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On quantum twist maps and spectral properties of Floquet operators

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ABSTRACT. – Quantum twist maps are introduced as the representatives of "kicked" quantum systems in the Heisenberg picture and their orbit structures are related to the various spectral types of the corresponding Floquet operators $\mathfrak{U}_V(\mathcal{T}, 0)$. By means of geometrical RAGE methods à la Enss and Veselić sufficient conditions for the absence of $\sigma_{ac}(\mathfrak{U}_V(\mathcal{T}, 0))$, respectively $\sigma_{\text{cont}}(\mathfrak{U}_V(\mathcal{T}, 0))$ are derived.

For the example of $\mathfrak{H}(t) = -id/d\theta + V(\theta) \cdot \sum_j \delta(t - j\mathcal{T})$, defined on $L^1(S^1, d\theta)$, the quasi-energy spectrum $\sigma(\mathfrak{U}_V(\mathcal{T}, 0))$ as well as the orbit structure of the twist map are determined for all $V \in \mathcal{C}^3(S^1)$ in case of $\mathcal{T}/2\pi \in \mathbb{Q}$, respectively for $\mathcal{T}/2\pi$ an irrational number of constant type. © Elsevier, Paris

Key words: Quantum twist maps, quasi-energies of kicked rotor, RAGE methods.

RÉSUMÉ. – On introduit les applications tordues quantiques en tant que représentantes de systèmes frappés quantiques dans la représentation de Heisenberg et la structure de leurs orbites est reliée aux divers types spectraux des opérateurs de Floquet correspondant $\mathbf{U}_V(\Gamma, 0)$. On dérive des conditions suffisantes pour l'absence de $\sigma_{\text{ac}}(\mathbf{U}_V(\Gamma, 0))$, respectivement $\sigma_{\text{cont}}(\mathbf{U}_V(\Gamma, 0))$ au moyen de méthodes RAGE géométriques à la

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Enss et Veselić. Le spectre de quasi-énergie $\sigma(\mathbf{U}_V(\Gamma,\, 0))$ ainsi que la structure des orbites de l'application tordue sont déterminés pour l'exemple $\mathbf{H}(t) = -i\, d/d\theta + V(\theta) \cdot \sum_j \delta(t - j\,\Gamma)$, défini sur $\mathbf{L}^2(\mathbf{S}^1,\, d\theta)$ pour tout $\mathbf{V} \in \mathbf{C}^3(\mathbf{S}^1)$ dans le cas $\Gamma/2\pi \in \mathbf{Q}$, respectivement $\Gamma/2\pi$ un nombre irrationnel de type constant.

1. QUANTUM TWIST MAPS

Quantum twist maps naturally emerge in the study of so-called "kicked" quantum systems. The latter are quantum systems under time-periodic external perturbations, which act in form of "$\delta(t - n\,\mathcal{T})$-pulses" (with the *Dirac δ-distribution* and $n \in \mathbb{Z}$, $\mathcal{T} > 0$, see [1] for a review and references). Quantum models of this kind are particularly suitable for numerical studies, however, the number of analytic discussions of the subject is limited, [1-6] for instance. Kicked quantum systems are often represented by a formal time-periodic family $\{\mathfrak{H}(t),\ t \in \mathbb{R}\}$ of Hamiltonians with

$$\mathfrak{H}(t) = H_0 + W \cdot \sum_{j \in \mathbb{Z}} \delta(t - j\,\mathcal{T}). \tag{1.1}$$

For convenience, the operator H_0 in (1.1) acts on $\mathcal{H} = L^2(\Omega,\, dx)$, with $\Omega = [a,\, b]$, $-\infty \leq a < b \leq \infty$, and is assumed self-adjoint with $\sigma(H_0) = \sigma_{disc}(H_0)$ and finitely generate eigenvalues E_m, $m \in \mathcal{M}$. The kick-potential W in general is a self-adjoint multiplication on $\mathcal{H}$ and $\mathcal{T} := 2\pi\nu \in \mathbb{R}^+$ is the kick-period. (For other models see [3], [6].)

Although there exist no self-adjoint realizations of $\{\mathfrak{H}(t),\ t \in \mathbb{R}\}$ on $\mathcal{H}$, the corresponding one-period propagator (the *Floquet operator*) is well-defined and sometimes used as the mathematical expression for kicked quantum systems:

$$\mathfrak{U}_W(\mathcal{T},\, 0) = \exp(-i\,\mathcal{T}\, H_0)\exp(-i\, W). \tag{1.2}$$

(For a discussion of (1.1), (1.2) in terms of the so-called extended Hilbert space formalism, see [7] and Section 2, for instance.)

To characterize the dynamics of the kicked quantum system in question, one often studies the evolutions $\Psi(N\,\mathcal{T}) := [\mathfrak{U}_W(\mathcal{T},\, 0)]^N\, \Psi_0$ of some initial state Ψ_0. In particular, numerical investigations of several models [8, 9 and references, for example] have led to new and important insights.

In the present article, however, we shall concentrate on a different point of view, usually referred as the *Heisenberg picture of quantum mechanics* [10] in order to define a quantum twist map as an automorphism of the *observables, i.e.* the selft-adjoint operators $f(H_0)$ and $g(W)$, generated by $\mathfrak{U}_W(\mathcal{T}, 0)$ on the "quantum phase space" $\mathcal{P}(\mathcal{H})$ such that

$$\left.\begin{aligned} [f(H_0)]_{n+1} &:= (\mathfrak{U}_W(\mathcal{T}, 0))^* [f(H_0)]_n \mathfrak{U}_W(\mathcal{T}, 0) \\ [g(W)]_{n+1} &:= (\mathfrak{U}_W(\mathcal{T}, 0))^* [g(W)]_n \mathfrak{U}_W(\mathcal{T}, 0) \end{aligned}\right\} \tag{1.3}$$

with $[f(H_0)]_0 := f(H_0)$ and $[g(W)]_0 := g(W)$. Typically, W is some bounded self-adjoint operator on $\mathcal{H}$ and $g(W)$ is bounded as well. However, $f(H_0)$ is assumed unbounded and the quantum phase space $\mathcal{P}(\mathcal{H})$ can be imagined as a "strip" in $\mathcal{S}(\mathcal{H}) \times \mathcal{BS}(\mathcal{H})$ with $\mathcal{S}(\mathcal{H})$ the space of self-adjoint operators on $\mathcal{H}$ and $\mathcal{BS}(\mathcal{H})$ the subspace of bounded self-adjoint operators on $\mathcal{H}$. Thus, the notion of a quantum twist map seems justified since repeated application $\mathfrak{U}_W(\mathcal{T}, 0)$ stretches and folds the orbits $([f(H_0)]_n, [g(W)]_n)$ defined by (1.3). (A more precise definition of "orbit" is given in Section 4).

The aim of this article is a simple characterization of the quantum dynamics represented by $\mathfrak{U}_W(\mathcal{T}, 0)$. At first we formulate an abstract "RAGE"-theorem for the Floquet operator $\mathfrak{U}_W(\mathcal{T}, 0)$ using the methods devised by Enss and Veselić. From that result sufficient conditions on the time evolution of W generated by H_0, which guarantee $\sigma_{ac}(\mathfrak{U}_W(\mathcal{T}, 0)) = \emptyset$, respectively $\sigma_{\text{cont}}(\mathfrak{U}_W(\mathcal{T}, 0)) = \emptyset$, are deduced. Afterwards, in Section 4, the notions of stable, strongly stable and unstable orbits of the quantum twist maps (1.3) are introduced and related to spectral properties of $\mathfrak{U}_W(\mathcal{T}, 0)$. Finally, in Section 5, the preceding findings are applied to the model sketched by $\mathfrak{H}(t) = -id/d\theta + V(\theta) \cdot \sum_j \delta(t - j\mathcal{T})$.

2. AN ABSTRACT RAGE-THEOREM IN EXTENDED HILBERT SPACE

This section prepares the ground for an application of results of Enss and Veselić [11] to $\mathfrak{U}_W(\mathcal{T}, 0)$. The deviation is necessary since the direct treatment of $\mathfrak{U}_W(\mathcal{T}, 0)$ à la Enss and Veselić runs into serious obstacles. First of all, the periodic family $\{\mathfrak{H}(t),\ t \in \mathbb{R}\}$ is not self-adjoint on $\mathcal{H}$, but equally disturbing is the fact that $\mathfrak{U}_W(\mathcal{T}, 0)$ *cannot* be represented in strongly continuous Floquet form, since $\mathfrak{U}_W(t, 0_+) = \exp(-itH_0)$ for all $0 < t < \mathcal{T}$. Thus, if the Floquet representation $\mathfrak{U}_W(t, 0) = \mathcal{P}(t)\exp(-itG)$ with $\mathcal{P}(\mathcal{T}) = \mathbb{I}$ would apply, $G \equiv H_0$ would follow. (Nevertheless, $\mathfrak{U}_W(\mathcal{T}, 0)$ is called Floquet operator).

Therefore, another representation of the kicked quantum system is needed. A convenient structure to describe the kicked quantum system is the extended Hilbert space formalism. We briefly recall some of the results:

Introduce the *extended Hilbert space* $\mathcal{H}_{ex}$ as the (closed) tensor product $\mathcal{H}_{ex} := L^2([0, \mathcal{T}], dt) \otimes L^2(\Omega, dx)$, with norm

$$\|\psi\|^2_{\mathcal{H}_{ex}} = \int_{\pi_\mathcal{T} := [0, \mathcal{T}]} dt \|\psi(t)\|^2_{L^2(\Omega, dx)} \tag{2.1}$$

and corresponding scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}_{ex}}$. In this set-up the unperturbed dynamics is represented by the self-adjoint, so-called Floquet Hamiltonian K_0 with

$$K_0 = \overline{-i\,\partial_t \otimes \mathbb{I} + \mathbb{I} \otimes H_0} \tag{2.2}$$

where

$$\mathcal{D}(-i\,\partial_t) = \{g \in L^2(\pi_\mathcal{T}) : \partial g/\partial t \in L^2(\pi_\mathcal{T}) \text{ and } g(0) = g(\mathcal{T})\}$$
$$-i\,\partial_t\, g := -i\,\partial g/\partial t \quad \forall g \in \mathcal{D}(-i\,\partial_t).$$

The operator K_0 is obviously self-adjoint on $\mathcal{H}_{ex}$ and its resolvent is represented by the strongly convergent expansion

$$(K_0 - z)^{-1} f = \sum_{n \in \mathcal{N}} \sum_{l \in \mathcal{L}} \int_{\pi_\mathcal{T} \times \Omega} d\hat{t}\, d\hat{x}\, g(z - E_l, \cdot, \hat{t})\, \overline{\psi_{n,l}(\hat{x})} f(\hat{t}, \hat{x})\, \psi_{n,l} \tag{2.3}$$

with $\{\psi_{n,l}, (n, l) \in \mathcal{N} \times \mathcal{L}\}$ the orthonormal basis of eigenfunctions to H_0 and $g(z - E_n)$ denoting Green's function to $-i\,\partial_t$. (For details see Lemma 2.1 below.)

The δ-kicks enter the formalism via the domain properties of the time-derivative. Formally, the "operator" $-i\,\partial/\partial t + W(\tilde{x})\,\delta(t - \mathcal{T})$, with $\tilde{x} \in \Omega$ fixed, generates translations along $\pi_\mathcal{T}$ with the jump condition $g(0) = \exp(-i\,W(\tilde{x}))\, g(\mathcal{T})$. Thus, a *singular* time-derivative can be introduced by the following self-adjoint operator on $\mathcal{H}_{ex}$:

$$\begin{aligned}
\mathcal{D}(-i\,\partial_{t,W}) = \{f \in \mathcal{H}_{ex} &: f(\cdot, x) \in \mathcal{D}(-i\,\partial_{t,W(x)}) \text{ a.e.},\\
&\mathcal{D}(-i\,\partial_{t,W(x)}) = \{g \in L^2(\pi_\mathcal{T}) : \partial g/\partial t \in L^2(\pi_\mathcal{T})\\
&\text{and } g(0) = \exp(-i\,W(x))\, g(\mathcal{T})\}\},\\
-i\,\partial_{t,W} f := -i\,\partial f/\partial t &\quad \forall f \in \mathcal{D}(-i\,\partial_{t,W}),
\end{aligned}$$

$$\exp(-\mu\,\partial_{t,W})\,\Psi(t, x) = \exp\left\{-i\,W(x)\,(\operatorname{sign} k) \sum_{j=1}^{\infty} \Theta(|k| - j)\right\} \times \begin{cases} \Psi(t - \hat{\mu}, x) \text{ if } t \geq \hat{\mu}_+ \\ \exp(-i\,W(x))\,\Psi(t + \mathcal{T} - \hat{\mu}, x) \text{ if } t \leq \hat{\mu}_- \end{cases} \tag{2.4}$$

for all $\mu = k\mathcal{T} + \hat{\mu}$, $k \in \mathbb{Z}$, $0 \leq \hat{\mu} < \mathcal{T}$ and $\Theta(\nu) := 1$ for all $\nu \geq 0$ and $\Theta(\nu) = 0$ otherwise. Together with $\overline{\mathbb{I} \otimes H_0}$ that singular time-derivative constitutes the Floquet Hamiltonian K_W for the kicked quantum system as demonstrated in the sequel. As a preparatory step, we introduce several intermediate operators:

(i) The minimal symmetric operator $\dot{k}$ on $\mathcal{H}_{ex}$ is defined as

$$\mathcal{D}(\dot{k}) = \mathcal{C}_0^\infty((0, \mathcal{T}) \times \dot{\Omega}), \quad \dot{k}f := [-i\,\partial/\partial t + h_0]\,f$$
$$\forall f \in \mathcal{D}(\dot{k}),\ \dot{\Omega} := (a, b).$$

where h_0 is the symbolic differential operator defining $\mathbb{I} \otimes H_0$, *i.e.* $H_0\,\psi := h_0\,\psi$ for all $\psi \in \mathcal{D}(H_0)$. General principles [12, for instance] imply that $\dot{k}^*\,\phi = (-i\,\partial/\partial t + h_0)\,\phi$ for all $\phi \in \mathcal{D}(\dot{k}^*)$.

(ii) The operator closure of $\dot{K}$ is denoted by $\dot{K}$, with the former given by

$$\left.\begin{array}{c} \mathcal{D}(\dot{K}) = \{\phi \in \mathcal{H}_{ex} : \partial\phi/\partial t \in \mathcal{H}_{ex},\ \phi \in \mathcal{D}(\overline{\mathbb{I} \otimes H_0}) \\ \text{and } \phi(0_+, x) = \phi(\mathcal{T}-, x) = 0 \text{ a.e.} \\ \dot{K}\,\phi := (-i\,\partial/\partial t + h_0)_\phi \quad \forall\,\phi \in \mathcal{D}(\dot{K}). \end{array}\right\} \tag{2.5}$$

Information about $\operatorname{ran}(\dot{K} - \bar{z})^\perp$ is collected in

LEMMA 2.1. – *The closed symmetric operator* $\dot{K} = \dot{K}^{**}$ *defined in* (2.5) *has deficiency indices equal to infinity and the defect space* $\ker(\dot{K}^* - z)$ *is spanned by the orthonormal system*

$$\mathcal{S}(z) := \{\Phi_{n,l}(z, \mathcal{T};\, t, x) = i c_{n,l}(z, \mathcal{T}) \exp[i(z - E_n)\,t]\,\psi_{n,l}(x)\}$$

with $(E_n, \psi_{n,l})$ *the* (n^{th} *eigenvalue,* $(n, l)^{th}$ *eigenfunction) of* H_0 *and* $c_{n,l}(z, \mathcal{T})$ *the normalization.*

Proof. – From $\dot{k} \subset \dot{K} \subset \dot{K}^* \subset \dot{k}^*$ we infer that $\dot{K}^*$ is densely defined and acts via $(-i\,\partial/\partial t + h_0)$ as well. Therefore $\overline{\dot{K}} =: \dot{K} = \dot{K}^{**}$.

It is obvious that $\mathcal{S}(z) \subset \ker(\dot{K}^* - z) \Rightarrow \overline{\operatorname{lin\,span}(\mathcal{S}(z))} \subset \ker(\dot{K}^* - z)$. The latter indeed represents an equality as seen from the following: The kernel $g(z - E_n;\, t, \hat{t})$ of the resolvent $(-i\,\partial_t - (z - E_n))^{-1}$ is

$$g(z - E_n;\, t, \hat{t}) = i\exp[i(z - E_n)(t - \hat{t})] \times \left\{\begin{array}{c} \exp(i(z - E_n)\mathcal{T})(1 - \exp(i(z - E_n)\mathcal{T}))^{-1} \text{ if } \hat{t} > t \\ (1 - \exp(i(z - E_n)\mathcal{T}))^{-1} \text{ if } \hat{t} < t \end{array}\right. \tag{2.6}$$

Assume that there exists some function $\Psi \in \ker(\dot{K}^* - z)$ such that $\Psi \notin \overline{\operatorname{lin\,span}(\mathcal{S}(z))}$. Then $\langle \Phi_{n,l}(z, \mathcal{T}), \Psi\rangle_{\mathcal{H}_{ex}} = 0$ for all $(n, l) \in \mathcal{N} \times \mathcal{L}$ and we infer from (2.6), respectively the form of $\Phi_{n,l}(z, \mathcal{T})$ that

$$\langle \overline{g(\bar{z} - E_n;\, 0,\, \cdot)},\ \tilde{\Psi}_{n,l} := \int_\Omega dx\, \overline{\psi_{n,l}(x)}\, \Psi(\cdot,\, x)\rangle_{L^2(\pi_\tau, dt)} = 0 \qquad \forall (n,\, l) \in \mathcal{N} \times \mathcal{L}. \tag{2.7}$$

Hence, from (2.6) and (2.7) it follows that $(-i\,\partial_t - (\bar{z} - E_n))^{-1}\,\tilde{\Psi}_{n,l}(s = 0) = 0$, *i.e.* $\tilde{\Psi}_{n,l} \in \operatorname{ran}(-i\,\dot{\partial}_t - (\bar{z} - E_n))$, where $-i\,\dot{\partial}_t$ is defined on proper functions vanishing at $t = 0$. That feature, together with the fact that every $\xi \in \mathcal{H}_{ex}$ is represented as $\xi = \sum_{(n,l)} \langle \overline{\psi_{n,l}},\, \xi(\,\cdot\,)\rangle_{L^2(\Omega)} \otimes \psi_{n,l}$, yields $\Psi \in \operatorname{ran}(\dot{K} - \bar{z})$ in contradiction to the assumption $\Psi \in \ker(\dot{K}^* - z)$. $\square$

Having prepared the prerequisites, we introduce a self-adjoint realization K_W of the formal operator $-i\,\partial/\partial t + h_0 + W(x) \cdot \delta(t - \mathcal{T})$.

LEMMA 2.2. – *Let* $W \in \mathcal{D}(H_0)$. *Then the self-adjoint Floquet Hamiltonian* K_W *on* $\mathcal{H}_{ex}$ *is the operator closure of* $\dot{K}_W$, *defined on* $\mathcal{D}(\overline{\mathbb{I} \otimes H_0}) \cap \mathcal{D}(-i\,\partial_{t,W})$ *with* $\dot{K}_W\,\Psi := -i\,\partial\Psi/\partial t + H_0\,\Psi$.

Proof. – From Lemma 2.1 we infer $\dot{K}_W^* \subset \dot{K}^*$. Yet, integration by parts assures that $\operatorname{lin\,span}\mathcal{S}(z)$ is not contained in $\mathcal{D}(\dot{K}_W^*)$. Therefore, the operator $\dot{K}_W$ has deficiency indices $(0,\, 0)$. Owing to $W \in \mathcal{D}(H_0)$ the boundary conditions are compatible. $\square$

According to *Stone's theorem* the Floquet Hamiltonian K_W generates the unitary one-parameter group $\mathcal{U}_W = \{\exp(-i\,\mu\,K_W)\}_{\mu \in \mathbb{R}}$ and the following statement connects the latter to $\mathfrak{U}_W(\mathcal{T},\, 0)$.

LEMMA 2.3. – *Define* K_W *as in Lemma* 2.2. *Then the monodromy operator* $\mathfrak{U}_W(\mathcal{T},\, 0)$ *introduced in* (1.2) *and* $\mathcal{U}_W$ *are related according to*

$$[\exp(-i\,(k\,\mathcal{T})\,K_W)\,\Psi](\mathcal{T}) = [\exp(-i\,\mathcal{T}\,H_0)\exp(-i W)]^k\,\Psi(\mathcal{T})$$

for all $\Psi \in \mathcal{D}(K_W)$ *and* $k \in \mathbb{Z}$.

Proof. – On account of Lemma 2.2 the Trotter product formula applies as

$$\exp(-i\,\tau\,K_W) = \operatorname*{s-lim}_{n\to\infty}\,[\exp(-i\,\tau\,(\overline{\mathbb{I} \otimes H_0})/n) \cdot \exp(-\tau\,\partial_{s,W}/n)]^n. \tag{2.8}$$

The decomposability of $\overline{\mathbb{I} \otimes H_0}$ and (2.4) allow the explicit computation of the nth-term at the RHS of (2.8). In particular, at $\tau = \mathcal{T}$ we find on $\mathcal{D}(\dot{K}_W)$

$$\begin{aligned}&[\exp(-i\,\mathcal{T}\,(\overline{\mathbb{I} \otimes H_0})/n) \cdot \exp(-\mathcal{T}\,\partial_{s,W}/n)]^n\,\psi(t)\\ &\quad = \exp(-i\,\mathcal{T}\,H_0)\exp(-i\,W)\,\psi(t)\end{aligned}$$

for all $\psi \in \mathcal{D}(\dot{K}_W)$ and almost all $(1 - n^{-1})\mathcal{T} < t < \mathcal{T}$ and therefore

$$\lim_{n \to \infty} [\exp(-i\mathcal{T}(\overline{\mathbb{I} \otimes H_0})/n) \cdot \exp(-\mathcal{T}\partial_{s,W}/n)]^n \psi(\mathcal{T}-) = \exp(-i\mathcal{T} H_0) \exp(-iW) \psi(\mathcal{T}-)$$

in $L^2(\Omega)$-sense. Note that the "kick condition" $\psi(0_+) = \exp(-iW)\psi(\mathcal{T}-)$ holds on the entire $\mathcal{D}(K_W)$ and the pointwise limit $n \to \infty$ of (2.8) exists due to dominated convergence. Thus, the claim follows by closure and the extension to any number of kicks is straightforward using the group properties of $\mathcal{U}_W$. □

Remark that Lemma 2.3 demonstrates that $\sigma(K_W)$ and $\sigma(\mathfrak{U}_W(\mathcal{T}, 0))$ are closely related. For instance, if $\lambda \in \sigma_{pp}(K_W)$ with eigenfunction(s) Ψ_λ, we obtain an eigenvalue equation for $\mathfrak{U}_W(\mathcal{T}, 0)$:

$$\exp(-i\mathcal{T}\lambda)\Psi_\lambda(\mathcal{T}-) = \exp(-i\mathcal{T} H_0)\exp(-iW)\Psi_\lambda(\mathcal{T}-).$$

We shall need a sufficiently explicit representation of $(K_W - z)^{-1}$. Therefore recall Krein's resolvent formula [12] as

$$(K_W - z)^{-1} - (K_0 - z)^{-1} = \sum_{(m,k)} \langle \sum_{(n,l)} \overline{\lambda_{mn}^{kl}(z, W)\Phi_{n,l}(\bar{z}, \mathcal{T})}, \cdot \rangle_{\mathcal{H}_{ex}} \Phi_{m,k}(z, \mathcal{T}) \tag{2.9}$$

for all $z \in \mathbb{C}\backslash\mathbb{R}$. The numbers $\lambda_{mn}^{kl}(z, W)$ are uniquely determined by the domain properties of K_W, respectively K_0, and the choice of the orthonormal set $\{\Phi_{n,l}(z, \mathcal{T})\}$ from Lemma 2.1. Note that $\sum_{(n,l)} \overline{\lambda_{mn}^{kl}(z, W)\Phi_{n,l}(\bar{z}, \mathcal{T})} \in \ker(\dot{K}^* - \bar{z})$ for each fixed pair (m, k).

To apply the Enss-Veselić results to the present situation, introduce a family of orthogonal projections on $\mathcal{H}_{ex}$ by

$$P(H_0) = \left\{ \mathcal{P}_M(H_0) = \mathbb{I} \otimes \mathbb{P}_M(H_0), \; \mathbb{P}_M(H_0) := \sum_{\substack{n \in \mathcal{N} \\ |n| \le M}} \mathbb{P}_n(H_0) \right\}_{M \in \mathbb{N}} \tag{2.10}$$

with $\mathbb{P}_n(H_0)$ the finite-dimensional orthogonal projection onto the n-th eigenspace of H_0. Evidently,

$$\|\mathcal{P}_M(H_0)\|_{\mathcal{B}(\mathcal{H}_{ex})} = 1 \quad \text{and} \quad s\text{-}\lim_{M \to \infty} \mathcal{P}_M(H_0) = \mathbb{I}.$$

LEMMA 2.4. – *Define K_W as in Lemma* 2.2 *and $\mathcal{P}_M(H_0)$ by* (2.10). *Then $\mathcal{P}_M(H_0)(K_W - i)^{-1}$ is compact for all $M \in \mathbb{N}$.*

Proof. – On account of Krein's formula (2.9) we only have to deal with the operators $[\mathbb{I} \otimes \mathbb{P}_n(H_0)](K_0 - i)^{-1}$ since $\langle \sum_{(n,l)} \overline{\lambda^{kl}_{mn}(z, W)\, \Phi_{n,l}(\bar{z}, \mathcal{T}),\, f_j\rangle_{\mathcal{H}_{ex}}} \underset{j\to\infty}{\to} 0$ for all $(m, k) \in \mathcal{M} \times \mathcal{K}$ and every $f_j \underset{j\to\infty}{\overset{w}{\to}} 0$. However, as

$$[\mathbb{I} \otimes \mathbb{P}_n(H_0)](K_0 - i)^{-1} = (-i\,\partial_t + E_n - i)^{-1} \otimes \mathbb{P}_n$$

and both factors on the RHS are compact, the claim follows immediately. □

Lemma 2.4 allows the adaptation of the "abstract RAGE-Theorem" of Enss and Veselić [11, Theorem 3.2] to characterize elements of $\mathcal{H}_{pp}(K_W)$ and $\mathcal{H}_{\text{cont}}(K_W)$, respectively. To this end we introduce the following subspaces of $\mathcal{H}_{ex}$:

Define $\mathcal{M}^b_+(P)$ as the set of $\Psi \in \mathcal{H}_{ex}$ for which

$$\lim_{M\to\infty} \sup_{k\in\mathbb{N}} \|(\mathbb{I} - \mathcal{P}_M(H_0)) \exp(-i(k\mathcal{T}) K_W)\,\Psi\|_{\mathcal{H}_{ex}} = 0. \tag{2.11}$$

(For $(-k) \in \mathbb{N}$ introduce $\mathcal{M}^b_-(P)$ in the same way.) In addition, denote by $\mathcal{M}^f_\pm(P)$ the sets with

$$\begin{aligned}\mathcal{M}^f_\pm(P) = \{\Psi \in \mathcal{H}_{ex} :\ \lim_{n\to\pm\infty} n^{-1} \sum_{k=0}^{n\mp 1} \|\mathcal{P}_M(H_0))\\ \times \exp(-i(k\mathcal{T}) K_W)\,\Psi\|_{\mathcal{H}_{ex}} = 0 \quad \forall M \in \mathbb{N}\}.\end{aligned} \tag{2.12}$$

Combining several of the Enss-Veselić [11] and above findings we formulate the main result concerning the spectral decompositions of K_W and $\mathfrak{U}_W(\mathcal{T}, 0)$.

THEOREM 2.5. – *Define* $\mathfrak{U}_W(\mathcal{T}, 0)$ *by* (1.2), K_W *as in Lemma* 2.2, $P(H_0)$, $\mathcal{M}^b_\pm(P)$ *and* $\mathcal{M}^f_\pm(P)$ *by* (2.10)-(2.12), *respectively. Then*

$$\text{(i)}\quad \mathcal{M}^b_-(P) = \mathcal{M}^b_+(P) = \mathcal{H}_{pp}(K_W)$$

$$\text{(ii)}\quad \mathcal{M}^f_-(P) = \mathcal{M}^f_+(P) = \mathcal{H}_{\text{cont}}(K_W)$$

$$\text{(iii)}\quad \sigma(\mathfrak{U}_W(\mathcal{T}, 0)) = \sigma_{\text{cont}}(\mathfrak{U}_W(\mathcal{T}, 0)) \cup \overline{\sigma_{pp}(\mathfrak{U}_W(\mathcal{T}, 0))}.$$

Proof. – Claims (i) and (ii) follow from [11, Theorem 2.3 and Theorem 3.2] applied to K_W. Empty residual spectrum of $\mathfrak{U}_W(\mathcal{T}, 0)$ is implied by unitary. (Here $\lambda \in \sigma_{\text{cont}}(\mathfrak{U}_W(\mathcal{T}, 0))$ if λ is not an eigenvalue and $\operatorname{ran}(\lambda - \mathfrak{U}_W(\mathcal{T}, 0))$ is dense in $\mathcal{H}$.) □

3. AN ABSTRACT RAGE-THEOREM FOR $\mathfrak{U}_W(\mathcal{T}, 0)$

Based on the results for K_W and $\mathfrak{U}_W(\mathcal{T}, 0)$ in Theorem 2.5, we now characterize point and continuous spectral subspaces of $L^2(\Omega)$ with respect to $\mathfrak{U}_W(\mathcal{T}, 0)$. To this end, define $\mathcal{H}_{pp}(\mathfrak{U}_W(\mathcal{T}, 0))$ as the closed linear span of the eigenvectors of $\mathfrak{U}_W(\mathcal{T}, 0)$ and $\mathcal{H}_{\text{cont}}(\mathfrak{U}_W(\mathcal{T}, 0))$ as its orthogonal complement with respect to $\mathcal{H}$. Accordingly, $\mathfrak{P}_{pp}(\mathfrak{U}_W(\mathcal{T}, 0))$ and $\mathfrak{P}_{\text{cont}}(\mathfrak{U}_W(\mathcal{T}, 0))$ denote the corresponding orthogonal projections.

THEOREM 3.1. – *Let* $\Psi \in \mathcal{H}_{\text{cont}}(K_W) \cap \mathcal{D}(\dot{K}_W)$ *and define* $\mathbb{P}_M(H_0)$ *by* (2.10). *Then* $\Psi(\mathcal{T}) \in \mathcal{H}_{\text{cont}}(\mathfrak{U}_W(\mathcal{T}, 0)) \cap \mathcal{D}(H_0)$ *and* $\lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} \|\mathbb{P}_M(H_0)\,\mathfrak{U}_W(T, 0)^k\,\Psi(\mathcal{T})\|_{\mathcal{H}}^2 = 0$ *for all* $M \in \mathbb{N}$.

Proof. – Lemma 2.3 guarantees that $\Phi \in \operatorname{lin\,span}\{\Psi_\lambda = \text{eigenfunction to } K_W\} \Rightarrow \Phi(\mathcal{T}) \in \operatorname{lin\,span}\{\psi_\mu = \text{eigenfunction to } \mathfrak{U}_W(\mathcal{T}, 0)\}$. Norm-closure in the sense of (2.1) and the expansion theorem provide $\Phi \in \mathcal{D}(\dot{K}_W) \cap \mathcal{H}_{pp}(K_W) \Rightarrow \Phi(\mathcal{T}) \in \mathcal{D}(H_0) \cap \mathcal{H}_{pp}(\mathfrak{U}_W(\mathcal{T}, 0))$. Now the first claim is deduced from the definition of the scalar product in (2.1) and the t-continuity of Φ and Ψ.

A computation similar to Lemma 2.3 provides the existence of unitary operators $\mathcal{W}(t, t - k\mathcal{T})$ on $\mathcal{H}$ such that

$$[\exp(-i(k\mathcal{T})K_W)\Psi](t) = \mathcal{W}(t, t - k\mathcal{T})\Psi(t)$$

almost everywhere on π_τ and for all $k \in \mathbb{Z}$. In particular, $\mathcal{W}(t, t - \mathcal{T}) = s\text{-}\lim_{n\to\infty}\{\exp(-i\mathcal{T}H_0 j/n)\cdot\exp(-iW)\cdot\exp(-i\mathcal{T}H_0(n - j/n)\}$ for $t/\mathcal{T} \in ((j-1)/n, j/n)$ with $1 \le j \le n$. In addition, we remark that $\mathcal{W}(t, t - k\mathcal{T}) = [\mathcal{W}(t, t - \mathcal{T})]^k$, *i.e.* the unitary family $\{\mathcal{W}(t, t - k\mathcal{T}), t \in (0, \mathcal{T})\}$ is strongly differentiable and for all $\Psi \in \mathcal{D}(\dot{K}_W)$, $k \in \mathbb{Z}$, it follows that $\|[\mathcal{P}_M(H_0)\exp(-i(k\mathcal{T})K_W)\Psi](\,\cdot\,)\|_{\mathcal{H}}$ is differentiable on $(0, \mathcal{T})$.

From the assumption on Ψ, (3.1) and Theorem 2.5, we infer for all intervals $\mathcal{I} \subset [0, \mathcal{T}]$

$$\lim_{n\to\infty}\int_{\mathcal{I}} dt\, n^{-1}\sum_{k=0}^{n-1}\|\mathbb{P}_M(H_0)\,\mathcal{W}(t, t - k\mathcal{T})\Psi(t)\|_{\mathcal{H}}^2 = 0$$

which, together with the continuity of the norms, implies a.e. on π_τ that

$$\lim_{n\to\infty} n^{-1}\sum_{k=0}^{n-1}\|\mathbb{P}_M(H_0)\,\mathcal{W}(t, t - k\mathcal{T})\Psi(t)\|_{\mathcal{H}}^2 =: \alpha(M, t) = 0. \quad (3.1)$$

Remark that $d(\|\mathcal{W}(\cdot, \cdot - k\mathcal{T})\Psi(\,\cdot\,)\|_{\mathcal{H}}^2)/dt = d\|\Psi(\,\cdot\,)\|_{\mathcal{H}}^2/dt < \beta$ everywhere and if $\alpha(\tilde{M}, \hat{t}) > 0$ for some $(\tilde{M}, \hat{t})$, then $\alpha(M, \hat{t}) > 0$ for all $M \ge \tilde{M}$ is implied. However, the latter and (3.1) would yield

$d\alpha(M, t)/dt|_{t=\hat{t}} = \infty$ for all $M \geq \tilde{M}$ in contradiction to the boundedness of $d\alpha(M = \infty, \cdot)/dt$. Therefore, (3.1) must be valid fo all $t \in [0, \mathcal{T}]$, which proves the second claim of the theorem. □

COROLLARY 3.2. – *Let* $\psi \in \mathcal{D}(H_0)$. *Then* $\psi \in \mathcal{H}_{cont}(\mathfrak{U}_W(\mathcal{T}, 0))$ *implies that* $\lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} \|\mathbb{P}_M(H_0)\mathfrak{U}_W(\mathcal{T}, 0)^k \psi\|_{\mathcal{H}}^2 = 0$ *for all* $M \in \mathbb{N}$.

Proof. – Assume $n_j^{-1} \sum_{k=0}^{n_j-1} \|\mathbb{P}_M(H_0)\mathfrak{U}_W(\mathcal{T}, 0)^k \psi\|_{\mathcal{H}}^2 > \mathcal{C} > 0$ for some subsequence $\{n_j, j \in \mathbb{N}\}$ with $n_j \to \infty$, some $M \in \mathbb{N}$ and note that every $\psi \in \mathcal{D}(H_0)$ can be identified with $\Psi(\mathcal{T})$, $\Psi \in \mathcal{D}(\dot{K}_W)$, since there exists some $\phi \in \mathcal{D}(\dot{K}_W)$, $\phi(\mathcal{T}) = 1$ such that $\Psi := \psi\phi \in \mathcal{D}(\dot{K}_W)$ and $\Psi(\mathcal{T}) \equiv \psi$. Hence, from Theorem 3.1 we conclude that $\mathfrak{P}_{pp}(\mathfrak{U}_W(\mathcal{T}, 0))\psi \neq 0$ in contradiction to the assumption of $\psi \in \mathcal{H}_{\text{cont}}(\mathfrak{U}_W(\mathcal{T}, 0))$. □

Theorem 3.1 allows the determination of sufficient conditions for the absence of $\sigma_{ac}(\mathfrak{U}_W(\mathcal{T}, 0))$, respectively $\sigma_{\text{cont}}(\mathfrak{U}_W(\mathcal{T}, 0))$. To this end, rewrite powers of $\mathfrak{U}_W(\mathcal{T}, 0)$ as

$$\mathfrak{U}_W(\mathcal{T}, 0)^k = \exp[-i\mathcal{T}H_0] \prod_{j=1}^{k} \times \exp\{-i[\exp(i(k-j)\mathcal{T}H_0)W\exp(-i(k-j)\mathcal{T}H_0)]\}. \tag{3.2}$$

That relation follows from insertion of $\exp[-ij\mathcal{T}H_0]\exp[ij\mathcal{T}H_0] = \mathbb{I}$ into $\mathfrak{U}_W(\mathcal{T}, 0)^k$ and regrouping. Abbreviate the unperturbed evolution (*i.e.* generated by H_0) of the kick-potential W over $(k-j)$-periods by W_{k-j}. Then there are the following characterizations of $\sigma(\mathfrak{U}_W(\mathcal{T}, 0))$:

THEOREM 3.3. – *Define* $\mathfrak{U}_W(\mathcal{T}, 0)$ *by* (1.1) *and* (1.2). *Then*

(i) *If* $T_L = w\text{-}\lim_{l\to\infty} \prod_{j=1}^{k_l} \exp(-iW_{k_l-j})$ *is unitary for some (unbounded) subsequence* $\{k_l \in \mathbb{N}\}_{l\in\mathbb{N}}$, *then* $\mathfrak{U}_W(\mathcal{T}, 0)$ *is pure singular.*

(ii) *If* $T := w\text{-}\lim_{k\to\infty} \prod_{j=1}^{k} \exp(-iW_{k-j})$ *is unitary, then* $\mathfrak{U}_W(\mathcal{T}, 0)$ *is pure point.*

Proof. – (i) Assume that $\sigma_{ac}(\mathfrak{U}_W(\mathcal{T}, 0)) \neq \emptyset$. The spectral theorem for unitaries, the Randon-Nikodym theorem and the Riemann-Lebesgue lemma imply that $\mathfrak{U}_W(\mathcal{T}, 0)^k \underset{k\to\infty}{\overset{w}{\to}} 0$ on $\mathcal{H}_{ac}(\mathfrak{U}_W(\mathcal{T}, 0))$. Note that $\exp[-ik_l\mathcal{T}E_m] \underset{l\to\infty}{\to} \exp(-i\xi_m)$ for all $E_m \in \sigma(H_0)$. Hence $s\text{-}\lim_{l\to\infty} \exp[-ik_l\mathcal{T}H_0]$ exists and together with the assumption on

T_L we deduce $\mathfrak{U}_W(\mathcal{T},\,0)^{k_l} \underset{l\to\infty}{\overset{s}{\to}} (s\text{-}\lim_{l\to\infty} \exp[-i\,k_j\,\mathcal{T}\,H_0])\,T_L$ in contradiction to $\mathfrak{U}_W(\mathcal{T},\,0)^{k_l} \underset{l\to\infty}{\overset{w}{\to}} 0$ on $\mathcal{H}_{ac}(\mathfrak{U}_W(\mathcal{T},\,0))$, since the assumption of weak convergence implies $T_L = s\text{-}\lim_{k\to\infty} \prod_{j=1}^{k_l} \exp(-\mathrm{i}\,W_{k_l-j})$ as well.

(ii) As in (i) we infer $T = s\text{-}\lim_{k\to\infty} \prod_{j=1}^{k} \exp(-\mathrm{i}\,W_{k-j})$. Set $\Lambda_k := \prod_{j=1}^{k} \exp(-\mathrm{i}\,W_{k-j}) - T$ and with (3.2)

$$\begin{aligned}&\sup_k \|(\mathbb{I} - \mathbb{P}_M(H_0))\,\mathfrak{U}_W(\mathcal{T},\,0)^k\,\psi\|_{\mathcal{H}}\\ &\quad\le \sup_k \|(\mathbb{I} - \mathbb{P}_M(H_0))\,\Lambda_k\,\psi\|_{\mathcal{H}} + \|(\mathbb{I} - \mathbb{P}_M(H_0))\,T\,\psi\|_{\mathcal{H}}\end{aligned} \tag{3.3}$$

as well as $\|(\mathbb{I} - \mathbb{P}_M(H_0))\Lambda_k\psi\|_{\mathcal{H}} \le \|\Lambda_k\psi\|_{\mathcal{H}} \underset{k\to\infty}{\to} 0$ uniformly in $M \in \mathbb{N}$ follow. Hence,

$$\begin{aligned}&\lim_{M\to\infty} \sup_k \|(\mathbb{I} - \mathbb{P}_M(H_0))\,\Lambda_k\,\psi\|_{\mathcal{H}}\\ &\quad= \sup_k \lim_{M\to\infty} \|(\mathbb{I} - \mathbb{P}_M(H_0))\,\Lambda_k\,\psi\|_{\mathcal{H}} = 0.\end{aligned} \tag{3.4}$$

As

$$\left.\begin{aligned}&n^{-1} \sum_{k=0}^{n-1} \|\mathbb{P}_M(H_0)\,\mathfrak{U}_W(\mathcal{T},\,0)^k\,\psi\|_{\mathcal{H}}^2\\ &\quad= \|\psi\|_{\mathcal{H}}^2 - n^{-1} \sum_{k=0}^{n-1} \|(\mathbb{I} - \mathbb{P}_M(H_0))\,\mathfrak{U}_W(\mathcal{T},\,0)^k\,\psi\|_{\mathcal{H}}^2\\ &\text{and}\quad n^{-1} \sum_{k=0}^{n-1} \|(\mathbb{I} - \mathbb{P}_M(H_0))\,\mathfrak{U}_W(\mathcal{T},\,0)^k\,\psi\|_{\mathcal{H}}^2\\ &\quad\le \sup_k \|(\mathbb{I} - \mathbb{P}_M(H_0))\,\mathfrak{U}_W(\mathcal{T},\,0)^k\,\psi\|_{\mathcal{H}}^2,\end{aligned}\right\} \tag{3.5}$$

relations (3.3) to (3.5) imply

$$n^{-1} \sum_{k=0}^{n-1} \|\mathbb{P}_M(H_0)\,\mathfrak{U}_W(\mathcal{T},\,0)^k\,\psi\|_{\mathcal{H}}^2 > \mathcal{C} > 0$$

for all $M \ge \tilde{M}$ and all $\psi \in \mathcal{H}$. In particular, for all $\phi \in \mathcal{D}(H_0)$ Corollary 3.2 implies that $\mathfrak{P}_{pp}(\mathfrak{U}_W(\mathcal{T},\,0))\,\phi \ne 0$. Thus $\phi \in \mathcal{D}(H_0) \cap \mathcal{H}_{\text{cont}}(\mathfrak{U}_W(\mathcal{T},\,0))$ is impossible. □

4. THE DYNAMICS OF QUANTUM TWIST MAPS

To derive detailed information about the twist maps (1.3), we assume the following convenient properties: (A 1) $\mathcal{D}(f(H_0)) \supset \mathcal{D}(H_0)$, (A 2) $\sigma(f(H_0))$ is discrete and unbounded and (A 3) $\exp(-iW)\mathcal{D}(H_0) \subset \mathcal{D}(H_0)$. In addition, we also assume that $g(W)$ is bounded and everywhere defined on $\mathcal{H}$. Then the map (1.3) can be iterated without domain problems and the following expressions are well-defined on $\mathcal{D}(H_0)$:

$$\left.\begin{aligned} [f(H_0)]_{n+1} &= f(H_0) + \sum_{j=1}^{n+1} \\ &\quad \times [\mathfrak{U}_W(\mathcal{T}, 0)^*]^j [f(H_0), \mathfrak{U}_W(\mathcal{T}, 0)]\, \mathfrak{U}_W(\mathcal{T}, 0)^{j-1} \\ [g(W)]_{n+1} &= g(W) + \sum_{j=1}^{n+1} \\ &\quad \times [\mathfrak{U}_W(\mathcal{T}, 0)^*]^j [g(W), \mathfrak{U}_W(\mathcal{T}, 0)]\, \mathfrak{U}_W(\mathcal{T}, 0)^{j-1} \end{aligned}\right\} \tag{4.1}$$

with $[f(H_0), \mathfrak{U}_W(\mathcal{T}, 0)] := f(H_0)\,\mathfrak{U}_W(\mathcal{T}, 0) - \mathfrak{U}_W(\mathcal{T}, 0)\, f(H_0)$.

In contrast to "classical" iterative maps, the question concerning the choice of the proper topology arises for the operator map. In order to relate quantum and classical maps, expectation values of quantum observables are desirable. However, based on the findings in Chapters 4 and 5, we propose the use of the strong topology in the study of (4.1).

DEFINITION 4.1. – *An orbit $\mathcal{O}(\psi_0)$ of the quantum twist map* (4.1) *is defined as the set $\{[f(H_0)]_n\,\psi_0,\ g(W)]_n\,\psi_0$ with $\psi_0 \in \mathcal{D}(H_0),\ n \in \mathbb{N}\}$ and is called*

(i) *stable if $\{\|[f(H_0)]_n\,\psi_0\|_{\mathcal{H}}\}_{n\in\mathbb{N}}$ is bounded,*

(ii) *strongly stable if $\mathcal{O}(\psi_0)$ is stable and*

$$s\text{-}\lim_{n_j\to\infty} \{[f(H_0)]_{n_j}\,\psi_0,\ [g(W)]_{n_j}\,\psi_0\}$$

exists for a subsequence $\{n_j(\psi_0) \in \mathbb{N}\}_{j\in\mathbb{N}}$,

(iii) *unstable if $\mathcal{O}(\psi_0)$ fails to fullfill* (i).

If every orbit $\mathcal{O}(\psi_0)$ is (strongly) stable then the quantum twist map (4.1) *is called (strongly) stable.*

Before relating the different orbits of (4.1) to the various spectral types of $\mathfrak{U}_W(\mathcal{T}, 0)$, we recall an important implication of the RAGE-theorem. (See also [13], [4].)

LEMMA 4.2. – *Let $f(H_0)$ such that* (A 1), (A 2) *hold and assume $\psi_0 \in \mathcal{D}(H_0)$. If $\mathfrak{P}_{\text{cont}}(\mathfrak{U}_W(\mathcal{T}, 0))\,\psi_0 \neq 0$ and $\mathfrak{U}_W(\mathcal{T}, 0)^k\,\psi_0 \in \mathcal{D}(f(H_0)^2)$ for all $k \in \mathbb{N}$, then there exists a subsequence $\{k_j(\psi_0) \in \mathbb{N}\}$ such that*

$$\lim_{j\to\infty} \langle \mathfrak{U}_W(\mathcal{T}, 0)^{k_j}\,\psi_0,\ f(H_0)^2\,\mathfrak{U}_W(\mathcal{T}, 0)^{k_j}\,\psi_0\rangle_{\mathcal{H}} = \infty.$$

Proof. [13]. – Let $\psi_c = \mathfrak{P}_{\text{cont}}(\mathfrak{U}_W(\mathcal{T}, 0))\,\psi_0 \neq 0$ and $\psi_{pp} = \mathfrak{P}_{\text{pp}}(\mathfrak{U}_W(\mathcal{T}, 0))\,\psi_0$. Corollary 3.2 implies that

$$\lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} \|\mathbb{P}_M(H_0)\,\mathfrak{U}_W(\mathcal{T}, 0)^k\,\psi_c\|_{\mathcal{H}}^2 = 0 \qquad \text{for all} \quad M \in \mathbb{N}.$$

Define $\langle \alpha_k \rangle_n := n^{-1} \sum_{k=0}^{n-1} \alpha_k$. Then $\langle \|\mathbb{P}_M(H_0)\,\mathfrak{U}_W(\mathcal{T}, 0)^k\,\psi_0\|_{\mathcal{H}}^2 \rangle_n \leq \langle (\|\mathbb{P}_M(H_0)\,\mathfrak{U}_W(\mathcal{T}, 0)^k\,\psi_c\|_{\mathcal{H}} + \|\mathbb{P}_M(H_0)\,\mathfrak{U}_W(\mathcal{T}, 0)^k\,\psi_{pp}\|_{\mathcal{H}})^2 \rangle_n$ yields

$$\limsup_{n\to\infty} \langle \|\mathbb{P}_M(H_0)\,\mathfrak{U}_W(\mathcal{T}, 0)^k\,\psi_0\|_{\mathcal{H}}^2 \rangle_n \leq \|\psi_{pp}\|_{\mathcal{H}}^2$$

and

$$\begin{aligned}\|\psi_0\|_{\mathcal{H}}^2 &= \langle \|\mathfrak{U}_W(\mathcal{T}, 0)^k\,\psi_0\|_{\mathcal{H}}^2 \rangle_n = \langle \|\mathbb{P}_M(H_0)\,\mathfrak{U}_W(\mathcal{T}, 0)^k\,\psi_0\|_{\mathcal{H}}^2 \rangle_n \\ &\quad + \langle \|(\mathbb{I} - \mathbb{P}_M(H_0))\,\mathfrak{U}_W(\mathcal{T}, 0)^k\,\psi_0\|_{\mathcal{H}}^2 \rangle_n\end{aligned}$$

provides

$$\liminf_{n\to\infty} \langle \|(\mathbb{I} - \mathbb{P}_M(H_0))\,\mathfrak{U}_W(\mathcal{T}, 0)^k\,\psi_0\|_{\mathcal{H}}^2 \rangle_n \geq \|\psi_c\|_{\mathcal{H}}^2.$$

Hence,

$$\begin{aligned}&\liminf_{n\to\infty} \langle \langle \mathfrak{U}_W(\mathcal{T}, 0)^k\,\psi_0,\, f(H_0)^2\,\mathfrak{U}_W(\mathcal{T}, 0)^k\,\psi_0 \rangle_{\mathcal{H}} \rangle_n \\ &\geq \inf_{|n|>M} f(E_n)^2 \|\psi_c\|_{\mathcal{H}}^2.\end{aligned}$$

As M is arbitrary, the claim immediately follows with

$$\inf_{|n|>M} f(E_n)^2 \underset{M\to\infty}{\longrightarrow} \infty. \quad \square$$

In view of Definition 4.1 several conclusions can be drawn from the results of Chapter 3 and Lemma 4.2. Sufficient conditions for *global stability* are discussed in the following

THEOREM 4.3. – (i) *Let* $f(H_0)$ *such that* (A1) *and* (A2) *hold, let* $\psi_0 \in \mathcal{D}(H_0)$, $\mathfrak{U}_W(\mathcal{T}, 0)^k\,\psi_0 \in \mathcal{D}(f(H_0)^2)$ *for all* $k \in \mathbb{N}$ *and* $\mathcal{D}(f(H_0)^2)$ *be* $\mathcal{H}$*-dense. Then, if the quantum twist map* (4.1) *is stable for all* $\psi_0 \in \mathcal{D}(H_0)$, $\mathfrak{U}_W(\mathcal{T}, 0)$ *is pure point.*

(ii) *Let* $W := w\text{-}\lim_{M\to\infty} \prod_{m=1}^{M} \exp(-i\,W_{M-m})$ *be unitary and* $W\,\mathcal{D}(H_0) \subset \mathcal{D}(H_0)$. *Then the quantum twist map* (4.1) *is strongly stable.*

Proof. (i) The stability assumption implies $\mathcal{C}(\psi_0) \geq \|[f(H_0)]_k\,\psi_0\|_{\mathcal{H}}^2 = \langle \mathfrak{U}_W(\mathcal{T}, 0)^k\,\psi_0,\, f(H_0)^2\,\mathfrak{U}_W(\mathcal{T}, 0)^k\,\psi_0 \rangle_{\mathcal{H}}$. Now the claim is deduced from Lemma 4.2.

(ii) Note that

$$W = s\text{-}\lim_{M\to\infty} \prod_{m=1}^{M} \exp(-\mathrm{i}\, W_{M-m})$$

implies $s\text{-}\lim_{M\to\infty} (\prod_{m=1}^{M} \exp(-\mathrm{i}\, W_{M-m}))^* = W^*$. (All operators involved are unitary.) As

$$[f(H_0)]_M = \left(\prod_{m=1}^{M} \exp(-i\, W_{M-m})\right)^* f(H_0) \left(\prod_{m=1}^{M} \exp(-i\, W_{M-m})\right),$$

the strong convergence of $[f(H_0)]_n \psi_0$ immediately follows. The discreteness of H_0 yields $\exp(\pm i j_l \mathcal{T} H_0) \underset{l\to\infty}{\overset{s}{\to}} \mathfrak{T}$ for some subsequence $\{j_l \in \mathbb{N}\}$, see Theorem 3.3. Hence, $[g(W)]_{n_j} \psi_0$ is strongly convergent. $\square$

Remark. – (i) The converse of Theorem 4.3 (i) might not be true, since expectations of $f(H_0)^2$ might grow unbounded even for pure point $\mathfrak{U}_W(\mathcal{T}, 0)$, cf. [9] as well.

(ii) The first assumption in Theorem 4.3 (ii) already means pure point $\mathfrak{U}_W(\mathcal{T}, 0)$ according to Theorem 3.3. Thus, $\sigma_{\text{cont}}(\mathfrak{U}_W(\mathcal{T}, 0))) = \emptyset$ is not necessarily equivalent to strong stability of (4.1).

What about unstable dynamics? The next result shows that there might be a countable set $\mathcal{E}(H_0)$ of perturbation periods $\mathcal{T}$ such that every orbit $\mathcal{O}(\psi_0)$ is unstable and $\mathfrak{U}_W(\mathcal{T}, 0)$ is pure absolutely continuous ("resonances").

THEOREM 4.4. – *Assume that $\sigma(H_0)$ and $\mathcal{T}$ are such that $\mathcal{T} E_n = 2\pi p_n/q$ for all E_n, $(p_n, q) \in \mathbb{Z}^2$ and $([f(H_0)]_q - f(H_0))\psi_0 \neq 0$ for all $\psi_0 \in \mathcal{D}(H_0)$. Then all orbits $\mathcal{O}(\psi_0)$ of (4.1) are unstable and $\sigma(\mathfrak{U}_W(\mathcal{T}, 0)) = \sigma_{ac}(\mathfrak{U}_W(\mathcal{T}, 0))$ iff $\prod_{m=1}^{q} \exp(-i\, W_{q-m})$ is pure absolutely continuous.*

Proof. – A short computation demonstrates that

$$[f(H_0)]_{kq} = f(H_0) + k \sum_{j=1}^{q} [\mathfrak{U}_W(\mathcal{T}, 0)^*]^j [f(H_0), \mathfrak{U}_W(\mathcal{T}, 0)]\, \mathfrak{U}_W(\mathcal{T}, 0)^{j-1}$$

$$[g(W)]_{kq} = \left(\left(\prod_{m=1}^{q} \exp(-i\, W_{q-m})\right)^*\right)^k g(W) \left(\prod_{m=1}^{q} \exp(-i\, W_{q-m})\right)^k$$

Hence, for $([f(H_0)]_q - f(H_0))\psi_0 \neq 0$, the divergence of the map is obvious.

As $\mathcal{T} E_n = 2\pi p_n/q$, (3.2) yields

$$\mathfrak{U}_W(\mathcal{T}, 0)^{kq} = \left(\prod_{m=1}^{q} \exp(-i W_{q-m})\right)^k \qquad \text{for all} \quad k \in \mathbb{Z}.$$

Hence, for all $\phi \in \mathcal{H}$ it follows that

$$\langle \phi, \mathfrak{U}_W(\mathcal{T}, 0)^{kq} \phi \rangle_{\mathcal{H}} = \int_0^{2\pi} \exp(\mathrm{i}\,\lambda\, k)\, d\langle \phi, G(\lambda)\, \phi \rangle_{\mathcal{H}}$$

where $G(\lambda)$ denotes the spectral family of $(\prod_{m=1}^{q} \exp(-i W_{q-m})$. Now the second claim follows from $\sigma(\mathfrak{U}_W(\mathcal{T}, 0)^q) = \sigma_{ac}(\mathfrak{U}_W(\mathcal{T}, 0)^q) \Leftrightarrow \sigma(\mathfrak{U}_W(\mathcal{T}, 0)) = \sigma_{ac}(\mathfrak{U}_W(\mathcal{T}, 0))$. $\square$

To compare classical and quantum dynamics, however, global results as above are not always satisfactory. As classical mechanics is essentially local, knowledge about the fate of individual trajectories in the sense of Definition 4.1 is desirable. For instance, strong of stability $\mathcal{O}(\psi_0)$ for some initial condition ψ_0 represents a quantum analogue to the existence of an invariant KAM-curve. Yet, using conventional methods, it seems to be hard to investigate single orbits without fulfilling global requirements of the type used in Theorems 4.3 to 4.5.

For the time being, we have to resort to explicit, sufficiently "simple" examples like the *toy model* in the next section, the *simplified quantum Fermi accelerator* [15], [16] or the related *quantum Pustylnikov model* [17].

5. THE TOY MODEL – AN EXAMPLE

This section contains applications of the above results to the family of formal Hamiltonians

$$\mathfrak{H}(t) = L - V(\theta) \cdot \sum_{j \in \mathbb{Z}} \delta(t - j\,T) \tag{5.1}$$

defined on $L^2(S^1, d\theta)$, where the self-adjoint $L := -\mathrm{i}\, d/d\theta$ acts on suitable 2π-periodic functions and V is a sufficiently smooth multiplication. (The notion "toy model" was coined by *J. S. Howland* and refers to the significant difference between (5.1) and the "real" kicked rotor, which contains L^2 rather than L, cf. [7].) In [2] rigorous results for a certain class of potentials were derived. In the sequel we basically recover some of these findings for a wide range of kick-potentials V. As a consequenc of Sections 2-4 the proofs of these statements are simplified.

The family (5.1) generates the one-period propagator $\mathfrak{U}_V(\mathcal{T}, 0) = \exp(-i\mathcal{T} L)\exp(i V)$ and the corresponding twist map reads

$$\left.\begin{aligned} L_{n+1} &= L_0 + \sum_{j=0}^{n} [\mathfrak{U}_V(\mathcal{T}, 0)^*]^j V_{\theta,0}\,\mathfrak{U}_V(\mathcal{T}, 0)^j \\ &=: L_0 + \sum_{j=0}^{n} V_{\theta,j} \\ V_{\theta,n+1} &= V_{\theta,0}(\cdot + (n+1)\mathcal{T}) \end{aligned}\right\} \tag{5.2}$$

(with $L_0 := L$, $V_{\theta,0} := dV/d\theta$) on account of the unitary translation group $\{\exp(-i(k\mathcal{T})L),\, k \in \mathbb{Z}\}$. It is obvious that the dynamics of the toy model is determined by the number theoretical properties of the kick period $\mathcal{T}$.

(A) Resonances

As $\sigma(L) = \mathbb{Z}$, any $\mathcal{T} = 2\pi p/q$ with $(p, q) \in \mathbb{N}^2$ falls into the class discussed in Theorem 4.4. Furthermore, $W_m = -V(. + 2\pi mp/q)$ and the assumption on $\prod_{m=1}^{q} \exp(-i W_{q-m})$ is obviously true for any $V \in \mathcal{C}^1(S^1)$, piecewise strictly monotone. Therefore, toy models of this kind are characterized by instability. (Cf. [2, Sect. 1.3, Theorem 2] as well.)

(B) Irrational stability

The (in-)stability of the irrational toy model depends on the (un-) boundedness of the sequence

$$\left\{\sum_{k=0}^{K} (dV/d\theta)(\theta + 2k\pi\nu),\ \theta \in S^1\right\}_{K\in\mathbb{N}}.$$

That feature, in turn, is determined by the number theoretical properties of $\nu \in \mathbb{R}\backslash\mathbb{Q}$, which are manifested in the specific way the unit interval is filled by the fractional parts $\{k\nu\}_1$ for $k \in \mathbb{N}$. (Here $\{k\nu\}_1 := k\nu - [k\nu]_1$ with the integer part $[k\nu]_1 \in \mathbb{N}$.)

A detailed analysis, which is the content of [18], provides the tools needed (see below) to prove the stability of the toy map (5.2) for all kick-potentials $V \in \mathcal{C}^3(S^1)$ in case of ν being an irrational of constant type. Numbers of that kind are characterized by the boundedness of the partial quotients in their continued fraction representation, [19] for example. In [20] it is shown that if ν is an irrational of constant type, a diophantine condition $|\nu - p/q| \geq k/q^{2+\sigma}$, with $k > 0$, $\sigma > 0$, for all $p, q \in \mathbb{Z}$, is fulfilled. In addition, we remark that the set of irrationals of constant type has Lebesgue measure zero, [19] for example.

The basic number theoretical property employed in the study of the toy model below is expressed in the following

PROPOSITION 5.1 [18]. – *Define* $\pi_{\min}^K := \inf_k \pi_k^K = \inf_{k \le K} (\{m_k \nu\}_1 - \{m_{k-1} \nu\}_1)$ *for the ordered fractional parts* $\{m_k \nu\}_1$ *and in the same way introduce* $\pi_{\max}^K := \max_k \pi_k^K$. *Then the sequence* $\{\pi_{\max}^K / \pi_{\min}^K\}_{K \in \mathbb{N}}$ *is bounded if and only if the irrational number* ν *is of constant type.*

(The proof of this statement is contained in [18].) Finally, the main result of this section.

THEOREM 5.1. – *Let* $\mathcal{T} = 2\pi\nu$, ν *irrational of constant type, define* $\mathfrak{U}_V(\mathcal{T}, 0)$ *by* (5.1) *with* $V \in C^3(S^1)$, *the toy map by* (5.2) *and* $\{\pi_{\max}^K / \pi_{\min}^K\}_{K \in \mathbb{N}}$ *as in Proposition* 5.1. *Then* $\mathfrak{U}_V(\mathcal{T}, 0)$ *is pure point and the quantum twist map* (5.2) *is strongly stable.*

Proof. – Fix $\theta \in S^1$, replace without restriction $V_{\theta, 0}(\theta + 2\pi m \nu)$ by $\mathcal{V}(m\nu)$, $\mathcal{V} \in \mathcal{C}^2([0, 1])$, and remark that $\int_0^1 dx\, \mathcal{V}(x) = 0$ holds. We can rearrange $\sum_{m=1}^M \mathcal{V}(m\nu)$ such that $\sum_{k=1}^M \mathcal{V}(m_k \nu)$, respectively $(-\sum_{k=1}^M |\mathcal{V}(m_k \nu)|)$, are ordered by $\{m_k \nu\}_1 > \{m_{k-1} \nu\}_1$ and series over positive and negative summands alternate. (Here $\{\alpha\}_1$ is the fractional part of $\alpha \in \mathbb{R}$ w.r. to $[0, 1]$.) For simplicity, only the case with two series is considered in the sequel.

Assume that $\mathcal{V}(x) \ge 0$ for all $x \in [0, a]$ and $\mathcal{V}(x) \le 0$ for all $x \in [a, 1]$. Then, for all $M \in \mathbb{N}$,

$$\sum_{m=1}^M \mathcal{V}(m\nu) = \sum_{k=1}^K \mathcal{V}(m_k \nu) - \sum_{l=1}^L |\mathcal{V}(m_l \nu)|$$

with $m_k \nu \in [0, a]$ and $m_l \nu \in [a, 1]$. Hence, using $\int_0^1 dx\, \mathcal{V}(x) = 0$,

$$\sum_{m=1}^M \mathcal{V}(m\nu) = \sum_{k=1}^K \mathcal{V}(m_k \nu) - (\pi_{\min}^M)^{-1} \int_0^a dx\, \mathcal{V}(x) - \sum_{l=1}^L |\mathcal{V}(m_k \nu)| + (\pi_{\min}^M)^{-1} \int_a^1 dx |\mathcal{V}(x)|.$$

It suffices to discuss only the contribution from the interval $[0, a]$, where we have

$$\sum_{k=1}^K \mathcal{V}(m_k \nu) - (\pi_{\min}^M)^{-1} \int_0^a dx\, \mathcal{V}(x) \le (\pi_{\max}^M / \pi_{\min}^M)(\pi_{\max}^K)^{-1} \times \left(\sum_{k=1}^K \mathcal{V}(m_k \nu)\, \pi_k^K - \int_0^a dx\, \mathcal{V}(x) \right). \tag{5.3}$$

The Cauchy polygon method [21] sets the convergence rate of the Riemann sum to the integral over a $\mathcal{C}^1([a, b])$-function as of order of $|\pi| := \max_k (x_k - x_{k-1})$ for any partition of $[a, b]$. Together with Proposition 5.1 that proves the boundedness of the RHS of (5.3) for $M \to \infty$.

The above arguments are independent of $\theta \in S^1$. Thus, we infer that the sequence of multiplications $\{\sum_{m=1}^M V_{\theta,0}(\cdot + m\mathcal{T})\}_{M\in\mathbb{N}}$ is norm-bounded, implying the stability of the quantum twist map (5.2). Now Theorem 4.3 (i) yields $\sigma(\mathfrak{U}_V(\mathcal{T}, 0)) = \overline{\sigma_{pp}(\mathfrak{U}_V(\mathcal{T}, 0))}$ as $(\mathfrak{U}_V(\mathcal{T}, 0)^k \psi_0 \in \mathcal{D}(L^2)$ for all $k \in \mathbb{Z}$ and all $\psi_0 \in \mathcal{D}(L^2)$. The latter follows from application of the above procedure to $d^2 V/d\theta^2$, which is justified on account of $V \in \mathcal{C}^3(S^1)$. Finally, strong stability of (5.2) stems from $\sum_{m=1}^{M_j} V_{\theta,0}(\cdot + m\mathcal{T}) \underset{j\to\infty}{\overset{s}{\longrightarrow}} \mathbb{V} \in \mathcal{H}$ for some subsequence $\{M_j \in \mathbb{N}\}$ due to its θ-uniform convergence indicated by (5.3). $\square$

What about $\mathcal{T}/2\pi$ diophantine, but not of the above type or a Liouville number [22], *i.e.* an irrational extremly well approximated by rationals?

In the former case J. S. Howland recently remarked [23] that for analytic kick-potentials the kick sequence $\{\sum_{m=1}^M V_{\theta,0}(\cdot - m\mathcal{T})\}_{M\in\mathbb{N}}$ is indeed bounded, *i.e.* $\mathfrak{U}_V(\mathcal{T}, 0)$ is pure point as seen from the arguments found at the end of the Proof of Theorem 5.2. For Liouville numbers there is no uniform behaviour and the (in-) stability depends on the individual character of $\mathcal{T}/2\pi$. Using Baire category arguments similar to [5, 24], however, the existence of a singular continuous $\mathfrak{U}_V(\mathcal{T}, 0)$ for (i) a certain $\mathbb{R}$-dense G_δ-set of normalized kick periods and (ii) a particular class of kick potentials can be shown [2, 5]. We conjecture that in these cases the toy map is unstable.

Finally, remark that via the identification $(-i\, d/d\theta, V(\theta)) \to (\mathfrak{L}, V(\theta))$, where $\mathfrak{L}$ is the classical angular momentum and V is the classical kick-potential, the classical toy map essentially looks like (5.2) and the (in-)stability of its orbits is determined by the same criteria as in the quantum case. Therefore, the (un-)boundedness of the sequence of kicks $\{\sum_{m=1}^M V_{\theta,0}(\cdot - m\mathcal{T})\}$ for different periods governs the dynamics of the classical model as well.

That feature exactly reflects the physical mechanism of energy exchange between the rotor and the external kicks. If the kick frequency is "close" to characteristic rotation numbers of the system, an unlimited amount of energy might be transferred. On the other hand, boundedness of the force $\sum_{m=1}^\infty V_{\theta,0}(\cdot - m\mathcal{T})$ indicates that the rotor can absorb only a limited amount of external energy.

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