_{-A}^{A}\exp\left(-\frac{i\xi_1}{2t}(-x+\xi_2+\xi_3-\xi_4)\right)d\xi_1\right)$$
$$= \frac{2\pi}{(4\pi t)^2}\int\int\int \delta\left(\frac{-x+\xi_2+\xi_3-\xi_4}{2t}\right)v(t,\xi_2)v(t,\xi_3)\bar{v}(t,\xi_4)$$
$$\times \exp\left(-\frac{i}{4t}(x^2-\xi_2^2-\xi_3^2+\xi_4^2)\right)d\xi_2 d\xi_3 d\xi_4$$
$$= \frac{1}{4\pi t}\int\int v(t,\xi_2)v(t,\xi_3)\bar{v}(t,\xi_2+\xi_3-x)$$
$$\times \exp\left(-\frac{i}{4t}(x^2-\xi_2^2-\xi_3^2+(\xi_2+\xi_3-x)^2)\right)d\xi_2 d\xi_3$$

Now we make the change of variables $\xi_2 = x-y$, $\xi_3 = x-z$, then we have $\xi_2+\xi_3-x = x-y-z$ and $x^2-\xi_2^2-\xi_3^2+(\xi_2+\xi_3-x)^2 = 2yz$. Therefore we obtain

$$U(-t)(|U(t)v|^2U(t)v)$$
$$= \frac{1}{4\pi t}\int\int v(t,x-y)v(t,x-z)\bar{v}(t,x-y-z)\exp\left(-\frac{iyz}{2t}\right)dydz$$
$$= \frac{1}{4\pi t}\int\int v(t,x-y)v(t,x-z)\bar{v}(t,x-y-z)dydz + Q(t,x), \quad (3.12)$$

where

$$Q(t,x) = \frac{1}{4\pi t}\int\int v(t,x-y)v(t,x-z)\bar{v}(t,x-y-z)\left(\exp\left(-\frac{iyz}{2t}\right)-1\right)dydz.$$

Substituting (3.12) into (3.11) and taking the Fourier transform, we obtain

$$i\hat{v}_t(t,p) + \frac{ia}{4\pi t}ip|\hat{v}(t,p)|^2\hat{v}(t,p) - ap\hat{Q}(t,p) = 0. \quad (3.13)$$

In order to eliminate the second term of the left hand side of the equation (3.13) we make the change of dependent variable

$$\hat{v}(t,p) = \hat{w}(t,p)\exp\left(\cdot-\frac{ipa}{4\pi}\int_1^t |\hat{v}(\tau,p)|^2\frac{d\tau}{\tau}\right). \quad (3.14)$$

Then we have by (3.13)

$$i\hat{w}_t(t,p) = ap\hat{B}(t,p)\hat{Q}(t,p), \tag{3.15}$$

where

$$\hat{B}(t,p) = \exp\left(\frac{ipa}{4\pi}\int_1^t |\hat{v}(\tau,p)|^2 \frac{d\tau}{\tau}\right).$$

Integrating (3.15) with respect to t from 1 to t, we have for $n = 0, 1$

$$(ip)^n\hat{w}(t,p) = (ip)^n\hat{w}(1,p) - i\int_1^t ap(ip)^n\hat{B}(\tau,p)\hat{Q}(\tau,p)d\tau. \tag{3.16}$$

By a simple calculation

$$\begin{aligned}
&\|p^{n+1}\hat{Q}(t,p)\|_\infty \le \|\partial_x^{n+1}Q(t,x)\|_1\\
&\le \frac{C}{t}\int\left|\partial_x^{n+1}\int\int v(t,x-y)v(t,x-z)\bar{v}(t,x-y-z)\right.\\
&\qquad\qquad \left.\times(e^{-iyz/2t}-1)dydz\right|dx\\
&\le \frac{C}{t}\int|\frac{yz}{t}|^\beta\{|\partial_x^{n+1}v(t,x-y)||v(t,x-z)||v(t,x-y-z)|\\
&+|v(t,x-y)||\partial_x^{n+1}v(t,x-z)||v(t,x-y-z)|\\
&+|v(t,x-y)||v(t,x-z)||\partial_x^{n+1}v(t,x-y-z)|\\
&+3n(|\partial_x v(t,x-y)||\partial_x v(t,x-z)||v(t,x-y-z)|\\
&+|v(t,x-y)||\partial_x v(t,x-z)||\partial_x v(t,x-y-z)|\\
&|\partial_x v(t,x-y)||v(t,x-z)||\partial_x v(t,x-y-z)|)\}dxdydz,
\end{aligned} \tag{3.17}$$

where we take β satisfying $0 \le \beta < 1/4$. Since

$$\begin{aligned}
|yz| &= |\big((x-z)-(x-y-z)\big)\big((x-y)-(x-y-z)\big)|\\
&\le C\big((x-y)^2+(x-z)^2+(x-y-z)^2\big)
\end{aligned}$$

we have by (3.17), Hölder's inequality and the identity $U(t)xU(-t) = J$

$$\begin{aligned}
\|p^{n+1}\hat{Q}(t,p)\|_\infty &\le Ct^{-1-\beta}\\
&\big(\|v\|_1^2\||x|^{2\beta}\partial_x^{n+1}v(t)\|_1 + \|\partial_x^{n+1}v\|_1\||x|^{2\beta}v\|_1\|v\|_1\\
&+n(\||x|^{2\beta}v\|_1\|v_x\|_1^2 + \||x|^{2\beta}v_x\|_1\|v_x\|_1\|v\|_1)\big)\\
&\le Ct^{-1-\beta}\|v(t)\|_{2,1}^3 = Ct^{-1-\beta}\|U(-t)u(t)\|_{2,1}^3\\
&\le Ct^{-1-\beta}\sum_{0\le j\le 2}\|U(-t)\partial_x^j u(t)\|_{0,1}\\
&\le Ct^{-1-\beta}(\|u(t)\|_{2,0} + \sum_{0\le j\le 2}\|xU(-t)\partial_x^j u(t)\|)\\
&= Ct^{-1-\beta}(\|u(t)\|_{2,0} + \sum_{0\le j\le 2}\|J\partial_x^j u(t)\|).
\end{aligned} \tag{3.18}$$

Integration by parts and Lemma 3.1 give

$$\sum_{0\le j\le 2}\|J\partial_x^j u(t)\|\le C(\|u(t)\|_{3,0}+\sum_{1\le j\le 2}\|J^j u(t)\|_{1,0})$$
$$\le C(1+t)^{C\epsilon}(\|u_0\|_{3,0}+\|u_0\|_{1,2}). \tag{3.19}$$

Hence (3.18) and (3.19) imply

$$\|p^{n+1}\hat{Q}(t,p)\|_\infty\le Ct^{-1-\beta+C\epsilon}(\|u_0\|_{3,0}+\|u_0\|_{1,2}). \tag{3.20}$$

We apply (3.20) to (3.16) to obtain

$$\begin{aligned}\|\mathcal{F}(\partial_x^n u)(t)\|_\infty &= \|\mathcal{F}(U(-t)\partial_x^n u)(t)\|_\infty=\|p^n\hat{v}(t,p)\|_\infty=\|p^n\hat{w}(t,p)\|_\infty\\
&\le \|p^n\hat{v}(1,p)\|_\infty + C(\|u_0\|_{3,0}+\|u_0\|_{1,2})\int_1^t \tau^{-1-\beta+C\epsilon}d\tau\\
&\le \|\mathcal{F}(U(-1)\partial_x^n u)(1)\|_\infty + C(\|u_0\|_{3,0}+\|u_0\|_{1,2})\\
&\le C(\|U(-1)\partial_x^n u(1)\|_1+\|u_0\|_{3,0}+\|u_0\|_{1,2})\\
&\le C(\|u(1)\|_{1,0}+\|xU(-1)\partial_x^n u(1)\|+\|u_0\|_{3,0}+\|u_0\|_{1,2})\end{aligned}$$

if we take β satisfying $\beta > C\epsilon$, where we have used the fact that $|\hat{B}(t,p)|=1$. We now apply Lemma 3.1 to the above inequality to get

$$\|\mathcal{F}(\partial_x^n u)(t)\|_\infty\le C(\|u_0\|_{3,0}+\|u_0\|_{1,2})\quad\text{for}\quad t>1. \tag{3.21}$$

From (3.10) and (3.21) it follows that

$$\sum_{0\le j\le 1}\|\partial_x^j u(t)\|_\infty\le Ct^{-1/2}(\|u_0\|_{3,0}+\|u_0\|_{1,2})\quad\text{for}\quad t>1. \tag{3.22}$$

We have the lemma by (3.9) and (3.22).

We are now in a position to prove our main results in this paper.

Proof of Theorem 1.1. – We have by Lemma 3.1 and Lemma 3.2

$$|||\,u\,|||_{X_T}\le C(\|\,u_0\,\|_{3,0}+\|\,u_0\,\|_{1,2})=C\varepsilon'\quad\text{for}\quad t\in[-T,T].$$

We take ε' satisfying $C\varepsilon'\le\varepsilon$. Then a standard continuation argument yields the result.

Proof of Theorem. 1.2. – By (3.16) and (3.20) we have

$$\begin{aligned}|\hat{w}(t,p)-\hat{w}(s,p)| &\le C\int_s^t\|p\hat{Q}(\tau,p)\|_\infty d\tau\\
&\le C(\|u_0\|_{3,0}+\|u_0\|_{1,2})\int_s^t\tau^{-1-\beta+C\epsilon}d\tau\\
&= C(s^{-\beta+C\epsilon}-t^{-\beta+C\epsilon})(\|u_0\|_{3,0}+\|u_0\|_{1,2}). \qquad (3.23)\end{aligned}$$

Hence there exists a unique function $W \in L^\infty$ such that

$$W(p) = \lim_{t\to\infty} \hat{w}(t,p) \quad \text{in} \quad L^\infty.$$

We let $t \to \infty$ in (3.23). Then we have

$$\begin{aligned} \|W - \hat{w}(t)\|_\infty &\le C \int_t^\infty \|p\hat{Q}(\tau,p)\|_\infty d\tau \\ &\le C(\|u_0\|_{3,0} + \|u_0\|_{1,2}) \int_t^\infty \tau^{-1-\beta+C\epsilon} d\tau \\ &= C(\|u_0\|_{3,0} + \|u_0\|_{1,2}) t^{-\beta+C\epsilon}, \end{aligned} \tag{3.24}$$

from which with (3.14) the first estimate (1.1) in Theorem 1.2 follows. We let

$$\begin{aligned} \Psi(t,p) &= \int_1^t \frac{-iap}{4\pi\tau}(|\hat{w}(\tau,p)|^2 - |\hat{w}(t,p)|^2)d\tau \\ &= \frac{iap}{4\pi}\int_1^t \log\tau \frac{\partial}{\partial\tau}|\hat{w}(\tau,p)|^2 d\tau. \end{aligned}$$

Then by (3.16)

$$\Psi(t,p) = \frac{iap}{4\pi}\int_1^t 2\mathrm{Re}(-iap\hat{B}(\tau,p)\hat{Q}(\tau,p)\bar{\hat{w}}(\tau,p)) \log\tau d\tau.$$

In the same way as in the proof of (3.23) we get

$$\begin{aligned} |\Psi(t,p) - \Psi(s,p)| &\le C \int_s^t \|p^2\hat{Q}(\tau,p)\|_\infty \|\hat{w}\|_\infty \log\tau d\tau \\ &\le C(s^{-\beta+C\epsilon} - t^{-\beta+C\epsilon})(\|u_0\|_{3,0} + \|u_0\|_{1,2})^2. \end{aligned} \tag{3.25}$$

This implies that there exists a unique function $\Phi \in L^\infty$ such that

$$\Phi(p) = \lim_{t\to\infty} \Psi(t,p).$$

We let $t \to \infty$ in (3.25). Then we have

$$\|\Phi - \Psi(t,p)\|_\infty \le Ct^{-\beta+C\epsilon}(\|u_0\|_{3,0} + \|u_0\|_{1,2})^2. \tag{3.26}$$

We easily find that the following identity holds

$$\begin{aligned} &\frac{-iap}{4\pi}\int_1^t |\hat{w}(\tau,p)|^2 d\tau = -i|W(p)|^2 \frac{ap}{4\pi}\log t + \Phi(p) \\ &+ (\Psi(t,p) - \Phi(p)) + (|\hat{w}(t,p)|^2 - |W(p)|^2)\frac{-iap}{4\pi}\log t. \end{aligned} \tag{3.27}$$

Applying (3.24) and (3.26) to (3.27), we obtain the estimate (1.2). By (1.1) and (1.2) we have

$$\|\mathcal{F}(U(-t)u)(t) - W\exp\left(-\frac{iap}{4\pi}|W|^2\log t + \Phi\right)\|_\infty \leq C(\|u_0\|_{3,0} + \|u_0\|_{1,2})^2 t^{-\beta+C\epsilon}. \tag{3.28}$$

The asymptotic formula (1.3) follows from (3.28), (2.11) and the identity

$$\begin{aligned} u(t,x) &= \frac{e^{ix^2/4t}}{\sqrt{4\pi it}}\int e^{-ixy/2t}U(-t)u(t,y)\{1+(e^{iy^2/4t}-1)\}dy \\ &= \frac{e^{ix^2/4t}}{\sqrt{4\pi it}}(\mathcal{F}U(-t)u(t))\left(t,\frac{x}{2t}\right) + R_0(t,x), \end{aligned}$$

where

$$R_0(t,x) = \frac{e^{ix^2/4t}}{\sqrt{4\pi it}}\int e^{-ixy/2t}U(-t)u(t,y)(e^{iy^2/4t}-1)dy.$$

This completes the proof of Theorem 1.2.

Proof of Corollary 1.3. – The desired estimate follows from (3.28) and the fact that $|W\exp\Phi| = |W|$. This completes the proof of Corollary 1.3.

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