ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 67, nº 4 (1997), p. 447-462
<http://www.numdam.org/item?id=AIHPA_1997__67_4_447_0>

Ann. Inst. Henri Poincaré,

Vol. 67, n° 4, 1997, p. 447-462. *Physique théorique*

Statistical independence of operator algebras

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ABSTRACT. – In the paper we investigate statistical independence of C^*-algebras and its relation to other independence conditions studied in operator algebras and quantum field theory. Especially, we prove that C^*-algebras A_1 and A_2 are statistically independent if and only if for every normalized elements $a \in A_1$ and $b \in A_2$ there is a state φ of the whole algebra such that $\varphi(a) = \varphi(b) = 1$. As a consequence we show that logical independence (see [17, 18]) implies statistical independence and that statistical independence implies independence in the sense of Schlieder. We prove that the reverse implications are not valid. Further, independence of commuting algebras is shown to be equivalent to independence of their centers. Finally, results on independence of commuting algebras are generalized to the context of Jordan-Banach algebras.

Key words: Operator algebras, independence of states, logical independence of real rank zero C^*-algebras, strict locality, algebraic quantum field theory.

RÉSUMÉ. – Nous examinons, dans cet article, la notion d'indépendance statistique de C^*-algebres ainsi que la comparaison avec les autres définitions données en théorie des algèbres d'opérateur ou en théorie quantique des champs. Nous démontrons en particulier que deux C^*-algèbres A_1 et A_2 sont statistiquement indépendantes si et seulement si pour toute paire $a \in A_1$ et $b \in A_2$ il existe un état φ de l'algèbre totale tel que $\varphi(a) = \varphi(b) = 1$. En conséquence, nous montrons que

[1] The author would like to thank the Grant Agency of the Czech Republic and von Humboldt Foundation for the support of his research (Grant No. 201/96/0117).

l'indépendance statistique implique l'indépendance au sens de Schlieder (voir [17, 18]). Nous démontrons que l'implication inverse est fausse. De plus, l'indépendance de deux algèbres commutant entre elles est équivalente à l'indépendance de leurs centres. Enfin, nous généralisons aux algèbres de Jordan-Banach les résultats concernant l'indépendance de deux algèbres commutant entre elles.

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1. INTRODUCTION AND PRELIMINARIES

Let A be a (unital) C^*-algebra. Two C^*-subalgebras A_1 and A_2 of an algebra A are said to be *statistically independent* if we can always find a state of the "global" algebra A with arbitrarily prescribed values on "local" subalgebras A_1 and A_2.

The aim of this note is to provide a lucid characterization of statistical independence and to establish hitherto unknown relations with other independence conditions arising in the algebraic approach to quantum theory.

From the mathematical standpoint statistical independence is a condition postulating the existence of all simultaneous extensions of states defined independently on given subalgebras. Therefore, this property is interesting from the point of view of the theory of states on operator algebras. Besides, this concept has also received a great deal of attention because of its relevance to quantum physics. It is commonly assumed in the mathematical foundations of quantum theory [9, 12, 21] that the system of observables of a given quantum system is formed by an operator algebra A, while the ensemble of real states of the system is given by the state space of the operator algebra A. In this context the statistical independence embodies naturally the independence of the corresponding physical subsystems. (Any preparation, *i.e.* any state, of the subsystem modelled by one algebra cannot affect any preparation of the subsystem given by another algebra.) For that reason the notion of statistical independence has been firstly introduced and studied by R. Haag and D. Kastler [5] in the context of algebraic quantum field theory. According to their axiom of independence the local algebras corresponding to spacelike separated regions in the Minkowski space-time, should be "completely uncoupled", *i.e.* statistically independent. Following this approach many various conditions of independence have been studied in the context of the algebraic quantum field theory as well as in general operator algebra setting [2, 7, 11, 17, 18, 19, 20, 22, 23]. In particular, an important progress has been made by H. Roose [19], who proved that commuting algebras A_1 and A_2 are statistically independent if and only if they are independent in the sense of Schlieder (see [20]) (*i.e.* if $ab \neq 0$, whenever elements $a \in A_1$ and $b \in A_2$ are non-zero). We shall call this type of independence the S-independence. The proof is based on interesting tensor product technique. In recent papers of M. Redei [17, 18] the following property of independence motivated by the quantum logic approach has been introduced: Two von Neumann subalgebras M_1 and M_2 of a von Neumann algebra M are *logically independent* if the infimum $e \wedge f$ is non-zero for any pair of non-zero projections $e \in M_1$ and $f \in M_2$. In the quantum logic interpretation it means that no non-trivial proposition on the system given by M_1 should imply – or be implied – by any non-trivial proposition on the system corresponding to M_2 (see [17, 18] for more detailed discussion).

Natural question arises (explicitly formulated e.g. in [7, 17, 18, 23]) of what is the relationship of the statistical independence, logical independence and S-independence in the category of C^*-algebras. We contribute to this problem.

In Theorem 2.1 it is proved that statistical independence is equivalent to the fact that any pair of positive normalized elements taken from subalgebras in question can be exposed by the same state of the global algebra. It can be also expressed by saying that given two positive normalized observables corresponding to two subsystems we can always prepare the whole system such that the expectation value of both observables is one. This seems to be a simplification from both mathematical and physical standpoint. As a consequence we extend some results of [23] and show that the principle of locality coincides with statistical independence in the category of C^*-algebras. Further we show that logical independence implies statistical independence and that statistical independence implies S-independence. By meas of counterexamples we show that the reverse implications are not valid in general.

Finally, in case of commuting algebras all notions of independence considered in this note coincide. Theorem 2.8 says that the independence of commuting algebras is equivalent to the independence of their centers, *i.e.* to the independence of the "classical" parts of systems. (The well-known result of Murray and von Neumann [13, Corollary of Theorem III] can be then viewed as a corollary.)

In the appendix we leave the context of C^*-algebras and extend results of H. Roose [19] to Jordan-Banach algebras. (As some physicists and mathematicians argue Jordan algebras are more appropriate for modelling quantum systems than associative algebras – see e.g. [7].) Unlike Roose's result which is heavily based on C^*-tensor products, our approach avoids tensor product technique and it is based on using direct compactness arguments. (The theory of tensor products of JB-algebras is not complete, yet – see [8] for special case of JC-algebras.)

Let us now define basic notions and fix the notation. For basic facts on operator algebras we refer to the monographs [10, 14].

Throughout the paper all C^*-algebras considered are unital with a unit 1. Further, all inclusions of C^*-algebras $B \subset A$ considered have the same unit. For a given subset S of a C^*-algebra A let S^+ denote its positive part, *i.e.* $S^+ = \{a \in S \mid a \geq 0\}$. The symbol $C^*(S)$ denotes the unital C^*-subalgebra of A generated by the subset $S \subset A$. The symbol $Z(A)$ will be reserved for the center of an algebra A. A linear functional ϱ on A is said to be a state if it is positive (*i.e.* if $\varrho(a) \geq 0$, whenever $a \geq 0$) and normalized (*i.e.* if $\varrho(1) = 1$). If A is an C^*-algebra of operators acting on a Hilbert space H, then the state $\omega_x(a) = (ax, x)$, $(a \in A)$, where $x \in H$, $\|x\| = 1$, is called a vector state of A. The convex set $S(A)$ of all states of an algebra A is called the state space of A. Endowed with the weak*-topology $S(A)$ is a compact Hausdorff space.

If M is a von Neumann algebra we denote by $P(M)$ the set of all projections in M. Equipped with the ordering $e \leq f \iff ef = f$ and orthocomplementation $e^\perp = 1 - e$, $P(M)$ becomes an orthomodular, complete lattice.

By the symbol $M_n(C)$ we shall mean the C^*-algebra of all $n \times n$ matrices. Finally, the linear span of set S in some linear space will be denoted by spS.

1.1. DEFINITION. – *Let A_1 and A_2 be C^*-subalgebras of a C^*-algebra A. We say that the pair (A_1, A_2) is*

(i) *statistically independent if for every state φ_1 of A_1 and every state φ_2 of A_2 there is a state φ of A extending both φ_1 and φ_2,*

(ii) *strictly local if for every state φ_1 of A_1 and every normalized positive element $a \in A_2$* (i.e. *a is positive and $\|a\| = 1$) there is state φ of A extending φ_1 and such that $\varphi(a) = 1$,*

(iii) *S-independent if $ab \neq 0$ whenever $a \in A_1$ and $b \in A_2$ are non-zero.*

Moreover, assume that A_1 and A_2 are von Neumann subalgebras of a von Neumann algebra A. Then the pair (A_1, A_2) is

(iv) *logically independent (see* [17, 18]*) if $e \wedge f \neq 0$, whenever $e \in P(A_1)$ and $f \in P(A_2)$ are non-zero.*

The above stated conditions are the main independence properties considered in this paper. Besides, their W^*-versions are often investigated in the literature. Let M_1 and M_2 be von Neumann subalgebras of a von Neumann algebra M. The pair (M_1, M_2) is said to be *W^*-independent* if for every normal state φ_1 and φ_2 of M_1 and M_2, respectively, there is a normal state of M extending both φ_1 and φ_2. The following concept was introduced in [11], where its physical meaning is explained: the pair (M_1, M_2) is strictly local if, given any normal state φ_1 of M_1 and any non-zero projection $e \in M_2$, we can always find a normal state φ of M extending φ_1 and such that $\varphi(e) = 1$.

2. RESULTS

In the first part of this section we focus on the statistical independence of (not necessarily commuting) C^*-algebras. One of the main results is the following characterization of statistically independent C^*-algebras.

2.1. THEOREM. – *A pair (A_1, A_2) of C^*-subalgebras of a C^*-algebra A is statistically independent if and only if for every positive normalized elements $a \in A_1$ and $b \in A_2$ there is a state φ of A such that $\varphi(a) = \varphi(b) = 1$.*

Proof. – One implication being trivial, it is enough to prove that assuming the existence of a state on A attaining its norm at independently chosen normalized and positive elements of A_1 and A_2, we can always find a common extension φ of given states φ_1 and φ_2 of A_1 and A_2, respectively.

In the first step we restrict our attention to the the case of A being a separable C^*-algebra. Moreover, assume that both φ_1 and φ_2 are pure states. Since the left kernels $\mathcal{L}_{\varphi_i} = \{x \in A \mid \varphi_i(x^*x) = 0\}$ $(i = 1, 2)$ are separable closed left ideals, there are strictly positive, normalized elements x_i's, $0 \leq x_i \leq 1$, of $\mathcal{L}_{\varphi_i} \cap \mathcal{L}^*_{\varphi_i}$ [14, Prop. 3.10.6]. Since $(\mathcal{L}_{\varphi_i})^+ = (Ker\ \varphi_i)^+$, we have $\varphi_i(x_i) = 0$. Setting now $c_i = 1 - x_i$, for $i = 1, 2$, we get normalized positive elements ($\varphi_i(c_i) = 1$) which are determining for states φ_1 and φ_2 in the following sense: If ψ is, for instance, any state of A_1 fulfilling $\psi(c_1) = 1$ then $\psi = \varphi_1$. Indeed, $\psi(c_1) = 1$ implies $\psi(x_1) = 0$ and so $\psi | \mathcal{L}_{\varphi_1} \cap \mathcal{L}^*_{\varphi_1} = 0$ by strict positivity of x_1. Thus,

$$(Ker\ \varphi_1)^+ = \mathcal{L}^+_{\varphi_1} \subset (Ker\ \psi)^+$$

and therefore $\mathcal{L}_{\varphi_1} \subset \mathcal{L}_\psi$. As $\mathcal{L}_{\varphi_1}$ is a maximal proper norm closed left ideal in A [14, Prop. 3.13.6], $\mathcal{L}_{\varphi_1} = \mathcal{L}_\psi$. It follows from the one-to-one surjective correspondence between pure states and maximal proper closed left ideals that $\psi = \varphi_1$ [14, Prop. 3.13.6].

According to the assumption there is a state φ of A such that $\varphi(c_1) = \varphi(c_2) = 1$. Having established determinacy of pure states φ_1 and φ_2 by elements c_1 and c_2, respectively, we see immediately that $\varphi|A_1 = \varphi_1$ and $\varphi|A_2 = \varphi_2$. In other words, φ is the desired extension.

Assume now that both φ_1 and φ_2 is a convex combination of pure states. Let us write

$$\varphi_1 = \sum_{i=1}^{n} \alpha_i \varrho_i, \quad \varphi_2 = \sum_{j=1}^{m} \beta_j \psi_j,$$

where $\varrho_1, \ldots, \varrho_n$ are pure states of A_1; $\psi_1, \ldots, \psi_m$ are pure states of A_2; $\alpha_i, \beta_j \geq 0$, $(i = 1, \ldots, n;\ j = 1, \ldots, m)$, and $\sum_{i=1}^{n} \alpha_i = \sum_{j=1}^{m} \beta_j = 1$. Employing the previous reasoning we can find states $\tilde{\varrho}_1, \ldots, \tilde{\varrho}_n$ of A such that

$$\tilde{\varrho}_i|A_1 = \varrho_i, \quad \tilde{\varrho}_i|A_2 = \psi_1 \quad (i = 1, \ldots, n).$$

Putting

$$\tilde{\psi}_1 = \sum_{i=1}^{n} \alpha_i \tilde{\varrho}_i$$

we can get a state of A with

$$\tilde{\psi}_1|A_1 = \varphi_1; \quad \tilde{\psi}_1|A_2 = \psi_1.$$

Similarly we can find states $\tilde{\psi}_2, \ldots, \tilde{\psi}_m$ of A such that

$$\tilde{\psi}_j|A_1 = \varphi_1 \text{ while } \tilde{\psi}_j|A_2 = \psi_j \quad (j = 2, \ldots, m).$$

Letting now $\varphi = \sum_{j=1}^{m} \beta_j \tilde{\psi}_j$ we see that φ is the desired extension of both φ_1 and φ_2.

Finally, let φ_1 and φ_2 be arbitrary states of A_1 and A_2, respectively. By the Krein-Milman theorem both φ_1 and φ_2 are weak*-limit points of nets $(\varphi_\alpha)_{\alpha \in I}$ and $(\psi_\beta)_{\beta \in J}$ consisting of convex combinations of pure states of A_1 and A_2, respectively. Considering now a net $(\varphi_{\alpha,\beta})_{\alpha \in I, \beta \in J}$ consisting of states $\varphi_{\alpha,\beta}$ extending simultaneously φ_α and ψ_β, we can employ the compactness of the state space of A and get the weak*-limit point φ of the net $\varphi_{\alpha,\beta}$. It can be easily verified that φ is the desired extension of φ_1 and φ_2.

Having proved separable case let us now turn to the case when A is arbitrary. Denote by $\mathcal{S}$ the system of all finite non-empty subsets of $A_1 \cup A_2$. For any $S \in \mathcal{S}$ we put

$$F_S = \{\varphi \in S(A) \mid \varphi \mid S \cap A_1 = \varphi_1, \quad \varphi|S \cap A_2 = \varphi_2\}.$$

Since any C^*-algebra A_S generated by the set S, $S \in \mathcal{S}$, is separable, there is a state of A_S extending both $\varphi_1|A_S \cap A_1$ and $\varphi_2|A_S \cap A_2$. In other words, the sets F_S, $S \in \mathcal{S}$, are non-empty closed sets in the compact space $S(A)$. Moreover, inclusion

$$F_{\cup_{i=1}^n S_i} \subset \cap_{i=1}^n F_{S_i}$$

says that the system $(F_S)_{S \in \mathcal{S}}$ enjoys the finite intersection property. Hence, $F = \cap_{S \in \mathcal{S}} F_S \neq \emptyset$ by compactness and any state $\varphi \in F$ is the required extension of φ_1 and φ_2. The proof is completed.

As an immediate corollary of Theorem 2.1 we get that the statistical independence and the principle of locality coincide in the category of C^*-algebras (unlike the case of von Neumann algebras treated e.g. in [11]). Also, Theorem 2.1 can be reformulated in operator-theoretic terms.

2.2. COROLLARY. – *Let* (A_1, A_2) *be a pair of* C^**-subalgebras of a* C^**-algebra* A. *The following conditions are equivalent:*

(i) (A_1, A_2) *is statistically independent,*

(ii) (A_1, A_2) *is strictly local.*

Moreover, when considered A in its universal representation as an algebra acting on the universal Hilbert space H_u, *then the above stated conditions are equivalent to*

(iii) *For every positive normalized elements* $a \in A_1$ *and* $b \in A_2$ *there is a common eigenvector* $\xi \in H_u$ *of* a *and* b *corresponding to eigenvalue one.*

Proof. – Since every state on an algebra A taken in its universal representation is a vector state it follows from Theorem 2.1 that algebras A_1 and A_2 are statistically independent if and only if for any positive, normalized elements $a \in A_1$ and $b \in A_2$ there is a unique vector $\xi \in H_u$ with

$$(a\xi, \xi) = (b\xi, \xi) = 1.$$

Therefore, $a\xi = b\xi = \xi$ and the proof is completed.

It has been proved in [23] that W^*-strict locality of two commuting von Neumann algebras implies their statistical independence. It also follows immediately from Theorem 2.1 that this result can be extended to all pairs.

In the sequel we explain the position of statistical independence and S-independence of C^*-algebras. The following Proposition has been proved in special case of mutually commuting algebras in [17, 18].

2.3. PROPOSITION. – *Any statistically independent pair* (A_1, A_2) *of* C^*-*subalgebras of a* C^*-*algebra* A *is* S-*independent.*

Proof. – Let $a \in A_1$ and $b \in A_2$ be non-vanishing elements of statistically independent algebras A_1 and A_2. Since $ab = 0$ would imply $a^*abb^* = 0$, we may assume that both a and b are positive, normalized elements. Let us choose a state φ_1 of A_1 such that $\varphi_1(a) = 1$ and φ_1 is multiplicative (*i.e.* pure) state on a commutative algebra generated by a. Let φ_2 be an arbitrary state of A_2 with $\varphi_2(b) = 1$. By the statistical independence we can take a common extension φ of φ_1 and φ_2 to the whole of A. Since $\varphi(a^2) = \varphi(a)^2 = 1$ (*i.e.* φ is definite at a) $\varphi(ax) = \varphi(a)\varphi(x)$ for every $x \in A$ (see e.g. [Lemma 2, p. 33]). Especially,

$$\varphi(ab) = \varphi(a)\varphi(b) = 1$$

implies that $ab \neq 0$. This concludes the proof.

2.4. REMARK. – It is not true in general that S-independence imply statistically independence. It can be demonstrated by the following simple example. Set $A = M_2(C)$ and let $A_1 = C^*(p)$, $A_2 = C^*(q)$, where p and q are one-dimensional, non-commuting projections. Since $A_1 = sp\{p, 1-p\}$, $A_2 = \{q, 1-q\}$ are two-dimensional it can be easily seen that A_1 and A_2 are S-independent. Nevertheless there is no state on A taking value one at p and q simultaneously. Therefore (A_1, A_2) is not a statistically independent pair.

We now turn to the relationship between statistical and logical independence of von Neumann algebras.

The following theorem and counterexample show that logical independence always implies statistical independence while the reverse implication is not valid. It answers in the negative Problem 1 posed in [18].

2.5. THEOREM. – *Every logically independent pair* (M_1, M_2) *of von Neumann subalgebras of a von Neumann algebra* M *is statistically independent.*

Proof. – According to Theorem 2.1 it suffices to prove that given positive normalized elements $a \in M_1$, and $b \in M_2$ of logically independent von Neumann subalgebras M_1 and M_2, there is a state φ of M with $\varphi(a) = \varphi(b) = 1$.

By the spectral theorem there are sequences $(p_n) \subset M_1$ and $(q_n) \subset M_2$ of non-zero spectral projections of a and b, respectively, satisfying the following inequalities for all $n \in N$:

$$ap_n \geq (1 - 1/n)p_n,$$
$$bq_n \geq (1 - 1/n)q_n.$$

The logical independence entails $p_n \wedge q_n \neq 0$. So we can take states φ_n's of M such that

$$\varphi_n(p_n \wedge q_n) = 1 \quad \text{for all } n \in N.$$

Since $a \geq ap_n$, $b \geq bq_n$ we have

$$\varphi_n(a) \geq (1 - 1/n)\varphi_n(p_n) = 1 - 1/n$$
$$\varphi_n(b) \geq (1 - 1/n)\varphi_n(q_n) = 1 - 1/n.$$

Applying now compactness of the state space of M we can choose a weak*-cluster point φ of (φ_n). Then, of course, $\varphi(a) = \varphi(b) = 1$ as it has been required. This concludes the proof.

REMARK. – Unlike von Neumann algebras projection structures of C^*-algebras do not form a lattice in general. Nevertheless, it should be remarked that the definition of logical independence (Definition 1.1. (iv)) can be reasonably extended from von Neumann algebras to all C^*-algebras in the following way: We say that a pair (A_1, A_2) of C^*-subalgebras of a C^*-algebra A is *logically independent* if for every couple of non-zero projections $p \in A_1$, $q \in A_2$ there is a non-zero projection $r \in A$ such that $r \leq p$ and $r \leq q$.

Then Theorem 2.5 holds in the more general context of C^*-algebras of real rank zero. Let us recall that a C^*-algebra is said to be of real rank zero if every self-adjoint element of A is a norm limit of self-adjoint elements with finite spectrum (see [4, 15]). For indeed, the proof of Theorem 2.5 can be easily adopted to show that any logically independent pair of C^*-algebras which have real rank zero is statistically independent. Besides von Neumann algebras and AW^*-algebras the class of C^*-algebras of real rank zero comprises many important examples of C^*-algebras (AF-algebras, Bunce-Deddens algebra, Cuntz's algebras, etc.) and it is intensively studied presently (see e.g. [3, 4]).

2.7. COUNTEREXAMPLE. – *A von Neumann algebra* $M = l^\infty \otimes M_5(C)$ *contains two-dimensional subalgebras* M_1 *and* M_2 *which are statistically independent but not logically independent.*

Proof. – First, let us observe by simple linear algebra arguments that for any $n \in N$ we can find a basis $\{u_n, v_n, x, y_n, z_n\}$ of a five-dimensional Hilbert space H_5 consisting of unit vectors such that the following conditions hold for all $n \in N$:

(1) $||u_n - x|| < 1/n$,

(2) $y_n \perp sp\{u_n, v_n\}$,

(3) $v_n \perp x$,

(4) $z_n \perp sp\{u_n, v_n, x, y_n\}$.

We shall use the symbol P_V for an orthogonal projection of H_5 onto a given subspace $V \subset H_5$. Setting

$$e_n = P_{sp\{u_n,v_n\}}, \quad f_n = P_{sp\{x,y_n\}}$$

for each $n \in N$ we get a sequence of projections in the full matrix algebra $M_5(C)$ with the following positions

$$\begin{aligned} e_n \wedge f_n &= 0 \\ e_n \wedge f_n^\perp &\geq P_{sp\{v_n\}} \neq 0 \\ e_n^\perp \wedge f_n &\geq P_{sp\{y_n\}} \neq 0 \\ e_n^\perp \wedge f_n^\perp &\geq P_{sp\{z_n\}} \neq 0. \end{aligned}$$

Moreover, $P_{sp\{x\}} e_n P_{sp\{x\}} = \lambda_n P_{sp\{x\}}$ for some $0 \leq \lambda_n \leq 1$. Then

$$\begin{aligned} |1 - \lambda_n| &= ||P_{sp\{x\}} - \lambda_n P_{sp\{x\}}|| \\ &= ||P_{sp\{x\}} - P_{sp\{x\}} e_n P_{sp\{x\}}|| \\ &= ||P_{sp\{x\}}(1 - e_n) P_{sp\{x\}}|| \\ &\leq ||(1 - e_n) P_{sp\{x\}}||. \end{aligned}$$

Since

$$||(1 - e_n) P_{sp\{x\}}|| = ||x - e_n x|| \leq 1/n,$$

by (1), we have

$$\lambda_n \geq 1 - 1/n.$$

In other words, for a vector state ω_x of $M_5(C)$ we see that, for all $n \in N$,

$$\omega_x(f_n) = 1,$$

while

$$\omega_x(e_n) = \omega_x(P_{sp\{x\}} e_n P_{sp\{x\}}) = \lambda_n \geq 1 - 1/n.$$

In the sequel we shall identify the tensor product algebra M with the l^∞-direct sum $\sum_{n=1}^\infty \oplus M_5(C)$ by means of the assignment $(a_n)_{n=1}^\infty \otimes b \to \sum_{n=1}^n \oplus\ a_n b$. Define now projections

$$e = \sum_{n=1}^\infty \oplus\ e_n$$

$$f = \sum_{n=1}^\infty \oplus\ f_n.$$

By the previous considerations,

$$e \wedge f = 0,\ \text{while}\ e^\perp \wedge f \neq 0, e \wedge f^\perp \neq 0,\ \text{and}\ e^\perp \wedge f^\perp \neq 0.$$

Therefore, we can always take a state of M attaining value one at $e^\perp \wedge f$ and the same applies to projections $e \wedge f^\perp$ and $e^\perp \wedge f^\perp$. We shall now construct a state φ of M such that $\varphi(e) = \varphi(f) = 1$.

To this end, let us take an arbitrary state φ_1 of l^∞ vanishing on c_0. Put $\varphi = \varphi_1 \otimes \omega_x$. As $1 \otimes P_{sp\{x\}} \leq f$ and $\varphi(1 \otimes P_{sp\{x\}}) = 1$ we have $\varphi(f) = 1$. On the other hand, $\varphi(1 \otimes P_{sp\{x\}}) = 1$ implies the following inequalities:

$$\begin{aligned}
\varphi(e) &= \varphi((1 \otimes P_{sp\{x\}})e(1 \otimes P_{sp\{x\}})) \\
&= \varphi\Big(\sum_{n=1}^\infty \oplus\ P_{sp\{x\}} e_n P_{sp\{x\}}\Big) = \varphi\Big(\sum_{n=1}^\infty \oplus \lambda_n P_{sp\{x\}}\Big) \\
&\geq \varphi\Big(\sum_{n=1}^\infty \oplus\ (1 - 1/n) P_{sp\{x\}}\Big) = \varphi\big((1 - 1/n)_{n=1}^\infty \otimes P_{sp\{x\}}\big) \\
&= \varphi_1\big((1 - 1/n)_{n=1}^\infty\big).
\end{aligned}$$

Since $\varphi_1((1/n)_{n=1}^\infty) = 0$ by the assumption, we see finally that $\varphi(e) = 1$. Put

$$M_1 = sp\{e, 1 - e\}, \quad M_2 = sp\{f, 1 - f\}.$$

By the previous reasoning we can verify easily (Theorem 2.1 or simple calculations) that (M_1, M_2) forms a statistically independent pair of two-dimensional subalgebras of M. On the other hand the fact that $e \wedge f = 0$ says that the pair (M_1, M_2) is not logically independent. The proof is complete.

Summing up our discussion we can conclude that the following chain of implications holds

$$\text{logical independence} \implies \text{statistical independence} \implies S\text{-independence}.$$

None of these implications can be reversed in general. Therefore the logical independence is the strongest independence condition. Nevertheless, if subalgebras in question mutually commute, then it is straightforward to see that the S-independence implies the logical independence. (This implication is due to the obvious fact that the infimum of commuting projections is their product.) Consequently, all concepts of independence considered in this note coincide in case of commuting von Neumann algebras. Further, in this important case the independence can be characterized by the position of centers of algebras under consideration. Before formulating this result let us therefore examine the independence of mutually commuting abelian subalgebras (classical case). More specifically, let M be an abelian von Neumann algebra and let M_1 and M_2 be von Neumann subalgebras of M such that M is generated by M_1 and M_2. Let Z, X and Y be the hyperstonean compact spaces of M, M_1 and M_2 correspondingly. Then the pair (M_1, M_2) is statistically independent if and only if Z is homeomorphic to the product $X \times Y$. Indeed, by [19, Theorem 2] (M_1, M_2) is statistical independent pair exactly when M is isomorphic to the tensor product $M_1 \bar{\otimes} M_2$. Hence, representing abelian von Neumann algebras in question by the algebras of continuous functions on their hyperstonean spaces we get (see e.g. [10, Chap. 11])

$$M \cong C(Z) \cong C(X) \otimes C(Y) \cong C(X, C(Y)) \cong C(X \times Y).$$

Thus, $Z \cong X \times Y$ ([10, 3.4]).

2.8. THEOREM. – *Let (M_1, M_2) be a pair of mutually commuting von Neumann subalgebras of a von Neumann algebra M. The following conditions are equivalent*

(i) *(M_1, M_2) is logically (statistically, S-) independent,*

(ii) *$(Z(M_1), Z(M_2))$ is logically (statistically, S-) independent,*

(iii) *$C^*(Z(M_1) \cup Z(M_2))$ is isometrically isomorphic to the C^*-tensor product $Z(M_1) \otimes Z(M_2)$,*

(iv) *the pure state space of the von Neumann algebra $W^*(Z(M_1) \cup Z(M_2))$ generated by $Z(M_1) \cup Z(M_2)$ is homeomorphic to the product of pure state spaces of $Z(M_1)$ and $Z(M_2)$.*

Proof. – The implication (i)$\Longrightarrow$(ii) being trivial we concentrate on implication (ii)$\Longrightarrow$(i). The rest of the proof follows from discussion preceeding this theorem.

Assume that the centers $Z(M_1)$ and $Z(M_2)$ are statistically independent. Consider non-zero projections $e \in M_1$ and $f \in M_2$. Denote by $c(e)$ and

$c(f)$ the central cover of projections e and f with respect to algebras M_1 and M_2, respectively. Then

$$c(e) = \bigvee_{u \in U(M_1)} u^* e u,$$

where u runs through the unitary group $U(M_1)$ of M_1 [14, Lemma 2.6.3]. Now

$$c(e)f = (\bigvee_{u \in U(M_1)} u^* e u) f = \bigvee u^* e u f \neq 0$$

if and only if there is a $u_0 \in U(M_1)$ such that $u_0^* e u_0 f \neq 0$. Since $u_0^* e u_0 f = u_0^* e f u_0$, we see that

$$ef \neq 0 \text{ if and only if } c(e)f \neq 0.$$

By symmetry

$$ef \neq 0 \text{ if and only if } c(e)c(f) \neq 0.$$

Nevertheless, $c(e)c(f)$ is always non-zero by the logical independence of $Z(M_1)$ and $Z(M_2)$. The proof is completed.

Assume that the central projection lattice of a von Neumann subalgebra M_1 of a von Neumann algebra M is an atomic lattice. Let M_2 be a von Neumann subalgebra in the commutant of M_1 such that $p \wedge q \neq 0$, whenever p is a non-zero central projection of M_2 and q is an atom in $Z(M_1)$. Then $(Z(M_1), Z(M_2))$ is logically independent and so by the virtue of Theorem 2.8 (M_1, M_2) is logically independent, too. Therefore, Theorem 2.8 can be viewed as a generalization of the classical result saying that the pair of commuting von Neumann algebras is S-independent provided at least of them is a factor [13, Corollary of Theorem III].

In the above discussion we mentioned the fact that W^*-strict locality implies logical independence. In view of Theorem 2.8 we can add that in case of commuting algebras even W^*-strict locality of their centers is sufficient for their logical independence. The converse implication is not valid. As an counterexample we can take the pair (M, M') consisting of a type I factor acting on a separable Hilbert space and its commutant. This pair is logically independent but not W^*-strict local. Indeed, according to [11] W^*-strict locality would imply that M is a type III factor.

APPENDIX

The aim of this appendix is to characterize the statistical independence in the realm of Jordan-Banach algebras.

We shall briefly recall basic definitions (see the monograph [6]). The JB-algebra A is a real Banach algebra with a Jordan product $\circ$ such that the following conditions are satisfied for all $a, b \in A$

(i) $a \circ b = b \circ a$,

(ii) $a \circ (a^2 \circ b) = a^2 \circ (a \circ b)$,

(iii) $||a^2|| = ||a||^2$,

(iv) $||a^2|| \leq ||a^2 + b^2||$.

Throughout this note all JB algebras considered are unital, e.i. admitting a unit element 1 with respect to a product $\circ$. Important examples of JB-algebras are self-adjoint parts of C^*-algebras endowed with a Jordan product $x \circ y = 1/2(xy + yx)$. Elements a, b in a JB algebra A are said to be *operator commuting* if $a \circ (b \circ x) = b \circ (a \circ x)$ for all $x \in A$. Subalgebras A_1 and A_2 of A are operator commuting if a and b operator commute for all $a \in A_1$ and $b \in A_2$. Observe that C^*-algebras commute if and only if the Jordan algebras formed canonically by their self-adjoint parts are operator commuting. The non-negative part A^+ of a JB-algebra A is the set $A^+ = \{a^2 \mid a \in A\}$. A state $\varphi \in A^*$ is a positive, normalized linear functional on A. A subspace U of A is called a *quadratic ideal* if $2a \circ (a \circ x) - a^2 \circ x \in U$ for all $a \in U$ and $x \in A$.

The conditions of independence transfer from C^*-algebra case immediately. Let A_1 and A_2 be JB-subalgebras of a JB-algebra A. We say that the pair (A_1, A_2) is *statistically independent* if for every pair of states $(\varphi_1, \varphi_2) \in A_1^* \times A_2^*$ there is a common state extension to A. The pair (A_1, A_2) is said to be S-independent if $a \circ b \neq 0$, whenever $a \in A_1$ and $b \in A_2$ are non-zero elements.

If ϱ is a state of a JB-algebra A then its left kernel $\mathcal{L}_\varrho = \{a \in A \mid \varrho(a^2) = 0\}$ is a quadratic ideal in A. The mapping $\varrho \to \mathcal{L}_\varrho$ is a one-to-one surjective correspondence between the set of pure states in A and the set of all maximal proper closed quadratic ideals in A. Using this fact precisely as in the proof of Theorem 2.1 and replacing other C^*-arguments in this proof by their straightforward Jordan versions, we can immediately generalize Theorem 2.1 to JB-algebras now. We are ready to extend Roose's results [19] to the context of Jordan algebras.

THEOREM. – *Let A_1 and A_2 be operator commuting JB- subalgebras of a JB-algebra A. The pair (A_1, A_2) is statistically independent if and only if it is S-independent.*

Proof. – Assume that (A_1, A_2) is S-independent. In view of the preceeding discussion it suffices to prove that taking non-negative elements $a \in A_1$ and $b \in A_2$ with $||a|| = ||b|| = 1$, we can always find a state φ of A with $\varphi(a) = \varphi(b) = 1$. The JB-subalgebra $JB(a,b)$ generated by a and b is associative. Since every associative JB-algebra is a self-adjoint part of an abelian C^*-algebra (see e.g. [6]) we can consider abelian C^*-algebras $\mathcal{A}, \mathcal{A}_1, \mathcal{A}_2$ such that $\mathcal{A}_{sa} = JB(a,b)$, $(\mathcal{A}_1)_{sa} = JB(a)$ and $(\mathcal{A}_2)_{sa} = JB(b)$. The S-independence of $JB(a)$ and $JB(b)$ implies the S-independence of $\mathcal{A}_1$ and $\mathcal{A}_2$. Denote by $P(\mathcal{A}_1)$ and $P(\mathcal{A}_2)$ the pure state space of $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. (By a pure state space we mean the compact Hausdorff space of all multiplicative states of a given algebra topologized by the weak*-topology.) By [19, Lemma 2] the S-independence of $\mathcal{A}_1$ and $\mathcal{A}_2$ means that the set $\{(\varrho|\mathcal{A}_1, \varrho|\mathcal{A}_2) \mid \varrho \text{ is a multiplicative state of } \mathcal{A}\}$ is weak*-dense in the product space $P(\mathcal{A}_1) \times P(\mathcal{A}_2)$. Let us now take multiplicative states ϱ_1 and ϱ_2 of $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, such that $\varrho_1(a) = \varrho_2(b) = 1$. Employing the previous reasoning we can find a net (ϱ_α) of multiplicative states of $\mathcal{A}$ such that $\varrho_\alpha|\mathcal{A}_1 \to \varrho_1$, $\varrho_\alpha|\mathcal{A}_2 \to \varrho_2$ in the sense of weak*-topology. Extending each ϱ_α from $JB(a,b)$ to the whole of A and passing to a weak*-cluster point of the resulting net in the state space of A, we get the desired state.

The reverse implication is the same as for C^*-algebras.

In the conclusion of this note let us remark that whenever JB-subalgebras A_1 and A_2 of a JB-algebra A are statistically independent then any couple of pure states φ_1, φ_2 of A_1 and A_2, respectively, has a common pure state extension over A. Indeed, it can be verified easily that the set F of all extensions of both φ_1 and φ_2 forms a weak*-closed face in the state space of A. Therefore any extreme point of F will be the desired pure extension. This assertion has been proved for commuting C^*-algebras in [19] by means of tensor products.

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(Manuscript received April 4th, 1996;
Revised version received May 20th, 1996.)