ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 68, nº 1 (1998), p. 1-16

<http://www.numdam.org/item?id=AIHPA_1998__68_1_1_0>

Ann. Inst. Henri Poincaré,

Vol. 68, n° 1, 1998, p. 1-16. *Physique théorique*

About poles of the resolvent, in a model for a harmonic oscillator coupled with massless scalar bosons

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ABSTRACT. – We present a Hamiltonian coupling a harmonic oscillator to a continuum of bosons for which the poles of the resolvent matrix elements are not in one-to-one correspondence with the eigenstates of the isolated oscillator. © Elsevier, Paris.

RÉSUMÉ. – Nous présentons un modèle de couplage d'un oscillateur harmonique avec un continuum de bosons dans lequel il n'y a pas de correspondance biunivoque entre les pôles des éléments de matrice de la résolvante de l'hamiltonien et les états propres de l'oscillateur isolé. © Elsevier, Paris.

1. INTRODUCTION

Rigorous mathematical treatments of the coupling of a harmonic oscillator (with Hamiltonian a^*a) to a continuum of massless scalar bosons (with Hamiltonian H_{rad}) are interesting since they may provide information valid for the atom-radiation interaction; such treatments have to take 2 aspects of the problem into account: the number of bosons may be arbitrary and the energy parameter classically describing the bosons continuously varies from 0. Friedrichs' model [1] incorporated only one photon. In [2], at the end of a study wich follows Friedrichs' one, Brownell sets a problem in which many photons are taken into account but does not treat it, by lack of interest, he writes. Indeed it is quite complicated. Nevertheless the result expected is the following: every eigenvalue of the unperturbed Hamiltonian but the lowest one disappear when a small coupling with the bosons is introduced. This is proved to be true in some models, as for instance the one studied by Arai in [3], for the harmonic oscillator. See also [4] on the subject. In ref [6] a similar result is announced for the atom and the electromagnetic field. The eigenvalues are expected to become complex numbers, poles of certain matrix elements of the resolvent of the Hamiltonian. We are personally interested in obtaining some more information about these poles. This information will be seeked in the following way; in the same way that the search for an eigenvector of an operator can be made easier if one considers its restriction to a subspace in which the eigenvector happens to lie, so may a pole of a matrix element of the resolvent of H be detected by considering only some restriction of this operator. We have thus to find subspaces of the Hilbert space of the system which permit to get interesting results about the spectrum of the full Hamiltonian.

We are particularly interested in the question in the following aspect. Is the structure of the unperturbed oscillator's spectrum conserved in the coupling, being understood that the poles of the matrix elements of the Hamiltonian's resolvent are shifted in the complex plane ? To be more accurate, is there a one-to-one correspondence between the unperturbed oscillator's energy levels and the poles of the matrix elements of the perturbed resolvent ? It is generally admitted that, at least when it is weak, the coupling of an atom with the photon continuum shifts the atomic energy levels in the complex plane; atomic levels thus get a certain width. However, the following argument will show that the situation is perhaps not so simple.

If one studies, as an example for a Hamiltonian describing the above oscillator's coupling, the following operator

$$H(\lambda, \mu) = a^*a \otimes 1 + 1 \otimes \mu H_{rad} + \lambda(a^* \otimes c(g) + a \otimes c^*(g)), \quad (1)$$

and is interested in its resolvent's poles, one can see that, for $\mu = 0$, the above mentioned one-to-one correspondence does not exist strictly speaking; this fact will be recalled in the proof of Proposition 2 below. The Hamiltonian is then $a^*a \otimes 1 + \lambda(a^* \otimes c(g) + a \otimes c^*(g))$, consisting of a part $a^*a \otimes 1$ and a perturbation $\lambda(a^* \otimes c(g) + a \otimes c^*(g))$ representing the coupling of the oscillator with the zero energy bosons. The states consisting of the oscillator in state $a^*\Omega_{osc}$ with 0,1,... or N photons all have the same (unperturbed) energy, and the coupling, for each of these states, shifts that energy by a different amount. The oscillator in state $a^*\Omega_{osc}$ with 0 photon has its energy shifted at a value which is the pole in z of $< a^*\Omega_{osc} \otimes \Omega_{rad} \mid G(\lambda, z) \mid a^*\Omega_{osc} \otimes \Omega_{rad} >$ in the neighbourhood of $z = 1$; the oscillator in state $a^*\Omega_{osc}$ with 1 photon has its energy shifted at a different value, which is the pole in z of $< (a^*)^2\Omega_{osc} \otimes \Omega_{rad} \mid G(\lambda, z) \mid (a^*)^2\Omega_{osc} \otimes \Omega_{rad} >$ in the neighbourhood of $z = 1$; $((a^*)^2\Omega_{osc} \otimes \Omega_{rad}$ is coupled to $a^*\Omega_{osc} \otimes c^*(g))$. Such displacements are analogous to those called A.C.Stark shifts or light shifts, which depend on the number of photons present together with the atom. But it must be noted that in our problem there is no exterior electromagnetic field. Of course, if the one-to-one correspondence is to be saved, this fact may provide a way of selecting peculiar poles among the existing ones: certainly, the state with 0 photon may be favoured. However, this way of proceeding must not conceal the fact that the number of poles is infinitely greater than the number of eigenvalues of the isolated oscillator's Hamiltonian. The unperturbed oscillator's energy levels are not only shifted in the complex plane, they are also split. (It must also be noted that the operator $H(\lambda, 0)$ is not bounded from below - see formula (13)).

These remarks on the spectrum structure of the coupled system when $\mu = 0$, incite us to undertake the study of the Hamiltonian when $\mu \neq 0$. If a continuity argument can be made rigorous, when μ is small we would expect a situation similar to the one when $\mu = 0$. That would later make interesting the exact treatment of the physical $\mu = 1$ case, as the question could be asked whether there is still a splitting (now into several complex poles) as in the $\mu = 0$ case. If it were not possible to get the positions of the different poles exactly, we might perhaps at least obtain information about them, even qualitative, such as their number.

At present, the complete treatment of that question is not yet our goal; to start off with, we will only study the Hamiltonian (1) for small μ with the aim of showing that at least two different poles are expected near the energy of the first excited level of the oscillator.

The continuity with respect to μ is not obvious because the operator $H(\lambda, \mu)$ is not a relatively bounded perturbation of $H(\lambda, 0)$. We overcome this obstacle by restricting our analysis onto the subspace $E_{n,\kappa}$, the eigenspace of $N_\kappa^{(tot)}$ corresponding to the eigenvalue n, where

$$N_\kappa^{(tot)} := a^*a \otimes 1 + 1 \otimes \int_{|p| \leq \kappa} c_p^* c_p$$

is the total number of particles (oscillator excitations plus photons) with support in $\{|p| \leq \kappa\}$. Note that if we choose $supp\ g \subseteq \{|p| \leq \kappa\}$

then $H(\lambda.\mu)$ commutes with $N_\kappa^{(tot)}$, and thus $H(\lambda.\mu)$ leaves the subset $E_{n.\kappa}$ invariant. We analyse the restriction of $H(\lambda.\mu)$ onto $E_{1,\kappa}$ and $E_{2,\kappa}$, respectively. This has two advantages. First we can explicitly compute the resolvent of the operator $H(\lambda.0)$ on these subspaces. Secondly, $1 \otimes H_{rad}$ is bounded on these subspaces. Thus the methods of regular perturbation theory do apply to $H(\lambda.\mu)$ restricted to $E_{n,\kappa}$ in spite of the unboundedness of $1 \otimes H_{rad}$ when acting on the full Hilbert space.

2. NOTATIONS

The Hamiltonian (1) acts in the Hilbert space $\mathcal{H} = \mathcal{H}_{osc} \otimes \mathcal{H}_{rad}$, where $\mathcal{H}_{osc} = L^2(\mathbb{R})$ and a^* is the operator creating one degree of excitation. The Hilbert space for the radiation is the Fock space $\mathcal{F}_b$ built with the 1-boson ("photon") space $L^2(\mathbb{R})$. $c^*(g)$ and $c(g)$ are the creation and annihilation operators for the radiation state consisting of one boson in state g. We suppose g real, satisfying $||g||_2 = 1$ and supp.$g \subset [0, \infty[$. Let us recall the definition of H_{rad} : let $\mathcal{F}_n$ be the n-boson space and $\mathcal{D}_n$ the subspace in $\mathcal{F}_n$ of functions φ_n of n variables such that, for $0 \leq i \leq n$, $p \longrightarrow |p_i| \; \varphi_n(p_1, \ldots, p_n)$ is in $L^2(\mathbb{R}^n)$; this last function is denoted by $h_i^{(n)}(\varphi_n)$. We set

$$\mathcal{D}(H_{rad}) :$$

$$= \{\varphi = (\varphi_1, \ldots, \varphi_n \ldots) \in \mathcal{F}_b;\ \varphi_n \in \mathcal{D}_n,\ \sum_{n=1}^{\infty} || \sum_{i=1}^{n} h_i^{(n)}(\varphi_n)||^2 < \infty\},$$

and define H_{rad} by $(H_{rad}\varphi)_n := \sum_{i=1}^n h_i^{(n)}(\varphi_n)$. $H(\lambda, \mu)$ is defined on $\mathcal{S} \otimes \mathcal{D}(H_{rad})$. Although the spectrum of an operator may depend strongly on its domain of definition, we need not be precise on that question for the moment, and we will not examine the question of the self-adjointness of $H(\lambda.\mu)$ either. Indeed, as we said in the introduction, calculations following in this paper will only concern restrictions of operators to subspaces of $\mathcal{H}$ where $H(\lambda.\mu)$ is bounded and self-adjoint. (In these subspaces, there are at most 2 "particles", and their states have a bounded energy; the vectors are explicitly given after equation (5) below.) Considering these restrictions will be sufficient to establish the interesting properties of the full operator.

We denote by $G(\lambda.\mu.z) = [z - H(\lambda.\mu)]^{-1}$, the resolvent for H, which we suppose defined for $\Im z > 0$.

We set

$$G_i(\lambda,\mu,z) = <(i!)^{-\frac{1}{2}}(a^*)^i\Omega_{osc}\otimes\Omega_{rad} \mid \times [z - H(\lambda,\mu)]^{-1} \mid (i!)^{-\frac{1}{2}}(a^*)^i\Omega_{osc}\otimes\Omega_{rad}> \quad (2)$$

We introduce operators R_i, often called level shift operators, defined by

$$R_i(\lambda,\mu,z) = \lambda V + \lambda^2 V Q_i\, [z - Q_i H_0(\mu) Q_i - \lambda Q_i V Q_i]^{-1} Q_i V, \quad (3)$$

where we again suppose the existence of the inverse, for $\Im z > 0$, and where

$$V = a^*\otimes c(g) + a\otimes c^*(g)$$
$$H_0(\mu) = a^*a\otimes 1 + 1\otimes \mu H_{rad}$$
$$Q_i = 1- \mid (i!)^{-\frac{1}{2}}(a^*)^i\Omega_{osc}\otimes\Omega_{rad}><(i!)^{-\frac{1}{2}}(a^*)^i\Omega_{osc}\otimes\Omega_{rad} \mid .$$

Setting

$$R_{i,i}(\lambda,\mu,z) = <(i!)^{-\frac{1}{2}}(a^*)^i\Omega_{osc}\otimes\Omega_{rad} \mid R_i(\lambda,\mu,z) \mid (i!)^{-\frac{1}{2}}(a^*)^i\Omega_{osc}\otimes\Omega_{rad}>, \quad (4)$$

one gets [5]

$$G_i(\lambda,\mu,z) = [z - E_i - R_{i,i}(\lambda,\mu,z)]^{-1}. \quad (5)$$

These formulas will only be used on 2 subspaces left invariant by H; they are the subspaces we talked about in the introduction, and are defined in the following way:

Definition. – First let E_1 denote the subspace generated by vectors of the form $a^*\Omega_{osc}\otimes\Omega_{bos}$ and $\Omega_{osc}\otimes\varphi_1$ with $\varphi_1\in\mathcal{D}_1$, and E_2 the one generated by vectors of the form $2^{-\frac{1}{2}}(a^*)^2\Omega_{osc}\otimes\Omega_{bos}$, $a^*\Omega_{osc}\otimes\varphi_1$ and $\Omega_{osc}\otimes\varphi_2$, $\varphi_1\in\mathcal{D}_1$, $\varphi_2\in\mathcal{D}_2$. E_1 (resp. E_2) is the eigenspace of N^{tot} associated to the eigenvalue 1 (resp. 2). We then denote by $E_{1,\kappa}$ the subset of E_1 obtained by requiring the function φ_1 to have its support contained in $[0,\kappa]$; similarly, $E_{2,\kappa}$ is defined by requiring the same property for φ_1, and supp. $\varphi_2\subset [0,\kappa[\times[0,\kappa[$.

Let us set $d(\lambda) = 1/2(\sqrt{1+4\lambda^2}-1)$. We have $d(\lambda) < \lambda^2$ if $\lambda\neq 0$ and $d(\lambda)\sim\lambda^2$, for small λ. The splitting of the oscillator's energy levels will be measured by that quantity.

3. STATEMENT AND PROOF OF THE MAIN PROPOSITION

As was said in the introduction, the purpose of the present study is to show that, at least for small μ , each of the functions $z \longrightarrow G_i(\lambda, \mu, z)$, $i = 1, 2$, has, in a neighbourhood of $z = 1$, a unique pole $z_i(\lambda, \mu)$ which is simple, and that $z_1(\lambda, \mu)$ and $z_2(\lambda, \mu)$ are distincts. The proof will of course rest on the fact that $G_1(\lambda, 0, .)$ and $G_2(\lambda, 0, .)$ have two distinct poles $z_1(\lambda, 0) = 1 + d(\lambda)$ and $z_2(\lambda, 0) = 1$ in the neighbourhood of $z = 1$. (This will be recalled below in the course of the proof of Proposition 2). To set our point, it will then be sufficient to establish the continuity of $z_i(\lambda, \mu)$ with respect to the parameter μ at $\mu = 0$.

THE MAIN PROPOSITION. – *Let λ be in $]0, 1]$. Let the support of g be bounded and κ be a real such that $g(p) = 0$, if $|p| \notin [0, \kappa]$. Then $\exists \mu_1(\lambda)$ and a neighbourhood $\mathcal{V}$ of $z = 1$ s.t. $\forall \mu$, $0 \leq \mu < \mu_1(\lambda)$, each of the 2 functions $z \longrightarrow G_1(\lambda, \mu, z)$ and $z \longrightarrow G_2(\lambda, \mu, z))$ has exactly 1 pole, respectively $z_1(\lambda, \mu)$ and $z_2(\lambda, \mu)$, in this neighbourhood, these 2 poles being distinct.*

In order to obtain the existence of the poles of $G_i(\lambda, \mu, .)$ near $z_i(\lambda, 0)$, we will use the following expression of Hurwitz theorem.

Let $f(\mu, z)$ be a function which is, for all μ s.t. $0 \leq \mu < \mu_0$, analytic in a disc $D(z_0, R)$, with center z_0 and radius R, not depending on μ. Let us suppose that $\mu \longrightarrow f(\mu, z)$ is continuous at 0, uniformly for $z \in D(z_0, R)$, and that $z \longrightarrow f(0, z)$ does not vanish in D exept at z_0, this zero being simple. Then there exists a function η, defined in $]0, R[$ and taking its values in $\mathbb{R}^+$, such that: $\forall \epsilon$ s.t. $0 < \epsilon < R$, $\forall \mu \in [0, \eta(\epsilon)[$, the function $z \longrightarrow f(\mu, z)$ has a unique zero which is simple in the disc $D(z_0, \epsilon)$. Let us denote it by $z(\mu)$; moreover, the function $\mu \longrightarrow z(\mu)$ is right-continuous at $\mu = 0$.

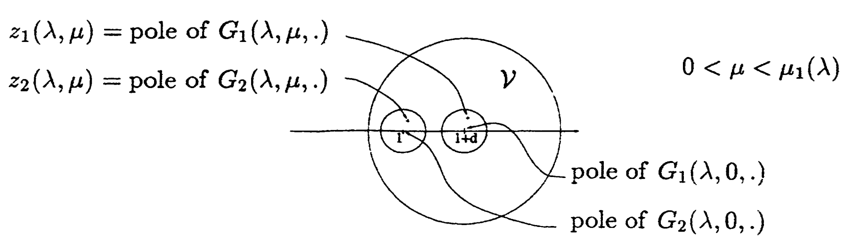

Fig. 1.

Let us set $f_i(\lambda, \mu, z) = z - i - R_{i,i}(\lambda, \mu, z)$. To apply this theorem to $f = f_i = G_i^{-1}$, let us first show that, λ being fixed, $R_{1,1}(\lambda, \mu, .)$ is analytic for z in a neighbourhood of $1 + d$, and $R_{2,2}(\lambda, \mu, .)$ analytic for

z in a neighbourhood of 1. Then we will prove that the 2 functions $\mu \longrightarrow R_{i,i}(\lambda, \mu, z)$ are continuous at 0, uniformly for z in these neighbourhoods.

PROPOSITION 1. – *Let λ be in $]0, 1]$. Let the support of g be bounded and κ such that $g(p) = 0$, $\forall p$, $|p| > \kappa$. Then there exists a constant C such that $z \longrightarrow R_{1,1}(\lambda, \mu, z)$ is analytic in $D_1 = D(1 + d(\lambda), \lambda^2)$, the disc with center $1 + d(\lambda)$ and radius λ^2, for μ satisfying $0 \leq \mu < \kappa^{-1}/2$;*
$z \longrightarrow R_{2,2}(\lambda, \mu, z)$ is analytic in $D_2 = D(1, \lambda^2/2)$ for $0 \leq \mu < C\kappa^{-1}\lambda^2$.
The functions $\mu \longrightarrow R_{1,1}(\lambda, \mu, z)$ (resp. $\mu \longrightarrow R_{2,2}(\lambda, \mu, z)$) are continuous at 0, uniformly for z in D_1 (resp. D_2).

Proof. – The case $i = 1$ can be treated separately thanks to the simplicity of the expression

$$R_{1,1}(\lambda, \mu, z) = \lambda^2 \int_0^\infty g^2(p)\ (z - \mu p)^{-1}\ dp. \tag{7}$$

We suppose $\lambda \in]0, 1]$. The inequality $1 + d(\lambda) - \lambda^2 > 1/2$ implies that, for $0 \leq \mu\kappa < 1/2$, there exists $a_1 > 0$ such that

$$\forall z \in D(1 + d(\lambda), \lambda^2), \quad \forall p \in \text{supp. } g, \quad |z - \mu p| > a_1;$$

consequently, $z \longrightarrow R_{1,1}(\lambda, \mu, z)$ is analytic in $D(1+d(\lambda), \lambda^2)$. The uniform continuity in the μ variable follows from

$$R_{1,1}(\lambda, \mu, z) - R_{1,1}(\lambda, 0, z) = \mu \frac{\lambda^2}{z} \int_0^\infty p\ g^2(p)\ (z - \mu p)^{-1}\ dp,$$

which implies

$$\forall z \in D(1 + d(\lambda), \lambda^2), \quad |R_{1,1}(\lambda, \mu, z) - R_{1,1}(\lambda, 0, z)| < 2\mu\lambda^2\kappa/a_1.$$

Let us examine now the $i = 2$ case.

a) First let us show that $z \longrightarrow R_{2,2}(\lambda, \mu, z)$ is analytic in D_2. Unlike in the preceding case, we have to go back to the definition of $R_2(\lambda, \mu, z)$ (formula (3)), which gives formally

$$\begin{aligned} R_{2,2}(\lambda, \mu, z) = 2\lambda^2 &< a^*\Omega_{osc} \otimes g \mid \\ &[1 - \mu L(\lambda, z) Q_2 H_{rad} Q_2]^{-1} L(\lambda, z) \mid a^*\Omega_{osc} \otimes g >. \end{aligned} \tag{8}$$

where we have set

$$L(\lambda, z) = [z - Q_2 H(\lambda, 0) Q_2]^{-1}$$

and used $V\big(2^{-\frac{1}{2}}(a^*)^2\Omega_{osc}\otimes\Omega_{rad}\big) = \sqrt{2}\lambda a^*\Omega_{osc}\otimes g$. Without trying to define the inverse operators in all $\mathcal{H}$, we will give a meaning to this formula by using the fact that $a^*\Omega_{osc}\otimes g$ is in $E_{2,\kappa}$, $E_{2,\kappa}$ being invariant by $H(\lambda,\mu)$, H_{rad} and Q_2.

$[z - Q_2H(\lambda,0)Q_2]_{\downarrow E_2}$ has an inverse for $\Im z > 0$, which will be denoted by $L_2(\lambda,z)$. We shall see that $[1-\mu L_2(\lambda,z)Q_2H_{rad}Q_2]_{\downarrow E_{2,\kappa}}$ has also an inverse for $\Im z > 0$. In order to obtain the analyticity properties of $R_{2,2}$, we both have to obtain analyticity properties of L_2 and to control the existence of the inverse of $[1-\mu L_2(\lambda,z)Q_2H_{rad}Q_2]_{\downarrow E_{2,\kappa}}$, when z crosses the real axis. To this end, let us give L_2 explicitly on decomposable elements in E_2 and look for a bound for the norm of its restriction to E_2', the subset of E_2 that these decomposable elements generate, by finite linear combinations.

For $\Im z > 0$, setting $q_1(\lambda,z) = z(z-1)-\lambda^2$ and $q(\lambda,z) = z(z-1)-2\lambda^2$, we get:

$$L_2(\lambda,z)\big((a^*)^2\Omega_{osc}\otimes\Omega_{rad}\big) = z^{-1}(a^*)^2\Omega_{osc}\otimes\Omega_{rad}; \tag{9a}$$

if φ_1 is proportional to g,

$$\begin{aligned} &L_2(\lambda,z)(a^*\Omega_{osc}\otimes\varphi_1)\\ &\quad = q^{-1}(\lambda,z)\ \big(za^*\Omega_{osc}\otimes\varphi_1 + \lambda\sqrt{2}\Omega_{osc}\otimes\varphi_1\vee g\big), \end{aligned} \tag{9b}$$

$$\begin{aligned} &L_2(\lambda,z)(\Omega_{osc}\otimes\varphi_1\vee g)\\ &\quad = q^{-1}(\lambda,z)\big(\lambda\sqrt{2}a^*\Omega_{osc}\otimes\varphi_1 + (z-1)\Omega_{osc}\otimes\varphi_1\vee g\big); \end{aligned} \tag{9c}$$

if φ_1 is orthogonal to g,

$$\begin{aligned} &L_2(\lambda,z)(a^*\Omega_{osc}\otimes\varphi_1)\\ &\quad = q_1^{-1}(\lambda,z)\big(za^*\Omega_{osc}\otimes\varphi_1 + \lambda\sqrt{2}\Omega_{osc}\otimes\varphi_1\vee g\big), \end{aligned} \tag{9b'}$$

$$\begin{aligned} &L_2(\lambda,z)(\Omega_{osc}\otimes\varphi_1\vee g)\\ &\quad = q_1^{-1}(\lambda,z)\big(\lambda/\sqrt{2}a^*\Omega_{osc}\otimes\varphi_1 + (z-1)\Omega_{osc}\otimes\varphi_1\vee g\big); \end{aligned} \tag{9c'}$$

if φ_1 and ψ_1 are orthogonal to g,

$$L_2(\lambda,z)(\Omega_{osc}\otimes\varphi_1\vee\psi_1) = z^{-1}\Omega_{osc}\otimes\varphi_1\vee\psi_1.$$

We used the notation $f\vee g = 2^{-1}(f\otimes g + g\otimes f)$.

Note that $L_2(\lambda,z)\neq G(\lambda,0,z)_{\downarrow E_2}$. If $|z-1|\leq\frac{1}{2}\lambda^2$, $|q(\lambda,z)|\geq\lambda^2$ and $|q_1(\lambda,z)|\geq\lambda^2/10$; therefore, $L_2(\lambda,z)_{\downarrow E_2'}$ may be analytically continued in

$D(1, \frac{1}{2}\lambda^2)$. Let us denote this operator by $L_2'(\lambda, z)$. We are now looking for an upper bound for its norm.

The 4 linear subspaces of E_2 generated respectively by the 4 sets of vectors

$$\{(a^*)^2\Omega_{osc} \otimes \Omega_{rad}\} \qquad \{a^*\Omega_{osc} \otimes g, \ \Omega_{osc} \otimes g \vee g\}$$

$$\{a^*\Omega_{osc} \otimes \varphi_1, \ \Omega_{osc} \otimes \varphi_1 \vee g; \ \varphi_1 \perp g\}, \quad \{\Omega_{osc} \otimes \varphi_1 \vee \psi_1; \ \varphi_1, \ \psi_1 \perp g\}$$

are invariant by $L_2'(\lambda, z)$. On each of these orthogonal subspaces, the norm of $L_2'(\lambda, z)$ is bounded respectively by 2, $\lambda^{-2}(2+2\lambda\sqrt{2}+\lambda^2/2)$, $10\lambda^{-2}(2+2\lambda\sqrt{2} + \lambda^2/2)$, 2. Therefore, with $b(\lambda) = 11(2 + 2\sqrt{2}\lambda + \lambda^2)$, we then get: $\forall z \in D(1, \frac{1}{2}\lambda^2)$, $||L_2'(\lambda, z)|| \leq 2 + b(\lambda)\lambda^{-2}$.

As $L_2'(\lambda, z)$ is bounded, formulas (9) allow to define its extension to E_2 and, as the bound is uniform for z in $D(1, \frac{1}{2}\lambda^2)$, the extension is analytic in that disc; it is the continuation of $L_2(\lambda, z)$.

We will now see the use of our hypothesis that the support of g is bounded. Even on E_2, H_{rad} is unbounded and thus we cannot make $\mu \ L_2(\lambda, z)Q_2H_{rad}Q_2$ small by taking μ small. If g had a non-compact support, we would then have to try and define the matrix elements of $[1 - \mu L_2(\lambda, z)Q_2H_{rad}Q_2]^{-1}$ by other means, for instance by an analytic continuation. Indeed, in the $i = 1$ case, by supposing also analyticity properties of g near the real axis, it is possible to move the integration contour in (7); but it is not so simple in the $i = 2$ case. So let us suppose that the support of g is bounded.

Since $||(H_{rad})_{\downarrow E_{2,\kappa}}|| \leq \sqrt{5} \ \kappa$, $H(\lambda, \mu)_{\downarrow E_{2,\kappa}}$ is bounded and self-adjoint ; $z - H(\lambda, \mu)_{\downarrow E_{2,\kappa}}$ is thus invertible for $\Im z > 0$ and so is $[1 - \mu L_2(\lambda, z)Q_2H_{rad}Q2]_{\downarrow E_{2,\kappa}}$. From

$$\begin{aligned} ||L_2(\lambda, z)Q_2H_{rad}Q_2 \downarrow_{E_{2,\kappa}} || &= ||L_2(\lambda, z)(H_{rad})_{\downarrow E_{2,\kappa}}|| \\ &\leq ||L_2(\lambda, z)|| ||(H_{rad})_{\downarrow E_{2,\kappa}}||, \end{aligned}$$

one deduces that

$$[1 - \mu L_2(\lambda, z)Q_2H_{rad}Q2]_{\downarrow E_{2,\kappa}}$$

is invertible for $z \in D(1, \frac{1}{2}\lambda^2)$ if

$$\mu < (\sqrt{5}\kappa)^{-1}(b(\lambda) + 2\lambda^2)^{-1}\lambda^2 \ . \tag{11}$$

Since $\exists b_0 > 0 \, : \, b_0^{-1} < (2\lambda^2 + b(\lambda))^{-1}$, it follows that the first part of Proposition 1 is true if $C = (2\sqrt{5}b_0)^{-1}$. Then we have

$$\left\| \left([1 - \mu L_2(\lambda, z) Q_2 H_{rad} Q_2]_{\downarrow E_{2,\kappa}} \right)^{-1} \right\| \leq 2 \, .$$

Moreover, for these values of μ, $z \to \, < 2 \mid R_2(\lambda, \mu, z) \mid 2 >$ is analytic in the disc $D(1, \frac{1}{2}\lambda^2)$.

b) Let us now get to the continuity property. We have

$$\begin{aligned} &< 2^{-\frac{1}{2}} (a^*)^2 \Omega_{osc} \otimes \Omega_{rad} \mid R(\lambda, \mu, z) \\ &\quad - R(\lambda, 0, z) \mid 2^{-\frac{1}{2}} (a^*)^2 \Omega_{osc} \otimes \Omega_{rad} >= (2\lambda^2 \mu) . \\ &< a^* \Omega_{osc} \otimes g \mid [1 - \mu L_2(\lambda, z) Q_2 H_{rad} Q_2]^{-1} L_2(\lambda, z) \\ &\quad \times Q_2 H_{rad} Q_2 L_2(\lambda, z) \mid a^* \Omega_{osc} \otimes g > . \end{aligned} \tag{12}$$

Let us fix ϵ, arbitrary but > 0, and try to satisfy

$$\begin{aligned} &| < 2^{-\frac{1}{2}} (a^*)^2 \Omega_{osc} \otimes \Omega_{rad} \mid R(\lambda, \mu, z) - R(\lambda, 0, z) \mid \\ &\quad 2^{-\frac{1}{2}} (a^*)^2 \Omega_{osc} \otimes \Omega_{rad} > | \leq \epsilon . \end{aligned} \tag{13}$$

We have

$$\begin{aligned} &| < 2^{-\frac{1}{2}} (a^*)^2 \Omega_{osc} \otimes \Omega_{rad} \mid R(\lambda, \mu, z) - R(\lambda, 0, z) \mid 2^{-\frac{1}{2}} (a^*)^2 \Omega_{osc} \otimes \Omega_{rad} > | \\ &\quad \leq 2\lambda^2 \mu \, \| [1 - \mu L_2(\lambda, z) Q_2 H_{rad} Q_2]^{-1}_{\downarrow E_{2,\kappa}} \| . \| L_2(\lambda, z) Q_2 H_{rad} Q_2 L_2(\lambda, z)_{\downarrow E_{2,\kappa}} \\ &\| L_2(\lambda, z) Q_2 H_{rad} Q_2 L_2(\lambda, z)_{\downarrow E_{2,\kappa}} \| \\ &\quad = \| L_2(\lambda, z) H_{rad} L_2(\lambda, z)_{\downarrow E_{2,\kappa}} \| \leq \| L_2(\lambda, z) \|^2 \| (H_{rad})_{\downarrow E_{2,\kappa}} \| . \end{aligned}$$

(13) will be verified if, besides (11), μ satisfies

$$4\lambda^2 \mu (2 + b/\lambda^2)^2 \sqrt{5} \, \kappa < \epsilon \, ,$$

which is realized if

$$0 \leq \mu < \left(\frac{1}{4\sqrt{5} \, \kappa} \frac{1}{(2\lambda^2 + b(\lambda))^2} \lambda^2 \right) \epsilon . \tag{14}$$

We thus get the continuity we looked for and the proof of Proposition 1 is achieved. ■

From the preceding results we shall infer information about the poles of the functions $G_i(\lambda, \mu, .)$ in the neighbourhood of $z_i(\lambda, 0)$.

PROPOSITION 2. – *Let g be as in Proposition 1 and $\lambda \in]0,1]$. For $0 < \epsilon < \lambda^2/2$, there exists a function $\mu_1(\lambda,\epsilon)$ such that, for $0 \leq \mu < \mu_1(\lambda,\epsilon)$,*
$z \rightarrow G_1(\lambda,\mu,z)$ has, for z in the neighbourhood $D(1+d(\lambda),\epsilon)$ of $1+d(\lambda)$, a unique pole $z_1(\lambda,\mu)$, which is simple.
$z \rightarrow G_2(\lambda,\mu,z)$ has, for z in the neighbourhood $D(1,\epsilon)$ of 1, a unique pole $z_2(\lambda,\mu)$, which is simple.
$\forall i,\ \mu \rightarrow z_i(\lambda,\mu)$ is continuous at $\mu = 0$; $z_1(\lambda,0) = 1+d(\lambda)$, $z_2(\lambda,0) = 1$.

Note. – $z_1(\lambda,.)$ is real valued for small positive μ.

Proof. – From the analyticity of $R_{i,i}(\lambda,\mu,.)$ we have just obtained (in the disc D_i, for $0 \leq \mu < C\kappa^{-1}\lambda^2$), one deduces that of $f_i(\lambda,\mu,.)$. In the same way, one gets the continuity of $f(\lambda,.,z)$ at 0, uniform for $z \in D_i$. We can therefore apply Hurwitz theorem provided we prove that $f_i(\lambda,0,.)$ has a unique zero, simple, in D_i, *i.e.* that, in that disc, $G_i(\lambda,0,.)$ has a unique, simple, pole. Let us now give a proof of that fact.

The spectrum of the Hamiltonian $H(\lambda,0)$ consists of all the real numbers of the form $s_+\ (1+d(\lambda)) - s_-\ d(\lambda)$, where s_+ and s_- are non-negative integers. This follows from the fact that, with θ given by $\tan 2\theta = 2\lambda$, the transformation

$$\begin{aligned} \beta_+ &= \cos\theta\ \alpha \otimes 1 + \sin\theta\ 1 \otimes c(g) \\ \beta_- &= -\sin\theta\ \alpha \otimes 1 + \cos\theta\ 1 \otimes c(g), \end{aligned}$$

leads to the following form for the Hamiltonian:

$$H(\lambda,0) = (1+d(\lambda))\ \beta_+^*\beta_+ - d(\lambda)\ \beta_-^*\beta_-. \tag{15}$$

Since the βs satisfy

$$[\beta_+,\beta_+^*] = [\beta_-,\beta_-^*] = 1, \qquad [\beta_-,\beta_+] = 0,$$

$H(\lambda,0)$ appears as the difference of 2 harmonic oscillator Hamiltonians; their energies are respectively $1+d(\lambda)$ and $d(\lambda)$.

As we are here focussing on a neighbourhood of 1, we have but to consider the values $1+(1-n)d$, with n a non-negative integer. An easy computation gives

$$G_1(\lambda,0,z) = z[z(z-1) - \lambda^2]^{-1}, \tag{16}$$

which has 2 poles, at $z = 1+d(\lambda)$ and $z = -d(\lambda)$. We denote by $z_1(\lambda,0)$ the one close to 1.

We have

$$R_{1,1}(\lambda, 0, z) = \lambda^2 z^{-1}, \quad \text{for } \Im z > 0. \tag{17}$$

Besides,

$$G_2(\lambda, 0, z) = \frac{1}{(z-1)} \frac{z(z-1) - 2\lambda^2}{z(z-2) - 4\lambda^2}, \tag{18}$$

which has 3 poles. The one close to 1 is denoted by $z_2(\lambda, 0)$; it is exactly 1, which does not depend on λ. The 2 others are respectively $1 + \sqrt{1 + 4\lambda^2}$ and $1 - \sqrt{1 + 4\lambda^2}$. We have

$$R_{2,2}(\lambda, 0, z) = 2\lambda^2 z[z(z-1) - 2\lambda^2]^{-1}, \quad \text{for } \Im z > 0. \tag{19}$$

Note that, for $\mu = 0$, the perturbative series in powers of λ for R_i and G_i can be summed easily, the summation giving the above results; it is not the case if $\mu \neq 0$.

The preceding calculations establish what is important for us: the poles $z_1(\lambda, 0) = 1 + d(\lambda)$ and $z_2(\lambda, 0) = 1$ are isolated in the discs D_1 and D_2. Proposition 2 is then going to follow from applying Hurwitz theorem to the 2 functions $f_i(\lambda, \mu, z)$.

For $i = 1$, we take $\mu_0 = \mu_{0,1}(\lambda) = \kappa^{-1}/2$, $z_0 = 1 + d(\lambda)$ and $R = \lambda^2$. As $1 + d(\lambda)$ is the only zero of $f_1(\lambda, 0, z)$ in D_1, a simple one, Proposition 1 and the Hurwitz theorem imply that there exists a function η_1 such that: $\forall \epsilon$ s.t. $0 < \epsilon < \lambda^2$, $\forall \mu \in [0, \eta_1(\epsilon)[$, the function $z \to f_1(\lambda, \mu, z)$ has a unique zero $z_1(\lambda, \mu)$ which is simple, in the disc $D(1 + d(\lambda), \epsilon)$.

For $i = 2$, we take $\mu_0 = \mu_{0,2}(\lambda) = C\kappa^{-1}\lambda^2$, $z_0 = 1$ and $R = \frac{1}{2}\lambda^2$. As 1 is the only zero of $f_2(\lambda, 0, z)$ in D_2, and is simple, Proposition 1 and the Hurwitz theorem imply that there exists a function η_2 such that $\forall \epsilon$ s.t. $0 < \epsilon < \frac{1}{2}\lambda^2$, $\forall \mu \in [0, \eta_2(\epsilon)[$, the function $z \to f_2(\lambda, \mu, z)$ has a unique zero $z_2(\lambda, \mu)$ which is simple, in the disc $D(1, \epsilon)$

For $0 < \epsilon < \lambda^2/2$, taking $\mu_1(\lambda, \epsilon) = \inf \{\eta_1(\epsilon), \eta_2(\epsilon)\}$, we get Proposition 2. ■

Although it is not useful for our main proof, let us now show that $z_1(\lambda, \mu)$ is real-valued on $[0, \mu_1(\lambda, \epsilon)[$.

If $0 \leq \mu < \mu_{0,1}(\lambda)$, $x \in [1 + d(\lambda) - \lambda^2, 1 + d(\lambda) + \lambda^2]$, and $p \in$ supp. (g), then $x - \mu p \neq 0$. We have

$$\frac{\partial f_1}{\partial x}(\lambda, \mu, x) = 1 + 2\lambda^2 \int_0^\infty (x - \mu p)^{-2}\, g^2(p)\, dp > 0,$$

$$f_1(\lambda, \mu, 1) < 0,$$

and

$$f_1(\lambda, \mu, 2) = 1 - \lambda^2 \int_0^{\infty} g^2(p)\ (2 - \mu p)^{-1}\ dp\ > 1 - \lambda^2\ .$$

Thus $f_1(\lambda, \mu, 2)$ is strictly positive if $\lambda \leq 1$. Therefore, for $0 \leq \mu < \mu_{0,1}(\lambda)$, $x \rightarrow f_1(\lambda, \mu, x)$ has one and only one (real) zero between 1 and 2; thus if $0 \leq \mu < \mu_1(\lambda, \epsilon)$ $(\leq \mu_{0,1}(\lambda))$, this zero is the one we found in the neighbourhood $D(1 + d(\lambda), \epsilon)$.

Proof of the main Proposition. – We just have to collect the preceding results. Let us set $\mathcal{V} = D(1 + d(\lambda), \lambda^2)$; it is a neighbourhood of 1. Let us choose $\epsilon_1(\lambda)$ satisfying $0 \leq \epsilon_1(\lambda) < \lambda^2/2$, and such that the discs $D(1, \epsilon_1)$ and $D(1 + d(\lambda), \epsilon_1)$ are disjoint and contained in $\mathcal{V}$.

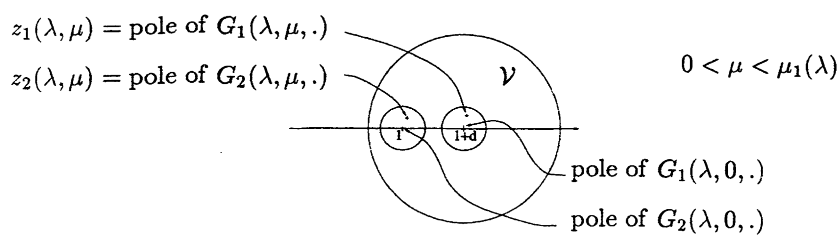

Fig. 2.

Setting $\mu_1(\lambda) = \mu_1(\lambda, \epsilon_1)$, given by Proposition 2, we see that, for $0 \leq \mu < \mu_1(\lambda)$, $G_1(\lambda, \mu, .)$ (resp. $G_2(\lambda, \mu, .)$) has a pole $z_1(\lambda, \mu)$ (resp. $z_2(\lambda, \mu)$) in the disc $D(1 + d(\lambda), \epsilon_1)$ (resp. $D(1, \epsilon_1)$). As a consequence, these two poles are distinct and our main proposition is proved. ∎

4. DISCUSSION

We have to underline that this splitting of the pole of the unperturbed resolvent when the coupling is introduced has just been proved with the hypothesis that the support of the function g is compact. We wish now to comment on that point.

When $i = 1$ and supp. $g \subset [0, \kappa]$, we saw that $G_1(\lambda, \mu, x)$ is real when x is real, ≥ 1 and $\mu < C\kappa^{-1}\lambda^2$. With such values, the pole in the x-variable is an eigenvalue of the Hamiltonian, corresponding to a stable state. If g has a non-compact support, then the integrant in formula (7) has a pole at $p = z/\mu$, for every real positive value of z and all $\mu > 0$.

Nevertheless, assuming analyticity properties for g allows to perform an analytic continuation of $R_{1,1}(\lambda, \mu, z)$ (and thus of $G(\lambda, \mu, z)$) in the lower half-plane of the z-variable. G will then take complex values for real z. One may expect to be able to prove, with a method analogous to the one we used here, that, in this case, G_1 has a unique pole, complex this time, in the considered region.

For the $i = 2$ case, without the compacity hypothesis, we will also have to use analytic continuations, but with other techniques. We leave that for a later work.

Besides, we want to point out the following : Information about the poles could be seeked by looking at the terms in the series of R in powers of λ. Unfortunately, one must be very careful doing so, as, unless a detailed study has been done, it is not correct to attribute properties that the individual terms may have to the sum of the series. For example, if $\mu = 0$, whether g has a compact support or not, each term in the expansion of $R_{2,2}$ has a pole at $z = 1$, whereas the sum of the series does not have a pole at that point; it has 2 poles at $1/2 \pm 1/2\sqrt{1 + 8\lambda^2}$. Also, if g has compact support and μ is small, we saw that $R_{2,2}$ is analytic in a neighbourhood of $z = 1$, whereas the term in second order of the expansion in powers of λ^2,

$$\int \frac{|g(p)|^2}{z - 1 - \mu p}\, dp, \tag{20}$$

has a branch point at $z = 1$.

5. CONCLUSION AND PERSPECTIVES

We proved that, for small μ, the poles of the resolvent of the oscillator's Hamiltonian are split when the coupling with the radiation is considered.

To obtain this result, we had to use a hypothesis on the function g describing the 1-photon state, but we think that this restriction is not crucial. We expect the 2 poles we considered to become complex if the support of g is no more compact, and our result, that they are distinct, to be still true in that case.

Now the question is: is the result still valid for $\mu = 1$?

If the answer to that question is yes, it seems that, extended to atom-radiation interaction, it would be in discrepancy with the usual picture of atomic states; indeed, at least according to the usual way of seeing it, only 1 pole of the complete Hamiltonian resolvent is expected for each energy level of the isolated atom. So, should we expect, in order to save that picture, the different poles that would be associated to such a level for small μ to amalgamate in one unique pole when $\mu = 1$? This might occur when μ gets greater than a certain critical value. Or do the poles keep distinct ? If this is the case, one might argue that it is because of the 2 important features of the problem we spoke about in the introduction: there exist photons with energy arbitrarily close to 0, and their number may be arbitrary (at least greater than 1). Indeed, the state consisting of an atom in a certain energy level without any photon is thus almost degenerated with the same atomic state accompanied by an arbitrary number of photons, if they have a small energy; that might be a reason why the unperturbed pole would then be split in many complex poles as in the $\mu = 0$ case, when the coupling is introduced. Now, this degeneracy occurs if states $a^*\Omega_{osc} \otimes c_p^*\Omega_{rad}$ are taken into account and, physically, they may go to states $\Omega_{osc} \otimes c_p^* c_{p'}^* \Omega_{rad}$, 2-photon states. This is why we think that the phenomenon we are thinking of cannot be seen in models in which at most 1 photon is present.

Should we admit that an entire family of an infinite number of poles is in correspondence with one unique eigenvalue of the isolated oscillator, we have then to cope with a situation which seems new to me. Indeed, whereas several widths may be associated to a unique atomic level, each one corresponding to a particular desexcitation channel, up to now, to my knowledge, it has not been considered that several energies, or poles, could be associated to such a level. We insist that there is no exterior field to be made responsible for that, exept the vacuum. Thus having the energy of an unstable state depend on the number of photons it will emit in the future might seem rather strange, even if it is true that the very definition of such an energy was already a problem, due to the width of the level. We think that the present study may cast new light on the subject. From another point of view, the question may be presented in the following way: are calculated or experimental profiles to be found peaked at different energies, according to how many photons are emitted? This problem was already examined in ref [7] in a perturbative way.

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*(Manuscript received on October 24th, 1995;
Revised version received on May 14th, 1996.)*