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# Semi-classical trace formula and clustering of eigenvalues for Schrödinger operators

by

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ABSTRACT. – This paper is devoted to certain semi-classical asymptotics of a Schrödinger type operator $A(h)$ in the vicinity of a regular value $E$ of its principal symbol $a_0(x, \xi)$. We investigate the semi-classical behaviour of the number $N_{E+rh,c}(h)$ of all eigenvalues $\lambda_j(h)$ of $A(h)$ situated in the interval $[E + rh - ch, E + rh + ch]$, where the energy shift parameter $r$ and the size constant $c > 0$ are both bounded. The behaviour of $N_{E+rh,c}(h)$ for small $h$ depends on an oscillating term $Q(h, r)$ which is related to the periodic trajectories of the Hamiltonian vector field $H_{a_0}$ on the energy hypersurface $\Sigma = \{(x, \xi) : a_0(x, \xi) = E\}$. If $Q(h, r)$ is uniformly continuous in $r$ for any $0 < h \leq h_0$, we obtain asymptotics of the counting function $N_{E+rh,c}(h)$ as $h$ tends to zero. On the other hand, the points of discontinuity of $Q(h, r)$ in $r$ may give rise to a clustering of eigenvalues of $A(h)$ near the energy level $E$. Such jumps of the function $Q$ in $r$ are described in terms of a suitable quantization condition. In particular, if $a_0$ is analytic in a neighborhood of $\Sigma$ and the energy surface is connected and of contact type we obtain a complete description of the asymptotics of $N_{E+rh,c}(h)$. Moreover, we

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obtain a new semi-classical trace formula giving for any $\rho(\tau) \in \mathcal{S}(\mathbf{R})$ with Fourier transform $\hat{\rho}(t) \in C_0^\infty(\mathbf{R})$ the asymptotics of

$$\sum_{\lambda_j(h)\leq\lambda} \rho\Big(\frac{E-\lambda_j(h)}{h}\Big)$$

in terms of certain dynamic and topological characteristics of the periodic trajectories of $H_{a_0}$ on $\Sigma$ without any additional clean intersection assumptions. © Elsevier, Paris.

*Key words:* Eigenvalues, periodic trajectories, clustering, quantization.

RÉSUMÉ. – L'article est consacré à l'étude des asymptotiques semi-classiques de Schrödinger type opérateur $A(h)$ au voisinage d'une valeur régulière $E$ du symbole principal $a_0(x, \xi)$. On examine le comportement semi-classique de la fonction de comptage $N_{E+rh,c}(h)$ des valeurs propres $\lambda_j(h)$ de $A(h)$ incluses dans l'intervalle $[E + rh - ch, E + rh + ch]$, où le paramètre de la translation de l'énergie $r$ et la constante $c > 0$ sont bornés. Le comportement de $N_{E+rh,c}(h)$ pour $h$ petit dépend d'un terme oscillant $Q(h, r)$ associé aux trajectoires périodiques du flot hamiltonien $H_{a_0}$ sur la surface d'énergie $\Sigma = \{(x, \xi) : a_0(x, \xi) = E\}$. Si $Q(h, r)$ est uniformement continu par rapport à $r$ pour tout $0 < h \leq h_0$, on obtient une asymptotique de la fonction de comptage $N_{E+rh,c}(h)$ quand $h$ tend vers zero. D'autre part, les points de discontinuité de $Q(h, r)$ par rapport à $r$ pourraient impliquer la concentration de valeurs propres de $A(h)$ près du niveau d'énergie $E$. Les sauts de la fonction $Q(h, r)$ sont déterminés par une condition convenable de quantification. En particulier, dans le cas où $a_0$ est analytique au voisinage de $E$ et la surface d'énergie est convexe et du type de contact, on obtient une description complète de l'asymptotique de $N_{E+rh,c}(h)$. De plus, on prouve une nouvelle formule de trace semi-classique qui donne pour toute $\rho(\tau) \in \mathcal{S}(\mathbf{R})$ ayant transformation de Fourier $\hat{\rho}(t) \in C_0^\infty(\mathbf{R})$ l'asymptotique de

$$\sum_{\lambda_j(h)\leq\lambda} \rho\Big(\frac{E-\lambda_j(h)}{h}\Big)$$

liée avec les caractéristiques dynamiques et topologiques des trajectoires périodiques de $H_{a_0}$ sur $\Sigma$ sans aucune hypothèse sur le comportement de ces trajectoires. © Elsevier, Paris.

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## 1. INTRODUCTION

This paper is concerned with the asymptotics of the eigenvalues of Schrödinger type operators $A(h)$ near a fixed energy level. The $h$-pseudodifferential operators

$$(A(h)\varphi)(x) = \int \exp(\imath\langle x-y,\xi\rangle)a\Big(\frac{x+y}{2}, h\xi; h\Big)\varphi(y)dyd\xi,$$

$$\varphi(x) \in \mathcal{S}(\mathbf{R}^n),$$

we deal with are obtained by Weyl quantization of smooth real-valued symbols

$$a(x,\xi;h) = \sum_{j=0}^{N} a_j(x,\xi)h^j.$$

Following [15], we say that the operator $A(h)$ is $h$-admissible, if the following assumptions are satisfied:

$(H_1)$ The function $a_0(x,\xi) - \gamma_0$ is a temperate weight for some $\gamma_0 \in \mathbf{R}$, *i.e.* there exist $C_0 > 0$ and $N_0 > 0$ such that

$$0 < a_0(x,\xi) - \gamma_0 \leq C_0(a_0(y,\eta) - \gamma_0)\Big(1 + |x-y| + |\xi - \eta|\Big)^{N_0}$$

for all $(x,\xi)$, $(y,\eta) \in T^*(\mathbf{R}^n)$,

$(H_2)$ For $0 \leq j \leq N$ and all multiindices $\alpha, \beta$ we have

$$|\partial_x^\alpha \partial_\xi^\beta a_j| \leq C_{\alpha,\beta,j}(a_o - \gamma_0).$$

Let us set

$$\lambda_0 = \lim_{|x|+|\xi|\to\infty} \inf a_0(x,\xi).$$

Under the conditions $(H_1)$ and $(H_2)$ the operator $A(h)$ admits a self-adjoint extension in $L^2(\mathbf{R}^{2n})$ for any $0 < h \leq h_0$ which also will be denoted by $A(h)$. Moreover, for each $\lambda < \lambda_0$ the spectrum of $A(h)$ in $(-\infty, \lambda]$ is formed by finitely many isolated eigenvalues $\lambda_j(h)$, $j = 1, \dots, p(\lambda, h)$, of finite multiplicity [15], [16].

Next we suppose that $E < \lambda_0$ is a regular value of $a_0$, and set

$$\Sigma = \{(x,\xi) \in T^*(\mathbf{R}^n) : a_0(x,\xi) = E\}.$$

Then $\Sigma$ is a smooth compact hypersurface and the differential $da_0$ does not vanish on it. Our aim is to study the semi-classical asymptotics $(h \searrow 0)$ of the function

$$N_{E+rh,c}(h) = \sharp\{j : \left|\lambda_j(h) - E - rh\right| < ch\},$$

counting with multiplicities the eigenvalues of the operator $A(h)$ in the shifted energy interval $[E + rh - ch, E + rh - ch]$. Here, the parameter $r$ of the energy shift and the size constant $c$ are both bounded, $|r| \leq r_0$, $0 < c \leq c_0$, and we allow $h$ to vary in the interval $0 < h \leq h_0$ with $E + (r_0 + c_0)h_0 < \lambda_0$.

We shall be mainly interested in the semi-classical behaviour of the counting function $N_{E+rh,c}(h)$ of the Schrödinger operator $A(h) = -h^2\Delta + V(x)$ having a symbol

$$a_0(x,\xi) = |\xi|^2 + V(x), \tag{1.1}$$

where $V \geq \gamma_1$ is a smooth real-valued potential. In this case we do not need the assumptions $(H_1)$ and $(H_2)$. On the other hand, it is natural to formulate the main results in the more general setting for $h$-admissible operators to include for instance the magnetic Schrödinger operator or other special operators in quantum mechanics.

The asymptotic of the counting function $N_{E+rh,c}(h)$ is closely related to the behaviour of the periodic trajectories of the Hamiltonian flow $\Phi^t = \exp(tH_{a_0})$ on the energy surface $\Sigma$. A point $\nu \in \Sigma$ is called periodic, if there exists $T > 0$ such that

$$\Phi^T(\nu) = \nu. \tag{1.2}$$

For any periodic point $\nu$ denote by

$$\gamma(\nu) = \{\Phi^t(\nu) : 0 \leq t \leq T(\nu)\} = \{(x(t), \xi(t)) : 0 \leq t \leq T(\nu)\}$$

the primitive periodic trajectory issuing from $\nu$, where $T(\nu) = T_\gamma$ is the period of $\gamma(\nu)$ defined as the smallest positive number $T$ for which (1.2) holds. Let $\Pi$ be the set of all periodic points on $\Sigma$. If $\Pi$ has Lebesgue measure zero in $\Sigma$, the following Weyl type asymptotic holds (*see* [2] and [14])

$$\lim_{h \searrow 0} \left[h^{n-1} N_{E,c}(h)\right] = \frac{2c}{(2\pi)^n} \mu(\Sigma), \tag{1.3}$$

where $\mu(\Sigma) = \int_\Sigma |\nabla a_0(x, \xi)|^{-1} dS$ is the Liouville measure of $\Sigma$ and $dS$ is the induced Lebesgue measure on $\Sigma$.

The asymptotic behaviour of $N_{E+rh,c}(h)$ could be quite complicated when the set of periodic points $\Pi$ is of a positive Lebesgue measure in $\Sigma$. Then it depends on certain dynamic and topological characteristics of the corresponding closed trajectories on $\Sigma$. For any periodic point $\nu$ let

$$S(\nu) = \int_{\gamma(\nu)} \xi dx = \int_0^{T(\nu)} \xi(t) \dot{x}(t) dt$$

be the action along the corresponding primitive periodic trajectory $\gamma(\nu)$. Next, we denote by $m(\gamma) \in \mathbf{Z}_4$ a suitable Maslov index related to $\gamma(\nu)$ and set

$$q_c(\nu) = \frac{\pi}{2} m(\gamma).$$

Let

$$s(\nu) = \int_0^{T(\nu)} a_1(x(t), \xi(t)) dt,$$

where $a_1$ is the subprincipal symbol of $A(h)$ and let $q(\nu) = s(\nu) - q_c(\nu)$.

The semi-classical behaviour of $N_{E+rh,c}(h)$ changes completely if the flow $\Phi^t(\nu)$ is totally periodic on $\Sigma$ which means that any point of $\Sigma$ is periodic and there is a positive constant $T$ such that

$$\exp(TH_{a_0})(\nu) = \nu, \forall \nu \in \Sigma.$$

If in addition $\Sigma$ is connected, then $q_c(\nu) = q$ is constant, and there exists $S > 0$ such that $S(\nu) = S$ for any $\nu \in \Sigma$. It turns out that (1.3) is not true any more, instead of asymptotics one gets a kind of clustering of eigenvalues of $A(h)$ at certain energies which are $\mathcal{O}(h)$ close to $E$. These energies can be written explicitly in terms of $T$. $S$ and $q$ by means of a quantization condition.

The asymptotics of the eigenvalues of elliptic operators on manifolds in the case when all points are periodic with common period have been examined by several authors (*see* [9], [6], [20], [12], [28]). The semi-classical asymptotics of $N_{E+rh,c}(h)$ in the totally periodic case have been investigated in [2], [7], [21]. Brummelhuis and Uribe [2] proved for the operator $A(h) = -h^2\Delta + V(x)$ that

$$\lim_{h_k \searrow 0}\left[h_k^{n-1} N_{E+\frac{q}{T}h_k,c}(h_k)\right] = \frac{\mu(\Sigma)}{(2\pi)^{n-1}T}, \quad h_k = \frac{S}{2\pi k}, \quad k \in \mathbf{N}, \tag{1.4}$$

uniformly with respect to $0 < c \leq c_0$. Note that the right-hand side of (1.4) is independent of $c$ in contrast to (1.3).

Recently, Dozias [7] performed a detailed analysis on the distribution of the eigenvalues of $A(h)$ when the flow is totally periodic on $\Sigma$ assuming that $s(\nu)$ is constant. The main result in [7] says that the spectrum of $A(h)$ around $E$ is clustered in certain intervals of the size of $h^2$. More precisely, set

$$I_k(h) = \left[E + r(k,h)h - Ch^2, E + r(k,h)h + Ch^2\right], C > 0,$$

where the integer $k$ and the shift parameter $r$ are determined by the quantization condition

$$-\frac{S}{T} + \frac{q}{T}h + \frac{2\pi k}{T}h = r(k,h)h, \quad |r(k,h)| \leq r_0.$$

For any $r_0$ fixed, consider the set $\mathcal{M} = \{(k,h) \in \mathbf{Z} \times (0,h_0] : |r(k,h)| \leq r_0\}$. Taking $(k,h) \in \mathcal{M}$, Dozias proves in [7] that

$$\sharp\{j : \lambda_j(h) \in I_k(h)\} = \frac{h^{1-n}\mu(\Sigma)}{(2\pi)^{n-1}T} + \mathcal{O}(h^{2-n}). \tag{1.5}$$

Let us remark that in contrast to the totally periodic case the Hamiltonian flow on the energy surface could be periodic but with variable periods of the trajectories (*see* [31] and Example 9.5).

Motivated by (1.3), (1.4) and (1.5), we say that there is a clustering of eigenvalues of $A(h)$ near the energy level $E$ if there exists a bounded function $r(h)$, $|r(h)| \leq r_0$, $h \in (0,h_0]$, and positive constants $C_1$ and $c_0$ such that

$$\liminf_{h \searrow 0} \left(h^{n-1} N_{E+r(h)h,c}(h)\right) \geq C_1, \quad \forall\, c \in (0,c_0]. \tag{1.6}$$

Our aim in this work is, without any additional assumptions on the structure of the set $\Pi$ of periodic points, to describe the semi-classical behaviour of $N_{E+rh,c}(h)$ in terms of the continuity properties of a certain oscillating function $Q(h,r)$ which will be defined bellow. We shall obtain either semi-classical asymptotics of $N_{E+rh,c}(h)$ or clustering near $E$ in the sense of (1.6), where the energy shift parameter $r(h)$ will be determined by a suitable quantization condition.

Define the residium $[z]_{2\pi}$ by

$$z = [z]_{2\pi} + 2\pi k, \ -\pi < [z]_{2\pi} \leq \pi, \ k \in \mathbf{Z},$$

and introduce the oscillating function

$$Q(h,r) = (2\pi)^{-n} \int_{\Pi} \left[\pi - h^{-1}S(\nu) + q(\nu) - rT(\nu)\right]_{2\pi} T(\nu)^{-1} d\nu, \quad (1.7)$$

where $d\nu = |\nabla a_0|^{-1} dS$ stands for the Liouville measure on $\Sigma$, $dS$ being the induced Lebesgue measure. The function $Q(h,r)$ has the following important property. The limits

$$Q(h,r \pm 0) = \lim_{\varepsilon \searrow 0} Q(h, r \pm \varepsilon)$$

exist for any $0 < h \leq h_0$ and any $r$. Moreover,

$$Q(h,r+0) - Q(h,r-0) = (2\pi)^{1-n} \int_{\Omega_{h,r}} \frac{d\nu}{T(\nu)}, \quad (1.8)$$

where

$$\Omega_{h,r} = \{\nu \in \Pi : \ h^{-1}S(\nu) - q(\nu) + rT(\nu) \equiv 0(2\pi)\}.$$

Thus $Q(h,r)$ has a jump at $r$ for given $h$, if the Liouville measure of the set $\Omega_{h,r}$ is positive. The function $Q(h,r)$ could be considered as a semi-classical analog of the oscillating function introduced by Guriev and Safarov [12], and Safarov [28], [29], who investigated the distribution of the eigenvalues at high energies for elliptic operators in compact manifolds with or without boundary.

Denote by $\Pi_+$ the set of points $\nu \in \Pi$ of a positive Lebesgue density in $\Pi$. It is said that $\nu \in \Pi_+$, if for any neighborhood $U$ of $\nu$ in $\Sigma$ the Liouville measure $\mu(U \cap \Pi)$ is positive. By definition, the complement to $\Pi_+$ in $\Pi$ is of a Lebesgue measure zero, hence, one can replace $\Pi$ with $\Pi_+$

in (1.7). From now on we suppose that $A(h)$ is a $h$-admissible operator such that the following condition holds:

$$h^{-1}S(\nu) + rT(\nu) - q(\nu) \geq \omega_0 > 0, \ \forall \nu \in \Pi_+, \ 0 < h \leq h_0. \tag{1.9}$$

This assumption is satisfied for instance for any hypersurface $\Sigma$ of contact type (*see* Section 8). On the other hand, $\Sigma$ is of contact type if the symbol $a_0(x, \xi)$ is *strictly $\xi$-convex* in a neighborhood $U$ of $\Sigma$, which means that

$$\left\langle \frac{\partial a_0}{\partial \xi}, \xi \right\rangle > 0, \quad \forall (x, \xi) \in U, \quad \xi \neq 0$$

(*see* [1], [17]). Obviously, the principal symbol (1.1) of the Schrödinger operator $-h^2\Delta + V(x)$ is strictly $\xi$-convex, hence, (1.9) holds at any regular energy hypersurface of $a_0(x, \xi) = |\xi|^2 + V(x)$. We are ready to formulate our main result.

THEOREM 1.1. – *Let $A(h)$ be a $h$-admissible operator, and let $E < \lambda_0$ be a regular value of $a_0$. Suppose that for any $r \in [-r_0, r_0]$ condition* (1.9) *is satisfied with a constant $\omega_0$ independent of $h$ and $r$. Then there exist $c_0 > 0$, $h_0 > 0$ and $\varepsilon_0 > 0$ such that for all $h \in (0, h_0]$, $|r| \leq r_0$, $0 < c \leq c_0$ and $0 < \varepsilon \leq \varepsilon_0$ we have*

$$\begin{aligned} h^{1-n}\Big[Q(h, r + c - \varepsilon) - Q(h, r - c + \varepsilon)\Big] - C_0\varepsilon h^{1-n} - o_\varepsilon(h^{1-n}) \\ \leq N_{E+rh,c}(h) - \frac{2c}{(2\pi)^n}\mu(\Sigma)h^{1-n} \\ \leq h^{1-n}\Big[Q(h, r + c + \varepsilon) - Q(h, r - c - \varepsilon)\Big] + C_0\varepsilon h^{1-n} + o_\varepsilon(h^{1-n}) \end{aligned}$$

*with $C_0 > 0$ independent of $h$, $c$, $\varepsilon$.*

Hereafter $o_\varepsilon(h^{1-n})$ means that for any fixed $\varepsilon$ the limit of the function $h^{n-1}o_\varepsilon(h^{1-n})$ is zero as $h \searrow 0$.

As a consequence we obtain either semi-classical asymptotics of the function $N_{E+rh,c}(h)$ or clustering near $E$.

COROLLARY 1.2. – *Let $A(h)$ be a $h$-pseudodifferential operator satisfying the asumptions of Theorem 1.1 and let $Q(h, r)$ be uniformly continuous with respect to $r \in [r_1, r_2]$, $r_1 < r_2$, for $h \in (0, h_0]$. Let $r_1 < R_1 < R_2 < r_2$. Then there exists $c_0 > 0$ such that for any $r \in [R_1, R_2]$, $0 < c \leq c_0$, and $h \in (0, h_0]$, we have*

$$N_{E+rh,c}(h) = \frac{2c}{(2\pi)^n}\mu(\Sigma)h^{1-n} + h^{1-n}\Big(Q(h, r+c) - Q(h, r-c)\Big) + o_c(h^{1-n}).$$

In particular, if $\mu(\Pi) = 0$ we have $Q = 0$ and (1.3) holds. If the function $Q$ is discontinuous with respect to $r$, its jumps may cause clustering near $E$. This happens for example, when there exists a bounded function $r(h)$, $h \in (0, h_0]$, *independent of* $\nu$, such that the quantization condition

$$h^{-1}S(\nu) + r(h)T(\nu) - q(\nu) \equiv 0\,(\mathrm{mod}\, 2\pi),\ 0 < h \leq h_0.$$

is fulfilled for any $\nu$ in a subset $\Pi^0 \subset \Pi$ of a *positive Lebesgue measure*. To satisfy the quantization condition above, it is enough to find an integer $p$ and a subset $\Pi^1 \subset \Pi$ of a positive Lebesgue measure, such that the quantity

$$r(h) = \Big([q(\nu) - h^{-1}S(\nu)]_{2\pi} + 2\pi p\Big)\, T(\nu)^{-1},\ 0 < h \leq h_0, \qquad (1.10)$$

is independent of $\nu \in \Pi^1$. Then (1.8) leads to

COROLLARY 1.3. – *Let $A(h)$ be a h-admissible operator satisfying the assumptions of Theorem* 1.1. *Suppose that there exist an integer p and a subset $\Pi^1$ of $\Pi$ of a positive Lebesgue measure such that $r(h)$ determined by* (1.10) *does not depend on $\nu \in \Pi^1$. Then for any $c > 0$ we have*

$$\liminf_{h \searrow 0} \left(h^{n-1} N_{E+r(h)h,c}(h)\right) \geq (2\pi)^{1-n} \int_{\Pi^1} \frac{d\nu}{T(\nu)}.$$

If $\Sigma$ is connected and totally periodic with a common period $T > 0$, we have $\Pi^1 = \Sigma$, $T(\nu) = T$, $S(\nu) = S$, and $q(\nu) = q$. Then we obtain (1.6) for any fixed $p$ taking the energy shift parameter $r(h)$ as in (1.10).

Let $\Sigma$ be connected and of contact type. As it was mentioned above, it is enough to suppose $a_0$ to be $\xi$-convex. Assume that the principal symbol $a_0$ of $A(h)$ is analytic in a neighborhood of $\Sigma$ and the subprincipal symbol $a_1$ is zero. We get a complete description of the asymptotics of $N_{E+rh,c}(h)$ in that case. First of all (1.9) holds (*see* Section 8), thus Theorem 1.1 can be applied. On the other hand, according to Theorem 1.2 in [24], either the Lebesgue measure of $\Pi$ is zero, or each $\nu \in \Sigma$ is periodic. We have $Q = 0$ in the first case which yields (1.3). In the second one, there exists an analytic function $T_0(\nu)$, $\nu \in \Sigma$, such that

$$\exp(T_0(\nu)H_{a_0})(\nu) = \nu,$$

and $S(\nu) = S$, $q(\nu) = q$ are constants for almost any $\nu$. If $T_0(\nu) = T$ is a constant function we obtain clustering choosing $r(h)$ as in (1.10). Now suppose that $T_0(\nu)$ is different from a constant. This case is a bit more complicated. First, for $r = 0$ we get "weak" clustering at $E$, namely

$$N_{E,c}(h_k) \sim (2\pi h_k)^{1-n}\Big(\int_\Sigma \frac{d\nu}{T(\nu)} + o(c)\Big), \ h_k = \frac{S}{q+2\pi k}, \ k \in \mathbf{N}. \tag{1.11}$$

On the other hand, if $r_1 < r_2$ and $0 \notin [r_1, r_2]$, the function $Q(h, r)$ turns to be uniformly continuous with respect to $r$ in that interval for any $h \in (0, h_0]$. Thus Corollary 1.2 holds and we have semi-classical asymptotics for $N_{E+rh,c}(h)$ for any $r \in [r_1, r_2]$. Moreover, taking into account the identity

$$Q(S(h^{-1}S + 2\pi k)^{-1}, r) = Q(h, r), \ k \in \mathbf{Z},$$

we observe that clustering in the sense of (1.6) is not possible. For more details in the contact case we refer to Section 8.

Consider the Schrödinger operator $A(h) = -h^2\Delta + V(x)$, where $V \geq \gamma_1$ is a smooth real-valued potential. Then $a_0$ is strictly $\xi$-convex, hence, $\Sigma$ is of contact type for any regular value

$$E < \tilde{\lambda}_0 = \lim_{|x|\to\infty} \inf V(x)$$

of $a_0(x, \xi) = |\xi|^2 + V(x)$. We shall show in Section 9 that the conclusions of Theorem 1.1, and Corollaries 1.2 and 1.3 remain valid without assuming $(H_1)$ and $(H_2)$.

The proof of Theorem 1.1 involves the following three steps.

1. First, we choose a smooth and real-valued cut-off function $f \in C_0^\infty((-\infty, \lambda_0))$, having support near $E$, $f = 1$ in a neighborhood of $E$, and consider the counting function

$$\sigma_h(\lambda) = \sum_{\lambda_j(h) \leq \lambda} f^2(\lambda_j(h))$$

of the operator $f^2(A(h))$, which is of a finite rank for any $0 < h \leq h_0$. Then we have

$$N_{E+rh,c}(h) = \sigma_h(E + rh + ch) - \sigma_h(E + rh - ch)$$

for any $|r| \leq r_0$, $0 < c \leq c_0$ and $0 < h \leq h_0$, where $h_0$ is sufficiently small. Making use of a suitable Tauberian theorem (*see* Theorem 2.1), we reduce the proof of Theorem 1.1 to the semi-classical asymptotics of the convolution $(\sigma_h' * \varrho_{\delta h})(E + rh)$. where $\sigma_h'$ is the derivative of $\sigma_h$, $\varrho_\delta(\lambda) = \delta^{-1}\varrho(\lambda\delta^{-1})$, $\delta > 0$, and $\varrho(\lambda)$ is an even function with a Fourier transform $\hat{\varrho}(t) \in C_0^\infty(\mathbf{R})$.

2. The main ingredient in the proof is a semi-classical trace formula which holds without any additional assumptions on the behaviour of the periodic trajectories on $\Sigma$.

Using the functional calculus developed in [16], we can write the convolution $(\sigma_h' * \varrho_{\delta h})(E + rh)$ modulo $\mathcal{O}(h^\infty)$ as the trace

$$\begin{aligned}(\sigma_h' * \varrho_{\delta h})&(E + rh)\\ = (2\pi h)^{-1}\mathrm{tr} \int &\exp(ith^{-1}(E + rh))\hat{\rho}(\delta t)g(x, hD_x)\\ &\times \exp\Big(-ith^{-1}A(h)\Big)g(x, hD_x)dt, \qquad (1.17)\end{aligned}$$

where $g(x, hD_x)$ is a self-adjoint $h$-admissible pseudodifferential operator representing $f(A(h))$. Moreover, the symbol of $g(x, hD_x)$ is localized in a small neigborhood of $\Sigma$, and it is equal to one near $\Sigma$. We consider the operator $\hat{\rho}(\delta t)g(x, hD_x)\exp\Big(-ith^{-1}A(h)\Big)g(x, hD_x)$ as a Fourier integral operator with a large parameter $h^{-1}$, having a compactly supported symbol, and related to a Lagrangian manifold $\Lambda$ (*see* Section 3). To obtain the asymptotics of (1.12), we have to take into account all periodic trajectories of $H_{a_0}$ in $\Sigma$ with periods $T(\nu) \leq T/\delta$ for some fixed $T > 0$. Without any clean intersection assumptions, the analysis of the corresponding oscillatory integrals may become rather difficult, because degenerate phase functions could appear. To overcome this obstacle, we make use of the so called *absolutely periodic* points and *absolutely periodic* trajectories. Our definition of absolutely periodic points is addapted to the semi-classical analysis in Sections 4 and 5, where a suitable (microlocal) representation of $g(x, hD_x)\exp\Big(-ith^{-1}A(h)\Big)g(x, hD_x)$ is given. Let $\nu^0$ be a periodic point and let $\gamma$ be the periodic trajectory passing through $\nu^0$. Denote by $P_\gamma$ the Poincaré map related to $\gamma$. We say that $\nu^0$ is *absolutely periodic* if there exists a positive integer $m$ such that the map $P_\gamma^m(z) - z$ is flat at $z(\nu^0) = 0$ in any local coordinates $z$. This means that any derivative of $P_\gamma^m(z) - z$ vanishes at $z = 0$. Let $m(\gamma)$ be the smallest positive $m$ having this property. Then the multiple periodic trajectory $\gamma^a = m(\gamma)\gamma$ will be called *absolutely periodic* and its period $T^a(\nu^0) = T_\gamma^a = m(\gamma)T_\gamma$ will be called *absolute period* of $\nu^0$. Notice that the definition of absolutely periodic points we give here is different from that in [24]. Moreover, certain points could be absolutely periodic with one period and only "simply" periodic with another period. For this reason we make a microlocalization with respect to $t$ and $x$. The importance of the absolutely periodic points for this kind of problems has been pointed out by Guriev and Safarov [12] and Safarov [28], [29]. The case when the periodic points outside a set of measure zero form a clean manifold has been considered by Zelditch [33]. We show in Lemma A.1 that the periodic points which are not absolutely periodic form a set of a Liouville measure zero and this plays a crucial role in the analysis of the trace formula in Section 5. On the other hand, the analysis of the absolutely periodic points is the main novelty with the previous works on semi-classical trace formulae provided clean intersection assumptions (*see* [2], [23], [22], [7]).

Our goal in Section 4 is the construction of suitable phase functions representing microlocally $\Lambda$ in special symplectic coordinates. Namely, we examine the representation of the Lagrangian manifold $\Lambda_0$ obtanied from $\Lambda$ after conjugation with canonical relations related to symplectic transformations (*see* Section 4 for a precise definition). For example, in the vicinity of a periodic point $\nu^0$, the Lagrangian manifold $\Lambda_0$ can be parametrized by a phase function

$$\psi(t, x, y, \eta) = (x_n - y_n - t)\eta_n + \langle x' - y', \eta' \rangle + L(x', \eta),$$

$$x = (x', x_n), \ \eta = (\eta', \eta_n).$$

This representation simplifies the calculus of the corresponding oscillatory integrals in (1.12). First, we can apply a stationary phase argument with respect to $t, \eta_n$. Secondly, the critical points of the function $L(x', y', E)$ coincide with the fixed points of the Poincaré map $P^{(\nu)}$ related to the periodic trajectories $\gamma(\nu)$ passing through $\nu$ close to $\nu^0$. Finally, the critical values of $L(x', y', E)$ are just the actions on the corresponding periodic trajectories $\gamma(\nu)$. The density related to the phase function $\psi$ can be further simplified at the absolutely periodic points which allows to write down the contribution in the trace of the absolutely periodic trajectories in an invariant form. Since the measure of the periodic points which are not absolutely periodic is zero, we prove that the contribution of the periodic trajectories to the trace (1.12) is given by the expression

$$(2\pi h)^{-n} \sum_{k \in \mathbf{Z}\setminus 0} \int_{\Pi} \exp\Big(ik(h^{-1}S(\nu)+rT(\nu)-q(\nu)\Big)\hat{\rho}(k\delta T(\nu))d\nu + o_\delta(h^{-n}). \tag{1.13}$$

Another applications of the idea to work with special phase functions are given in [4].

In particular, taking $\delta = 1$ and $r = 0$, we obtain a Gutzwiller's semi-classical trace formula without any additional assumptions on the periodic trajectories.

THEOREM 1.4. – *Suppose that $E < \lambda < \lambda_0$ and let $E$ be a regular value of $a_0$. Then for any function $\varrho(\tau) \in \mathcal{S}(\mathbf{R})$ with a Fourier transform $\hat{\varrho}(t) \in C_0^\infty(\mathbf{R})$ we have*

$$\sum_{\lambda_j(h)\leq\lambda} \varrho\Big(\frac{E-\lambda_j(h)}{h}\Big) = \hat{\varrho}(0)\frac{\mu(\Sigma)}{(2\pi)^n}h^{1-n}$$
$$+ (2\pi)^{-n}h^{1-n}\int_\Pi \sum_{k\in\mathbf{Z}\setminus 0} \exp\Big(\imath k(h^{-1}S(\nu) - q(\nu))\Big)\hat{\varrho}(kT(\nu))d\nu + o_\varrho(h^{1-n}). \quad (1.14)$$

*where the eigenvalues in the left-hand side are counted with their multiplicities.*

Note that no clean interesection condition is required in Theorem 1.4. This is the main difference with the trace formula in [23] (*see* also [13], [11], [2], [22], [7]). On the other hand, our statement is weaker than those cited above in the sense that our remainder term is only $o_\rho(h^{1-n})$, while a complete asymptotic expansion in powers of $h$ holds if a clean intersection condition is met.

3. In Section 6 we obtain another representation of the leading singularity in (1.3) involving the oscillating function $Q(h,r)$. Here, we are inspired by an idea used by Safarov [28], [29] for the analysis of the asymptotics at infinity of the counting function of the eigenvalues of elliptic operators on compact manifolds.

The paper is organized as follows. Section 2 is devoted to a Tauberian type theorem and a localization argument based on the functional calculus developed in [16]. In Section 3 we collect certain facts about Fourier integral operators with a large parameter in the sense of Duistermaat [8] and we write the Schwartz kernel of the propagator $\exp(-ith^{-1}A(h))$ microlocally near $\Sigma$ as an oscillatory integral with a large parameter $h^{-1}$ related to a Lagrangian manifold $\Lambda$. Suitable phase functions representing microlocally the Lagrangian manifold $\Lambda_0$ are constructed in Section 4. The trace formula concerning (1.12) is proved in Section 5. A simple application of this formula yields Theorem 1.4. In Section 6 we establish another representation of the leading singularity making use of the function $Q(h,r)$. The main results are proved in Section 7. The particular case of an energy surface of contact type is considered in Section 8. Finally, in Section 9 we apply our results to the Schrödinger operator $A(h) = -h^2\Delta + V(x)$, and discuss some examples of Schrödinger operators with smooth potentials for which clustering takes place without assuming that the flow on $\Sigma$ is totally periodic.

The principal results of this work have been annonced in [25].

## 2. TAUBERIAN THEOREM AND LOCALIZATION

Consider a pseudodifferential operator $A(h)$

$$(A(h)\varphi)(x) = \int \exp(\imath\langle x-y,\xi\rangle)a\left(\frac{x+y}{2}, h\xi; h\right)\varphi(y)dyd\xi$$

with a Weyl symbol satisfying $(H_1)$ and $(H_2)$.

To pass from the semi-classical trace formula to the asymptotics of $N_{E+rh,c}(h)$ we need a suitable Tauberian theorem. Let $\sigma_h(\lambda), h \in (0, h_0]$, be a family of non-decreasing and non-negative functions in $\mathbf{R}$ and $\lambda_1 < E < \lambda_2 < \lambda_0$. Suppose that $\sigma_h(\lambda)$ has the following properties:

(i) $\sigma_h(\lambda) = 0$ for $\lambda \leq \lambda_1$,

(ii) $\sigma_h(\lambda) = \sigma_h(\lambda_2)$ for $\lambda \geq \lambda_2$,

(iii) $\sigma_h(\lambda) \leq Kh^{-n}$, $\forall h \in (0, h_0]$, $\forall \lambda \in \mathbf{R}$.

Let $\rho(\lambda) \in \mathcal{S}(\mathbf{R})$, $\rho(-\lambda) = \rho(\lambda)$, and $\rho(\lambda) \geq 0$, $\forall\lambda \in \mathbf{R}$, while $\rho(\lambda) \geq \delta_0 > 0$ for $|\lambda| \leq \varepsilon_0$. We assume that the Fourier transform of $\rho(\lambda)$ satisfies

$$\hat{\rho}(t) \in C_0^\infty(\mathbf{R}), \quad \text{supp } \hat{\rho} \subset [-\delta_1, \delta_1], \quad \delta_1 > 0, \quad \hat{\rho}(0) = 1.$$

Introduce for $0 < \delta \leq 1$ the function

$$\rho_{\delta h}(\lambda) = \frac{1}{\delta h}\rho\left(\frac{\lambda}{\delta h}\right),$$

and notice that

$$\int \rho_{\delta h}(\lambda)\exp\left(-ith^{-1}\lambda\right)d\lambda = \hat{\rho}(\delta t).$$

THEOREM 2.1. – *Assume that there exist functions* $\gamma_0(\lambda)$, $\gamma_1(\lambda) \in C^1(\mathbf{R})$ *such that*

$$\frac{d}{d\lambda}\left(\sigma_h * \rho_h\right)(\lambda) = \gamma_0(\lambda)h^{-n} + \gamma_1(\lambda)h^{1-n} + \mathcal{O}_\lambda(h^{2-n}), \tag{2.1}$$

*where $\mathcal{O}_\lambda$ is locally uniform with respect to $\lambda$. Then for each $\varepsilon$. $0 < \varepsilon \leq 1$, each $\delta$. $0 < \delta \leq 1$, and $\lambda \in [\lambda_1 . \lambda_2]$, we have*

$$\Big(\sigma_h * \rho_{\delta h}\Big)(\lambda - \varepsilon h) - C_0\gamma_0(\lambda)\varepsilon^{-1}\delta h^{1-n} - C_1 h^{2-n} \leq \sigma_h(\lambda)$$

$$\leq \Big(\sigma_h * \rho_{\delta h}\Big)(\lambda + \varepsilon h) + C_0\gamma_0(\lambda)\varepsilon^{-1}\delta h^{1-n} + C_1 h^{2-n}$$

*where $C_0 > 0$, $C_1 > 0$ are independent of $\varepsilon . \delta . \lambda$ and $h$.*

*Proof.* – As in [26] we find constants $C_2 > 0$, $C_3 > 0$ such that for any $\lambda \in [\lambda_1 . \lambda_2]$ and any $\tau \in \mathbf{R}$ we have

$$|\sigma_h(\lambda + \tau h) - \sigma_h(\lambda)| \leq C_2\gamma_0(\lambda)\Big(1 + |\tau|\Big)h^{1-n} + C_3\Big(1 + |\tau|^2\Big)h^{2-n}.$$

Then for $\delta\eta \geq -\varepsilon$, we obtain

$$\sigma_h(\lambda) - \sigma_h(\lambda - \varepsilon h - \delta h\eta) \geq 0,$$

and consequently

$$\sigma_h(\lambda) - \Big(\sigma_h * \rho_{\delta h}\Big)(\lambda - \varepsilon h) = \int \Big(\sigma_h(\lambda) - \sigma_h(\lambda - \varepsilon h - \delta h\eta)\Big)\rho(\eta)d\eta$$

$$\geq \int_{\delta\eta \leq -\varepsilon} \Big(\sigma_h(\lambda) - \sigma_h(\lambda - \varepsilon h - \delta h\eta)\Big)\rho(\eta)d\eta$$

$$\geq -C_1'\gamma_0(\lambda)h^{1-n}\int_{\delta\eta\leq-\varepsilon}(1 + |\eta|)\rho(\eta)d\eta - C_4 h^{2-n}$$

$$\geq -C_0\gamma_0(\lambda)\varepsilon^{-1}\delta h^{1-n} - C_4 h^{2-n}$$

with constants $C_0 > 0, C_4 > 0$ depending only on $\rho(\lambda)$. This proves the left-hand side of the inequality in Theorem 2.1. For the right-hand one we apply a similar argument. ♠

Now fix $\varepsilon_1 > 0$, $\lambda_1 + \varepsilon_1 < E < \lambda_2 - \varepsilon_1 < \lambda_0$, and choose a smooth function

$$f(t) \in C_0^\infty((\lambda_1, \lambda_2))$$

such that $0 \leq f(t) \leq 1$, and $f(t) = 1$ for $\lambda_1 + \varepsilon_1 \leq t \leq \lambda_2 - \varepsilon_1$. Next we set

$$\sigma_h(\lambda) = \sum_{\lambda_j(h)\leq\lambda} f^2(\lambda_j(h)).$$

Assume that the principal symbol $a_0(x, \xi)$ has no critical values in the interval $[\lambda_1, \lambda_2]$.

We have

$$N_{E+rh,c}(h) = \sum_{\lambda_j(h)\in[E+rh-ch,E+rh+ch]} f^2(\lambda_j(h)) \tag{2.2}$$
$$= \sigma_h(E+rh+ch) - \sigma_h(E+rh-ch)$$

for any $0 < h \leq h_0$, where $h_0$ is sufficiently small. We shall show that $\sigma_h(\lambda)$ satisfies the assumptions of Theorem 2.1. Following the calculus in [16], one proves that $f(A(h))$ can be represented by a self-adjoint $h$-admissible pseudodifferential operator $g(x, hD_x)$ with a symbol

$$g(x,\xi;h) \sim \sum_{j=0}^{\infty} g_j(x,\xi)h^j,$$

where

$$g_0(x,\xi) = f\Big(a_0(x,\xi)\Big),$$
$$\text{supp } g_j \subset a_0^{-1}\Big((\lambda_1,\lambda_2)\Big),\ j\in \mathbf{N}.$$

Using the functional calculus we obtain

$$\frac{d}{d\lambda}\Big(\sigma_h * \rho_{\delta h}\Big)(\lambda) = \int \rho_{\delta h}(\lambda-\mu)d\sigma_h(\mu) =$$
$$\sum_{\lambda_j(h)\leq\lambda_2} f^2(\lambda_j(h))\varrho_{\delta h}(\lambda-\lambda_j(h)) = \mathrm{tr}\Big(f(A(h))\varrho_{\delta h}(\lambda-A(h))f(A(h))\Big),$$

and replacing $f(A(h))$ by $g(x, hD_x)$ we get

$$(\sigma_h' * \varrho_{\delta h})(\lambda) = (2\pi h)^{-1}\mathrm{tr}\int \exp(ith^{-1}\lambda)\,\hat{\rho}(\delta t)\,g(x,hD_x)$$
$$\times \exp\Big(-ith^{-1}A(h)\Big)g(x,hD_x)\,dt\ +\ \mathcal{O}(h^\infty), \tag{2.3}$$

where "tr" stands for the trace of the given trace class operator, and $\sigma_h'(\lambda)$ is the derivative of $\sigma_h(\lambda)$ in a distribution sense.

To check (2.1) we use an argument from [26]. More precisely, we have the following

PROPOSITION 2.2. – [26] *There exists $\delta_1 > 0$ sufficiently small such that for any $\lambda < \lambda_0$ we have*

$$\frac{d}{d\lambda}\Big(\sigma_h * \rho_h\Big)(\lambda)$$
$$= (2\pi h)^{-n}\int_{a_0(x,\xi)=\lambda} \frac{(g_0(x,\xi))^2}{|\nabla a_0(x,\xi)|}d\nu_\lambda + \gamma_1(\lambda,g)h^{1-n} + \mathcal{O}_\lambda(h^{2-n}).$$

*where $\nu_\lambda$ is the induced Lebesgue measure on the surface $\{(x.\xi) : a_0(x.\xi) = \lambda\}$, the function $\gamma_1(\lambda, g)$ is smooth with respect to $\lambda$, and $\mathcal{O}_\lambda$ is locally uniform with respect to $\lambda$.*

On the other hand, $\sigma_h(\lambda) = 0$ for $\lambda \leq \lambda_1$, and $\sigma_h(\lambda) = \sigma_h(\lambda_2)$ for $\lambda \geq \lambda_2$. Hence, $\sigma_h(\lambda)$ satisfies the assumptions of Theorem 2.1.

To estimate $(\sigma_h' * \rho_{\delta h})(\lambda)$ we shall use (2.3). First choose a function $\vartheta(t) \in C_0^\infty(\mathbf{R})$ such that

$$\text{supp } \vartheta(t) \subset (-\delta_1, \delta_1), \ \vartheta(t) = 1 \text{ for } |t| \leq \frac{\delta_1}{2}.$$

We define the functions $\psi_h(\lambda, \delta)$ and $\phi_h(\lambda, \delta)$ by

$$\phi_h(\lambda, \delta) = (2\pi h)^{-1} \int \exp\Big(ith^{-1}\lambda\Big)(1 - \vartheta(t))\hat{\varrho}(\delta t)dt,$$

$$\psi_h(\lambda, \delta) = (2\pi h)^{-1} \int \exp\Big(ith^{-1}\lambda\Big)\vartheta(t)\hat{\varrho}(\delta t)dt.$$

Then,

$$(\sigma_h' * \varrho_{\delta h})(\lambda) = (\sigma_h' * \psi_h)(\lambda) + (\sigma_h' * \phi_h)(\lambda),$$

and the convolution $(\sigma_h' * \psi_h)(\lambda)$ can be estimated using Proposition 2.2. In Section 5 we shall study the trace

$$(\sigma_h' * \phi_h)(\lambda) = (2\pi h)^{-1}\text{tr} \int \exp(ith^{-1}\lambda)\Big((1 - \vartheta(t))\hat{\rho}(\delta t)\Big)g(x, hD_x)$$
$$\times \exp\Big(-ith^{-1}A(h)\Big)g(x, hD_x)dt, \tag{2.4}$$

where $(1 - \vartheta(t))$ vanishes in a neighborhood of the origin and the Schwartz kernel of the operator

$$g(x, hD_x) \exp\Big(-ith^{-1}A(h)\Big)g(x, hD_x)$$

is a Fourier oscillatory integral in the sense of [8].

## 3. FOURIER INTEGRAL OPERATORS WITH LARGE PARAMETER

In this section we recall certain basic facts about the Fourier integral operators (F.I.O.) with a large parameter $\lambda = 1/h$ whose kernels are oscillatory integrals in the sense of Duistermaat [8] (*see* also [22]). Let $X$ be a smooth Riemannian manifold of dimension $d \geq 2$. Consider $T^*(X)$ with the standard symplectic two-form $\omega = -d\alpha$, where $\alpha$ is the canonical one-form. For a given smooth curve $\gamma$ in $T^*(X)$ (not necessarily closed) one defines the action by

$$A(\gamma) = \int_\gamma \alpha.$$

Let $\imath : \Lambda \longrightarrow T^*(X)$ be a compact connected imbedded Lagrangian submanifold. Suppose that $\Lambda$ is exact. This means that the Liouville class $[i^*\alpha]$ of $i^*\alpha$ in the first cohomology group $H^1(\Lambda; \mathbf{R})$ is trivial, *i.e.*

$$\imath^*\alpha = df,$$

for some smooth function $f$ on $\Lambda$. This condition is equivalent to the requirement

$$A(\gamma) = 0$$

for any loop $\gamma$ in $\Lambda$.

Locally the Lagrangian manifold $\Lambda$ can be defined by a non-degenerate phase function as follows. Let $\nu^0 = (x^0, \xi^0) \in \Lambda$, and let $x$ be local coordinates in a neighborhood $U_0$ of $x^0$. There is a smooth phase function $\Psi(x, \theta)$ in $U_0 \times \mathbf{R}^m$, $m \geq 0$, such that, if $m \geq 1$, then

$$\operatorname{rank} d_{(x,\theta)} d_\theta \Psi = m$$

on

$$C_\Psi = \{(x, \theta) \in U_0 \times \mathbf{R}^m : \ d_\theta \Psi = 0\},$$

and the map

$$\imath_\Psi : C_\Psi \ni (x, \theta) \longrightarrow (x, d_x \Psi(x, \theta)) \in \Lambda_\Psi \subset \Lambda \tag{3.1}$$

is a local diffeomorphism in $\Lambda$. Note that $C_\Psi$ is a smooth submanifold of $U_0 \times \mathbf{R}^m$ of dimension $m$. In case that $m = 0$ one takes $C_\Psi = U_0$.

Let $\nu^0 = (x^0, \xi^0) \in T^*(X)$, and let $u(x, h)$ be a smooth function in $X \times (0, h_0]$, $h_0 > 0$. We say that $\nu^0$ does not belong to the frequency set $\mathrm{WF}(u)$ of $u$ if there exist neighborhoods $U_0$, $V_0$, respectively of $x^0$ and $\xi^0$, and a function $\phi \in C_0^\infty(U_0)$ such that in any local coordinates in $U_0$ we have

$$\int e^{ih^{-1}\langle x, \xi\rangle} \phi(x) u(x, h) dx = \mathcal{O}(h^\infty), \ h \searrow 0,$$

uniformly with respect to $\xi \in V_0$.

To any exact Lagrangian manifold $\Lambda$ and any $s \in \mathbf{R}$ one can associate a class of oscillatory integrals $I^s(X, \Lambda; h)$. Namely, a smooth function $u(x, h)$ in $X \times (0, h_0]$, $h_0 > 0$, belongs to $I^s(X, \Lambda; h)$ if for any $\nu^0 = (x^0, \xi^0) \in \Lambda$ there exists a non-degenerated phase function $\Psi(x, \theta)$ and amplitude $b(x, \theta, h)$ such that the oscillatory integral

$$u^0(x, h) = (2\pi h)^{-\frac{1}{4}(d+2m)} \int_{\mathbf{R}^m} e^{ih^{-1}\Psi(x,\theta)} b(x, \theta, h) d\theta \tag{3.2}$$

represents microlocally $u(x, h)$ at $\nu^0$, which means that

$$\nu^0 \notin \mathrm{WF}(u - u^0).$$

The amplitude $b(x, \theta, h)$ is a smooth function in $U_0 \times \mathbf{R}^m \times (0, h_0]$ with a compact support with respect to $(x, \theta)$ which is independent of $h$ and asymptotically

$$b(x, \theta, h) = b_0(x, \theta) h^s + b_1(x, \theta) h^{s+1} + \cdots .$$

If $m = 0$ there is no integration with respect to $\theta$.

One can consider $u$ as a 1/2-density in $X$ multiplying it by the standard 1/2-density

$$(\det g_{ij}(x))^{-1/4} |dx|^{1/2},$$

where $ds^2 = \sum_{i,j} g_{ij}(x) dx_i dx_j$ is a Riemannian metric on $X$. Since $\Lambda$ is exact, the principal symbol $\sigma(u)$ of $u(x, h)$ is well defined as a product

$$\sigma(u) = e^{ih^{-1}f(\nu)} \sigma_1 \otimes \sigma_2, \ \nu \in \Lambda.$$

Here $\sigma_1$ and $\sigma_2$ are sections in the half-density bundle $\Omega_{1/2}(\Lambda)$ and the Maslov bundle $M(\Lambda)$, respectively, and $f$ is a smooth function representing the Liouville factor (cf. [8]). We have microlocally

$$f(\nu) = \tilde{\Psi}(\nu) = \imath_\Psi^*(\Psi_{|C_\Psi})(\nu), \ \nu \in \Lambda_\Psi \subset \Lambda.$$

Such function $f(\nu)$ exists globally on $\Lambda$ since the latter is exact. Indeed, we have locally

$$\begin{aligned} d\tilde{\Psi}(\nu) &= d\imath_\Psi^*(\Psi_{|C_\Psi})(\nu) = \imath_\Psi^* \left( \frac{\partial \Psi}{\partial x} dx + \frac{\partial \Psi}{\partial \theta} d\theta \right)_{|C_\Psi} (\nu) \\ &= \imath_\Psi^* \left( \frac{\partial \Psi}{\partial x} dx \right)_{|C_\Psi} (\nu) = \imath^* \alpha. \end{aligned}$$

Take another representation $u^1$ of $u$ given by (3.2) with a phase function $\Psi_1$. Then $\nu^0 \notin \mathrm{WF}(u^1 - u^0)$ and we obtain

$$d\tilde{\Psi} = \imath^*\alpha = d\tilde{\Psi}_1, \; \tilde{\Psi}(\nu^0) = \tilde{\Psi}_1(\nu^0)$$

which implies $\tilde{\Psi} = \tilde{\Psi}_1$ in a neighborhood of $\nu^0$. Since $\Lambda$ is exact, there exists a globally defined exponent $f$ of the Liouville factor. Note that the function $f$ is uniquely determined on $\Lambda$ modulo a constant, hence, it is enough to know $f$ at a single point in order to recover it on the whole manifold $\Lambda$.

For given smooth manifolds $X$ and $Y$ and a compact Lagrangian submanifold $\Lambda$ in $T^*(X \times Y)$ we denote by $I^s(X, Y, \Lambda; h)$ the family of Fourier integral operators (F.I.O.)

$$U_h : C^\infty(Y) \to C^\infty(X)$$

with a large parameter $h^{-1}$ whose Schwartz kernels belong to $I^s(X \times Y, \Lambda; h)$. A special case of F.I.O. with a large parameter is the parametrix $U_h(t, x)$ of the Schrödinger equation near a compact non-degenerated energy level of the corresponding classical Hamiltonian $a_0(x, \xi)$. The operator $U_h(t, x)$ is defined by

$$\left(\frac{h}{i}\frac{\partial}{\partial t} + A(h)\right)U_h(t, x) = \mathcal{O}(h^\infty), \; (t, x) \in \mathbf{R}^{n+1}, \tag{3.3}$$

$$U_h(0, x) = Q(x, D_x, h),$$

where $Q(x, D_x, h)$ is a pseudodifferential operator with a large parameter $h$ having a Schwartz kernel of the form

$$(2\pi h)^{-n} \int e^{ih^{-1}\langle x-y,\xi\rangle} q(x, \xi, h) d\xi.$$

The amplitude $q(x, \theta, h)$ is a classical symbol with respect to the large parameter $h^{-1}$ in $\mathbf{R}^{2n} \times (0, h_0]$ whose support with respect to $(x, \xi)$ is contained in a small compact neighborhood $\mathcal{O}$ of the energy level $\Sigma$ and $q = 1$ in a smaller neighborhood of $\Sigma$ for any $h \in (0, h_0]$.

For any fixed $T > 0$ one can solve globally (3.3) in $[-T, T] \times \mathbf{R}^n$ as in [8]. The kernel $U_h(t, x, y)$ of the operator $U_h(t, x)$ belongs to the class $I^{1/4}(\mathbf{R}^{n+1} \times \mathbf{R}^n, \Lambda; h)$ associated with the Lagrangian submanifold

$$\Lambda' = \Big\{(t, x, y, \tau, \xi, -\eta) \in T^*(\mathbf{R}^{n+1} \times \mathbf{R}^n) \; : \; \tau = -a_0(y, \eta),$$
$$\Phi^t(y, \eta) = (x, \xi), \; (t, y, \eta) \in [-T, T] \times \mathcal{O}\Big\}.$$

Taking $\mathcal{O}$ simply connected, we can assume that $\Lambda'$ is exact. According to Theorem 1.4.1 in [8], there exists an operator $U_h(t,x)$ which solves (3.3) and whose kernel is in $I^{1/4}(\mathbf{R}^{n+1} \times \mathbf{R}^n, \Lambda'; h)$ with a principal symbol

$$\sigma(u) = b(\nu)e^{ih^{-1}f(\nu)}\sigma_1 \otimes \sigma_2, \ \nu \in \Lambda.$$

Parametrizing $\Lambda$ by the projection $\pi : \Lambda' \longrightarrow \mathbf{R} \times T^*(\mathbf{R}^n)$, $\pi(t,x,y,\tau,\xi,-\eta) = (t,y,\eta)$, one can take the half-density $\sigma_1$ in the form

$$\sigma_1 = \pi^*\Big(|dt \wedge dy \wedge d\eta|^{1/2}\Big). \tag{3.4}$$

The exponent $f$ of the Liouville factor is given by the action

$$f(\pi^{-1}(t,y,\eta)) = \int_{\tilde{\gamma}} (\tau dt + \eta dy) = -a_0(y,\eta)t + \int_{\gamma_t} \eta dy$$

$$= -a_0(y,\eta)t + A(\gamma_t),$$

where

$$\tilde{\gamma}(t,y,\eta) = \Big\{(s,x,y,\tau,\xi,-\eta) \in T^*(\mathbf{R}^{n+1} \times \mathbf{R}^n) \ : \ \tau = -a_0(y,\eta),$$

$$\Phi^s(y,\eta) = (x,\xi), \ 0 \leq s \leq t\Big\},$$

and for any fixed $(y,\eta) \in \mathcal{O}$ we have

$$\gamma_t(y,\eta) = \{\Phi^s(y,\eta) \ : \ 0 \leq s \leq t\}.$$

Moreover, since $(y,\eta)$ are symplectic coordinates, the density $\sigma_1$ is invariant with respect to $\Phi^t$ and the transport equation for the principal symbol yields

$$b(\pi^{-1}(t,y,\eta)) = \exp\left(-i\int_0^t a_1(\Phi^s(y,\eta))ds\right),$$

$a_1(x,\xi)$ being the subprincipal symbol of $A(h)$. Thus the principal symbol $\sigma(U_h)$ of $U_h$ can be written in the form

$$\sigma(U_h)(t,y,\eta) = \exp\big(ih^{-1}(-a_0(y,\eta)t + A(\gamma_t))\big)$$

$$\times \exp\left(-i\int_0^t a_1(\Phi^s(y,\eta))ds\right)\sigma_1 \otimes \sigma_2. \tag{3.5}$$

## 4. ABSOLUTELY PERIODIC POINTS AND GENERATING FUNCTIONS

Our aim in this section is to find simple phase functions representing locally the Lagrangian manifold $\Lambda_0$ introduced below. Using suitable symplectic coordinates we obtain $\Lambda_0$ as the composition of the Lagrangian manifold $\Lambda$ given in the previous section by two symplectic transformations.

Let $\gamma$ be a primitive periodic trajectory of $H_{a_0}$ in $\Sigma$, and let $Y \subset \Sigma$ be a transversal section to $\gamma$ at $\nu^0 \in \gamma$. Consider the Poincaré map

$$P_\gamma : Y^0 \to Y,$$

associated to $\gamma$, $Y^0$ being a neighborhood of $\nu^0$ in $Y$. It is given by

$$P_\gamma(\nu) = \Phi^{T_0(\nu)}(\nu) \in Y, \ \nu \in Y^0,$$

where the return time function $T_0(\nu)$ is smooth and $T_0(\nu^0) = T(\nu^0) = T_\gamma$. For any positive integer $k$, we denote by $k\gamma$ the multiple periodic trajectory of period $T_{k\gamma} = kT_\gamma$, and set $P = P_\gamma^k$. Then

$$P : Y^0 \to Y$$

is the Poincaré map associated with $k\gamma$ in a suitable neighborhood $Y^0$ of $\nu^0$ in $Y$.

Below we fix the integer $k$ and we deal with the Poincaré map $P$. We are going to work in suitable local symplectic coordinates in a neighborhood of $\nu^0$ in $T^*(\mathbf{R}^n)$ which are defined as follows. First we set $\zeta_n = a_0(x, \xi)$ and then find vector functions $z = (z_1, ..., z_n), \zeta' = (\zeta_1, ..., \zeta_{n-1})$, such that

$$\{z_j, z_k\} = 0, \ \{z_j, \zeta_k\} = \delta_{jk}, \ \{\zeta_j, \zeta_k\} = 0,$$

$$z(\nu^0) = 0, \ \zeta'(\nu^0) = 0, \ \zeta_n(\nu^0) = E,$$

where $\delta_{jk} = 0$ for $j \neq k$, $\delta_{jj} = 1$. Denote by $\chi$ the corresponding symplectic transformation

$$\chi(z, \zeta) = (x(z, \zeta), \xi(z, \zeta)), \ (z, \zeta) \in U,$$

mapping a neighborhood $U$ of $(0, (0, E))$ in $T^*(\mathbf{R}^n)$ to a neighborhood of $\nu^0$, and set

$$\tilde{\chi}(t, z, \tau, \zeta) = (t, x(z, \zeta), \tau, \xi(z, \zeta)), \ (t, z, \tau, \zeta) \in V_1, \tag{4.1}$$

$V_1$ being a neighborhood of $(T_{k\gamma}, 0, E, (0, E))$ in $T^*(\mathbf{R}^{n+1})$. Consider the flow

$$\phi^t(z,\zeta) = (\chi^{-1}\Phi^t\chi)(z,\zeta), \ (t,z,\zeta) \in V,$$

where $V$ is a sufficiently small neighborhood of $(T_{k\gamma}, \chi^{-1}(\nu^0))$ in $\mathbf{R} \times U$ such that

$$\phi^t(z,\zeta) \in U, \ \forall (t,z,\zeta) \in V.$$

The sections

$$S = \{z_n = 0\} \cap U, \ S_0 = \{z_n = 0\} \cap U_0$$

are transversal to the flow $\phi^t(\nu)$, and we obtain a smooth map $R : S_0 \to S$ determined by

$$R(z', 0, \zeta) = \phi^{s(z',\zeta)}(z', 0, \zeta) \in S, \ \forall (z', 0, \zeta) \in S_0.$$

Here $U_0 \subset U$, $s(z',\zeta)$ is the corresponding return time function which is smooth in $S_0$ and normalized by $s(0,0,E) = T_{k\gamma}$. Using the invariance of the Hamiltonian vector fields under symplectic transformations, we get for any $(t,z,\zeta) \in V$ the equality

$$\begin{aligned}\phi^t(z,\zeta) &= \big(\chi^{-1} \circ \Phi^t \circ \exp(z_n H_{a_0}) \circ \chi\big)(z', 0, \zeta)\\ &= \Big(\chi^{-1} \circ \Phi^{t+z_n-s(z',\zeta)}\Phi^{s(z',\zeta)} \circ \chi\Big)(z', 0, \zeta)\\ &= \exp\Big((t + z_n - s(z',\zeta))H_{\zeta_n}\Big)R(z', 0, \zeta).\end{aligned}$$

Hence, we obtain

$$\phi^t(z,\zeta) = \exp\Big((t + z_n - s(z',\zeta))H_{\zeta_n}\Big)R(z', 0, \zeta), \ (t,z,\zeta) \in V. \quad (4.2)$$

Let $p_1, \dots, p_n, q_1, \dots, q_n$ stand for the coordinate projections in $T^*(\mathbf{R}^n)$ and

$$p' = (p_1, \dots, p_{n-1}), \ q' = (q_1, \dots, q_{n-1}).$$

According to (4.2), we have

$$\begin{aligned}(p' \circ \phi^t)(z,\zeta) &= (p' \circ R)(z', 0, \zeta),\\ (p_n \circ \phi^t)(z,\zeta) &= t + z_n - s(z',\zeta), \qquad (4.3)\\ (q' \circ \phi^t)(z,\zeta) &= (q' \circ R)(z', 0, \zeta),\\ (q_n \circ \phi^t)(z,\zeta) &= \zeta_n,\end{aligned}$$

for any $(t,z,\zeta) \in V$.

Consider the Lagrangian submanifold of $T^*(\mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^n)$ defined by

$$\Lambda_0' = \{(t, x, y, \tau, \xi, -\eta) : \tau = -\eta_n = -\xi_n,$$

$$\phi^t(y, \eta) = (x, \xi), \ (t, y, \eta) \in V\},$$

and set

$$\tilde{\nu}^0 = (T_{k\gamma}, 0, 0, -E, 0, E, 0, -E) \in \Lambda_0'.$$

The corresponding canonical relation

$$\Lambda_0 \subset T^*(\mathbf{R} \times \mathbf{R}^n) \times T^*(\mathbf{R}^n),$$

$$\Lambda_0 = \{((t, x, \tau, \xi), (y, \eta)) : (t, x, y, \tau, \xi, -\eta) \in \Lambda_0'\}$$

is a transversal composition of $\Lambda$ and the canonical relations $C_0$ and $C_1$ which are graphs of the symplectic transformations $\chi$ and $\tilde{\chi}^{-1}$. Namely,

$$\Lambda_0 = C_1^{-1} \circ \Lambda \circ C_0, \tag{4.4}$$

where

$$C_0 = \{(\chi(z, \zeta), (z, \zeta)) : (z, \zeta) \in U_0\},$$

$$C_1^{-1} = \{((t, x, \tau, \xi), \tilde{\chi}(t, x, \tau, \xi)) : (t, x, \tau, \xi) \in V_0\},$$

and $U_0$, $V_0$, are suitable neighborhoods of $(0, 0, E)$ and $(T_{k\gamma}, 0, E, 0, E)$ in $T^*(\mathbf{R}^n)$ and $T^*(\mathbf{R}^{n+1})$, respectively. As usually we shall denote by $C_0'$ and $C_1'^{-1}$ the corresponding Lagrangian manifolds in $T^*(\mathbf{R}^n \times \mathbf{R}^n)$ and $T^*(\mathbf{R} \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n)$, respectively, *i.e.*

$$C_0' = \{(z, y, \zeta, -\eta) : (z, \zeta, y, \eta) \in C_0\},$$

$$C_1'^{-1} = \{(t, x, s, z, \tau, \xi, -\tau', -\zeta) : (t, x, \tau, \xi; s, z, \tau', \zeta) \in C_1^{-1}\}.$$

Set $x' = (x_1, \dots, x_{n-1})$ and introduce the function

$$\psi(x', y', \eta) = -\langle y', \eta' \rangle + Q(x', \eta', \eta_n), \tag{4.5}$$

where $Q \in C^\infty(\mathbf{R}^{2n-1})$. We shall parametrize $\Lambda_0$ by a suitable phase function in a neighborhood of $\tilde{\nu}^0$ as follows.

PROPOSITION 4.1. – *There exist local coordinates $y'$ in a neighborhood of the origin in $\mathbf{R}^{n-1}$ and a smooth function $\psi$ of the form (4.5) such that*

*Lagrangian submanifold* $\Lambda'_0$ *can be parametrized in a neighborhood of* $\tilde{\nu}^0$ *by the phase function*

$$\Psi(t, x, y, \eta) = (x_n - y_n - t)\eta_n - \langle y', \eta' \rangle + Q(x', \eta', \eta_n).$$

*Proof.* – Set $\Sigma^0_{\eta_n} = \{\zeta_n = \eta_n\} \cap U_0$ and let $W$ be a sufficiently small neighborhood of the origin in $T^*(\mathbf{R}^{n-1})$ such that

$$W_{\eta_n} = \{(z', 0, \zeta', \eta_n) : (z', \zeta') \in W\}$$

is a subset of

$$\Sigma_{\eta_n} \cap S_0 = \{z_n = 0, \ \zeta_n = \eta_n\} \cap U_0$$

for $\eta_n$ in a fixed neighborhood of $E$. For such $\eta_n$ consider the map

$$P^0_{\eta_n} : W^0 \to W, \ P^0_{\eta_n}(z', \zeta') = \Big((p' \circ R)(z', 0, \zeta', \eta_n), (q' \circ R)(z', 0, \zeta', \eta_n)\Big), \tag{4.6}$$

where $W^0$ is a sufficiently small neighborhood of the origin in $W$. We define $W^0_{\eta_n}$ in the same way as $W_{\eta_n}$. Note that $P^0_E$ represents the Poincaré map of $k\gamma$. Moreover, $P^0_{\eta_n}$ is symplectic in $W^0$ with respect to the symplectic two-form

$$\sum_{j=1}^{n-1} dx_j \wedge d\xi_j,$$

and

$$\Lambda^0_{\eta_n} = \left\{(x', y', \xi', -\eta') : \ P^0_{\eta_n}(y', \eta') = (x', \xi'), \ (y', \eta') \in W^0\right\}$$

is a Lagrangian manifold in $W^0 \times W^0$, the latter being equipped with the symplectic two-form

$$\sum_{j=1}^{n-1} (dx_j \wedge d\xi_j + dy_j \wedge d\eta_j).$$

LEMMA 4.2. – *There exist local coordinates in a neighborhood of the origin in* $\mathbf{R}^{n-1}$ *and a smooth function* $\psi(x', y', \eta', \eta_n)$ *having the form (4.5) such that the Lagrangian manifold* $\Lambda^0_{\eta_n} \subset T^*(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1})$ *is parametrized by* $\psi$ *for any* $\eta_n$ *in a neighborhood* $\Xi$ *of* $E$. *Moreover, for any* $\eta_n \in \Xi$ *we have* $\det \psi_{x'\eta'} \neq 0$.

*Proof.* – According to Theorem 21.1.6 in [19], we can choose local coordinates so that the projection

$$\pi : \Lambda_E^0 \ni (x', \xi', y', \eta') \mapsto (x', \eta')$$

is a diffeomorphism. The same remains true for the projection $\Lambda_{\eta_n}^0 \mapsto (x', \eta')$ for $\eta_n$ sufficiently close to $E$. Therefore, we can represent $\xi', y'$ as functions of $(x', \eta', \eta_n)$, $\eta_n \in \Xi$ being a parameter. Since $\Lambda_{\eta_n}^0$ is a Lagrangian manifold, the one-form $\xi' dx' - y' d\eta'$ is exact on $\Lambda_{\eta_n}^0$. Hence there exists a smooth function $Q(x', \eta', \eta_n)$ such that $\xi' = \frac{\partial Q}{\partial x'}, y' = -\frac{\partial Q}{\partial \eta'}$. This completes the proof of Lemma 4.2. ♠

According to Lemma 4.2, there exist local coordinates in $\mathbf{R}^{n-1}$ and a smooth function $\psi$ of the form (4.5) such that

$$\Lambda_{\eta_n}^0 = \{(x', y', \psi_{x'}, \psi_{y'}) : \ \psi_{\eta'} = 0\},$$

where $\psi_{\eta'}$ stands for the partial derivatives of $\psi$ with respect to $\eta'$. Next we shall find a smooth function $g(\eta_n)$ such that the phase function

$$\Psi(t, x, y, \eta, \eta_n) = (x_n - y_n - t)\eta_n + \psi(x', y', \eta', \eta_n) + g(\eta_n) \tag{4.7}$$

parametrizes $\Lambda_0'$ in a neighborhood of $\tilde{\nu}^0$. Indeed, according to (4.3), we have

$$\Lambda_0' = \Big\{(t, x, y, \tau, \xi, -\eta) : \ \tau = -\eta_n = -\xi_n, \ x_n - y_n - t + s(y', \eta) = 0,$$

$$P_{\eta_n}^0(y', \eta') = (x', \xi'), \ (t, x, \eta) \in V\Big\}$$

$$= \Big\{(t, x, y, -\eta_n, \psi_{x'}, \eta_n, \psi_{y'}, -\eta_n) : \ \psi_{\eta'} = 0,$$

$$x_n - y_n - t + s(y', -\psi_{y'}, \eta_n) = 0\Big\}.$$

The one-form

$$\sigma = -t d\tau - x_n d\xi_n - y_n d\eta_n + \eta' dy' + \xi' dx'$$

is exact on $\Lambda_0'$. Moreover, for any $\eta_n$ close to $E$ the phase function $\psi(x', y', \eta', \eta_n)$ is non-degenerate, hence,

$$M_0 = \{(t, x, y, \eta', \eta_n) : \ \psi_{\eta'} = 0, x_n - y_n - t + s(y', -\psi_{y'}, \eta_n) = 0\}$$

is locally a smooth manifold, and $\eta_n$ is a free parameter on it. Denote by $j : M_0 \to \Lambda'_0$ the corresponding diffeomorphism

$$j(t, x, y, \eta', \eta_n) = (t, x, y, -\eta_n . \psi_{x'}, \eta_n . \psi_{y'}, -\eta_n).$$

Then, the pull-back

$$j^*(\sigma_{|\Lambda'_0}) = \left(s(y', -\psi_{y'}, \eta_n) - \frac{\partial \psi}{\partial \eta_n}\right)_{|M_0} d\eta_n + d(\psi_{|M_0})$$

is an exact form on $M_0$. Therefore, there exists a smooth function $g(\eta_n)$ such that

$$s(y', -\psi_{y'}, \eta_n) = \frac{\partial \psi}{\partial \eta_n} + g'(\eta_n) \text{ on } \{\psi_{\eta'} = 0\},$$

and we have proved that the phase function (4.7) parametrizes $\Lambda_0$ in a neighborhood of $\tilde{\nu}_0$. This completes the proof of the Proposition 4.1.♠

Note that the images $Y$ and $Y^0$ of $W_E$ and $W_E^0$ with respect to $\chi$ are both transversal to $\gamma$ at $\nu^0 \in \Sigma$, and

$$P = \chi \circ P_0 \circ \chi^{-1} : Y^0 \to Y$$

is the corresponding Poincaré map associated to $k\gamma$ in $\Sigma$. In particular,

$$P(\nu) = P_\gamma^k(\nu) = \Phi^{t(\nu)}(\nu), \ \nu \in W_1,$$

where $t(\nu) = s(z'(\nu), \zeta'(\nu), E)$, $t(\nu^0) = kT_\gamma$, is the corresponding return time function, and $\chi^{-1}(\nu) = (z(\nu), \zeta(\nu))$.

The density related to the phase function $\Phi$ can be simplified if $\nu^0$ is absolutely periodic. Recall that a periodic point $\nu^0$ is absolutely periodic if there exists a positive integer $m$ such that the map $P_\gamma^m(z) - z$ is flat at $z(\nu^0) = 0$, $P_\gamma$ being the Poincaré map related to the periodic trajectory $\gamma$ passing through $\nu^0$. Let $m(\gamma)$ be the smallest positive $m$ with this property. Fix a positive integer $q$ and set $k = qm(\gamma)$, $P = P_\gamma^k$. Then

$$P : Y^0 \to Y$$

is the Poincaré map associated to the multiple absolutely periodic trajectory $k\gamma = q\gamma^a$, where $Y^0$ is a suitable neighborhood of $\nu^0$ in $Y$. Obviously, these definitions depend neither on the choice of the section $Y$ nor on the local coordinates on it.

Consider $P$ in the locally symplectic coordinates $(z, \zeta)$. It is represented there by the map $P_E^0 : W^0 \to W$ introduced by (4.6). Hence,

$$R(z', 0, \zeta', E) - (z', 0, \zeta', E)$$

is flat at $\chi^{-1}(\nu^0) = (0, 0, 0, E)$, and in view of (4.3), the vector functions

$$(p' \circ \phi^t)(z, \zeta', E) - z', \text{ and } (q' \circ \phi^t)(z, \zeta', E) - \zeta', \ (t, z', 0, \zeta', E) \in V, \tag{4.8}$$

are flat at $(kT_\gamma, 0, 0, 0, E)$, too.

In particular, the vector function

$$P_E^0(y', \eta') - (y', \eta')$$

is flat at the origin and this property does not depend on the choice of the local coordinates. Using Lemma 4.2, we parametrize the Lagrangian manifold $\Lambda_{\eta_n}^0$ by

$$\psi(x', y', \eta) = -\langle y', \eta' \rangle + Q(x', \eta), \ Q \in C^\infty(\mathbf{R}^{2n-1}).$$

Set

$$Q(x', \eta) = \langle x', \eta' \rangle + L(x', \eta),$$

where $L$ is a smooth function. According to Proposition 4.1, the phase function

$$\Psi(t, x, y, \eta) = (x_n - y_n - t)\eta_n + \langle x' - y', \eta' \rangle + L(x', \eta)$$

yields a parametrization for the Lagrangian manifold $\Lambda_0'$, and

$$\Lambda_0' = \left\{ \left( t, x, x' + \frac{\partial L}{\partial \eta'}, x_n - t + \frac{\partial L}{\partial \eta_n}, -\eta_n, \eta' + \frac{\partial L}{\partial x'}, \eta_n, -\eta \right) : (t, x, \eta) \in V \right\}. \tag{4.9}$$

Set

$$T(y_n, x, \eta) = x_n - y_n + \frac{\partial L}{\partial \eta_n}(x', \eta).$$

The function

$$\tau(x', \eta) = T(x_n, x, \eta) = \frac{\partial L}{\partial \eta_n}(x', \eta)$$

does not depend on $x_n$, and we have

$$\left(x, \eta' + \frac{\partial L}{\partial x'}(x', \eta), \eta_n\right) = \phi^{\tau(x', \eta)}\left(x' + \frac{\partial L}{\partial \eta'}(x', \eta), x_n, \eta\right), \quad (x, \eta) \in U_0. \tag{4.10}$$

Taking in particular $x_n = 0$, and setting $L_0(x', \eta') = L(x', \eta', E)$, we obtain

$$\tau(x', \eta', E) = s\left(x' + \frac{\partial L_0}{\partial \eta'}(x', \eta'), \eta', E\right), \tag{4.11}$$

and

$$P^0\left(x' + \frac{\partial L_0}{\partial \eta'}(x', \eta'), \eta'\right) = \left(x', \eta' + \frac{\partial L_0}{\partial x'}(x', \eta')\right), \tag{4.12}$$

where $s(y', \eta)$ is the return time function related to $R$. The fixed points of $P^0$ are given by

$$\frac{\partial L}{\partial x'}(x', \eta', E) = 0, \quad \frac{\partial L}{\partial \eta'}(x', \eta', E) = 0,$$

hence, the set $\mathrm{Fix}(P^0)$ of fixed points of $P^0$ coincides with the set $\mathrm{Crit}(L_0)$ of all stationary points of the function $L_0(x', \eta')$ in $W^0$. Denote by $\mathrm{Fix}(P^0)^a$ the set of points $(x', \eta') \in W^0$, where the vector function $P^0(x', \xi') - (x', \xi')$ has a zero of infinite order. These points correspond to the absolutely periodic trajectories of $H_{a_0}$ situated in a neighborhood of $\gamma$ in $\Sigma$ and having periods close to $T_{k\gamma} = kT_\gamma$. Then the vector function

$$P^0\left(x' + \frac{\partial L_0}{\partial \eta'}(x', \eta'), \eta'\right) - \left(x' + \frac{\partial L_0}{\partial \eta'}(x', \eta'), \eta'\right)$$
$$= \left(-\frac{\partial L_0}{\partial \eta'}(x', \eta'), \frac{\partial L_0}{\partial x'}(x', \eta')\right)$$

has a zero of infinite order on $\mathrm{Fix}(P_0)^a$. In particular,

$$D_{x'}^\alpha D_{\eta'}^\beta L_0(x', \eta') = 0, \ \forall (x', \eta') \in \mathrm{Fix}(P^0)^a,$$

for any multiindices $\alpha$ and $\beta$, $|\alpha| + |\beta| \neq 0$, and we get

$$\det\left(I + \frac{\partial^2 L_0}{\partial x' \partial \eta'}\right) = 1 \tag{4.13}$$

on $\mathrm{Fix}(P^0)^a$.

Summarizing we obtain

PROPOSITION 4.3. – *Suppose that* $\nu^0$ *is absolutely periodic and* $k = qm(\gamma)$ *for some positive integer* $q$, *where* $\gamma$ *is the primitive periodic trajectory passing through* $\nu^0$ *and* $m(\gamma)\gamma$ *is the corresponding absolutely periodic trajectory. Then* $\Lambda_0'$ *can be parametrized in a neighborhood of* $\tilde{\nu}^0 = (kT_\gamma, 0, 0, -E, 0, E, 0, -E)$ *by the phase function*

$$\Psi(t, x, y, \eta) = (x_n - y_n - t)\eta_n + \langle x' - y', \eta' \rangle + L(x', \eta), \tag{4.14}$$

*where* $L$ *is a smooth function. Moreover,*

$$\mathrm{Fix}(P^0) = \mathrm{Crit}(L_0),$$

*and* $L_0(x', \eta') = L(x', \eta', E)$ *satisfies* (4.13) *on* $\mathrm{Fix}(P^0)^a$.

From now on we suppose that $\nu^0$ is a periodic point that may not be absolutely periodic. According to Proposition 4.1, there exist local coordinates $y'$ in a neighborhood of $\mathbf{R}^{n-1}$ such that the projection

$$\pi : \Lambda_E^0 \to \mathbf{R}^{2n-2}, \ \pi(x', y', \xi', -\eta') = (x', \eta')$$

is a diffeomorphism. Then, $\Lambda_0'$ is parametrized by the phase function (4.14) in a neighborhood of $\tilde{\nu}^0$. Moreover, the matrix

$$\left( I + \frac{\partial^2 L_0}{\partial x' \partial \eta'} \right)$$

is invertible.

We are going to explore the relation between the function $L_0(x', \eta')$ and the action

$$G(\nu) = A(\Gamma(\nu)) = \int_{\Gamma(\nu)} \xi dx,$$

where $\Gamma(\nu) = \{\Phi^t(\nu) : \ 0 \le t \le t(\nu)\}$, and $t(\nu) = s(z'(\nu), \zeta'(\nu), E)$, $\nu \in Y^0$, is the return time function of $P$. For any $(x', \eta') \in W^0$ set

$$\xi'(x', \eta') = \eta' + \frac{\partial L}{\partial x'}(x', \eta', E)$$

and

$$A^0(x', \eta') = A(\Gamma(\nu)), \ \nu(x', \eta') = (x', 0, \xi'(x', \eta'), E),$$
$$\nu'(x', \eta') = (x', \xi'(x', \eta')).$$

The function $L(x', \eta)$ is uniquely defined up to a constant and we normalize it by

$$L(0, 0, E) = A(\Gamma(\nu^0)) = kS(\nu^0).$$

PROPOSITION 4.4. – *There exists a smooth function $F(x', \xi')$ in $W^0$ such that*

$$L(x', \eta', E) - \left\langle \eta', \frac{\partial L}{\partial \eta'}(x', \eta', E) \right\rangle$$
$$= A^0(x', \eta') -$$
$$F\Big(P^0(\nu'(x', \eta'))\Big) + F\Big(\nu'(x', \eta')\Big), \ \forall (x', \eta') \in W^0.$$

*Proof.* – Set $\sigma^0 = \imath^* \sigma$, where $\sigma$ is the canonical one-form on $T^*(X)$ and $\imath : W \to T^*(\mathbf{R}^n)$ is the inclusion map. As in [10], we prove the following Poincaré - Cartan identity

$$P^* \sigma^0 - \sigma^0 = dG,$$

where $G(\nu) = A(\Gamma(\nu))$ (*see* Lemma A.3). Next we proceed as in [24], Proposition 2.3. Denote by

$$\chi_0 : W \to Y$$

the restriction of $\chi$ to $W$, and set $\alpha = \eta' dy'$. Then $\chi_0$ is exact symplectic, and there exists a smooth function $F$ such that

$$(\chi_0)^*(\sigma^0) = \alpha + dF(y', \eta').$$

Moreover, $P^0 = \chi_0^{-1} P \chi_0$. Hence,

$$(P^0)^*(\alpha) - \alpha = (\chi_0)^*(P^* \sigma^0 - \sigma^0) - (P^0)^* dF + dF$$
$$= d\Big(A^0(\chi_0(y', \eta')) - F(P^0(y', \eta')) + F(y', \eta')\Big).$$

On the other hand, using (4.12), we obtain

$$d\Big(A^0(x', \eta') - F(P^0(\nu'(x', \eta'))) + F(\nu'(x', \eta'))\Big)$$
$$= \left(\eta' + \frac{\partial L_0}{\partial x'}(x', \eta')\right) dx' - \eta' d\left(x' + \frac{\partial L_0}{\partial \eta'}(x', \eta')\right)$$
$$= d\left(L_0(x', \eta') - \left\langle \eta', \frac{\partial L_0}{\partial \eta'}(x', \eta') \right\rangle\right).$$

Taking into account the normalization of $L(x', \eta)$, we complete the proof of the assertion. ♠

As a consequence we obtain

COROLLARY 4.5. – *We have*

$$L_0(x', \xi') = A^0(x', \xi') \tag{4.15}$$

*on the set*

$$\mathrm{Fix}(P^0) = \mathrm{Crit}(L_0).$$

Fix $\lambda_1 < E < \lambda_2 < \lambda_0$ and suppose that all points in the interval $[\lambda_1, \lambda_2]$ are regular values of $a_0$. Fix $T > 0$ and consider the set

$$P_T = \{\nu \in a_0^{-1}([\lambda_1, \lambda_2]) : \ \nu \ \text{is absolutely periodic}$$
$$\text{with absolute period } T^a(\nu) \leq T\}.$$

We are going to prove that $P_T$ is closed in $a_0^{-1}\Big([\lambda_1, \lambda_2]\Big)$. For this purpose, introduce the set $K_{a,T} \in \mathbf{R}^{2n+1}$ of all $(\nu, t) \in a_0^{-1}([\lambda_1, \lambda_2]) \times [0, T]$ such that $\nu$ is an absolutely periodic point with a primitive period $T_\gamma = T(\nu) = t$ and an absolute period $T^a(\nu) = m(\gamma)T_\gamma \leq T$.

LEMMA 4.6. – *The set $K_{a,T}$ is compact in $\mathbf{R}^{2n+1}$, while $P_T$ is compact in $\mathbf{R}^{2n}$.*

*Proof.* – Assume that $(\nu_j, t_j) \in K_{a,T}$, $(\nu_j, t_j) \to (\nu_0, t_0)$. Let $\gamma_j$, $j = 1, \ldots,$ be the absolutely periodic trajectories passing through $\nu_j$. For any $t \in \mathbf{R}$ we have

$$\Phi^{t_j+t}(\nu_j) = \Phi^t(\nu_j)$$

which yields

$$\Phi^{t_0+t}(\nu_0) = \Phi^t(\nu_0).$$

Thus $\nu_0$ is a periodic point of $\Phi^t$ with a primitive period

$$T(\nu_0) \leq t_0 = \lim_{j \to \infty} T(\nu_j).$$

Denote by $\gamma_0$ the primitive periodic trajectory related to $\nu_0$. Obviously, $T(\nu) \geq c_0$ for each periodic point $\nu$ with some $c_0 > 0$, hence,

$$T \geq T^a(\nu_j) = m(\gamma_j)T(\nu_j) \geq m(\gamma_j)c_0.$$

Then $m(\gamma_j) \leq T/c_0$ and we may assume that $m(\gamma_j) = m_0 \in \mathbf{N}$, $\forall j \in \mathbf{N}$.

Next, let $Y \subset T^*(\mathbf{R}^n)$ be a section transversal to $\gamma_0$. Choose local coordinates $(z, s)$ in $Y$ so that $s(\nu) = a_0(\nu), z(\nu_0) = 0$. Assume that $a_0(\nu_0) = C_0$ with $C_0$ sufficiently close to $E$. Then $z$ are local coordinates on $Y \cap a_0^{-1}\Big([C_0 - \epsilon, C_0 + \epsilon]\Big)$ for any $\epsilon > 0$ small enough. Consider the map $R : Y_1 \longrightarrow Y$ given by

$$R(z, s) = \Phi^{T(z,s)}(z, s) \in Y, \quad (z, s) \in Y_1,$$

where $Y_1 = W_1 \times [C_0 - \epsilon, C_0 + \epsilon]$ and $\epsilon$ is taken sufficiently small. Here $T(z, s)$ is a smooth function corresponding to the return time. Moreover, we can assume that

$$T(0, C_0) = m_0 T(\nu^0) \leq T.$$

Thus, setting $C_j = a_0(\nu_j),\ j = 0, 1, \ldots,$ we obtain

$$R(z, C_j) = \Big(P_{\gamma_j}^{m_0}(z), C_j\Big), \quad j = 0, 1, \ldots,$$

where $P_{\gamma_j}^{m_0}(z)$ represent the Poincaré maps associated to the absolutely periodic trajectories $\gamma_j^a = m_0 \gamma_j$. Consequently, the vector functions $P_{\gamma_j}^{m_0}(z) - z$ are flat at $z = z(\nu_j)$, and we have

$$\partial_z^\alpha \Big(R(z, C_j) - (z, C_j)\Big)|_{z=z(\nu_j)} = 0, \quad \forall \alpha.$$

Passing to a limit as $j \to \infty$, we conclude that

$$\partial_z^\alpha \Big(R(z, C_0) - (z, C_0)\Big)|_{z=z_0} = 0, \quad \forall \alpha,$$

which implies

$$\partial_z^\alpha \Big(P_{\gamma_0}^{m_0}(z) - z\Big)|_{z=z_0} = 0, \quad \forall \alpha.$$

This shows that $\nu_0$ is an absolutely periodic point with an absolute period

$$T^a(\nu_0) = m(\gamma_0) T(\nu^0) \leq m_0 T(\nu_0) \leq T.$$

This shows that $K_{a,T}$ is compact. Finally, taking the projection into $\mathbf{R}^{2n}$, we deduce that $P_T$ is compact and the proof of Lemma 4.6 is complete. ♠

## 5. TRACE FORMULA

Our aim in this section is to obtain the leading term in the asymptotic of the convolution $(\sigma_h' * \phi_h)(E + rh)$, $|r| \leq r_0$, where $\phi_h$ was introduced in Section 2. Hereafter, we fix $r_0 > 0$, and to simplify the notations we drop $\delta$. The main result in this section is the following

THEOREM 5.1. – *Let $\hat{\phi}(t) \in C_0^\infty(\mathbf{R})$, and let $\hat{\phi}(t)$ vanish in a neighborhood of the origin. Then for any $|r| < r_0$ and $h \in (0, h_0]$, we have*

$$(\sigma_h' * \phi_h)(E + rh) = (2\pi h)^{-1}\mathrm{tr} \int \exp(ith^{-1}(E + rh))\hat{\phi}(t)g(x, hD_x)$$

$$\times \exp\Big(-ith^{-1}A(h)\Big)g(x, hD_x)dt$$

$$= (2\pi h)^{-n} \sum_{k \in \mathbf{Z}\setminus 0} \int_\Pi \hat{\phi}(kT(\nu))$$

$$\times \exp\Big(ik(h^{-1}S(\nu) + rT(\nu) - q(\nu))\Big)d\nu + o_\phi(h^{-n}),$$

*where $\phi_h(\lambda) = \frac{1}{h}\phi(\frac{\lambda}{h})$ and $o_\phi(h^{-n})$ does not depend on $r$.*

The idea of the proof of Theorem 5.1 is to consider the integral above as a Fourier integral operator with a large parameter $h^{-1}$. Making use of the local generating functions of $\Lambda_0$ introduced in the previous section we shall explore the corresponding oscillatory integrals representing the trace.

First we choose a suitable finite covering of

$$\Lambda_1' = \{(t, x, y, \tau, \xi, \eta) \in \Lambda' : \; a_0(y, \eta) \in [\lambda_1, \lambda_2]\},$$

where $\Lambda'$ was introduced in Section 3, $\lambda_1 < E < \lambda_2 < \lambda_0$, and each $\lambda \in [\lambda_1, \lambda_2]$ is a regular value of $a_0$. Fix $T > 0$, and suppose that the support of $\hat{\phi}(t)$ is contained in $[-T, T]$. Consider the set

$$\Lambda^\Pi = \{(t, x, y, \tau, \xi, \eta) \in \Lambda_1' : \; \nu = (y, \eta) \in \tilde{\Pi},$$

$$|t| = kT(\nu) \leq T \text{ for some } k \in \mathbf{N}\},$$

where $\tilde{\Pi}$ is the set of the periodic points of $H_{a_0}$ in $a_0^{-1}([\lambda_1, \lambda_2])$. Obviously, $\Lambda^\Pi$ is a closed subset of $\Lambda_1'$. Denote by $\Lambda_a^\Pi$ the set of all $(t, x, y, \tau, \xi, \eta) \in \Lambda^\Pi$ such that $\nu = (y, \eta) \in \tilde{\Pi}$ is absolutely periodic and $|t| = qT^a(\nu) \leq T$ for some positive integer $q$. According to Lemma 4.6, $\Lambda_a^\Pi$ is a closed subset of $\Lambda_1'$. First we choose a finite open covering

$$V_j, \; j = 1, \ldots, j_0,$$

of $\Lambda_a^\Pi \cap \Sigma$ and denote by $V$ the union of $V_j$. Shrinking the interval $[\lambda_1, \lambda_2]$ if necessary and using Lemma 4.6 again, we obtain $\Lambda_a^\Pi \subset V$. Next we choose a finite open covering

$$V_j,\ j = j_0 + 1, \dots, j_1,$$

of $\Lambda^\Pi \setminus V$ such that $V_j,\ j = j_0 + 1, \dots, j_1$, do not intersect $\Lambda_a^\Pi$. Then $V_j,\ j = 1, \dots, j_1$, is a finite covering of $\Lambda^\Pi$, and we suppose that the construction of the phase functions presented in the previous section works in any $V_j$. Finally, we get a finite open covering $V_j,\ j = 1, \dots, j_2$, of $\Lambda_1'$ such that $V_j,\ j = j_1 + 1, \dots, j_2$, do not intersect $\Lambda^\Pi$.

Choose a partition of the unity $\kappa_j(t)B_j(x, hD_x),\ j = 1, \dots, j_2$, in $\mathbf{R}^{n+1}$ subordinated to $V_j,\ j = 1, \dots, j_2$, that is

$$\mathrm{WF}\left(\sum_{j=1}^{j_2} \kappa_j B_j - \mathrm{Id}\right) \cap \left\{(t, x, \tau, \xi) :\ a_0(x, \xi) \in [\lambda_1, \lambda_2],\ |t| \leq T\right\} = \emptyset,$$

and

$$\{(t, x, y, \tau, \xi, \eta) :\ t \in \mathrm{supp}\kappa_j,\ -\tau \in [\lambda_1, \lambda_2],\ (y, \eta) \in \mathrm{WF}(B_j)\} \cap \Lambda_1' \subset V_j.$$

Here $\kappa_j(t)$ are smooth functions with compact supports and $B_j(x, hD_x)$ are pseudodifferential operators with a large parameter $h^{-1}$ having the form

$$B_j u(x) = (2\pi h)^{-n} \int e^{ih^{-1}\langle x-y, \xi\rangle} b^j(x, \xi, h) u(y) dy d\xi.$$

Introduce the operator

$$R_j(h) = (2\pi h)^{-1} \int e^{ih^{-1}t(E+rh)} \hat{\phi}(t) \kappa_j(t) g(x, hD_x)$$
$$\times \exp\Big(-ith^{-1}A(h)\Big) B_j(x, hD_x) g(x, hD_x) dt.$$

We consider separately three cases.

*Case* 1. – *Trace of* $R_j(h)$ *for* $1 \leq j \leq j_0$.

We can suppose that $V_j$ is a neighborhood of a point

$$\tilde{\nu}^0 = (qT^a(\nu), x^0, x^0, -E, \xi^0, -\xi^0) \in \Lambda_1' \cap \Sigma$$

such that $\nu^0 = (x^0, \xi^0) \in \Pi^a$ is absolutely periodic with an absolute period $T^a(\nu^0) \leq T$. Let $\gamma$ be the primitive periodic trajectory of $H_{a_0}$ in $\Sigma$ issuing from $\nu^0$. Then the multiple periodic trajectory $k\gamma = q\gamma^a$ is absolutely periodic. To simplify the notations we drop the index $j$.

Choose a Fourier integral operator $Q(h) \in I^0(\mathbf{R}^n \times \mathbf{R}^n, C_0'; h)$ with a large parameter $h^{-1}$ such that

$$\mathrm{WF}\Big(Q(h)Q(h)^* - Id\Big) \cap \mathrm{WF}(B) = \emptyset, \tag{5.1}$$

where $C_0'$ is the Lagrangian manifold corresponding to the canonical relation $C_0$ introduced in Section 4 as the graph of the symplectic transformation $\chi$. Namely,

$$C_0' = \{(z, y, \zeta, -\eta) \ : \ (z, \zeta, y, \eta) \in C_0\},$$

$$C_0 = \{(\chi(y, \eta), y, \eta)) \ : \ (y, \eta) \in U_0\},$$

where $U_0$ is a small neighborhood of $(0, (0, E))$. Then $\chi(U_0)$ will be a small neighborhood of $\nu^0$ and we assume that $\mathrm{WF}(B) \subset \chi(U_0)$. Set $Q_1(h) = \kappa_0(t)Q(h)\kappa_0(t)$, where $\kappa_0(t) \in C_0^\infty(\mathbf{R})$ and $\kappa_0(t) = 1$ in a neighborhood of supp$\kappa$. Then the operator $Q_1^*(h)$ is a Fourier integral operator in $I^0(\mathbf{R}^{n+1} \times \mathbf{R}^{n+1}, C_1'; h)$ with a large parameter $h^{-1}$ whose canonical relation $C_1$ is given by

$$C_1 = \{(t, x, \tau, \xi; t, z, \tau, \zeta) \in T^*(\mathbf{R} \times \mathbf{R}^n) \times T^*(\mathbf{R} \times \mathbf{R}^n) \ :$$

$$(z, \zeta) = \chi(x, \xi), \ (t, x, \tau, \xi) \in V_0\},$$

where $V_0$ is a small neighborhood of $(kT_\gamma, 0, 0, E)$. Choosing the support of $\kappa_0(t)$ small enough, we consider the operator

$$U_h^0(t) = Q_1^*(h)U_h^1(t)Q(h) \in I^{1/4}(\mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^n, \Lambda_0'; h)$$

associated to the Lagrangian manifold $\Lambda_0'$, where

$$U_h^1(t) = g(x, hD_x)\exp\Big(-ith^{-1}A(h)\Big)B(x, hD_x)g(x, hD_x).$$

Using (5.1) we obtain

$$\operatorname{tr} R(h) = \operatorname{tr}\Big(Q^*(h)R(h)Q(h)\Big) + \mathcal{O}(h^\infty) = \operatorname{tr} R^0(h) + \mathcal{O}(h^\infty), \tag{5.2}$$

where

$$R^0(h) = (2\pi h)^{-1}\int e^{ih^{-1}t(E+rh)}\,\kappa(t)\hat{\phi}(t)U_h^0(t)dt. \tag{5.3}$$

Recall that the canonical relation $\Lambda_0$ corresponding to $\Lambda_0'$ is a transversal composition of canonical relations $C_0, \Lambda$ and $C_1$ given by (4.4). Now we parametrize the Lagrangian manifold $\Lambda_0'$ by the projection $\tilde{\pi} : \Lambda_0' \to \mathbf{R} \times T^*(\mathbf{R}^n)$, $\tilde{\pi}(t, x, y, \tau, \xi, -\eta) = (t, y, \eta)$. Making use of (3.5), we obtain as in [18], Section 4.2, (*see* also Proposition 5 in [22]), the following

PROPOSITION 5.2. – *Under the parametrization of $\Lambda_0'$ by the coordinates $(t, y, \eta)$ the principal symbol $\sigma(U_h^0(t))$ of $U_h^0(t)$ devided by the corresponding Maslov section in a neighborhood of $\tilde{\nu}_0 = (T_{k\gamma}, 0, 0, -E, 0, E, 0, -E)$ is given by*

$$(\tilde{\pi}^{-1})^*(\sigma(U_h^0(t))) = e^{ih^{-1}F(t,y,\eta)}\tilde{u}_0(t, y, \eta)|dt \wedge dy \wedge d\eta|^{1/2},$$

*where*

$$F(t, y, \eta) = -\tilde{G}(\phi^t(y, \eta)) - t\eta_n + A\Big(\gamma_t(\chi(y, \eta))\Big) + \tilde{G}(y, \eta),$$

$$\phi^t = \chi^{-1} \circ \Phi^t \circ \chi,$$

$$\tilde{u}_0(t, y, \eta) = \exp\left(-i\int_0^t a_1(\Phi^s(\chi(y, \eta)))ds\right)b_0(\chi(y, \eta))w(\eta_n)^2\kappa_0^2(t),$$

*the smooth function $\tilde{G}$ represents the Liouville factor of $Q(h)$, $b_0(x, \xi)$ is the principal symbol of $B$ and $w(\eta_n) = g_0(\chi(y, \eta))$ is compactly supported and equal to 1 in a neighborhood of $E$.*

The proof of Proposition 5.2 will be given in the Appendix.

Parametrizing $\Lambda_0'$ by $(t, x, \eta)$ as in Proposition 4.3, the kernel of $U_h^0(t)$ becomes

$$U_h^0(t, x, y, h) = (2\pi h)^{-n} \int e^{ih^{-1}(\Psi(t,x,y,\eta)+C)}\theta(y)u(t, x, \eta, h)d\eta. \quad (5.4)$$

Here $\Psi$ is given by (4.14), $C$ is a constant coming from the Liouville factor, $\theta(y) \in C_0^\infty(\mathbf{R})$ is equal to 1 in a neighborhood of 0, and

$$u(t, x, \eta, h) = e^{i\pi m(t,x,\eta)/2}\Big[u_0(t, x, \eta) + hu_1(t, x, \eta) + \cdots\Big].$$

Here $u_j(t, x, \eta)$ are classical symbols with a compact support with respect to $(t, x, \eta)$ and $m(t, x, \eta) \in \mathbf{Z}_4$ is a Maslov index.

We are going to find the function $u_0$. Seting $\imath = \tilde{\pi} \circ i_\Psi$, and using Proposition 5.2, we obtain from the definition of the principal symbol of a Fourier integral operator with a large parameter that

$$u_0\sqrt{d_\Psi} = \imath^*\Big(\tilde{u}_0\sqrt{|dt \wedge dy \wedge d\eta|}\Big).$$

where

$$d_\Psi = \left|\frac{D(t,x,\Psi_\eta,\eta)}{D(t,x,y,\eta)}\right|^{-1} |dt\wedge dx\wedge d\eta| = |dt\wedge dx\wedge d\eta|.$$

On the other hand, the equation $\Psi_\eta(t,x,y,\eta) = 0$ has an unique solution $y = y(t,x,\eta) = (y'(x',\eta), y_n(t,x,\eta))$ which is given by

$$y' = x' + \frac{\partial L}{\partial \eta'}(x',\eta),\ y_n = x_n - t + \frac{\partial L}{\partial \eta_n}(x',\eta).$$

Then we have $\imath(t,x,\eta) = (t, y(t,x,\eta),\eta)$, and we obtain

$$u_0(t,x,\eta) = \tilde{u}_0(t,y(t,x,\eta),\eta)J(x',\eta),$$

where

$$J(x',\eta) = \det\left(Id + \frac{\partial^2 L}{\partial x'\partial \eta'}(x',\eta)\right)^{\frac{1}{2}}. \tag{5.5}$$

Hence, setting $\nu = \chi(y(t,x,\eta),\eta)$ we obtain

$$e^{i\pi m(t,x,\eta)/2}u_0(t,x,\eta) = v_0(t,x,\eta)$$

$$= e^{i\pi m(t,\nu)/2}\exp\Big(-i\int_0^t a_1\Big(\Phi^s(\nu)\Big)ds\Big)w^2(\eta_n)b_0(\nu)J(x',\eta)\kappa_0^2(t), \tag{5.6}$$

As in [3] and [27], we shall show that $C$ is zero. To do this consider the Liouville factor of $U_h^0(t)$ in the coordinates $(t,y,\eta)$ given by the parametrization of $\Lambda_0'$ by the projection $\tilde{\pi}$. Its exponent is given by $\tilde{\Psi}(\tilde{\pi}^{-1}(t,y,\eta)) + C$, where $\tilde{\Psi} = \imath_\Psi^*\Big(\Psi|_{C_\Psi}\Big)$. It is enough to make the computations at the point $(kT(\nu),\nu)$, $\nu = (0,0,E)$. We have

$$\tilde{\Psi}(kT(\nu),\nu)+C = -kT(\nu)E+L(\nu)+C = -kT(\nu)E+kS(\nu)+C, \tag{5.7}$$

since by definition $L(0,0,E) = kS(\nu^0)$. On the other hand, according to Proposition 5.2, the exponent of the Liouville factor of $U_h^0(t)$ is

$$-\tilde{G}(\phi^t(y,\eta)) - t\eta_n + A\Big(\gamma_t(\chi(y,\eta))\Big) + \tilde{G}(y,\eta). \tag{5.8}$$

Comparing (5.7) with the value of (5.8) at $(kT(\nu),\nu)$ we obtain $C = 0$.

We are ready to compute the trace

$$\operatorname{tr} R(h) = \operatorname{tr}\left(\int e^{ih^{-1}t(E+rh)}\kappa(t)\,\hat{\phi}(t)U_h^1(t)\,dt\right),$$

where $|r| \leq r_0$. According to (5.2), we have

$$\operatorname{tr} R(h) = \operatorname{tr} R^0(h) + \mathcal{O}(h^\infty)$$

$$= \operatorname{tr}\left(\int e^{ih^{-1}t(E+rh)}\kappa(t)\hat{\varrho}(t)\, U_h^0(t)\, dt\right) + \mathcal{O}(h^\infty).$$

To simplify the notations set

$$G(t,x,\eta) = e^{itr}\kappa(t)\hat{\phi}(t)v_0(t,x,\eta)\Big(J(x',\eta)\Big)^{-1},$$

where $v_0(t,x,\eta)$ is given by (5.6). Using the stationary phase argument with respect to $(t,\eta_n)$, we get

$$\operatorname{tr} R(h) = (2\pi h)^{-n}$$

$$\times\Big(\operatorname{tr}\int \exp\big(ih^{-1}(tE + (x_n - y_n - t)\eta_n + \langle x'-y',\eta'\rangle + L(x'\eta))\big)$$

$$\times G(t,x,\eta)J(x',\eta)dtd\eta + \mathcal{O}(h^\infty)\Big)$$

$$= (2\pi h)^{1-n}\Big(\operatorname{tr}\int \exp\big(ih^{-1}((x_n-y_n)E + \langle x'-y',\eta'\rangle + L_0(x',\eta'))\big)$$

$$\times G(T(y_n,x,\eta',E),x,\eta',E)J(x',\eta',E)d\eta' + \mathcal{O}(h)\Big),$$

where the functions $T(y_n,x,\eta)$ and $L_0(x',\eta')$ have been defined in the previous section. Then we obtain

$$\operatorname{tr} R(h) = (2\pi h)^{1-n}\int_W \exp\big(ih^{-1}L_0(x',\eta')\big)J(x',\eta',E) \tag{5.9}$$

$$\times\left(\int_{\mathbf{R}} G(\tau(x',\eta',E),x,\eta',E)dx_n\right)dx'd\eta' + o(h^{1-n}),$$

where $W$ is a neighborhood of the origin in $\mathbf{R}^{2n-2}$ and $\tau(x',\eta',E)$ satisfies (4.11).

Fix a $0<\beta<1$, set

$$W_h^1 = \{\varrho \in W:\ |\nabla L_0(\rho)| \leq h^\beta\},$$

and denote by $W_h^2$ the complement of $W_h^1$ in $W$. Here, $P^0 : W^0 \to W$ represents the Poincaré map in the coordinates $(x',\xi')$. According to Proposition 4.3, the set of the critical points of $L_0$ coincides with $\operatorname{Fix}(P^0)$. We can integrate (5.9) by parts in $W_h^2$ modulo a term $o(h^{1-n})$. Then, the domain of integration $W$ in (5.9) can be replaced by the set $\mathrm{Fix}(P^0)$. Furthermore, according to Lemma A.1, the Lebesgue measure of $\mathrm{Fix}(P^0)^a$ is equal to that of $\mathrm{Fix}(P^0)$, where $\mathrm{Fix}(P^0)^a$ was introduced in Section 4. Thus we can pass to integration over $\mathrm{Fix}(P^0)^a$ and the integrand in (5.9) can be simplified considerably. First we can get rid of the density $J(x', \eta', E)$. Indeed, for any $(x', \eta') \in \mathrm{Fix}(P^0)^a$ we have $J(x', \eta', E) = 1$, in view of (4.13) and (5.5).

On the other hand, for any $(y', \eta') \in \mathrm{Fix}(P^0)^a$, Corollary 4.5 yields

$$L_0(x', \eta') = A^0(x', \eta') = kS(\nu),$$

where $\nu = \chi(x', 0, \eta', E) \in \Pi$. Thus we obtain

$$\operatorname{tr} R(h) = (2\pi h)^{1-n} \int_{\mathrm{Fix}(P_0)^a} \exp\left(ih^{-1}A^0(x', \eta')\right)$$
$$\times \left(\int_{\mathbf{R}} G(\tau(x', \eta', E), x, \eta', E)dx_n\right)dx'd\eta' + o(h^{1-n}).$$

Using Lemma A.1 again, we can replace the domain of integration in the first integral by $\mathrm{Fix}(P^0)$. Moreover, for any $(x', \eta') \in \mathrm{Fix}(P^0)$, we have

$$\tau(x', \eta', E) = kT(\nu), \ k \in \mathbf{N}.$$

Consequently, the trace of $R(h)$ becomes

$$\operatorname{tr} R(h) = (2\pi h)^{1-n} \sum_{k \in \mathbf{Z}\backslash 0} \int_{\mathrm{Fix}(P^0)} \exp\left(ih^{-1}kS(\nu)\right)$$
$$\times \left(\int_{\mathbf{R}} G(kT(\nu), x, \eta', E)dx_n\right)dx'd\eta' + o(h^{1-n}).$$

Finally, taking into account (5.6), we obtain

$$\operatorname{tr} R(h) = (2\pi h)^{1-n} \sum_{k \in \mathbf{Z}\backslash 0} \int_{\mathrm{Fix}(P^0)}$$
$$\times \exp\Big(ik(h^{-1}S(\nu) + q_c(\nu) - s(\nu) + rT(\nu))\Big)$$
$$\times (\kappa\,\hat{\phi})(kT(\nu))b_0(\nu)dy_n dy' d\eta' + o_T(h^{1-n}),$$

where we have used the notations

$$\nu = \chi(y', y_n, \eta', E) = \Phi^{y_n}(\chi(y', 0, \eta', E)) \in \Pi \cap \operatorname{supp} b_0,$$

and

$$s(\nu) = \int_0^{T(\nu)} a_1(\Phi^s(\nu))ds, \; q_c(\nu) = \frac{\pi}{2} m(T(\nu), \nu);$$

It remains to return to the coordinates $(x, \xi) = \chi(y, \eta)$ using the invariance of the Liouville measure with respect to symplectic transformations.

Denote by $\Omega$ the Liouville form on $\Sigma$. It is defined as the pull-back $j^*\Omega_0$, where $j : \Sigma \longrightarrow T^*(X)$ is the natural inclusion map and $(n!) da_0 \wedge \Omega_0 = \omega^n$ in a neighborhood of $\Sigma$, $\omega$ being the standard symplectic form on $T^*(X)$. Since $\chi^*(a_0) = \eta_n$, and

$$\chi^*(\omega) = \sum_{j=1}^{n} dy_j \wedge d\eta_j,$$

we obtain

$$\chi^*(\Omega) = dy_1 \wedge \cdots \wedge dy_n \wedge d\eta_1 \wedge \cdots \wedge d\eta_{n-1}.$$

On the other hand, $d\nu = |\Omega|$, where $d\nu$ stands for the Liouville measure on $\Sigma$ and $|\Omega|$ is the density related to $\Omega$. Finally, we obtain

$$\operatorname{tr} R(h) = (2\pi h)^{1-n} \sum_{k \in \mathbf{Z}\backslash 0} \int_{\Pi} \exp\Big(ik(h^{-1}S(\nu) - q(\nu) + rT(\nu))\Big) \tag{5.10}$$
$$\times (\kappa\,\hat{\phi})(kT(\nu)) b_0(\nu) d\nu + o_T(h^{1-n}),$$

where $q(\nu) = s(\nu) - q_c(\nu)$.

*Case* 2. – *Trace of* $R_j(h)$ *for* $j_0 < j \leq j_1$.

As above consider the trace of the operator $R^0(h)$ introduced by (5.3). We can suppose that $\Lambda_0 \cap V_j$ is parametrized by the phase function

$$\Psi(t, x, y, \eta) = (x_n - y_n - t)\eta_n + \psi(x, y, \eta),$$

where $\psi$ is given by (4.5). Setting

$$\psi(x, y, \eta) = \langle x' - y', \eta' \rangle + L(x', \eta),$$

and integrating by parts with respect to $(t, \eta_n)$, we obtain

$$\operatorname{tr} R(h) = (2\pi h)^{1-n} \int_{\mathrm{Crit}(L_0)} \exp\left(ih^{-1}L_0(x', \eta')\right) J(x', \eta', E)$$
$$\times \left(\int_{\mathbf{R}} G(\tau(x', \eta', E), x, \eta', E) dx_n\right) dx' d\eta' + o(h^{1-n}).$$

Fix $(x'_0, \eta'_0) \in \mathrm{Crit}(L_0)$ and set $\nu = \chi(x'_0, 0, \eta'_0, E)$. According to Proposition 4.4, $(x'_0, \eta'_0) \in \mathrm{Fix}(P^0)$, and we have

$$\tilde{\nu} = (kT(\nu). x'_0, 0, y'_0, 0, -E, \eta'_0, E, \eta'_0, E) \in \Lambda^{\Pi} \cap V_j,$$

where

$$kT(\nu) = \tau(x'_0, \eta'_0, E) \in \operatorname{supp} \kappa, \ k \in \mathbf{N}.$$

In particular, the periodic trajectory $k\gamma(\nu)$ is not absolutely periodic, hence the vector function

$$P^0(y', \eta') - (y', \eta')$$

has a zero of finite order at $(y'_0, \eta'_0)$. Consequently, the vector function

$$P^0\left(x' + \frac{\partial L_0}{\partial \eta'}(x', \eta'), \eta'\right) - \left(x' + \frac{\partial L_0}{\partial \eta'}(x', \eta'), \eta'\right)$$
$$= \left(-\frac{\partial L_0}{\partial \eta'}(x', \eta'), \frac{\partial L_0}{\partial x'}(x', \eta')\right)$$

has a zero of finite order at $(y'_0, \eta'_0)$. Therefore, the Lebesgue measure of $\mathrm{Fix}(P^0) = \mathrm{Crit}(L_0)$ in $\Sigma$ is zero, and we get

$$\operatorname{tr} R(h) = o_T(h^{1-n}).$$

In the same way replacing 1 by $J(x', \eta', E)$ and using the argument of Case 1, we obtain

$$(2\pi h)^{1-n} \int_{\mathrm{Fix}(P^0)} \exp\left(ih^{-1} L_0(x', \eta')\right) J(x', \eta', E)$$
$$\times \left(\int_{\mathbf{R}} G(\tau(x', \eta', E), x, \eta', E) dx_n\right) dx' d\eta' = o_T(h^{1-n}).$$

Finally, using the invariance of the Liouville measure we write $\operatorname{tr} R(h)$ in the form (5.10).

*Case* 3. – *Trace of* $R_j(h)$ *for* $j_1 < j \leq j_2$.

Fix $j$ as above. Then there are no periodic trajectories $\gamma$ issuing from $\mathrm{WF}(B_j)$ and such that $kT_\gamma \in \operatorname{supp} \kappa_j$ for some $k \in \mathbf{Z}$. Representing $R(h)$ as an oscillatory integral and integrating by parts, we obtain

$$\operatorname{tr} R(h) = \mathcal{O}_T(h^\infty).$$

Finally, summing up we obtain

$$(\sigma'_h * \phi_h)(E + rh) = (2\pi h)^{1-n} \sum_{k \in \mathbf{Z}\setminus 0} \int_{\Pi}$$

$$\times \exp\Big(ik(h^{-1}S(\nu) - q(\nu) + rT(\nu))\Big)\, \hat{\phi}(kT(\nu))d\nu + o_T(h^{1-n}). \quad (5.11)$$

The proof of the Theorem 5.1 is complete. ♠

Using Theorem 5.1, it is easy to obtain the Gutzwiller's trace formula given in Theorem 1.4.

*Proof of Theorem* 1.4. – Fix $\epsilon > 0$ so that $E + \epsilon < \lambda$ and take a function $\chi(t) \in C_0^\infty(-\infty, \lambda)$ such that $\chi(t) = 1$ for $E - \epsilon \leq t \leq E + \epsilon$. Then

$$\sum \chi^2(\lambda_j(h))\rho\Big(\frac{E - \lambda_j(h)}{h}\Big) = \operatorname{tr}\ \Big(\chi(A(h))\rho\Big(\frac{E - A(h)}{h}\Big)\chi(A(h))\Big)$$

$$= (2\pi)^{-1} \operatorname{tr}\ \int \chi(A(h)) \exp(ith^{-1}E)\hat{\rho}(t) \exp\Big(ith^{-1}A(h)\Big)\chi(A(h))dt$$

and we can apply the argument of Theorem 5.1 with $\delta = 1, r = 0$ and $g(x, hD_x)$ representing $\chi(A(h))$. For the singularity at 0 we apply Proposition 2.2. To deal with the term

$$\sum_{\lambda_j(h) \leq \lambda} \Big(1 - \chi^2(\lambda_j(h))\Big)\rho\Big(\frac{E - \lambda_j(h)}{h}\Big),$$

notice that $|E - t| > \epsilon$ on the support of $(1 - \chi^2(t))$, hence

$$\rho\Big(\frac{E - t}{h}\Big) = \mathcal{O}(h^N), \quad \forall N \text{ for } |E - t| > \epsilon.$$

Taking into account that $\sharp\{\lambda_j(h) \leq \lambda\} = \mathcal{O}(h^{-n})$, we complete the proof of the assertion. ♠

Now, let us turn back to the end of Section 2. Recall that $\varrho(\tau)$ is an even function, $\hat{\varrho}(0) = 1$, the support of $\hat{\varrho}(t)$ is contained in a small interval $[-\delta_1, \delta_1]$, $\delta_1 > 0$, and

$$\phi_h(\lambda, \delta) = (2\pi h)^{-1} \int \exp(ith^{-1}\lambda)(1 - \vartheta(t))\hat{\rho}(\delta t)dt.$$

$$\psi_h(\lambda, \delta) = (2\pi h)^{-1} \int \exp(ith^{-1}\lambda)\vartheta(t)\hat{\rho}(\delta t)dt.$$

We introduce the function

$$\chi_h(\lambda, \delta) = (2\pi h)^{-1} \int \exp(ith^{-1}\lambda)(it)^{-1}(1 - \vartheta(t))\hat{\rho}(\delta t)dt.$$

Then,

$$\Big(\sigma_h * \phi_h\Big)(\lambda) = h\frac{d}{d\lambda}\Big(\sigma_h * \chi_h\Big)(\lambda)$$

and we obtain

$$(\sigma_h * \rho_{\delta h})(E + rh) = \int_{-\infty}^{E+rh} \frac{d}{d\lambda}(\sigma_h * \psi_h)(\lambda)$$

$$+h\frac{d}{d\lambda}(\sigma_h * \chi_h)(E + rh) = I_1 + I_2.$$

The term $I_1$ can be treated using Proposition 2.2. On the other hand, applying Theorem 5.1 with $\phi_h(\lambda)$ replaced by $\chi_h(\lambda, \delta)$, we obtain

$$I_2 = -i(2\pi)^{-n}h^{1-n} \int_\Pi \sum_{k\in\mathbf{Z}\setminus 0} k^{-1}T^{-1}(\nu)\hat{\rho}(\delta kT(\nu))$$

$$\times \exp\Big(ik(h^{-1}S(\nu) + rT(\nu) - q(\nu))\Big)d\nu + o_\delta(h^{1-n})$$

$$= 2(2\pi)^{-n}h^{1-n} \int_\Pi T^{-1}(\nu)\sum_{k=1}^{\infty} k^{-1}\hat{\rho}(\delta kT(\nu))$$

$$\times \sin\Big(k(h^{-1}S(\nu) + rT(\nu) - q(\nu))\Big)d\nu + o_\delta(h^{1-n}).$$

Consequently, using the above argument for $r = r_1, r_2$, we obtain the following

COROLLARY 5.3. – *For any $-r_0 \le r_1 \le r_2 \le r_0$, and $h \in (0, h_0]$, we have*

$$(\sigma_h * \varrho_{\delta h})(E + r_2h) - (\sigma_h * \varrho_{\delta h})(E + r_1h) = h^{1-n}(r_2 - r_1)(2\pi)^{-n}\mu(\Sigma)$$

$$+2(2\pi)^{1-n}h^{1-n} \int_\Pi T^{-1}(\nu)\sum_{k=1}^{\infty} k^{-1}\hat{\rho}(\delta kT)$$

$$\times\Big(\sin\Big(k(h^{-1}S + r_2T - q)\Big) - \sin\Big(k(h^{-1}S + r_1T - q)\Big)\Big)d\nu + o_\delta(h^{1-n}). \tag{5.12}$$

*where $o_\delta(h^{1-n})$ does not depend on $r$.*

## 6. REPRESENTATION OF THE LEADING SINGULARITY

Our aim in this section is to estimate the leading terms of the equality (5.12). Notice that the integration in (5.12) can be taken over $\Pi_+$, so we shall concentrate our attention on the term

$$\int_{\Pi_+} T(\nu)^{-1} \sum_{k=1}^{\infty} k^{-1} \hat{\rho}(\delta k T(\nu)) \sin\Big(k(h^{-1}S(\nu) + rT(\nu) - q(\nu))\Big) d\nu.$$

We make the following assumption

$$h^{-1}S(\nu) + rT(\nu) - q(\nu) \geq \omega_0 > 0, \quad \forall \nu \in \Pi_+, \tag{6.1}$$

which is satisfied for any compact hypersurface $\Sigma$ of contact type (*see* Section 8).

PROPOSITION 6.1. – *Suppose that (6.1) is satisfied. Then for any* $0 < \varepsilon \leq 1$, $0 < \delta \leq 1$ *and* $|r| \leq r_0$, *we have*

$$\begin{aligned}
&(2\pi)^{-n} h^{1-n} \int_{\Pi_+} [\pi - h^{-1}S(\nu) + q(\nu) - (r - \varepsilon)T(\nu)]_{2\pi} T(\nu)^{-1} d\nu - \\
&\qquad C_1 \varepsilon h^{1-n} - C_0 \varepsilon^{-1} \delta h^{1-n} - o_\delta(h^{1-n}) \\
&\leq 2(2\pi)^{-n} h^{1-n} \int_{\Pi_+} T(\nu)^{-1} \sum_{k=1}^{\infty} k^{-1} \hat{\rho}(\delta k T) \sin\Big(k(h^{-1}S + rT - q)\Big) d\nu \\
&\leq (2\pi)^{-n} h^{1-n} \int_{\Pi_+} [\pi - h^{-1}S(\nu) + q(\nu) - (r + \varepsilon)T(\nu)]_{2\pi} T(\nu)^{-1} d\nu \\
&\qquad + C_1 \varepsilon h^{1-n} + C_0 \varepsilon^{-1} \delta h^{1-n} + o_\delta(h^{1-n})
\end{aligned}$$

*with constants* $C_0 > 0$ *and* $C_1 > 0$ *independent of* $\varepsilon$, $\delta$ *and* $h$.

*Proof.* – Set $\lambda = h^{-1}$ and denote

$$z(\lambda, \nu) = \lambda S(\nu) + rT(\nu) - q(\nu).$$

Fix $\nu \in \Pi_+$ and write $T$, $S$, $q$ and $z$ instead of $T(\nu)$, $S(\nu)$, $q(\nu)$ and $z(\lambda, \nu)$. Recall that $\varrho$ is an even function. Then for $S \neq 0$ we get

$$\sum_{k=1}^{\infty} k^{-1} T^{-1} \hat{\rho}(\delta k T) \sin(kz)$$

$$= -\frac{z}{2T} + \frac{S}{2T}\int_{-rT+q/S}^{\lambda} \sum_{k=-\infty}^{\infty} \hat{\rho}(\delta kT) e^{ik(\mu S+rT-q)} d\mu.$$

Setting

$$\psi(\tau) = \rho\Big(\tau + \frac{\mu S + rT - q}{\delta T}\Big),$$

we have

$$\hat{\rho}(\delta kT) e^{ik(\mu S+rT-q)} = \hat{\psi}(\delta kT).$$

By the Poisson formula we obtain

$$\sum_{k=-\infty}^{\infty} \hat{\psi}(\delta kT) = 2\pi\delta^{-1}T^{-1}\sum_{k=-\infty}^{\infty} \psi\Big(\frac{2\pi k}{\delta T}\Big)$$

$$= \frac{2\pi}{\delta T}\sum_{k=-\infty}^{\infty} \rho\Big(\frac{2\pi k + \mu S + rT - q}{\delta T}\Big).$$

Then

$$\sum_{k=1}^{\infty} T^{-1}k^{-1}\hat{\rho}(\delta kT)\sin(kz) = -\frac{z}{2T} + \Big(\frac{\pi}{T}\Big)\sum_{k=-\infty}^{\infty} I^{\delta}_{z,k},$$

where

$$I^{\delta}_{z,k} = \int_{\frac{2\pi k}{\delta T}}^{(2\pi k+z)/\delta T} \rho(\eta)d\eta.$$

Taking a sequence $S_m \neq 0$ going to 0, we obtain the same result when $S = 0$.

Setting

$$I^{\delta}_{z}(\nu) = \sum_{k\in\mathbf{Z}} I^{\delta}_{z,k}(\nu),$$

we get

$$2(2\pi)^{-n}h^{1-n}\int_{\Pi_+} T(\nu)^{-1}\sum_{k=1}^{\infty} k^{-1}\hat{\rho}(\delta kT)\sin\Big(k(h^{-1}S + rT - q)\Big)d\nu$$

$$= (2\pi)^{-n}\lambda^{n-1}\int_{\Pi_+}\Big[2\pi I^{\delta}_{z(\lambda,\nu)}(\nu) - z(\lambda,\nu)\Big]T^{-1}(\nu)d\nu. \qquad (6.2)$$

For $\tau \in \mathbf{R}$ we define $-\pi < [\tau]_{2\pi} \leq \pi$ by the equality

$$\tau = [\tau]_{2\pi} + 2k\pi, k \in \mathbf{Z}.$$

Introduce the function $\operatorname{sgn}(z) = 1$ for $z \geq 0$ and $\operatorname{sgn}(z) = -1$ for $z < 0$. The following Lemma is a counterpart of Lemma 1 in [28].

LEMMA 6.2. – *For each* $\nu \in \Pi_+$, *each* $0 < \varepsilon \leq 1$ *and each* $0 < \delta \leq 1$ *we have*

$$-\varepsilon T(\nu) + \Big[\pi - z(\lambda, \nu) + \varepsilon T(\nu)\Big]_{2\pi} - C_0 \varepsilon^{-1} \delta T(\nu)$$

$$\leq 2\pi I^{\delta}_{z(\lambda,\nu)}(\nu) - z(\lambda, \nu) + 2\pi \operatorname{sgn}(z(\lambda, \nu)) \int_{|z|/\delta T}^{\infty} \rho(\eta) d\eta$$

$$\leq \varepsilon T(\nu) + \Big[\pi - z(\lambda, \nu) - \varepsilon T(\nu)\Big]_{2\pi} + C_0 \varepsilon^{-1} \delta T(\nu)$$

*with a constant* $C_0 > 0$ *independent of* $\varepsilon$, $\delta$ *and* $\nu \in \Pi_+$.

*Proof.* – To simplify the notations we drop the variables $\nu$ and $\lambda$ writing for example $T$ and $z$ instead of $T(\nu)$ and $z(\lambda, \nu)$. It is easy to see that the assertion holds for any $C_0 > 0$ if $z = 0$. Suppose that $z > 0$. Let

$$\theta(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Let us introduce

$$\theta_{z,k,\varepsilon} = \theta\Big(z + \varepsilon T - 2\pi k\Big), \quad k \in \mathbf{N}.$$

A simple argument shows that

$$2\pi \sum_{k=1}^{\infty} \theta_{z,k,\varepsilon} = \Big(z + \varepsilon T - \pi\Big) + [\pi - z - \varepsilon T]_{2\pi}.$$

Consequently, the proof is reduced to the following inequalities:

$$\sum_{k=1}^{\infty} \theta_{z,k,-\varepsilon} - C_0' \varepsilon^{-1} \delta T + 1/2$$

$$\leq I^{\delta}_z + \int_{z/\delta T}^{\infty} \rho(\eta) d\eta \leq \sum_{k=1}^{\infty} \theta_{z,k,\varepsilon} + C_0' \varepsilon^{-1} \delta T + 1/2.$$

Clearly, for each $\kappa > 0$ we have

$$\sum_{k=1}^{\infty} \int_{k\kappa}^{\infty} \rho(\eta) d\eta \leq C_1 \sum_{k=1}^{\infty} \frac{1}{(1 + k\kappa)^2} \leq \frac{C_2}{\kappa}.$$

Then

$$\sum_{k=1}^{\infty} I^{\delta}_{z,k} \leq \sum_{k=1}^{\infty} \int_{\frac{2\pi k}{\delta T}}^{\infty} \rho(\eta) d\eta \leq \frac{C_2}{2\pi} \delta T \leq C_3 \varepsilon^{-1} \delta T$$

with $C_3 > 0$ independent of $\varepsilon$, $z$, $\delta$ and $T$.

On the other hand,

$$I^{\delta}_{z,0} - 1/2 = - \int_{z/\delta T}^{\infty} \rho(\eta) d\eta$$

and it remains to treat the terms $I^{\delta}_{z,k}$ with $k \leq -1$.

Assume that

$$z + \varepsilon T + 2\pi k \geq 0, \quad k = -1, \dots, -m,$$

$$z + \varepsilon T + 2\pi k < 0, \quad k = -(m+1), \dots$$

Then we obtain

$$2\pi k + z < 2\pi \Big( k + (m+1) \Big) - \varepsilon T < 2\pi \Big( k + (m+2) - 1 \Big)$$

for all $k \leq -(m+2)$. This implies

$$\sum_{k=-(m+2)}^{-\infty} I^{\delta}_{z,k} \leq \sum_{k=-(m+2)}^{-\infty} \int_{-\infty}^{(2\pi k + z)/\delta T} \rho(\eta) d\eta$$

$$< \sum_{k=-1}^{-\infty} \int_{-\infty}^{\frac{2\pi}{\delta T} k} \rho(\eta) d\eta \leq C_4 \varepsilon^{-1} \delta T$$

with a constant $C_4 > 0$ independent of $\varepsilon$, $\delta$, $z$ and $T$.

To deal with the term with $k = -(m+1)$, notice that

$$-2\pi(m+1) + z < -\varepsilon T,$$

hence

$$I^{\delta}_{z,-(m+1)} \leq \int_{-\infty}^{-\frac{\varepsilon}{\delta}} \rho(\eta) d\eta \leq C_5 \int_{-\infty}^{-\frac{\varepsilon}{\delta}} \frac{1}{(1-\eta)^2} d\eta \leq C_6 \varepsilon^{-1} \delta T$$

with $C_6 > 0$ independent of $\varepsilon$, $\delta$, $z$ and $T$.

Finally,

$$\sum_{k=-1}^{-m} I^{\delta}_{z,k} \leq m \int_{-\infty}^{\infty} \rho(\eta) d\eta = m = \sum_{k=1}^{\infty} \theta_{z,k,\varepsilon}.$$

Taking the sum of the above estimates we obtain the right-hand equality in Lemma 6.2. For the left-hand one we follow a similar argument. Assume that

$$z - \varepsilon T + 2\pi k \geq 0, \quad k = -1, \dots, -m,$$
$$z - \varepsilon T + 2\pi k < 0, \quad k = -(m+1), \dots$$

Then we have

$$\sum_{k=-(m+1)}^{-\infty} I^{\delta}_{z,k} \geq - \sum_{k=-1}^{-\infty} \int_{-\infty}^{2\pi k/\delta T} \rho(\eta) d\eta \geq -C_7 \varepsilon^{-1} \delta T.$$

On the other hand, for $k = -1, \dots, -m$ we have

$$z + 2\pi k \geq \varepsilon T + 2\pi(m + k).$$

This yields

$$\sum_{k=-1}^{-m} I^{\delta}_{z,k} - m \geq \sum_{k=-1}^{-m} \int_{\frac{2\pi k}{\delta T}}^{(\varepsilon T + 2\pi(m+k))/\delta T} \rho(\eta) d\eta - m$$

$$\geq - \sum_{k=1}^{\infty} \int_{-\infty}^{-\frac{2\pi k}{\delta T}} \rho(\eta) d\eta - \sum_{k=0}^{\infty} \int_{\frac{\varepsilon}{\delta} + \frac{2\pi k}{\delta T}}^{\infty} \rho(\eta) d\eta \geq -C_8 \varepsilon^{-1} \delta T$$

with $C_8 > 0$ independent on $\varepsilon$, $\delta$, $z$ and $T$. This proves the Lemma for $z > 0$.

Suppose that $z < 0$. Then, $I^{\delta}_{z} = -I^{\delta}_{-z}$, and using the equality $[z]_{2\pi} = -[2\pi - z]_{2\pi}$ for $[z]_{2\pi} \neq \pi$ as well as the assertion for $-z > 0$, we complete the proof. ♠

Going back to the proof of the Proposition 6.1, it remains to estimate

$$M(h) = \int_{\Pi_+} \int_{|z|/\delta T}^{\infty} \rho(\eta) d\eta d\nu, \tag{6.3}$$

where $z(\lambda, \nu) = \lambda S(\nu) + rT(\nu) - q(\nu)$, $|r| \leq r_0$, and $\lambda = h^{-1}$. Using (6.1), we conclude that

$$z(h^{-1}, \nu) \geq \omega_1 > 0, \quad \forall \nu \in \Pi_+, \ 0 < h \leq h_0,$$

and

$$M(h) \leq \mu(\Sigma) \int_{\omega_1/\delta}^{\infty} \varrho(\eta) d\eta \leq C_1 \delta, \tag{6.4}$$

$\mu(\Sigma)$ being the Liouville measure of $\Sigma$. Using (6.4) and Lemma 6.2 we prove the proposition. ♠

## 7. PROOF OF THE MAIN RESULTS

In this section we prove Theorem 1.1 and Corollaries 1.2 and 1.3.

*Proof of Theorem.* 1.1. – Fix $r_0 > 0$ and $c_0$ and take $|r| \leq r_0$ and $0 < c \leq c_0$. According to (2.2) we have

$$N_{E+rh,c}(h) = \sigma_h(E + rh + ch) - \sigma_h(E + rh - ch).$$

Applying Theorem 2.1, we obtain for each $\varepsilon$, $0 < \varepsilon \leq 1$, and each $\delta$, $0 < \delta \leq 1$, the estimate

$$\Big(\sigma_h * \rho_{\delta h}\Big)\Big(E + h(r + c - \varepsilon/2)\Big) - \Big(\sigma_h * \rho_{\delta h}\Big)\Big(E + h(r - c + \varepsilon/2)\Big)$$

$$-C_0\varepsilon^{-1}\delta h^{1-n} - \mathcal{O}(h^{2-n}) \leq N_{E+rh,c}(h) \leq C_0\varepsilon^{-1}\delta h^{1-n} + \mathcal{O}(h^{2-n})$$

$$+\Big(\sigma_h * \rho_{\delta h}\Big)\Big(E + h(r + c + \varepsilon/2)\Big) - \Big(\sigma_h * \rho_{\delta h}\Big)\Big(E + h(r - c - \varepsilon/2)\Big),$$

where $C_0 > 0$ is independent of $\varepsilon, \delta$ and $h$. Next applying Corollary 5.3 and Theorem 6.1 with $\varepsilon/2$, we get

$$(\sigma_h * \rho_{\delta h})\Big(E + h(r + c + \epsilon/2)\Big) - (\sigma_h * \rho_{\delta h})\Big(E + h(r - c - \epsilon/2)\Big)$$

$$= \frac{2c}{(2\pi)^n}\mu(\Sigma)h^{1-n} + h^{1-n}\Big[Q(h, r + c + \epsilon) - Q(h, r - c - \epsilon)\Big]$$

$$+C_1\epsilon h^{1-n} + C_2\epsilon^{-1}\delta h^{1-n} + o_\delta(h^{1-n})$$

with some constants $C_1 > 0, C_2 > 0$ independent on $\epsilon, \delta, h$. For fixed $\varepsilon$ we take $\delta = \varepsilon^2$ and this completes the proof of the right-hand inequality in Theorem 1.1. For the left-hand one we use a similar argument. ♠

*Proof of Corollary* 1.2. – Suppose that the function $Q(h, r)$ is uniformly continuous with respect to $r$ in the interval $[r_1, r_2]$ for any $h \in (0, h_0]$. Let $r_1 < R_1 < R_2 < r_2$. Fix $c_0 > 0$ such that $r_1 + 2c_0 \leq R_1$ and $R_2 + 2c_0 \leq r_2$. Then fix $0 < c \leq c_0$ and take $\delta > 0$ arbitrary. We are going to find $h_1 > 0$ such that the inequality

$$\left|h^{n-1} N_{E+rh,c}(h) - \frac{2c}{(2\pi)^n}\mu(\Sigma) - \Big(Q(h, r + c) - Q(h, r - c)\Big)\right| \leq \delta$$

holds for any $0 < h \leq h_1$ and any $r \in [R_1, R_2]$. First using the uniform continuity of $Q$ with respect to $r$ we find $0 < \varepsilon < \frac{\delta}{4C_0}$, where $C_0$ is the constant of Theorem 1.1, so that

$$\Big|Q(h, r + c) - Q(h, r + c \pm \varepsilon)\Big| < \delta/4, \quad \Big|Q(h, r - c) - Q(h, r - c \pm \varepsilon)\Big| < \delta/4$$

for any $r \in [R_1, R_2]$. Then with $\varepsilon$ fixed as above we find $h_1$ such that

$$\left|o_\varepsilon(h^{1-n})\right| \leq \frac{\delta}{4}h^{1-n}, \ \forall h \in (0, h_1].$$

Applying the inequalities of Theorem 1.1, we obtain the assertion of Corollary 1.2. ♠

*Proof of Corollary* 1.3. – We follow the same arguments as above. Suppose that there exists a subset $\Pi^1 \subset \Pi$ of a positive Lebesgue measure in $\Sigma$ and an integer $p$ such that the quantity (1.10) does not depend on $\nu \in \Pi^1$. Now we choose $\varepsilon = c/2$ and take $r = r(h)$, $0 < h \leq h_1$, in Theorem 1.1. We fix $0 < \delta < \int_{\Pi_1} \frac{d\nu}{T(\nu)}$ and find some $T > 0$ such that the Liouville measure of the set

$$\{\nu \in \Pi : \ T(\nu) \geq T\}$$

is less than $\frac{\delta T_0}{4}$, where $T_0 = \min_{\nu\in\Pi} T(\nu)$. This is possible since

$$\mu(\Pi) = \sum_{m=0}^{\infty} \mu(\Pi_m), \quad \Pi_m = \{\nu \in \Pi : m \leq T(\nu) < m+1\}.$$

We can assume that $T(\nu) < T, \ \ \forall \nu \in \Pi_1$. Fix $c$ so that $0 < c \leq c_1$, where

$$c_1 = \min\Big(\frac{\pi\delta}{4\mu(\Sigma)}, \ \frac{\pi}{4T}, \ \frac{\delta}{4C_0}\Big),$$

and set $U = \{\nu \in \Pi : \ T(\nu) < T\}$. Next for any $h \in (0, h_1]$ we divide $U$ into two components $U_1^h$ and $U_2^h$ as follows. We say that $\nu \in U_1^h$ if there exists an integer $k$ such that

$$2\pi k \leq -h^{-1}S(\nu) + q(\nu) - (r(h) \pm c/2)T(\nu) < 2\pi(k+1),$$

and we define $U_2^h$ as the complement to $U_1^h$ in $U$. We have

$$-h^{-1}S(\nu) + q(\nu) - r(h)T(\nu) \equiv 0 \ (\mathrm{mod}\, 2\pi), \ \forall \nu \in \Pi^1, \ h \in (0, h_1].$$

hence,

$$\Pi^1 \subset U_2^h, \ h \in (0, h_1]. \tag{7.1}$$

Consider the function

$$G(h, \nu) = \left[\pi - h^{-1}S(\nu) + q(\nu) - (r(h) + c/2)T(\nu)\right]_{2\pi}$$

$$-\left[\pi - h^{-1}S(\nu) + q(\nu) - (r(h) - c/2)T(\nu)\right]_{2\pi}.$$

Since $Tc_0 \leq \pi/4$, we have for any $h \in (0, h_1]$ the equalities

$$|G(h.\nu)| = cT(\nu) \leq cT, \ \nu \in U_1^h,$$

$$G(h,\nu) = 2\pi - cT(\nu) \geq 2\pi - cT, \nu \in U_2^h.$$

Then, using (7.1), we obtain

$$Q(h, r(h) + c/2) - Q(h, r(h) - c/2)$$

$$\geq (2\pi)^{1-n}\Big(\int_{U_2^h} \frac{d\nu}{T(\nu)} - T_0^{-1}\mu(\Pi \setminus U) - \frac{c}{\pi}\mu(U)\Big)$$

$$\geq (2\pi)^{1-n}\Big(\int_{\Pi_1} \frac{d\nu}{T(\nu)} - \delta/2\Big)$$

for any $h \in (0, h_1]$. Taking $h_1(c)$ eventually smaller we can arrange the inequality

$$\left|o_{c/2}(h^{1-n})\right| \leq \frac{\delta}{4}h^{1-n}.$$

Thus applying the left-side inequality of Theorem 1.1 we obtain

$$\lim_{h\searrow 0} \inf \left(h^{1-n}N_{E+r(h)h,c}(h)\right) \geq (2\pi)^{1-n}\Big(\int_{\Pi_1} \frac{d\nu}{T(\nu)} - \delta\Big)$$

for all $c \in (0, c_1]$. Notice that for fixed $h > 0$ the function $N_{E+r(h)h,c}(h)$ is increasing with respect to $c$ so we have the same result for all $c > 0$. Since $\delta$ can be taken arbitrary small, we complete the proof of the assertion. ♠

## 8. ENERGY SURFACES OF CONTACT TYPE

The pair $(\Sigma, \sigma)$ of a smooth compact manifold $\Sigma$ of dimension $2n - 1$, $n \geq 2$, and one-form $\sigma$ on it is called contact manifold if $\sigma$ is a contact form, that is, the exterior product

$$\sigma \wedge (d\sigma)^{n-1}$$

is a volume form on $\Sigma$. The corresponding Reeb vector field $\Xi$, called as well contact vector field (cf. [10]), is determined uniquely by the inner product

$$\imath(\Xi)\sigma = 1, \ \imath(\Xi)d\sigma = 0. \tag{8.1}$$

Let $E < \lambda_0$ be a regular value of $a_0$ and let $\Sigma$ be the corresponding compact energy surface. The pull-back $j^*\omega$ of the canonical symplectic two-form $\omega$ of $T^*(\mathbf{R}^n)$ via the inclusion map $j : \Sigma \to T^*(\mathbf{R}^n)$ is a degenerated two-form on $\Sigma$. Following A. Weinstein [32], we call $\Sigma$ a hypersurface of contact type, if there exists a contact one-form $\sigma$ on it such that $d\sigma = j^*\omega$. Since the kernel of $j^*\omega$ is spanned by the restriction of the Hamiltonian vector field $H_{a_0}$ on $\Sigma$, the hypersurface $\Sigma$ is of contact type, if and only if there exists one-form $\sigma$ on $\Sigma$ such that $d\sigma = j^*\omega$ and

$$b(\varrho) = \imath(H_{a_0})\sigma(\varrho) \neq 0 \ , \ \forall \ \varrho \in \Sigma. \tag{8.2}$$

If (8.2) holds, the restriction of $H_{a_0}$ at $\Sigma$ and the contact vector field $\Xi$ are related by

$$H_{a_0}(\nu) = b(\nu)\Xi(\nu), \ \ \nu \in \Sigma. \tag{8.3}$$

Following [1] (*see* also [17]), we say that $a_0(x,\xi)$ is *stricly $\xi$-convex* in $U \in \mathbf{R}^{2n}$, if

$$\left\langle \frac{\partial a_0}{\partial \xi}(x,\xi), \xi \right\rangle > 0, \ \ \forall (x,\xi) \in U, \ \xi \neq 0.$$

If $a_0(x,\xi)$ is strictly $\xi$-convex in a neighborhood $U$ of $\Sigma$, the hypersurface $\Sigma$ is of contact type, and (8.2) holds taking $\sigma = \alpha\xi dx + df$ with suitable $\alpha > 0$ and $f \in C^\infty(\mathbf{R}^{2n})$ (*see* [1], Example 1).

We are going to show that condition (1.9) is satisfied if $\Sigma$ is of contact type. Let $\nu \in \Pi$ and let $\gamma(\nu)$ be the primitive periodic trajectory of $H_{a_0}$ associated to $\nu$ and having period $T(\nu)$. According to (8.2), $\gamma(\nu)$ is as well a primitive periodic trajectory of $\Xi$ of certain period $\tau(\nu)$, and

$$T(\nu) = \int_0^{\tau(\nu)} \frac{1}{b(\exp(t\Xi)(\nu))} dt.$$

On the other hand, the one-form $\sigma - j^*(\xi dx)$ is exact on $\Sigma$, and (8.2) implies

$$\tau(\nu) = \int_{\gamma(\nu)} \sigma = \int_{\gamma(\nu)} \xi dx = S(\nu).$$

Hence,

$$T(\nu) = \int_0^{S(\nu)} \frac{1}{b(\exp(t\Xi)(\nu))} dt,$$

and there exist positive constants $C_1$ and $C_2$, such that

$$C_1 T(\nu) \leq S(\nu) \leq C_2 T(\nu). \ \forall \nu \in \Pi.$$

Since the infinimum of $T(\nu)$ on $\Pi$ is strictly positive, condition (1.9) is satisfied taking $h_0$ sufficiently small.

The following result has been proved in [24].

THEOREM 8.1. – *Let $E < \lambda_0$ be a regular value of the symbol $a_0(x, \xi)$. Suppose that $\Sigma$ is connected and of contact type. Assume that the symbol $a_o(x. \xi)$ is analytic in a neighborhood of $\Sigma$. Then either $\mu(\Pi) = 0$ or there exists an analytic function $\tilde{T}(\nu)$ in $\Sigma$ such that*

$$\exp\Big(\tilde{T}(\nu)H_{a_0}\Big)(\nu) = \nu, \ \forall \nu \in \Sigma. \tag{8.4}$$

The function $\tilde{T}(\nu)$ is not uniquely determined but using the analyticity of $\tilde{T}(\nu)$ we normalize it so that $\tilde{T}(\nu) = T(\nu)$ for almost any $\nu \in \Sigma$. Indeed, there exists $M_0 > 0$, and for any $\nu \in \Sigma$ there is a positive integer $1 \leq m(\nu) \leq M_0$, such that $\tilde{T}(\nu) = m(\nu)T(\nu)$. Denote by $M$ the largest integer $1 \leq M \leq M_0$ such that the equality $m(\nu) = M$ holds on a set of a positive Lebesgue measure and set $T_0(\nu) = \frac{1}{M}\tilde{T}(\nu)$. By analyticity, $\exp\Big(T_0(\nu)H_{a_0}\Big)(\nu) = \nu, \forall \nu \in \Sigma$, and it is easy to see that $T_0(\nu) = T(\nu)$ for almost any $\nu$.

Now we turn to Theorem 1.1 and Corollaries 1.2 and 1.3 which can be applied to $h$-admissible operator $A(h)$ since (1.9) holds. If $\mu(\Pi) = 0$, we have $Q = 0$ which yields (1.3). Suppose that (8.4) is satisfied. Then there exist constants $S$ and $q$ such that the equalities

$$S(\nu) = S, q(\nu) = q \tag{8.5}$$

hold for almost any $\nu \in \Sigma$. To prove it we take any $\nu_1$ and $\nu_2$ such that $\tilde{T}(\nu_j) = T(\nu_j)$, $j = 1, 2$. Then connecting these two points with a path $\alpha$ and using Stokes formula we obtain

$$S(\nu_1) - S(\nu_2) = \int_{\gamma(\nu_1)} \xi dx - \int_{\gamma(\nu_2)} \xi dx = \int_M \omega = 0,$$

since the manifold

$$M = \{\exp\Big(tH_{a_0}\Big)(\nu) : \ 0 \leq t \leq \tilde{T}(\nu), \ \nu \in \alpha\}$$

is isotopic. Hence, $S(\nu) = S(\nu_1)$ is constant almost everywhere. Moreover, $q(\nu) = q(\nu_1)$ is constant almost everywhere, too. As one can see from the proof, (8.5) holds for any connected energy surface of contact type as long as (8.4) is satisfied.

Now we have two possibilities. First suppose that $\tilde{T}(\nu) = T$ is a constant on $\Sigma$. Then choosing

$$r(h) = ([q - h^{-1}S]_{2\pi} + 2\pi p)T^{-1},$$

for some integer $p$, we conclude that Corollary 1.3 holds and there is a clustering near $E$.

Now suppose that $\tilde{T}(\nu)$ is not identically constant. Then there is "weak" clustering at $E$ in the following sense

$$N_{E,c}(h_k) \sim (2\pi h_k)^{1-n}\Big(\int_\Sigma \frac{d\nu}{T(\nu)} + o(c)\Big),\ h_k = \frac{S}{q + 2\pi k},\ k \in \mathbf{N}. \tag{8.6}$$

Indeed, applying (1.8) to $h = h_k$ and $r = 0$, and using Theorem 1.1 as in the proof of Corollary 1.2 we obtain (8.6). On the other hand, we have the following

PROPOSITION 8.2. – *Assume the conditions of Theorem 8.1 fulfilled and suppose that $\tilde{T}(\nu)$ is different from a constant. Then for any $r_1 < r_2$ such that $0 \notin [r_1, r_2]$ the function $Q(h, r)$ is uniformly continuous with respect to $r \in [r_1, r_2]$ for any $h \in (0, h_0]$.*

*Proof.* – Fix $\varepsilon > 0$. Since the function $\tilde{T}$ is analytic and different from a constant, there exists $\delta > 0$ such that the Liouville measure of the set

$$V_0 = \{\nu \in \Sigma : \ |d\tilde{T}(\nu)| < 2\delta\}$$

is less that $\varepsilon$. Moreover, since $\tilde{T}(\nu) = T(\nu)$ for almost any $\nu \in \Sigma$, we can suppose that $\tilde{T}(\nu) = T(\nu)$ outside $V_0$. Next, we take a finite covering $\{V_1, \dots, V_k\}$ of the complement to $V_0$ in $\Sigma$ such that

$$|dT(\nu)| \geq \delta,\ \nu \in V_j,\ j = 1, \dots, k.$$

Let $\kappa_j$, $j = 0, 1, \dots, k$, be a partition of the unity in $\Sigma$ subordinated to the covering $V_j$, $j = 0, \dots, k$. Fix $j \neq 0$ and choose smooth local coordinates $z$ in $V_j$ such that $z_1(\nu) = T(\nu)$ in $V_j$. Consider the function

$$Q_j(h, r) = (2\pi)^{-n} \int_\Sigma \Big[\pi - h^{-1}S + q - rT(\nu)\Big]_{2\pi} T(\nu)^{-1}\kappa_j(\nu)d\nu$$

$$= \int_{\mathbf{R}} \Big[\pi - h^{-1}S + q - rz_1\Big]_{2\pi} J_j(z_1)dz_1.$$

where $J_j(z_1)$ is a smooth function with compact support. Setting $y = rz_1$, $r \in [r_1, r_2]$, we obtain

$$Q_j(h, r) = \int_{\mathbf{R}} [\pi - h^{-1}S + q - y]_{2\pi} J_j\Big(\frac{y}{r}\Big)\frac{dy}{r},$$

which is uniformly continuous with respect to $r \in [r_1, r_2]$ for $h \in (0, h_0]$ since $0 \notin [r_1, r_2]$. Then there exists $\delta_1 > 0$ such that

$$|Q_j(h, r) - Q_j(h, r')| \leq \varepsilon(2k)^{-1}, \; j = 1, \dots, k$$

for any $r, \; r' \in [r_1, r_2], \; |r - r'| \leq \delta_1$ uniformly with respect to $h \in (0, h_0]$. Summing up we get

$$|Q(h, r) - Q(h, r')| \leq \varepsilon/2 + \frac{(2\pi)^{1-n}\epsilon}{T_0},$$

where $T_0 = \inf T(\nu)$ is introduced in the previous section. This proves the assertion. ♠

Now we can apply Corollary 1.2 for $r \in [r_1, r_2]$ and we have semi-classical asymptotics for $N_{E+rh,c}(h)$ for any $r$ in that interval. Moreover, taking into account the identity

$$Q(S(h^{-1}S + 2\pi k)^{-1}, r) = Q(h, r), \; k \in \mathbf{Z},$$

we observe that clustering in the sense of (1.6) is not possible.

## 9. APPLICATIONS TO THE SCHRÖDINGER OPERATOR AND EXAMPLES

Consider the Schrödinger operator $A(h) = -h^2\Delta + V(x)$, where $V(x) \geq \gamma_0$ is a smooth real-valued potential. Then $a_0(x, \xi) = |\xi|^2 + V(x)$ is obviously strictly $\xi$-convex, hence, $\Sigma$ is of contact type for any regular value

$$E < \lambda_0 = \lim_{|x|\to\infty} \inf V(x)$$

of $a_0(x, \xi)$. In this case we can drop the conditions $(H_1)$ and $(H_2)$. Indeed, as in Section 5 in [14], we can replace $V(x)$ by a smooth real-valued potential $\tilde{V}(x)$ without changing the asymptotics of the counting function $N_{E+rh,c}(h)$, where $\tilde{V}(x) = V(x)$ on $V^{-1}((-\infty, \lambda_1))$ and $\tilde{V} = \text{const} > \lambda_1$

on $V^{-1}([\lambda_2, \infty))$ for some $E < \lambda_1 < \lambda_2 < \lambda_0$. Applying Theorem 1.1 to the $h$-admissible operator $\tilde{A}(h) = -h^2\Delta + \tilde{V}(x)$ we obtain the following

THEOREM 9.1. – *Let $A(h)$ be the Schrödinger operator $-h^2\Delta + V(x)$ having a smooth real-valued potential $V \geq \gamma_0$, and let $E < \lambda_0$ be a regular value of its symbol $a_0(x,\xi)$. Then the conclusions of Theorem 1.1 and Corollaries 1.2 and 1.3 are valid.*

We are going to consider examples of potentials for which the set $\Pi$ does not coincide with the energy surface $\Sigma$ but nevertheless clustering takes place near $E$.

*Example* 9.2. – Fix a positive number $E$ and $\alpha \notin \mathbf{Q}$ such that $\sqrt{2}/2 < \alpha < 1$. Consider the spherically symmetric potential

$$V(x) = \phi(|x|^2)|x|^2, \ x = (x_1, \dots, x_n) \in \mathbf{R}^n, \ n \geq 2,$$

where the function $\phi \in C^\infty(\mathbf{R})$ satisfies $0 < \alpha^2 \leq \phi \leq 1$, and $\phi(t) = 1$ for $t \leq E^2/2 + a$, $\phi(t) = \alpha^2$ for $t \geq E^2/(2\alpha^2) - a > E^2/2 + a$, $a > 0$ being fixed sufficiently small. Set

$$A(h) = -h^2\Delta + V(x).$$

We are going to investigate the semi-classical asymptotics of the function $N_{E+rh,c}(h)$.

Set $a_0(x,\xi) = \xi^2 + V(x)$, and consider the flow of $H_{a_0}$ at the non-degenerated energy surface

$$\Sigma = \{(x,\xi) : a_0(x,\xi) = E^2\}.$$

Set

$$e_1 = (1, 0, \dots, 0), \ e_2 = (0, 1, 0, \dots, 0) \in \mathbf{R}^n,$$

fix $\delta > 0$, and consider the following sets of a positive Lebesgue measure

$$\Pi^1 = \left\{(x,\xi) \in \Sigma : \left|x - \frac{E}{\sqrt{2}}e_1\right| + \left|\xi - \frac{E}{\sqrt{2}}e_2\right| \leq \delta\right\},$$

$$\Pi^2 = \left\{(x,\xi) \in \Sigma : \left|x - \frac{E}{\sqrt{2}\alpha}e_1\right| + \left|\xi - \frac{E}{\sqrt{2}}e_2\right| \leq \delta\right\}.$$

LEMMA 9.3. – *There exists $\delta > 0$ such that $\Pi^j \subset \Pi$, $j = 1, 2$. Moreover, we have $T(\nu) = \pi$, $S(\nu) = \pi E^2$ for $\nu \in \Pi^1$ and $T(\nu) = \pi/\alpha$, $S(\nu) = \pi E^2/\alpha$, for $\nu \in \Pi^2$.*

*Proof.* – We choose $\delta > 0$ such that $4\delta(\sqrt{2}E + \delta) < a$. Let $\nu = (x^0, \xi^0) \in \Pi^1$ and

$$\exp(tH_{a_0})(\nu) = (x(t), \xi(t)).$$

We are going to show that the inequality

$$\left| |x(t)|^2 - \frac{E^2}{2} \right| < \frac{a}{2} \tag{9.1}$$

holds for any $t \in \mathbf{R}$. First notice that

$$\left| |x(0)|^2 - \frac{E^2}{2} \right| = \left| \langle x(0) - \frac{E}{\sqrt{2}} e_1, x(0) + \frac{E}{\sqrt{2}} e_1 \rangle \right| \leq \delta(\sqrt{2}E + \delta).$$

In the same way we get

$$\left| |\xi(0)|^2 - \frac{E^2}{2} \right| \leq \delta(\sqrt{2}E + \delta).$$

On the other hand, we obtain

$$\left| \langle x(0), \xi(0) \rangle \right| \leq \left| \langle x(0) - \frac{E}{\sqrt{2}} e_1, \xi(0) - \frac{E}{\sqrt{2}} e_2 \rangle \right|$$

$$+ \frac{E}{\sqrt{2}} |\langle e_1, \xi(0) \rangle| + \frac{E}{\sqrt{2}} |\langle e_2, x(0) \rangle| \leq \delta(\sqrt{2}E + \delta).$$

The above argument shows that inequality (9.1) holds for any $t$ in an interval $(-t_0, t_0)$, $t_0 > 0$. Let $(-T, T)$ be the maximal interval of that form such that (9.1) is valid for any $t \in (-T, T)$. Suppose that $T < +\infty$. Then $V(x) = |x|^2$ in a neighborhood of the curve $\{x(t) : t \in [-T, T]\}$, and we obtain

$$x(t) = \sin(2t)\xi(0) + \cos(2t)x(0),$$

$$\xi(t) = \cos(2t)\xi(0) - \sin(2t)x(0),$$

for any $t \in [-T, T]$. Hence,

$$\left| |x(t)|^2 - \frac{E^2}{2} \right| \leq \sin^2(2t) \left| |\xi(0)|^2 - \frac{E^2}{2} \right| + \cos^2(2t) \left| |x(0)|^2 - \frac{E^2}{2} \right|$$

$$+ |\sin(4t)| \left| \langle x(0), \xi(0) \rangle \right| \leq 2\delta(\sqrt{2}E + \delta) < \frac{a}{2}$$

for any $t \in [-T, T]$. Therefore, choosing $\epsilon > 0$ sufficiently small, (9.1) holds for any $t \in [-T - \epsilon, T + \epsilon]$, and we conclude that $T = +\infty$.

This proves the inclusion $\Pi^1 \subset \Pi$ and using the explicit formula for $\exp(tH_{a_0})(\nu)$, $\nu \in \Pi^1$, we compute the corresponding periods and the action. Applying the same argument to the set $\Pi^2$, we prove Lemma 9.3. ♠

Now we have two open subsets $\Pi^j$, $j = 1, 2$, of $\Pi$ with rationally independent periods. For any fixed $p \in \mathbf{Z}$ the quantization condition (1.10) reads

$$r_1(h) = \Big([q - h^{-1}\pi E^2]_{2\pi} + 2\pi p\Big)\pi^{-1} \text{ on } \Pi^1,$$

$$r_2(h) = \Big([q - h^{-1}\pi E^2/\alpha]_{2\pi} + 2\pi p\Big)\alpha\pi^{-1} \text{ on } \Pi^2,$$

and applying Corollary 1.3 we get clustering near $E$.

*Example* 9.4. – Let $E, \alpha \notin \mathbf{Q}$, and let $\phi$ be as in Example 9.2. Consider the operator $A(h) = -h^2\Delta + V(x)$, where

$$V(x) = \phi(|x'|^2)|x'|^2 + |x''|^2,$$

$$x = (x', x''),\ x' = (x_1, x_2),\ x'' = (x_3, ..., x_n),\ n \geq 3.$$

Let us define $\Pi^1$ and $\Pi^2$ as in Example 9.2, and set $\Sigma = \{(x, \xi) : a_0(x, \xi) = E^2\}$. As in the previous example we prove that $\Pi^1$ consists only of periodic points of $H_{a_0}$ and

$$\mu(\Pi) \geq \mu(\Pi^1) > 0.$$

In the same way we show that $V(x)$ coinsides with the potential $\alpha^2|x'|^2 + |x''|^2$ in a neighborhood of any integral curve

$$\exp(tH_{a_0})(x, \xi),\quad (x, \xi) \in \Pi^2, t \in \mathbf{R}.$$

As $\alpha \notin \mathbf{Q}$, the Liouville measure of the periodic points of $H_{a_0}$ in $\Pi^2$ is zero, hence

$$\mu(\Pi) \leq \mu(\Sigma \setminus \Pi^2) < \mu(\Sigma).$$

Nevertheless, applying Corollary 1.3, we get clustering near $E$ as in Example 9.2.

We are going to give an example of a $h$-admissible operator for which Proposition 8.2 and Corollary 1.2 hold.

*Example* 9.5. – Fix $E > 0$, and denote by $g(x)$ a smooth function in $\mathbf{R}^3$ such that $g(x) \geq 1$, $\forall x \in \mathbf{R}^3, g(x) = (1 + 2x_1^2 + x_2^2 + x_3^2)^{-1}$ for $|x| \leq 2E^2$, and $g(x) = 1$ for $|x| \geq 3E^2$. Consider the $h$-admissible operator $A(h)$

associated to the Hamiltonian $a_0(x, \xi) = g(x)(|\xi|^2 + |x|^2 - E^2)$. Consider the zero energy level of $a_0$

$$\Sigma = \{a_0 = 0\} = \{|\xi|^2 + |x|^2 = E^2\}.$$

Then $\Sigma = \Pi$ consists only of periodic points of $H_{a_0}$, and it is easy to see that for any $\nu = (x, \xi) \in \Sigma$ the corresponding action is $S(\nu) = \pi E^2$, while the primitive period is

$$T(x, \xi) = \int_0^{\pi} \frac{ds}{g(\cos(2s)\, x + \sin(2s)\, \xi)}.$$

Set $p_j = (x_j^2 + \xi_j^2)^{1/2}$, $j = 1, 2, 3$, and for $p_j > 0$ define $0 \leq \theta_j \leq \pi$ by $\cos(\theta_j) = x_j/p_j$. Then we have

$$T(x, \xi) = \int_0^{\pi} (1 + 2p_1^2 \cos^2(2s - \theta_1) + p_2^2 \cos^2(2s - \theta_2) + p_3^2 \cos^2(2s - \theta_3))ds$$

$$= \pi(1 + (E^2 + x_1^2 + \xi_1^2)/2).$$

Therefore, the period $T(x, \xi)$ is analytic and different from a constant in $\Sigma$. Hence, there is a "weak" clustering for the eigenvalues of $A(h)$ at the zero in the sense of (8.6). On the other hand, applying Proposition 8.2 and Corollary 1.2, we obtain asymptotics of $N_{hr,c}(h)$ in any interval $r \in [r_1, r_2]$, $0 < r_1 < r_2$.

## APPENDIX

**A.1.** For any measurable set $F$ of $\mathbf{R}^d$, $d \geq 1$, denote by $F_+$ the set of points of $F$ of a positive Lebesgue density in $F$ with respect to the Lebesgue measure $\sigma$ in $\Sigma$. By definition, $\nu \in F_\nu$ if $\nu \in F$ and $\sigma(U \cap F) > 0$ for any neighborhood $U$ of $\nu$. Obviously, the Lebesgue measure of $F \setminus F_+$ is zero and $(F_+)_+ = F_+$.

Let $W^0 \subset W$ be open neighborhoods of some $\nu^0 \in \mathbf{R}^d$, and let $P : W^0 \longrightarrow W$ be a smooth map. Denote by $\mathrm{Fix}(P)$ the set of the fixed points of $P$, and by $\mathrm{Fix}(P)^a$ the set of all $\nu \in W^0$ such that the map $P(z) - z$ is flat at $z = \nu$.

LEMMA A.1. – *We have*

$$\mathrm{Fix}(P)_+ \subset \mathrm{Fix}(P)^a.$$

*In particular,*

$$\sigma(\mathrm{Fix}(P)_+) = \sigma(\mathrm{Fix}(P)^a).$$

The proof of the assertion follows immediately from the following.

LEMMA A.2. – *Let $F \in W^0$ be a subset with a positive Lebesgue measure and let $f$ be a smooth function in $W^0$ such that $f(z) = 0$ for each $z \in F$. Then, $f$ is flat at $F_+$, that is*

$$\partial_z^\alpha f(z) = 0 \ \ \forall z \in F_+,\ \forall \alpha.$$

Lemma A.2 is proved in [24] (see Lemma 2.1).

**A.2.** Let $\gamma$ be a periodic (eventually multiple) trajectory of $H_{a_0}$ in $\Sigma$, and let

$$P : Y^0 \to Y,$$

be the corresponding Poincaré map. Here, $Y \subset \Sigma$ is a transversal section to $\gamma$ at $\nu^0 \in \gamma$, $Y^0$ is a neighborhood of $\nu^0$ in $Y$, and

$$P(\nu) = \Phi^{t(\nu)}(\nu) \in Y,\ \nu \in Y^0,$$

$t(\nu)$ being the return time function of $P$. Denote by

$$\imath : \Sigma \to T^*(\mathbf{R}^n),\quad \imath_0 : Y^0 \to \Sigma,$$

the corresponding inclusion mappings, and set

$$\sigma = \imath^*(\xi dx),\quad \sigma^0 = \imath_0^*(\sigma),$$

$\xi dx$ being the canonical one-form in $T^*(\mathbf{R}^n)$. As in Section 4 we set

$$G(\nu) = \int_{\Gamma(\nu)} \sigma,\quad \nu \in Y^0,$$

where $\Gamma(\nu) = \{\Phi^t(\nu) :\ 0 \le s \le t(\nu)\}$. We shall prove the following analog of the Poincaré-Cartan identity given in [10].

LEMMA A.3. – *We have*

$$P^*(\sigma^0) - \sigma^0 = dG.$$

*Proof.* – For each $s \in [0.1]$ define the inclusion map $f_s : Y^0 \to \Sigma$ by

$$f_s(\nu) = \exp(st(\nu)H_{a_0})(\nu), \quad \nu \in Y^0.$$

Set

$$\Sigma_0 = \{f_s(\nu) : \ 0 \leq s < 1, \ \nu \in Y^0\},$$

and define the vector field $\Xi$ on $\Sigma_0$ by

$$\Xi(\varrho) = t(\nu)H_{a_0}(\varrho), \quad \varrho = f_s(\nu) \in \Sigma_0.$$

Then we have

$$\imath(\Xi)d\sigma = \imath(H_K)d\sigma = 0 \quad \text{on } \Sigma_0,$$

where $\imath(\Xi)d\sigma$ stands for the corresponding inner product, and

$$K(\varrho) = t(\nu)(a_0(\varrho) - E), \quad \varrho = f_s(\nu) \in \Sigma_0.$$

Taking into account the relations

$$f_0 = \imath_0, \quad f_1 = \imath_0 \circ P,$$

we obtain

$$P^*(\sigma^0) - \sigma^0 = f_1^*(\sigma) - f_0^*(\sigma) = \int_0^1 \frac{d}{ds} f_s^* \sigma ds$$

$$= \int_0^1 f_s^* \Big(d(\imath(\Xi)\sigma) + \imath(\Xi)d\sigma\Big)ds =$$

$$d\left(\int_0^{t(\nu)} (\Phi^s)^* \Big(\imath(H_{a_0})\sigma\Big)\, ds\right) = dG(\nu),$$

which proves the assertion. ♠

**A.3.** The proof of Proposition 5.2 can be obtained from the general results in [22]. For the sake of completeness we present below a proof, using the fact that $U_h^1$ and $Q(h)$ are Fourier integral operators related to the graphs of canonical transformation.

*Proof of Proposition* 5.2. – First we consider the composition $U_h^1 \circ Q(h)$. As in Lemma 4.2 we can find local coordinates $z$ in a neighborhood of the origin in $\mathbf{R}^n$ such that the projections

$$\Lambda \ni (t, x, \tau, z, \zeta) \to (t, x, \zeta), \quad C_0 \ni (z, \zeta, y, \eta) \to (y, \zeta),$$

are local doffeomorphisms. Indeed, there exists a Lagrangian subspace $L$ in $T_0(T^*(\mathbf{R}^n))$ which is transversal to both $L_1 = (D(\Phi^{-t}))(V)$ and

$L_2 = ((D\chi)^{-1})(V)$, $V = T(T_0^*)$ being the tangent space to the fiber at the origin. Choosing local coordinates $z$ such that $L$ coincides with the horizontal space $\{(\delta z, 0) : \delta z \in \mathbf{R}^n\}$, we find phase functions

$$\Phi_1(t, x, z, \zeta) = \phi(t, x, \zeta) - \langle z, \zeta \rangle, \quad \Phi_2(z, y, \theta) = \langle z, \theta \rangle - \psi(y, \theta).$$

which parametrize locally the canonical relations $\Lambda$ and $C_0$. Then the Schwartz kernels of $U_h^1$ and $Q(h)$ have the form

$$U_h^1(t, x, z) = (2\pi h)^{-n} \int_{\mathbf{R}^n} e^{ih^{-1}\Phi_1(t,x,z,\zeta)} w(t, x, z, \zeta, h) d\zeta \qquad (A.1)$$

$$Q(h)(z, y) = (2\pi h)^{-n} \int_{\mathbf{R}^n} e^{ih^{-1}\Phi_2(z,y,\theta)} q(z, y, \theta, h) d\theta, \qquad (A.2)$$

where $w$ and $q$ are classical symbols with respect to $h^{-1}$ of order 0 having leading terms $w_0(t, x, z, \zeta)$ and $q_0(z, y, \theta)$. Now we can write locally the Schwartz kernel of the composition $U_h^1 \circ Q(h)$ as an oscillatory integral with a phase function

$$\Phi(t, x, y, z, \zeta, \theta) = \Phi_1(t, x, z, \zeta) + \Phi_2(z, y, \theta)$$

and amplitude $v = wq$ with a leading term

$$v_0(t, x, y, z, \zeta, \theta) = w_0(t, x, z, \zeta) q_0(z, y, \theta).$$

We parametrize the canonical relation $\Lambda \circ C_0$ of $U_h^1 \circ Q(h)$ by the projection

$$s : \Lambda \circ C_0 \ni (t, x, \tau, \xi, y, \eta) \to (t, y, \eta).$$

Then the half-density part $s^*\Big(\sigma(U_h^1 \circ Q(h))\Big)$ of the principal symbol $\sigma\Big(U_h^1 \circ Q(h)\Big)$ can be written in the coordinates $(t, y, \eta)$ in the form

$$p(t, y, \eta) |dt \wedge dy \wedge d\eta|^{1/2}.$$

We are going to explore the relation between $p$ and the half-density part of the principal symbols $\sigma(U_h^1)$ and $\sigma(Q(h))$ in the coordinates $(t, y, \eta)$. Let us take local coordinates $\lambda = (t, y, \theta)$ in

$$C_\Phi = \{(t, x, y, z, \zeta, \theta) : \Phi'_z = \Phi'_\zeta = \Phi'_\theta = 0\}.$$

Denote by $\lambda_\Phi : C_\Phi \to \mathbf{R}_t \times T^*(\mathbf{R}^n_y)$ the composition $\lambda_\Phi = s \circ \imath_\Phi$. Then, taking into account the equality

$$\left| \frac{D(t, y, \theta, \Phi'_z, \Phi'_\zeta, \Phi'_\theta)}{D(t, y, \theta, x, \zeta, z)} \right|^{-\frac{1}{2}} |dt \wedge dy \wedge d\theta|^{1/2}$$

$$= \left|\det \phi''_{x\zeta}\right|^{-1/2} |dt \wedge dy \wedge d\theta|^{1/2}$$

$$= \left|\det \phi''_{x\zeta}\right|^{-1/2} \left|\det \psi''_{y\theta}\right|^{-1/2} |dt \wedge dy \wedge d\eta|^{1/2},$$

we obtain

$$\lambda^*_\Phi \Big( w_0 \left|\det \phi''_{x\zeta}\right|^{-1/2} \left|\det \phi''_{x\zeta}\right|^{-1/2} \Big)(t, y, \eta) = p(t, y, \eta). \tag{A.3}$$

Moreover, taking into account (3.5) we write the half-density part of $\pi^*\Big(\sigma(U^1_h)\Big)$ in the form

$$g_0(\Phi^t(z, \zeta)) b(t, z, \zeta) g_0(z, \zeta) |dt \wedge dz \wedge d\zeta|^{1/2},$$

where $\pi : \Lambda \to \mathbf{R}_t \times T^*(\mathbf{R}^n_z)$ stands for the natural projection, $g_0$ is the principal symbol of the pseudodifferential operator $g(x, hD_x)$ and $b(t, z, \zeta)$ is defined in Section 3. Let us set

$$\lambda_{\Phi_1} = \pi_1 \circ \pi \circ \imath_\Phi : \ C_{\Phi_1} \to \mathbf{R}_t \times T^*(\mathbf{R}^n_y),$$

where $\pi_1(t, y, \eta) = (t, \chi^{-1}(z, \zeta))$. Then we have

$$\left| \frac{D(t, x, \zeta, \Phi'_{1\zeta})}{D(t, x, \zeta, z)} \right|^{-\frac{1}{2}} |dt \wedge dx \wedge d\zeta|^{1/2} = \left|\det \phi''_{x\zeta}\right|^{-1/2} |dt \wedge dz \wedge d\zeta|^{1/2},$$

which implies

$$\lambda^*_{\Phi_1} \Big( w_0 \left|\det \phi''_{x\zeta}\right|^{-1/2} \Big)(t, y, \eta) = g_0\Big(\Phi^t(\chi(y, \eta))\Big) b(t, \chi(y, \eta)) g_0(\chi(y, \eta)), \tag{A.4}$$

since $\chi$ is symplectic. Denote by $r_0(y, \eta)$ the half-density part of $s_2^*\Big(\sigma(Q(h))\Big)$, where $s_2 : C_0 \to T^*(\mathbf{R}^n_y)$ is the natural projection, and set $\lambda_{\Phi_2} = s_2 \circ \imath_{\Phi_2}$. Then we have

$$\lambda^*_{\Phi_2} \Big( q_0 \left|\det \psi''_{y\theta}\right|^{-1/2} \Big)(t, y, \eta) = r_0(y, \eta). \tag{A.5}$$

On the other hand, we have

$$C_\Phi = \{(t. x. y. z. \zeta. \theta) : \quad \theta = \zeta, \ (t, x, z, \zeta) \in C_{\Phi_1}, \ (z, y, \theta) \in C_{\Phi_2}\}.$$

and according to (A.3), (A.4) and (A.5), we obtain

$$\lambda_{\Phi}^{*}\left(w_0\left|\det\phi''_{x\varsigma}\right|^{-1/2}\left|\det\phi''_{x\varsigma}\right|^{-1/2}\right)(t,y,\eta)$$

$$=\lambda_{\Phi_1}^{*}\left(w_0\left|\det\phi''_{x\varsigma}\right|^{-1/2}\right)(t,y,\eta)\lambda_{\Phi_2}^{*}\left(q_0\left|\det\psi''_{y\theta}\right|^{-1/2}\right)(t,y,\eta).$$

Hence,

$$p(t,y,\eta)=g_0(\Phi^t(\chi(y,\eta)))b(t,\chi(y,\eta))g_0(\chi(y,\eta))r_0(y,\eta),$$

and in view of (5.1) we can suppose that $r_0(y,\eta)=1$ in a neighborhood of $\operatorname{supp}(bg_0)$. On the other hand, $g_0\left(\Phi^t(\chi(y,\eta))\right)=g_0(\chi(y,\eta))=w(\eta_n)$, where $w(\eta_n)=1$ for $\eta_n$ in a neighborhood of $E$. Finally, we obtain

$$p(t,y,\eta)=b(t,\chi(y,\eta))(w(\eta_n))^2.$$

In the same way we deal with the half-density part of the composition $Q_1(h)^*\circ(U_h^1\circ Q(h))$. To obtain the Liouville factor of the principal symbol of $Q_1(h)^*\circ U_h^1\circ Q(h)$, we note that the phase function

$$\Phi_3(t,s,x,z,\tau,\theta)=(t-s)\tau+\psi(x,\theta)-\langle z,\theta\rangle$$

parametrizes the canonical relation $C_1$ of $Q_1(h)^*$ in a neighborhood of

$$(\nu^0,\tilde{\chi}(\nu^0)),\ \nu^0=(kT(\nu),0,E,0,E).$$

The proof of Proposition 5.2 is complete. ♠

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