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Probability and quantum symmetries I. The theorem of Noether in Schrödinger's euclidean quantum mechanics

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ABSTRACT. – Noether's Theorem in classical Lagrangian mechanics, for Lagrangians quadratic in the velocities, is generalized to a class of $\mathbb{R}^3$-valued diffusion processes with diffusion matrix proportional to the identity. When the proportionality constant reduces to zero, both hypothesis and conclusions of our Theorem reduce to the classical ones. It is shown that this result can be interpreted along the line of Feynman's path integral approach to nonrelativistic quantum mechanics. The analytical continuation in time of the conclusion of this Euclidean Theorem provides a new Theorem on symmetries in regular quantum mechanics.

RÉSUMÉ. – Le Théorème de Noether de la mécanique lagrangienne classique, dans le cas des lagrangiens quadratiques par rapport aux vitesses,

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est étendu à une classe de processus de diffusion à valeurs dans $\mathbb{R}^3$, dont la matrice de diffusion est proportionnelle à l'identité. Lorsque la constante de proportionnalité se réduit à zéro, nous retrouvons le théorème usuel. Nous montrons que ce résultat s'interprète en terme d'intégrale de Feynman dans le cadre de la mécanique quantique non relativiste. Par prolongement analytique par rapport à la variable temporelle, ce résultat fournit un nouveau Théorème concernant les symétries de la mécanique quantique usuelle.

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0. INTRODUCTION

We prove a Theorem of Noether for a class of $\mathbb{R}^3$ - valued diffusion processes, regarded as rigorous counterparts of the diffusions used formally by Feynman for non - relativistic quantum particles in potentials. When Planck's constant $\hbar = 0$ the hypothesis and conclusion of our Theorem reduce to those of Noether's Theorem in classical Lagrangian mechanics, for the restricted class of Lagrangians considered by Feynman. The underlying probabilistic framework is due, in essence, to E. Schrödinger [26 a)]. It deals with the heat equation and not directly the Schrödinger's one so it is only an Euclidean analogy. However, it is a completely time symmetric approach, in contrast to the familiar Euclidean one, due to M. Kac [6-4]. The role of the first integrals of the classical equations of motion is played here by martingales with respect to the two sigma algebras involved in Schrödinger's approach (one for the past, one for the future information about the physical system). A purely probabilistic treatment of Noether's Theorem can be found in [25].

The organization of the present paper, focused on the relations with Feynman's path integral theory is the following: The first Chapter reviews the Theorem of E. Noether in classical Lagrangian mechanics, as well as the basic (formal) results of Feynman's calculus for functionals of quantum trajectories. No analogue of the classical Theorem of Noether has been obtained by Feynman, even heuristically.

Chapter 2 is an expository account of Schrödinger's Euclidean Quantum Mechanics needed for our purpose. The probabilistic concepts of action functional and associated critical diffusions, equation of motion and random variables counterpart of quantum observables are described. Some of the tools needed afterwards are also collected there. It is shown in what sense our probabilistic framework can be regarded as a mathematically rigorous version of Feynman's path integral approach and, in particular, why quantum constants of motion correspond to martingales.

Chapter 3 is devoted to Noether's Theorem. The value of a typical action functional of classical Lagrangian mechanics depends, in general, on the choice of the parameter along the trajectory. There is a classical procedure for transforming such action into a parameter invariant one (*i. e.* geometrically meaningful): this is to go over a parametric representation of trajectory or, equivalently, to extend by one the dimension of the domain of the action functional. Parameter invariance imposes some restrictions on the lagrangians and transformations. We shall see that the same is true in our probabilistic framework.

A natural generalization of the definition of invariance of the action under a Lie group of transformations is provided and, from this, it is shown that along each diffusion critical for the stochastic action, a certain function of space - time is a martingale. In this sense, the Lie algebra of the constants of motion for the classical dynamical system becomes, quantically, a collection of martingales of the underlying critical diffusions.

It is also shown that necessary conditions for the classical Noether's Theorem survive in a regularized way (using the probability measures of the underlying diffusions).

The conclusion of the Theorem is also proved, independentely, by a pure Lie group argument, together with a formula relating the quantum Hamiltonian of the system and the infinitesimal generator of the critical diffusions.

Chapter 4 provides the physical motivation of our work. It is shown that, after the proper analytical continuation in the time parameter, the probabilistic conclusion of the Noether Theorem turns into the existence of a family of quantum observables $N(t)$ which are constants of motion in the usual (L^2) sense. Even in the simplest cases, new quantum symmetries emerge from our approach, justifying *a posteriori* our Euclidean detour.

The final chapter is a discussion of the Euclidean results and provides a physical interpretation of some of the tools introduced. Our Theorem of Noether can be generalized in many ways and some of them are mentioned in the conclusion.

Finally, let us stress that, although we shall use some of the kinematical tools introduced by E. Nelson in his "Stochastic Mechanics" [18], Schrödinger's Euclidean framework has a completely different dynamical structure. In particular, it is not true, there, that classical constants of motion correspond to martingales. No rigorous Theorem of Noether has been found in this context. For recent developments in real time, *cf.* [33 - 34]. Other investigations of symmetries for stochastic dynamical systems include [37].

1. FEYNMAN PATH INTEGRALS AND THE MISSING THEOREM OF NOETHER

Let Q be the configuration manifold of a classical dynamical system. The paths are C^2 maps $q : \begin{array}{c} I \to Q \\ t \to q(t) \end{array}$ and the regular Lagrangian is

$$\begin{array}{c} L : TQ \times \mathbb{R} \to \mathbb{R} \\ (q, \dot{q}, t) \to L(q, \dot{q}, t) \end{array}$$

where TQ is the tangent bundle of Q. For simplicity, here, we will consider exclusively the configuration manifold $Q = \mathbb{R}^3$ of a single classical particle in space but our method will be considerably more general.

If our (unit mass) classical particle is subjected only to a force field of the form $F = -\nabla V$, for V the scalar potential, the simplest Lagrangian is of the form

$$L = \frac{1}{2}|\dot{q}|^2 - V(q, t). \tag{1.1}$$

for $|\cdot|$ the Euclidean norm. Let us introduce the action functional S, defined on a domain $\mathcal{D}_S \subset C^2([t_0, t_1], Q)$ and associated with the Lagrangian L,

$$S[q(); t_0, t_1] = \int_{t_0}^{t_1} L(q(s), \dot{q}(s), s)\, ds. \tag{1.2}$$

Hamilton's least action principle [1] says that among all regular trajectories between two fixed configurations $q(t_0) = q_0$, $q(t_1) = q_1$, at the extremities of $I = [t_0, t_1]$, the physical motion $\bar{q}$ is critical point of the action S, *i.e.* its variational (Gâteaux) derivative in any smooth direction δq cancels: $\delta S[\bar{q}](\delta q) = 0$. Equivalently $\bar{q} = \bar{q}(t)$ solves the Euler - Lagrange equations in Q

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \tag{1.3}$$

Bundles of solutions of this equation result in the same way from a variational principle involving the action (1.2) to which is added an (initial) boundary condition $S_0(q_0)$. This is what is needed for Hamilton-Jacobi theory [1].

Noether's Theorem is the second most important Theorem of classical Lagrangian mechanics. For the sake of future notations, let us describe it, in its simplest formulation [2]:

Let

$$\begin{aligned} U_\alpha \,:\, Q \times I &\to Q \times I \\ (q,\, t) &\to (Q\,(q,\, t;\, \alpha),\, \tau(q,\, t;\, \alpha)) \end{aligned} \tag{1.4}$$

a given one-parameter ($\alpha \in \mathbb{R}$) local group of transformations of the $(q,\, t)$ plane. It is assumed that the functions Q and τ are of class C^2 in all their variables and such that

$$Q\,(q,\, t;\, 0) = q,\; \tau\,(q,\, t;\, 0) = t. \tag{1.5}$$

Therefore, around $\alpha = 0$,

$$\begin{cases} Q\,(q,\, t;\, \alpha) = q + \alpha X\,(q,\, t) + o\,(\alpha) \\ \tau\,(q,\, t;\, \alpha) = t + \alpha T\,(q,\, t) + o\,(\alpha) \end{cases} \tag{1.6}$$

where $(T\,(q,\, t),\, X\,(q,\, t))$ is called tangent vector field of the family $\{U_\alpha\}$ (and T, X infinitesimal generators of the transformation (1.4)). Let $t \to q\,(t)$ be an arbitrary C^2 trajectory in Q. Then one shows that, if α is small enough, the image under U_α of the graph of $q\,()$ is the graph of a one - parameter family of trajectories in the $(Q,\, \tau)$ - space transformation. So the action (1.2) becomes $S\,[Q\,();\, \tau_0,\, \tau_1]$ under the transformation. Let $\Phi\,:\, Q \times I \to \mathbb{R}$ be a C^2 additional generator (the "divergence"). The action (1.2) is said to be divergence-invariant if, for any α small enough,

$$S\,[q\,();\, t_0,\, t_1] = S\,[Q\,(); \tau_0, \tau_1] - \alpha \int_{t_0}^{t_1} \frac{d\Phi}{dt}\,(q\,(t),\, t)\, dt + o\,(\alpha) \tag{1.7}$$

for any C^2 trajectory $q\,()$ in $\mathcal{D}_S$ and time interval $I = [t_0,\, t_1]$, and where the trajectories and interval of integration in the r.h.s. action of (1.7) result from the abovementioned one parameter family (*cf.* [2]).

Noether's Theorem says that, when (1.7) holds, the smooth function

$$\left[\left(\frac{\partial L}{\partial \dot q}\right) X + \left(L - \frac{\partial L}{\partial \dot q}\dot q\right) T - \Phi\right] (\bar q\,(t),\, t) \tag{1.8}$$

is constant on any critical point $\bar q\,(\cdot)$ of the action (1.2). The first factor of (1.8) defines the momentum observable $p = \frac{\partial L}{\partial \dot q}$ and the second one the energy $-H = L - \frac{\partial L}{\partial \dot{q}}\dot{q}$ associated with the starting Lagrangian L. As shown by E. Cartan [8], (1.8) can be regarded, when $\Phi = 0$, as the central geometrical object of classical Hamiltonian mechanics.

The simplest illustrations are quite familiar:

a) If the Lagrangian L is independent on the i-component of the configuration vector q, then the action (1.2) is invariant under the translation

$$U_\alpha \,:\, (q,\, t) \rightarrow (q + \alpha\, e_i,\, t) \tag{1.9}$$

where e_i is an unit vector along the axis i. Consequently, the invariance (1.7) holds for $X = e_i$, $T = \Phi = 0$ and, from (1.8), the i - component of the momentum p is conserved.

b) If L does not depend on the time t, the action (1.2) is invariant under

$$U_\alpha \,:\, (q,\, t) \rightarrow (q,\, t + \alpha),\ \textit{i.e.}\ \text{for}\ T = 1,\ X = \Phi = 0 \tag{1.10}$$

and (1.8) shows that the energy observable H is conserved.

c) For q in $Q = \mathbb{R}^3$, if L is invariant under a spatial counterclockwise rotation about the q^3-axis, *i.e.* of the form $R_3(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$, the action (1.2) is invariant under $U_\alpha : (q, t) \rightarrow (R_3(\alpha) q, t)$.

Then the relation (1.8) for $X = \begin{pmatrix} -q^2 \\ q^1 \\ 0 \end{pmatrix}$, $T = 0$ and $\Phi = 0$ shows that $(-q^2 p_1 + q^1 p_2)$ is constant. This is the conservation of the component I^3 of the total angular momentum.

Regular Quantum Mechanics is an Hamiltonian framework and, therefore, has little to tell us about Noether's Theorem. The only systematic lagrangian approach to Quantum Mechanics is Feynman's one [3]. It is mathematically inconsistent (*cf.* [4]) but, along the years, he provided us with a set of very powerful heuristic tools.

According to Feynman ([3], p. 172), every quantum mechanical law, for the system characterized classically by the action (1.2), follows from the formal integration by parts formula

$$\langle \delta\, F\,[\omega]\,(\delta\omega)\rangle_S = -\frac{i}{\hbar}\langle F\,\delta S\,[\omega]\,(\delta\omega)\rangle_S \tag{1.11}$$

where, in order to distinguish them from the classical trajectories, we denote now by $t \rightarrow \omega\,(t) \in Q$ the quantum ones in, say, the path space

$$\Omega^y = \{\omega\,(\cdot) \in C\,([t_0,\, t_1];\, Q)\ \text{s. t.}\ \omega\,(t_1) = y\} \tag{1.12}$$

F is, according to Feynman, an "arbitrary" functional on Ω^y, whose "expectation" $\langle\cdot\rangle_S$ is computed with respect to the "complex weight"

$$\exp\frac{i}{\hbar}S[\omega;\, t_0,\, t_1]\cdot\prod_{t_0\leq s\leq t_1} d\omega\,(s),$$

where $\hbar$ is Plank's constant, used as a probability measure on Ω^y. In his formula (1.11), Feynman denotes by $\delta G\,[\omega]\,(\delta\omega)$ the formal directional derivative of the functional G in the direction $\delta\omega$.

Here is a minimal summary of the formal consequences of (1.11), corresponding to the choice of a few functionals F and directions $\delta\omega$ (*cf.* [3], chapter 7 for details):

For the simplest lagrangian (1.1), the following "probabilistic" version of the Euler - Lagrange equation (1.3) holds:

$$\left\langle\left(\frac{\omega\,(t+\Delta t)-2\omega\,(t)+\omega\,(t-\Delta t)}{(\Delta t)^2}\right)\right\rangle_S = -\langle\nabla V\,(\omega\,(t))\rangle_S \quad (1.13)$$

The left hand side of Eq. (1.13) is a time discretized version of the second derivative along quantum paths. Feynman does not take the continuum limit $\Delta t \searrow 0$ in order to avoid divergences. Indeed, for another choice of functional of the paths in Eq. (1.11), he shows that ([3], (7-45))

$$\left\langle\omega_l(t)\frac{(\omega\,(t)-\omega\,(t-\Delta t))\,m}{\Delta t}\right\rangle_S - \left\langle\frac{(\omega\,(t+\Delta t)-\omega\,(t))\,m}{\Delta t}\,\omega_l(t)\right\rangle_S = i\,\hbar\delta_{lm} \quad (1.14)$$

where $l,\, m = 1,\, 2,\, 3$ are the components in $\mathbb{R}^3$.

Feynman interprets this relation as the space-time version of the basic commutation relations between the position Q and momentum quantum observables P in $L^2(\mathbb{R}^3)$, $[Q_l,\, P_m] = i\hbar\delta_{lm}$.

Notice that, if the continuum limit could be taken under the "expectations" $\langle\cdot\rangle_S$ of (1.14), two distinct time derivatives should coexist at the same time t, otherwise the Heisenberg principle would be violated. After a formal manipulation of (1.14), Feynman observes that the quantum paths $t \rightarrow \omega\,(t)$ are Brownian - like, expect for the imaginary factor i (r. h. of Eq. (1.14)) in the diffusion coefficient.

Feynman's heuristic diffusion theory also provides us with a formal algorithm to associate "random variable" (with respect to the "measure" underlying $\langle\cdot\rangle_S$) to regular quantum observable O in $L^2(Q)$. For the need of Noether Theorem, let us mention his (time discretized) momentum and energy "random variables" for the Lagrangian (1.1):

$$O = P = -i\hbar\nabla \ \text{ becomes } \ \frac{\omega(t+\Delta t) - \omega(t)}{\Delta t} \tag{1.15}$$

$$O = H = -\frac{\hbar^2}{2}\Delta + V$$
$$\text{becomes } \left(\frac{1}{2}\left(\frac{\omega(t+\Delta t) - \omega(t)}{\Delta t}\right)^2 - \frac{\hbar}{2}\frac{i}{\Delta t} + V(\omega(t))\right) \tag{1.16}$$

Although the non classical term $-\frac{\hbar}{2}\frac{i}{\Delta t}$ in the energy random variable (1.16) is reminiscent of the typical effects of Itô's stochastic calculus [5], we cannot interpret Feynman's original strategy along this line since its probabilistic content is, in fact, empty. The quotation marks above were, therefore, needed.

As it is well known since M. Kac, some probabilistic counterparts of Feynman's quantization procedure makes sense if the Schrödinger equation underlying the integration by parts formula is, first, replaced by the associated heat equation ("Euclidean" approach [6]). We shall recall in the next paragraph the relevant way to do this for our purpose, so that (1.11) becomes rigorous, as well as its consequences (1.13) and (1.14). Before this, let us stress that, although the whole point of Feynman's approach is to be a quantization of classical Lagrangian mechanics, no "probabilistic" version of the fundamental Theorem of Noether is provided there. As a matter of fact, the same holds true for the elementary classical concept of first integral (or constant of motion). Of course, when the classical dynamical system is conservative, the expectation of Feynman's energy random variable (1.16), for exemple, is constant in time, but one could expect a stronger characterization of first integrals holding true almost everywhere with respect to an underlying probability measure. We shall see that such characterization is precisely the one involved in our Noether's Theorem.

2. SCHRÖDINGER'S EUCLIDEAN QUANTUM MECHANICS AND FEYNMAN'S PATH INTEGRALS

Let $(\Omega, \mathcal{T}, P)$ be a Probability space, that is a triple where the sample space Ω is a set of all elementary events, $\mathcal{T}$ is the sigma algebra of the observable random events and P is a measure on $(\Omega, \mathcal{T})$ such that

$P(\Omega) = 1$. A Q- valued diffusion process on $(\Omega, \mathcal{T}, P)$, equipped with a non decreasing family of sigma -algebras $\mathcal{P}_t \subset \mathcal{T}$, $t \geq 0$, can be regarded as solution of an Itô's stochastic differential equation (S. D. E.) [5],

$$d\, z_t = \sigma\,(z_t)\, d\, W_t + B\,(z_t,\, t)\, dt \tag{2.1}$$

where σ is a given nonsingular real square matrix and B a given real vector field, both of the same dimension as Q. Moreover, W_t is a standard Q-valued Wiener process. By definition, a solution z_t is $\mathcal{P}_t$- measurable and for $t \geq 0$, almost surely,

$$z_t = z_0 + \int_0^t \sigma\,(z_s)\, dW_s + \int_0^t B\,(z_s,\, s)\, ds \tag{2.2}$$

where the first integral in the right hand side is an Itô (forward) stochastic integral, well defined under proper restrictions on σ [5]. Since the sample paths of W are of unbounded variation on any interval, so are those of z.

In order to make sense of an action functional like (1.2) along the paths of diffusions like (2.2), one introduces the infinitesimal generator of z_t, denoted by D, and defined for smooth f on $Q \times I$,by

$$E\,[f\,(z_{t+\Delta t},\, t) - f\,(z_t,\, t) \mid \mathcal{P}_t] = Df\,(z_t,\, t)\,\Delta t + o\,(\Delta t) \tag{2.3}$$

where $E[\ldots \mid \mathcal{P}_t] \equiv E_t$ is a conditional expectation, given the "history" $\mathcal{P}_t$, until the time t.

Eq. (2.2) implies that D is, formally, the operator

$$D = \frac{\partial}{\partial t} + B \cdot \nabla + \frac{1}{2}(\sigma \sigma^T)\Delta \tag{2.4}$$

where ∇ and Δ denote respectively the gradient and Laplacian in Q. The comparison with Feynman (1.14) shows that it is sufficient to take, as symmetric and nonnegatively defined diffusion matrix,

$$\sigma \sigma^T = \hbar \mathbb{I} \tag{2.5}$$

where $\mathbb{I}$ is the identity matrix in $Q = \mathbb{R}^3$ and σ^T is the transpose of the matrix σ. However, it is known since Cameron's work [7] that, in contrast with (1.14), the diffusion matrix must be real if we want to deal with well defined probability measures on paths space of continuous functions.

The forward generator (2.4), with diffusion matrix (2.5), is the appropriate regularization of the formal time derivative along the sample paths used by

Feynman: when $\hbar = 0$ it reduces to the ordinary derivative. For example, the definition (2.3) for $f(x, t) = x$ shows that the drift vector field B is the regularization of the a.s. divergent velocity $\frac{\mathrm{d}}{\mathrm{d}t} z_t$, *i.e.*

$$D\, z_t = B(z_t, t) \tag{2.6}$$

Given this interpretation of (2.4) (advocated by Nelson [10]), what about the Lagrangian L needed for an action functional like (1.2) ?

A priori, without further specification on the diffusions solving (2.1) (with $\sigma = \sqrt{\hbar}\mathbb{I}$), no canonical way to associate a Lagrangian L to diffusions is known. However, it has been shown that for a special class of diffusions, whose existence was suggested by E. Schrödinger and S. Bernstein [26], such a relation is intrinsic [9] (in the sense that a whole family of such diffusions with common dynamical properties is associated with the same Lagrangian L). Let us consider the quantum Hamiltonian observable H associated with the classical system described by (1.2), and simple Lagrangian (1.1). H is a self-adjoint extension of

$$-\frac{\hbar^2}{2}\Delta + V(q, t) \tag{2.7}$$

on $L^2(Q)$.

The potential V is a real - valued measurable function over $Q \times I$, belongs to the Kato class as a function of the space, and may also depend smoothly on the time parameter. In the time independent case, it is known that the integral kernel of $exp\,(-\frac{t}{\hbar}H)$ is jointly continuous in all its variables and strictly positive [9 b].

Then it has been shown that there is a large class of Q - valued ("Bernstein") diffusion solving simultaneously two stochastic differential equations, in the weak sense (meaning that the probability space, the filtrations and the Brownian motions are not given *a priori* [5]) namely, for $t \in I$,

$$\begin{cases} \mathrm{d}z_t = \sqrt{\hbar}\mathrm{d}W_t + D\, z_t\, \mathrm{d}t & (1) \\ \mathrm{d}_* z_t = \sqrt{\hbar}\mathrm{d}_* W_t + D_* z_t\, \mathrm{d}t & (2) \end{cases} \tag{2.8}$$

where the first equation is defined with respect to a nondecreasing filtration $\mathcal{P}_t$, $t \in I$, as (2.1), but (2.8) (2) refers, instead, to a non decreasing filtration $\mathcal{F}_t$, interpreted as the future information on our dynamical system. W_t and W_{*t} denote, respectively, $\mathcal{P}_t$ and $\mathcal{F}_t$ Q- valued Wiener processes. The notation d_*, borrowed from Nelson [10], denotes the infinitesimal backward differential $\mathrm{d}_* f(t) = f(t) - f(t - \mathrm{d}t)$ and D_* the associated generator,

$$E\left[f(z_t, t) - f(z_{t-\Delta t}, t - \Delta t) \mid \mathcal{F}_t\right] = D_* f(z_t, t)\Delta t + o(\Delta t) \quad (2.3^*)$$

like in (2.3) but with respect to the filtration $\mathcal{F}_t$. As an operator defined on C_0^∞ it can be written

$$D_* = \frac{\partial}{\partial t} + D_* z \cdot \nabla - \frac{\hbar}{2}\Delta \tag{2.9}$$

Notice the minus sign in front of the Laplacian, due to the use of backward Itô's calculus. The Bernstein diffusion associated with the Hamiltonian (2.7) are uniquely determined by the forward and backward drifts vector fields:

$$D\, z_t = \hbar \nabla \log \eta(z_t, t) \quad (2.10)(1)$$

$$D_*\, z_t = -\hbar \nabla \log \eta_*(z_t, t) \quad (2.10)(2)$$

when η and η_* are, respectively, regular positive solutions of heat equations in $L^2(\Omega)$, $t \in I$, with H as in (2.7),

$$\begin{cases} \hbar \dfrac{\partial \eta}{\partial t} = H\eta \\ -\hbar \dfrac{\partial \eta_*}{\partial t} = H\eta_* \end{cases} \tag{2.11}$$

namely the Euclidean versions of the Schrödinger equation underlying Feynman's approach, and of its complex conjugate, with appropriate positive boundary conditions. We will always assume, afterwards, that our regularity conditions on (2.10) are sufficient to guarantee existence and uniqueness of weak solutions of (2.8)(1) and (2). We shall impose the finite kinetic energy conditions

$$E_{t_0} \int_{t_0}^{t_1} |D\, z(s)|^2 ds < \infty \text{ and } E^{t_1} \int_{t_0}^{t_1} |D_* z(s)|^2 ds < \infty$$

for $[t_0, t_1] \subset I$, and that the drift (2.10)(1) and (2.10)(2) are Lipschitz continuous.

Those restrictions are much too strong (for example, they are not fulfilled by the free, *i.e.* $V = 0$, "Bernstein bridge" on $I = [t_0, t_1]$, starting from the distribution δ_x at time t_0 and ending in δ_z at time t_1) but they will be sufficient for our needs here.

The need of the two equations (2.11) for the construction of the underlying measures, instead of one as usually, is specific to Schrödinger Euclidean Quantum Mechanics ([9 b], [11]). The time interval I of existence of z_t depends on the boundary conditions of (2.11). Each result with respect to the filtration $\mathcal{P}_t$ has a counterpart with respect to $\mathcal{F}_t$ which is formally, its time reversal. The time reversibility of the whole construction is, actually, built in the use of (2.11).

If $\mu_W^\hbar$ denotes the Wiener measure with parameter $\hbar$ (*cf.* (2.8)) and initial distribution the law of z_0, then the measure of z, μ_z, is absolutely continuous with respect to $\mu_W^\hbar$, with Radon - Nikodym density

$$\rho[z] = \exp\left\{-\frac{1}{\hbar}\int_0^T V(z_t, t)\, dt\right\}\frac{\eta(z_T, T)}{\eta(z_0, 0)} \tag{2.12}$$

The proof uses Feynman - Kac formula, the theorem of Girsanov and the special form (2.10) of the drifts. With $\rho[z]$, one shows the existence of weak solutions of (2.8)(1), and the time reversed version of the functional $\rho[z]$ provides weak solutions of (2.8)(2).

Using the resulting measures μ_z, one shows the existence of a rigorous version of Feynman's integration by parts formula (1.11), via Malliavin's stochastic calculus of variations [28], [38], and valid for a wide class of functionals in associated Sobolev spaces. We shall only mention here those few consequences needed for the Noether Theorem (*cf.* [12] for details).

The relevant action functional (assumed to be finite) of any smooth Bernstein diffusion z is, henceforth, the conditional expectation

$$S[z(\cdot)] = E_t\int_t^T \bar{L}(z_s, D z_s, s)\, ds - E_t \log \eta_T(z_T) \tag{2.13}$$

for the lagrangian

$$\bar{L}(q, \dot{q}, s) = \frac{1}{2}|\dot{q}|^2 + V(q, s) \tag{2.14}$$

$[t, T] \subset I$, and η_T a smooth positive function in $L^2(\mathbb{R}^3)$.

The change of sign of V, with respect to Feynman's action (1.1) is familiar when Schrödinger's equation is replaced by the heat equation. Although our approach uses, therefore, the same data as the conventional Euclidean one (*cf.* [13], for example) we are going to see that its dynamical content is quite distinct.

The action (2.13) is associated with the nonlinear (uniformly parabolic) Hamilton-Jacobi-Bellman (HJB) equation:

$$\frac{\partial A}{\partial t} - \frac{1}{2}|\nabla A|^2 + \frac{\hbar}{2}\Delta A + V = 0, \quad (t, y) \in I \times \mathbb{R}^3$$

with final condition $A_T(y) = -\hbar \ln \eta_T(y)$, $y \in \mathbb{R}^3$.

If V and $A_T(y)$ are continuous and bounded, an unique regular solution (*i.e.* $\in C_b^{1,2}(I \times \mathbb{R}^3)$) exists [35]. Existence and uniqueness of continuous solutions is also known if V and A_T are limits of bounded continuous functions.

THEOREM 2.1. *cf.* [14 b]. – *If a regular solution A of* (HJB) *is known, then* $\forall (t, y) \in I \times \mathbb{R}^3$,

$$A(y, t) \leq E_{t,y}\left\{ \int_t^T \bar{L}(z_s, D z_s, s)\, ds - \hbar \log \eta_T(z_T) \right\}$$

for any z in a class of $\mathbb{R}^3$-valued diffusions on $(\Omega, \mathcal{T}, P)$, with $z_t = y$, z_s adapted to $\mathcal{P}_s$, $t \leq s \leq T$ and admitting a $\mathcal{P}_s$ adapted drift $D z_s$ defined by (2.3) *and such that* $E \int_I |D z_s|^m ds < \infty$, $\forall m \in N$. *It is also assumed that Dynkin formula holds for a set large enough in the domain of the generator of z_s.*

The inequality reduces to an equality only on the Markovian diffusion with drift (2.10) (1), *where η solves the first heat equation of* (2.11) *with final condition* $\eta(q, T) = \eta_T(q)$.

For other probabilistic characterizations of such diffusions (Föllmer, Wakolbinger, Cattiaux-Léonard) *cf.* [25]. When V is differentiable, one shows directly (*cf.* [12]) that the critical points of (2.13) solve almost surely

$$D D z_s = \nabla V(z_s, s), \quad s \in [t, T] \tag{2.15}$$

when

$$\begin{cases} z_t = y \\ D z_T = \hbar \nabla \log \eta_T(z_T) \end{cases}$$

For more about this *cf.* [27]. Eq. (2.15) can be regarded as a rigorous probabilistic version of Feynman's Euler Lagrange equation (1.13) for the Lagrangian (1.1).

Since Bernstein measures, in contrast with Feynman's ones, do exist, the continuum limit is well defined in Eq. (2.15).

In general, the minimum of the action (2.13) in the abovementioned class of admissible diffusions of the Theorem is not regular enough to be a classical solution of HJB equation. But it is always a "viscosity solution". This concept of weak solution is appropriate to the problem of singular perturbation describing now the relation between the classical ($\hbar = 0$) and non classical case ($\hbar \neq 0$). *cf.* [14a]. In particular, it provides, under weak restrictions, not only existence but also uniqueness of the solution of HJB equation with final boundary conditions. As mentioned before, this generally will not be needed for our present study of symmetries.

As when $\hbar = 0$, one proves easily that the gradient of HJB, in a generalized sense if needed, coincides with the equation of motion (2.15).

The probabilistic version of Feynman's algorithm to associate random variables to quantum observable is as follows. Let O be one of the densely defined, normal operators acting on the one-parameter family of Hilbert spaces which are the Euclidean counterparts of $L^2(Q)$ (*cf.* [9.b]), and η an element of the cone of positive vectors in those Hilbert spaces. As a function of t, this Euclidean quantum state solves (2.11)(1) in the strong L^2-sense. By construction, the probability density of a Bernstein diffusion at time t, $t \in I$, is

$$\eta^* \eta(q, t)\, dq = \mu_t^B(dq) \tag{2.16}$$

("Euclidean Born interpretation" cf. [9]). The forward random variable associated with the observable O in the Euclidean state $\eta \in \mathcal{D}_O$ at time $t \in I$ is defined by

$$o(z_t, t) = \left(\frac{O\eta}{\eta}\right)(z_t, t) \tag{2.17}$$

if the r.h.s. is μ_t^B-integrable. For example, the momentum and energy random variables are [9b], respectively,

$$B(z_t, t) = D z_t \tag{2.18}$$

$$\varepsilon(z_t, t) = \left(-\frac{1}{2}B^2 - \frac{\hbar}{2}\nabla \cdot B + V\right)(z_t, t) \tag{2.19}$$

Given the definition (2.3), we observe that, up to the change of sign of Euclidean origin, those random variables coincide with Feynman's formal ones in (1.15-16). In particular, the $\hbar$-dependent term in the energy can really be regarded, now, as a consequence of Itô's forward calculus. As before, we have taken advantage of the existence of the continuum $\lim\limits_{\Delta t \searrow 0}$. The reasoning leading to (2.17) is familiar in problems of foundations of quantum mechanics [15].

The same quantum observable can, as well, be associated with a random variable with respect to the decreasing filtration $\mathcal{F}_t$, involving positive solutions of the adjoint heat equation (2.11)(2). For example, the backward momentum and energy are

$$B_*(z_t, t) = D_* z(t) \tag{2.20}$$

$$\varepsilon_*(z_t, t) = \left(-\frac{1}{2}B_*^2 + \frac{\hbar}{2}\nabla \cdot B_* + V\right)(z_t, t) \tag{2.21}$$

Notice the change of sign in the non classical term of (2.21), with respect to the forward energy (2.19).

Let us stress that the coexistence of two filtrations in this construction is fundamental: the probabilistic version of some basic results of Feynman requires this. For example the rigorous version of Eq. (1.14) becomes

$$E[z_l(t) D_* z_m(t) - D\, z_m(t)\, z_l(t)] = \hbar \delta_{lm}, \quad l, m = 1, 2, 3. \qquad (\widetilde{1.14})$$

We shall also need a probabilistic counterpart of first integrals of the classical dynamical system. Let f be a C_0^∞ function of space - time and η a positive solution of the heat equation (2.11) (1), used in the forward stochastic differential equation (2.8) (1) for z_t.

Assuming that $(f\eta) \in \mathcal{D}_H$, Eqs. (2.4) and (2.11) imply that

$$Df(z_t, t) = \frac{1}{\eta(z_t, t)}\left(\frac{\partial}{\partial t} - \frac{1}{\hbar}H\right)(f\eta)(z_t, t), \quad t \in I \tag{2.22}$$

where H is the quantum Hamiltonian involved in (2.11). This formula is relevant for the forward filtration $\mathcal{P}_t$. Its time reversed version is [16]

$$D_* f_*(z_t, t) = \frac{1}{\eta_*(z_t, t)}\left(\frac{\partial}{\partial t} + \frac{1}{\hbar}H\right)(f\eta_*)(z_t, t), \quad t \in I \tag{2.23}$$

On the other hand, in regular Quantum Mechanics, and under proper restrictions on dense domains of the observables O and H in $L^2(Q)$, $O(\tau)$ is called a constant of motion if it satisfies [17]

$$i\hbar\frac{\partial}{\partial \tau}O(\tau) + [O(\tau), H] = 0 \tag{2.24}$$

where $[O, H]$ denotes the commutator $OH - HO$.

It was proved in [9 b] that the following Euclidean version of (2.24) makes sense as well, on dense domains, in Schrödinger's Euclidean framework,

$$\hbar\frac{\partial}{\partial t}O(t) + [O(t), H] = 0, \quad t \in I \tag{2.25}$$

for $O(t)$ the Euclidean counterpart of the quantum observable $O(\tau)$. Let $O(t)$ be an Euclidean constant of motion observable, in (2.25) sense. Using the definitions (2.17) and (2.22), it follows that the random variable o associated with the Euclidean observable O satisfies, a. s.

$$D\,o(z_t,\,t) = 0, \quad t \in I \tag{2.26}$$

when the left hand side is well defined. This is equivalent to say that $o(z_t,\,t)$ is a $\mathcal{P}_t$- martingale. Therefore the probabilistic counterpart of quantum constant of motion is a martingale, one of the cornerstones of stochastic analysis. Clearly, the analogous statement with respect to the backward filtration $\mathcal{F}_t$ is that the associated backward random variable $o_*(z_t,\,t)$ satisfies, when the left hand side exists (*cf.* [11, 16]):

$$D_* o_*(z_t,\,t) = 0, \quad t \in I \text{ a. s.} \tag{2.27}$$

When L is as in (2.14), for example, the following conservation of energy holds

$$D\varepsilon\,(z_t,\,t) = \frac{\partial V}{\partial t}\,(z_t,\,t), \quad t \in I, \text{ a. s.} \tag{2.28}$$

with respect to $\mathcal{P}_t$ and its analogue for ε_* holds with respect to $\mathcal{F}_t$.

We shall use constantly the following classical result, with the convention $E\,[\ldots \mid z_s = x] = E_{s,\,x}[\ldots]$:

Dynkin formula

Let $f\,:\,\mathbb{R}^3 \times I \to \mathbb{R}$ in the domain $\mathcal{D}_D$ of the infinitesimal generator D of (2.4). Suppose that f, $\frac{\partial f}{\partial t}$ and $D\,f$ are continuous on $R^3 \times I$ and such that $E_{s,\,x}|f\,(z_t,\,t)|$ and $E_{s,\,x}\int_s^u |D\,f\,(z_t,\,t)|\,dt$ are finite for $s < t$ in I. Then

$$E_{s,\,x}\int_s^u D\,f\,(z_t,\,t) = E_{s,\,x}\,f\,(z_u,\,u) - f\,(x,\,s) \tag{2.29}$$

Proof. – *cf.* [29]. The result holds more generally. □

We have now the tools needed to formulate our extention of the classical Theorem of Noether. For a different approach, see [25].

A review of Schrödinger's Euclidean Quantum Mechanics and of its relations with Feynman's path integral theory can be found in [31].

3. THE THEOREM OF NOETHER

Let the Hamiltonian H be fixed as in (2.7) and L the corresponding Lagrangian (2.14) (from now on we omit the line on L since only Euclidean formulation, unless explicitly mentioned, will be used).

Let us consider the action functional (2.13) on $I = [t_0, t_1]$. There, $\eta_{t_1}(q)$ denotes the positive final condition for (2.11) (1) on I needed to evaluate the action on a critical diffusion with drift (2.10) (1).

Like in classical mechanics, for a given equation of motion (2.15) the Lagrangian L is far from being unique:

LEMMA 3.0. – *Let L be the Lagrangian* (2.14) *whose critical points solve Eq.* (2.15) *a.e. on I. Let L′ be of the form*

$$L'(z, Dz, s) = L(z, Dz, s) + Df(z, s)$$

where $f : \mathbb{R}^3 \times I \to \mathbb{R}$ *is such that Dynkin formula holds but, otherwise, arbitrary. Then L′ leads to the same equation of motion* (2.15) *a. e. on I.*

Proof. – For $\tilde{L}(z, Dz, s) = Df(z, s)$, regarded as a Lagrangian of his own, the Euler-Lagrange equation $D\left(\frac{\partial \tilde{L}}{\partial Dz}\right) - \frac{\partial \tilde{L}}{\partial z} = 0$ reduces to an identity a. s.

Lemma 3.0 is a probabilistic counterpart of a classical gauge transformation *cf.* [19 b; 32].

In particular, choosing f appropriately in the lemma, the boundary condition of the stochastic action functional (2.13) at time t_1 disappears. Therefore, without loss of generality, we shall restrict ourselves to functionals of the form

$$S[z(\cdot)] = E_{t_0} \int_{t_0}^{t_1} L(z_s, Dz_s, s)\, ds, \qquad t_0 < t_1 \tag{3.1}$$

We call natural domain $\mathcal{D}_S$ the set of diffusions z solving (2.8) (1) and such that $S[z(\cdot)] < \infty$.

From the geometrical point of view, only line integrals independent of the parametrization of the classical trajectories are relevant. Let us define the analogue property for the functional (3.1):

DEFINITION 3.1. – The action S is parameter invariant on I iff

$$E_{t_0} \int_{t_0}^{t_1} L(z_s, Dz_s, s)\, ds = E_{\tau_0} \int_{\tau_0}^{\tau_1} L(Q_\tau, DQ_\tau, \tau)\, d\tau \tag{3.2}$$

$\forall z \in \mathcal{D}_S$, $\forall [t_0, t_1] \subseteq I$, where Q_τ is the time changed diffusion $Q_\tau = z_{f^{-1}(\tau)}$ where $\tau = f(t)$ is any C^1 orientation preserving transformation (i.e. $\dot{\tau} > 0$). DQ_τ denotes the drift of this time changed diffusion.

Our basic action functional with Lagrangian (2.14) is not parameter invariant, even when the potential $V = 0$. Consider, for example, the Bernstein diffusion critical point of (3.1) for a zero potential, and characterized by the solution of the free heat equation (2.11)(1),

$$\eta(q, t) = e^{\frac{1}{\hbar}\{(V_0, q) - \frac{1}{2}|V_0|^2 t\}}, \qquad t \in \mathbb{R} \tag{3.3}$$

where $(\cdot, \cdot)$ denotes the Euclidean scalar product in $\mathbb{R}^3$ and V_0 is a constant vector. The diffusion z_t associated via (2.10)(1) has the constant drift $Dz_t = V_0$. If this diffusion solves (2.8)(1) with the Lebesgue measure as distribution at time 0, theoretical physicists look at (3.3) as a plane wave solution of the Euclidean Schrödinger equation (2.11)(1) with $V = 0$, and probabilists as the (forward) "exponential martingale" for the Brownian motion. The computation of the action functional for $L(q, \dot{q}) = \frac{1}{2}|\dot{q}|^2$ is immediate:

$$S[z(\cdot)] = E_{x,t} \int_t^0 \frac{1}{2}|Dz_s|^2 \, ds = -\frac{1}{2}|V_0|^2 t, \qquad t < 0 \tag{3.4}$$

Consider the simplest time change $\tau = c\,s$ for c any positive constant. Then $DQ_\tau = Dz_s \dfrac{ds}{d\tau} = \dfrac{V_0}{c}$ and the r.h.s. of (3.2) becomes

$$E_{Q,\tau} \int_{ct}^0 \frac{1}{2}|DQ_\tau|^2 d\tau = -\frac{1}{c}\frac{1}{2}|V_0|^2 t \neq S[z(\cdot)]$$

In the classical case, there is a standard procedure to turn into a parameter invariant action the basic one initially given, say

$$S[Q] = \int_{\tau_0}^{\tau_1} L\left(Q_\tau, \frac{dQ_\tau}{d\tau}, \tau\right) d\tau.$$

This is to go over a smooth parametric representation

$$Q = Q(t), \qquad \tau = \tau(t), \qquad t_0 \leq t \leq t_1 \tag{3.5}$$

and to regard the action as defined in the extended configuration space (Q, τ), *i.e.*

$$S[Q, \tau] = \int_{t_0}^{t_1} L\left(Q_\tau, \frac{dQ/dt}{d\tau/dt}, \tau\right) \frac{d\tau}{dt} \, dt \tag{3.6}$$

The new integrand is, therefore,

$$\mathcal{L}\left(Q, \tau, \frac{dQ}{dt}, \frac{d\tau}{dt}\right) = L\left(Q, \frac{dQ/dt}{d\tau/dt}, \tau\right) \cdot \frac{d\tau}{dt}$$

In particular, it is positive-homogeneous of degree 1 in the velocities $\frac{dQ}{dt}$, $\frac{d\tau}{dt}$. The value of $S[Q, \tau]$ depends, now, only on the curve $Q(t)$, $\tau(t)$ and not on the particular parametric representation, by homogeneity of $\mathcal{L}$. Therefore, the new functional is parameter invariant. The same method will apply to make the stochastic action (3.1) parameter invariant.

Before doing this, let us notice the following relation with the above mentioned classical invariance under a one parameter group of transformations (*cf.* (1.6)-(1.7)), when the divergence $\Phi = 0$:

LEMMA 3.2. – *If $S[q]$ is parameter invariant on I, it is invariant under the one - parameter group of transformations $Q = q$, $\tau = t + \alpha T(t)$, $\forall T$ such that $\tau(t)$ is orientation preserving.*

A strong physical argument will be given later for justifying, in our probabilistic generalization, the form of time transformation given in Lemma 3.2 (to compare with (1.6)). This choice can also be proved to be necessary if we require invariance (*cf.* [25]).

We can now formulate the probabilistic counterpart of the invariance condition (1.7):

DEFINITION 3.3. – The basic stochastic action functional (3.1) is divergence invariant under the Lie group of transformations (1.6), for $T = T(t)$ such that τ is orientation preserving and divergence Φ such that Dynkin's formula holds if, for α small enough,

$$E_{t_0} \int_{t_0}^{t_1} L(z_t, Dz_t, t)\, dt = E_{\tau_0} \int_{\tau_0}^{\tau_1} L(Q_\tau, DQ_\tau, \tau)\, d\tau - \alpha\, E_{t_0} \int_{t_0}^{t_1} D\,\Phi(z_t, t)\, dt + o(\alpha)$$

$\forall [t_0, t_1] \subset I$ (or its image $[\tau_0, \tau_1]$ under time change).

If $t \rightarrow z_t$ is smooth, D reduces to the ordinary time derivative, the expectations disappear and this definition reduces to the invariance condition (1.7) used in the classical Noether Theorem.

A priori, however, it is not clear if our probabilistic generalization of the invariance condition can ever be satisfied. Among the difficulties to overcome, the first one is to construct explicitly a one-parameter family of diffusions such that the right hand side of the definition (3.3) can be compared with the starting (l.h.s.) action.

We are going to provide sufficient conditions for making sense of the invariance definition (3.3).

From (1.6), the determining equation of the process Q_τ in the right hand side action is now, for sufficiently small α,

$$Q(t+\alpha T(t)) = z_t + \alpha X(z_t, t), \quad t \in I$$

Equivalently, carrying all the α-dependence in the right hand side, one defines a one parameter family of processes indexed by t by

$$Q^\alpha(t) = z(t-\alpha T) + \alpha X(z(t-\alpha T), t-\alpha T), \quad t \in I \tag{3.7}$$

Since $z_t \equiv z(t)$ is an $\mathbb{R}^3$ valued diffusion solving (2.8) (1) with drift (2.10) (1), let us assume that $X(q, \tau)$ is continuous on $\mathbb{R}^3 \times I$, with values in $\mathbb{R}^3$ and continous partial derivatives.

Then, it follows from Itô's Theorem [5] that $X(z_t, t)$ is a three dimensional diffusion solving a known stochastic differential equation (with respect to the filtration $\mathcal{P}_t$).

Eq. (3.7) defines a family of processes such that $Q^0(t) = z(t)$. Another useful interpretation of our space - time transformation is that the probabilistic counterpart of the graph argument used in the classical Noether Theorem is the following:

PROPOSITION 3.4. – *For α small enough, the transformation*

$$Q = q + \alpha X(q, t) + o(\alpha)$$
$$\tau = t + \alpha T(t) + o(\alpha)$$

for $T : I \to \mathbb{R}$ of C^1 class with $\dot{T} > 0$ and $X : \mathbb{R}^3 \times I \to \mathbb{R}^3$ as before, carries a given $\mathbb{R}^3$-valued Bernstein diffusion

$$dz_t = B(z_t, t)\,dt + \sqrt{\hbar}\,dW_t, \quad t \in [t_0, t_1]$$

with drift $B(z, t) = \hbar\nabla \log \eta(z, t)$, into a one parameter family of diffusions $Q(\tau)$, indexed by τ in $[\tau(t_0), \tau(t_1)]$, solving weakly

$$\mathrm{d}Q(\tau) = (B+\alpha(DX - B\dot{T}))(Q(\tau), \tau)d\tau + \left(\mathbb{I} + \alpha\left(X_q - \frac{1}{2}\dot{T}\mathbb{I}\right)\right)\sqrt{\hbar}d\tilde{W}(\tau)$$

where $\tilde{W}$ is a $\mathcal{P}_\tau$ $\mathbb{R}^3$-valued Wiener process, $\mathbb{I}$ the 3×3 identity matrix and X_q the matrix $\left(\frac{\partial X^i}{\partial q^j}\right)$, $i, j = 1, 2, 3$ of derivatives of components.

Proof. – Consider the change of configuration alone. By Itô's Theorem $Q(z_t, t) \equiv Q_t$ solves the $\mathcal{P}_t$ stochastic differential equation

$$dQ_t = (Dz_t + \alpha DX(z_t, t))\,dt + (\mathbb{I} + \alpha X_q(z_t, t))\sqrt{\hbar}\,dW_t$$

where $\mathbb{I}$ is the 3×3 identity matrix. Now consider the deterministic change of time $t \to \tau = t + \alpha T(t)$ in this equation (*cf.* [5]). If $Q_t \equiv Q(\tau(t))$ and $\frac{d\tau}{dt} = 1 + \alpha \dot{T} > 0$, then $Q(\tau)$ solves

$$dQ(\tau) = \frac{DQ_t}{d\tau/dt} d\tau + \frac{\mathbb{I} + \alpha X_q}{\sqrt{d\tau/dt}} \sqrt{\hbar} d\tilde{W}(\tau)$$

where $\tilde{W}(\tau)$ is a $\mathcal{P}_\tau$ Wiener process. Introducing the first order term of the α expansions of $\frac{1}{1+\alpha\dot{T}}$ and $\frac{1}{\sqrt{1+\alpha\dot{T}}}$, the conclusion follows.

Remark. – Regarded as a one parameter family of drifts of diffusions, the drift of $Q(\tau)$ in Prop. 3.4, behaves as expected under our space - time transformation. Indeed, by differentiability of stochastic differential equations with respect to a parameter,

$$\frac{\partial}{\partial \alpha} |_{\alpha=0} DQ(\tau) \equiv \delta DQ_\tau = (DX - B\dot{T})(Q(\tau), \tau) \tag{3.8}$$

is the (forward) probabilistic generalization of the classical relation between infinitesimal variations with and without time change, often denoted by

$$\delta \dot{q} = \frac{d}{dt} \delta q - \dot{q} \frac{d}{dt} \delta t \tag{3.9}$$

in classical books on calculus of variations [8-19]. In Eq. (3.9), δq symbolizes a spatial variation like X, and δt a temporal one like T in Prop. 3.4. According to Eq. (3.9), under such a space - time transformation, the ordinary time derivative $\frac{d}{dt}$ along the trajectory and a variation δ do not commute anymore, as they do when $\delta t = 0$. We shall come back in Prop. 3.9 on the role of $\delta \dot{q}$.

From the measure theory viewpoint, however,the family of diffusions of Prop. 3.4 is not directly appropriate to mimic, starting from Definition 3.3, the argument leading to the classical Noether Theorem: measures corresponding to processes with different diffusion coefficients are not absolutely continuous with respect to each others (not comparable). Since each forward Bernstein diffusion is built from a positive solution η of Eq. (2.11) (1), let us restrict ourselves to space-time transformations leaving this equation invariant. More precisely, consider the transformations

$$\begin{cases} \tilde{\eta} = \eta + \frac{\alpha}{\hbar} \Phi(q, t) \eta + o(\alpha) \\ Q = q + \alpha X(q, t) + o(\alpha) \\ \tau = t + \alpha T(t) + o(\alpha) \end{cases} \tag{3.10}$$

where, in addition to those of Prop. 3.4, we allow a transformation of the solution η defining the diffusion z_t, involving a smooth real-valued infinitesimal generator Φ on $\mathbb{R}^3 \times I$. The transformation (3.10), with coefficients yet unspecified, are associated with the vector field (infinitesimal generator)

$$\mathcal{L} = X^i(q, t)\frac{\partial}{\partial q_i} + T(t)\frac{\partial}{\partial t} + \frac{\Phi}{\hbar}\eta(q, t)\frac{\partial}{\partial \eta} \tag{3.11}$$

(where Einstein's summation convention has been used, for $i = 1, 2, 3$) defined on the space of independent variables q, t and dependent variable η of Eq. (2.11) (1).

The problem is now to find X, T and Φ such that, for any positive regular solution η of

$$\hbar\frac{\partial \eta}{\partial t} = -\frac{\hbar^2}{2}\Delta_q \eta + V(q, t)\eta, \quad q \in \mathbb{R}^3, t \in I \tag{3.12}$$

then $\tilde{\eta}(Q, \tau)$ solves as well

$$\hbar\frac{\partial \tilde{\eta}}{\partial \tau} = -\frac{\hbar^2}{2}\Delta_Q \tilde{\eta} + V(Q, \tau)\tilde{\eta} \tag{3.13}$$

where Δ_Q denotes the Laplacian in the new configuration variable. This amounts to say that (3.12) is invariant under the twice extended local Lie group of transformations (3.10) and that we are looking for the symmetry group of this equation (*cf.* [20], for example). One shows that a necessary and sufficient condition for such invariance is the validity of some elementary partial differential equations for the coefficients X_i, T and Φ of $\mathcal{L}$, called the determining (or defining) equations of the group:

LEMMA 3.5. – *The determining equations of the above symmetry group of the heat equation (3.12), defined by the infinitesimal generator $\mathcal{L}$ are*

(1) $$\frac{\mathrm{d}T}{\mathrm{d}t} = 2\frac{\partial X^i}{\partial q^i}$$

(2) $$\frac{\partial X^i}{\partial t} = \frac{\partial \Phi}{\partial q^i}$$

(3) $$\frac{\partial \Phi}{\partial t} + \frac{\hbar}{2}\Delta\Phi = \frac{\mathrm{d}T}{\mathrm{d}t}V + X^i\frac{\partial V}{\partial q^i} + T\frac{\partial V}{\partial t}$$

(4) $$\frac{\partial X^i}{\partial q^j} + \frac{\partial X^j}{\partial q^i} = 0, \quad \textit{for } i = 1, 2, 3,\ j \neq i$$

Starting from the definitions (3.10), the proof consists in computing how the derivatives transform. The first (respectively second) extension refer to how the first (respectively second) partial derivatives are transformed under (3.10). For details of the computation in the one-dimensional free case $V = 0$, *cf.* [20].

The determining equations 1) to 4) are linear. For a given regular potential V they are always solvable in closed form. Indeed, as well known in Lie theory, they form an overdetermined system of equations. In the case of the heat equation (3.12) and for many physical potentials V of interest (free case, harmonic oscillator, Coulomb potential, etc...) their solutions are available in tables of Lie groups. When they are not, it is a simple, sometimes tedious, exercise to compute them. A few examples will be given later.

PROPOSITION 3.6. – *When the generators* X, T *and* Φ *of the transformation* (3.10) *satisfy the determining equations of the symmetry group of Eq.* (3.12), *the one-parameter family of diffusions of Prop.* (3.4) *solves weakly*

$$\mathrm{d}Q(\tau) = (B + \alpha(\nabla\Phi - X_q \cdot B))\,\mathrm{d}\tau + \sqrt{\hbar}\,\mathrm{d}\tilde{W}(\tau) \tag{3.15}$$

In particular, their respective probability measures are all absolutely continuous with respect to each other. Moreover, the invariance condition (3.3) *for the basic stochastic action functional is satisfied.*

Proof. – Equation (1) of the lemma implies that the α-coefficient of the martingale part is zero. On the other hand, using Eqs. 1) and 2), $DX - B\dot{T} = \nabla\Phi - X_q B$.

This proves the first assertion. Regarding the second one, the transformations (3.10) follow from the expansion around the identity $\alpha = 0$ of the relation between the (positive) regular solutions of the heat Eqs. (3.12) and (3.13):

$$\tilde{\eta}(q,t)e^{\frac{\alpha}{\hbar}\Phi(q,t)} = \eta(q + \alpha X(q,t), t + \alpha T(t))$$

Therefore

$$-\hbar\log\tilde{\eta}(q,t) - \alpha\Phi(q,t) = -\hbar\log\eta(q + \alpha X, t + \alpha T) + o(\alpha) \tag{$\star$}$$

Since $\tilde{\eta}$ solves (3.13), it follows from Th. 2.1 (via Fleming's "logarithmic transformation" $A(q,t) = -\hbar\log\tilde{\eta}(q,t)$ [14 b]) that the first term of the l.h.s. of ($\star$) can be interpreted, for $t = t_0$, as

$$E_{t_0}\int_{t_0}^{t_1} L(z_t, Dz_t, t)\,\mathrm{d}t,$$

with L the regular Lagrangian (2.14) associated with (3.13). In the conditions of Dynkin formula (2.29), the second term of the l.h.s. of $(\star)$ is $\alpha E_{t_0} \int_{t_0}^{t_1} D\Phi(z_t, t)dt$. In both cases, we have ignored (without lack of generality, by the lemma 3.0) the conditional expectations corresponding to boundary conditions.

Since η solves as well (3.12), the same Th. 2.1 enables us to reinterpret the r.h.s. of $(\star)$, with the same proviso, as

$$E_{t_0+\alpha T} \int_{t_0+\alpha T(t_0)}^{t_1+\alpha T(t_1)} L\,(z_t+\alpha X(z_t,t),\ D(z_t+\alpha X(z_t,t)),\ t+\alpha T)\,\mathrm{d}(t+\alpha T(t)).$$

By the determining equation of the process Q, *i.e.* $Q(t + \alpha T) = z_t + \alpha X(z_t, t)$, this reduces to

$$E_{\tau_0} \int_{\tau_0}^{\tau_1} L\,(Q_\tau,\ DQ_\tau,\ \tau)\,\mathrm{d}\tau$$

so that $(\star)$ means precisely that the invariance condition (3.3) is satisfied. This proves the second assertion.

Remark. – Here we have required the parameter invariance of the stochastic action functional (3.1) (see [25] for another approach). If we allow time-transformations $\tau = t + \alpha\, T\,(t,\, q)$ for $T\ :\ I \times \mathbb{R}^3 \to \mathbb{R}$ of C^1 class, then the symmetry group of Eq. (3.12) involves an additional determining equation, namely

$$\frac{\partial T}{\partial q^i} = 0, \quad i = 1,\, 2,\, 3. \tag{5}$$

It follows that $T = T\,(t)$ only, as anticipated in (3.10) is, in fact, a key necessary condition of our method. We shall come back later on the physical interpretation of this.

Now we are in the conditions to formulate our

Theorem of Noether

Suppose that the action functional (3.1) *is divergence invariant under the Lie group of transformations* (3.10), *namely, for any* $[t_0,\, {}^t 1] \subset I$,

$$E_{t_0} \int_{t_0}^{t_1} L\,(z_t,\ Dz_t,\ t)\,\mathrm{d}t = E_{\tau_0} \int_{\tau_0}^{\tau_1} L\,(Q_\tau,\ DQ_\tau,\ \tau)\,\mathrm{d}\tau - \alpha\, E_{t_0} \int_{t_0}^{t_1} D\Phi\,(z_t,\ t)\,\mathrm{d}t + o\,(\alpha) \tag{3.16}$$

where the generators X, T and Φ satisfy the determining equations of the symmetry group of the heat equation, as in Lemma 3.5, *and then the one parameter family of diffusions Q_τ is the one of Prop.* 3.6. *Then, along each Bernstein diffusion z_t, $t \in I$, critical point of the action* (3.1), *the process (using the summation convention)*

$$\{B^i(z_t,\, t)\, X^i(z_t,\, t) + \varepsilon\, (z_t,\, t)\, T\,(t) - \Phi\,(z_t,\, t)\}, \quad i = 1,\, 2,\, 3. \tag{3.17}$$

when it belongs to $\mathcal{D}_D$, is a $\mathcal{P}_t$ martingale, for B and ε the drift vector field and energy random variable (2.18-2.19).

We shall need the following consequence of Prop. 3.6 and Lemma 3.5:

LEMMA 3.7. – *Let z be a critical point of the left hand side action in Eq.* (3.16) *and therefore a solution of the regularized Euler-Lagrange equation* (2.15),

$$D\, D\, z_t = \nabla V\,(z_t, t) \ \text{ a. s. }, t \in [t_0,\, t_1]$$

Let Q_τ be the diffusion associated with the Lie group transformation of Prop. 3.4, *for α small enough. Then, if the determining equations of Lemma* 3.5 *hold, the diffusion Q_τ solve a.s.*

$$D\, D\, Q_\tau = \nabla V\,(Q_\tau, \tau) \ \text{ a. s. }, \tau \in [\tau\,(t_0),\, \tau\,(t_1)] \tag{3.18}$$

In other words, the form of the regularized Euler-Lagrange equations is preserved in the new space-time coördinates $(Q,\, \tau)$.

Proof. – The proof of Prop. 3.4 used the fact that

$$D\, Q\,(\tau) = \frac{D\, Q_t}{\mathrm{d}\tau/\mathrm{d}t}, \qquad \text{for } D\, Q_t = D z_t + \alpha\, D\, X\,(z_t,\, t)$$

and, as well, $\dfrac{\mathrm{d}\tau}{\mathrm{d}t} = 1 + \alpha\, \dot{T}$. Consequently, to the first order in α, we have found that $DQ\,(\tau) = B + \alpha\,(DX - B\,\dot{T})$. In the same way, since $T\,(t)$ is of bounded variation,

$$\begin{aligned} D\, D\, Q\,(\tau) &= \frac{D\, D\, Q_t}{\mathrm{d}\tau/\mathrm{d}t} \\ &= \frac{D\, D\, z + \alpha\,(D\, D\, X - D\, B.\dot{T} - B\,\ddot{T})}{1 + \alpha\dot{T}} \\ &= D\, D\, z + \alpha\,(D\, D\, X - D\, B.\dot{T} - B\,\ddot{T} - D\, D\, z\, \cdot \dot{T}) + o\,(\alpha^2). \end{aligned}$$

Also

$$\nabla V(Q,\tau) = \nabla V(z,t) + \alpha \Delta V.X + \alpha \frac{\partial^2 V}{\partial z \partial t} T + o(\alpha^2)$$

By Itô's calculus, one can compute the factor of α in the r.h.s. of the first relation. Using the determining equations of Lemma 3.5, including the gradient of the relation 3), and the fact that $DDz_t = \nabla V(z_t, t)$ a.s., one shows that this factor of α is identically zero. □

Remark. – One verifies that $Q(\tau)$ solving (3.15) can be interpreted, at the first order in α, as a Bernstein diffusion associated with the heat equation (3.13), *i.e.* (*cf.* (2.10) (1)):

$$dQ(\tau) = \hbar \nabla \log \tilde{\eta}(Q(\tau), \tau)\, d\tau + \sqrt{\hbar}\, d\tilde{W}(\tau) \tag{3.19}$$

for $\tilde{W}$ a certain $\mathbb{R}^3$-valued Wiener process.

Proof of the Theorem of Noether. – Consider the action $S^\alpha[Q]$ in the r.h.s. of (3.16), for any $Q(\tau)$ solving weakly (3.19).

The first step is to transform this action into a parameter invariant one, in analogy with the classical case (3.6):

$$E_{t_0} \int_{t_0}^{t_1} \left\{ L\left(Q^\alpha(t), \frac{DQ^\alpha(t)}{d\tau^\alpha/dt}, \tau^\alpha(t)\right) + \alpha D\Phi(Q^\alpha(t), \tau^\alpha(t)) \right\} \frac{d\tau^\alpha}{dt} dt \tag{3.20}$$

where the α-dependence has been made explicit in the change of variables defined by Prop. 3.4. The resulting action on the extended configuration space is denoted by $J^\alpha[Q^\alpha, \tau^\alpha]$. Clearly, $Q^0(t) = z_t$ and $\tau^0(t) = t$. Also $J^0[z, t]$ coincides with the left hand side action of the invariance condition (3.16). This condition can therefore be interpreted as

$$\frac{\partial J^\alpha}{\partial \alpha}[Q^\alpha, \tau^\alpha]\,|_{\alpha=0} \equiv \delta J[z_t, t](X, T) \equiv \delta J = 0$$

where, for t in $[t_0, t_1]$, $\varphi^\alpha : I \to \mathbb{R}$

$$\begin{cases} dQ^\alpha(t) = \hbar \nabla \log \tilde{\eta}(Q^\alpha(t), \tau^\alpha(t)) \varphi^\alpha dt + \sqrt{\hbar \varphi^\alpha} dW_t \\ d\tau^\alpha(t) = (1 + \alpha T)\, dt \equiv \varphi^\alpha dt \end{cases} \tag{3.21}$$

The first equation of (3.21) is just (3.19), read in the original time coordinate of the basic action functional, the second is the one already used in Prop. 3.4.

At the first order in α, Eqs. (3.21) are consistent with the conclusion of Prop. 3.4. For example,

$$\frac{\partial}{\partial\alpha}\hbar\nabla\log\tilde{\eta}\,(Q^{\alpha}(t),\,\tau^{\alpha}(t))\,|_{\alpha=0}=\frac{\partial}{\partial\alpha}\left(\frac{DQ_t^{\alpha}}{\varphi^{\alpha}}\right)|_{\alpha=0}$$
$$=D\,X\,(z_t,\,t)-B\,(z_t,\,t)\,\dot{T}\,(t),$$

as found in Eq. (3.8).

Using twice Itô's integration by parts formula, once for Dz_t and $X\,(z_t,\,t)$ and another time for $\varepsilon\,(z_t,\,t)$ and $T\,(t)$, the computation of δJ yields, finally,

$$\begin{aligned}\delta J =&E_{t_0}[B^i(z_t,\,t)\,X^i(z_t,\,t)+\varepsilon\,(z_t,\,t)\,T\,(t)-\Phi\,(z_t,\,t)\,|_{t=t_0}^{t=t_1}]\\ &-E_{t_0}\int_{t_0}^{t_1}(D\,D\,z_t-\nabla V\,(z_t,\,t))\,X\,(z_t,\,t)\,\mathrm{d}t\\ &-E_{t_0}\int_{t_0}^{t_1}\left(D\,\varepsilon\,(z_t,\,t)-\frac{\partial V}{\partial t}\,(z_t,\,t)\right)T\,(t)\,\mathrm{d}t\end{aligned}$$

But we know that the critical Bernstein diffusions $z_t,\ t\in I\subset[t_0,\,t_1]$ satisfy almost surely, by (2.15) and (2.28),

$$D\,D\,z_t=\nabla V\,(z_t,t)\ \text{and}\ D\,\varepsilon\,(z_t,\,t)=\frac{\partial V}{\partial t}(z_t,\,t)$$

Therefore, the condition $\delta J=0$ means that $\forall[t_0,\,t_1]\subset I$,

$$\begin{aligned}&E_{t_0}[B^i(z_{t_1},\,t_1)\,X^i(z_{t_1},\,t_1)+\varepsilon\,(z_{t_1},\,t_1)\,T\,(t_1)-\Phi\,(z_{t_1},\,t_1)]\\ &\quad=B^i(z_{t_0},\,t_0)\,X^i(z_{t_0},\,t_0)+\varepsilon\,(z_{t_0},\,t_0)\,T\,(t_0)-\Phi\,(z_{t_0},\,t_0)\end{aligned}\tag{3.23}$$

i.e. that the random variable (3.17) is a $\mathcal{P}_t$-martingale, $t\in I$.

Remark. – Generally, the drift B (*cf.* (2.10) (1)) of a Bernstein diffusion depends explicitly on $\hbar$. Let us denote by B^0 its $\lim_{\hbar\to 0}B\,(q,\,t)$ when it exists. When $\hbar=0$, D reduces to the ordinary derivative along a solution of the first order ODE: $\dot{q}=B^0(q,\,t)$. For B^0 a continuous and uniformly Lipschitzian vector field, this equation has a unique solution for a given initial condition. Analogously, the energy random variable $\varepsilon\,(z_t,\,t)$ reduces for $\hbar=0$ to $\varepsilon\,(q,\,t)=-\frac{1}{2}|\dot{q}|^2+V\,(q,\,t)$ *i.e.* the (Euclidean) energy associated with the Lagrangian (2.14). As suggested by Feynman, the classical limit of observables is simpler, here, than in regular quantum mechanics.

When $\hbar=0$, Eq. (3.23) defines an ordinary constant of motion along the smooth path $t\to q\,(t)$ and our conclusion coincides with the one of the classical Noether's Theorem 1.7 for the Lagrangian (2.14).

When $\hbar > 0$, the stochastic differential equation (2.8) (1) is a singular perturbation of this classical dynamical system, in Feynman's sense. It is well known that the path $t \to z_t$ converge in probability toward $q(t)$.

In the classical framework, one shows easily that the invariance condition (1.7), valid $\forall [t_0, t_1]$, is equivalent to a condition involving only the Lagrangian and not the action functional. The computation of $\frac{\partial}{\partial \alpha} |_{\alpha=0}$ provides us with the

PROPOSITION 3.9. – *A necessary condition for the validity of the classical invariance condition* (1.7) *is that the Lagrangian* L *of the action* (1.2) *satisfies*

$$\frac{\partial L}{\partial t} T + \frac{\partial L}{\partial q^i} X^i + \frac{\partial L}{\partial \dot{q}^i} \left(\frac{\mathrm{d}X^i}{\mathrm{d}t} - \dot{q}^i \frac{\mathrm{d}T}{\mathrm{d}t} \right) + L \frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\mathrm{d}\Phi}{\mathrm{d}t} \tag{3.24}$$

where the summation convention is used, $i = 1, 2, 3$.

According to Lie group theory, the 3 first terms on the l. h. s. of Eq. (3.24) constitute the first prolongation of the infinitesimal generator $T(t) \frac{\partial}{\partial t} + X_i(q, t) \frac{\partial}{\partial q^i}$ associated with the transformations (1.6) when $T = T(t)$ (*cf.* [20]). The factor of $\frac{\partial L}{\partial \dot{q}^i}$ corresponds to the symbolic expression (3.9).

The following regularization of (3.24) holds for the critical Bernstein diffusion z_t:

PROPOSITION 3.10. – *If* $L(q, \dot{q}, t)$ *denotes our basic Lagrangian* (2.14) *a necessary condition for the invariance condition* (3.16) *is that along the paths of the diffusion* z_t, $t \in I$, *of the Theorem,*

$$\frac{\partial L}{\partial t} T + \frac{\partial L}{\partial z^i} X^i + \frac{\partial L}{\partial (D z)^i} (D X^i - (D z)^i D T) + L D T = D \Phi \tag{3.25}$$

almost surely.

Proof. – Let us rewrite the determining equations of Lemma 3.5 as almost sure identities:

$$\begin{aligned} D T(t) &= 2 \frac{\partial X^i}{\partial q^i}(z_t, t) \\ D X(z_t, t) &= \frac{\partial X}{\partial t}(z_t, t) + X_q(z_t, t) B(z_t, t) + \frac{\hbar}{2} tr(X_{qq}) \end{aligned} \tag{3.26}$$

where $tr(X_{qq})$ denotes the trace of the 3×3 matrix whose elements are 3 vectors and X_q was defined in Prop. 3.4. By the determining equations 2) and 1) of Lemma 3.5,

$$D X(z_t, t) = \nabla \Phi(z_t, t) + X_q(z_t, t) B(z_t, t) \tag{3.27}$$

Therefore

$$D X^i - (D z)^i D T = (\nabla \Phi - X_q B)^i, \quad i = 1, 2, 3$$

Using also 4): $\frac{\partial X^i}{\partial q^j} = 0$, $i \neq j$, (3.25) reduces to

$$\frac{\partial V}{\partial t} T + \frac{\partial V}{\partial q^i} X^i = D \Phi - (D z)^i \nabla \Phi - 2 \frac{\partial X^i}{\partial q^i} V, \quad i = 1, 2, 3, \tag{3.28}$$

which coincides with 3) of Lemma 3.5. The necessity of (3.25) for the invariance condition (3.16), $\forall [t_0, t_1]$, is verified as in the classical case. *cf.* [25].

The dynamical caracterization of the critical diffusion z_t, *i. e.* the Euler Lagrange equation (2.15) and conservation of energy (2.28), together with the determining equations as above are, actually, sufficient to obtain directly the conclusion of the Theorem of Noether:

PROPOSITION 3.11. – *Let z_t, $t \in I$, a critical point of the action functional* (3.1). *Then the determining equations of Lemma* 3.5 *imply that, almost surely,*

$$D (B^i(z_t, t) X^i(z_t, t) + \varepsilon (z_t, t) T (t) - \Phi (z_t, t)) = 0 \tag{3.29}$$

Proof. – By definition of a critical diffusion $z_t, t \in I$, (2.15), $DDz_t = \nabla V (z_t, t)$ holds, almost surely, as well as, by (2.28), $D\varepsilon (z_t, t) = \frac{\partial V}{\partial t}(z_t, t)$. Itô's formula for the l.h.s. of (3.29) and the fact that $t \to T (t)$ is of bounded variation imply that the left hand side of (3.29) is

$$D B^i(z_t, t) X^i(z_t, t) + B^i(z_t, t) DX^i(z_t, t) + \hbar \frac{\partial B^j}{\partial q^i} \frac{\partial X^j}{\partial q^i}$$
$$+ D\varepsilon (z_t, t) T (t) + \varepsilon (z_t, t) DT (t) - D\Phi (z_t, t)$$

After substitution of (3.26)-(3.28), of the Euler-Lagrange equation and of the conservation of energy, the resulting expression is zero, almost surely.

We are going to consider some illustrations of our theorem of Noether. First, the probabilistic counterparts of the classical cases (1.9) and (1.10):

a) If the Lagrangian (2.14) of the action functional (3.1) is independent on the i-component q^i of the vector q, then (3.1) is invariant under the transformation

$$U_\alpha : (q, t) \to (q + \alpha e_i, t)$$

where e_i denotes the unit vector along the axis i. This means that the invariance condition (3.16) holds for the generator $X = e_i$ and $T = \Phi = 0$.

Then (3.17) says that the i-component of the drift vector field (probabilistic counterpart of the momentum for the Lagrangian (2.14)) is a martingale.

b) If the Lagrangian (2.14) (*i.e.* the potential V) is independent on the time t, then (3.1) is invariant under

$$U_\alpha : (q, t) \to (q, t + \alpha)$$

corresponding to $X = \Phi = 0$ and $T = 1$ in the invariance condition (3.16). Then (3.17) says that the energy random variable is a martingale. This we know already by Eq. (2.28) for $V = V(z_t)$.

c) Consider the free case, *i.e.* the Lagrangian (2.14) when $V = 0$. By construction of our Noether's Theorem, the generators X, T and Φ associated with the invariance condition (3.16) are the ones defining the symmetry group of heat equation (3.12). In the free case, the symmetry group is 13 dimensional. (In general, $\frac{1}{2}d(d+3)+4$ dimensional, where d is the dimension of the configuration space of the system.) The generators X, T and Φ are found in tables of Lie groups (*cf.* [20], for example). For Eq. (3.12), their computation dates back to Lie himself.

When $d = 1$, here is a list of these $\frac{1}{2}(1+3)+4 = 6$ generators:

	$X(q, t)$	$T(t)$	$\Phi(q, t)$
(1)	0	0	-1
(2)	0	1	0
(3)	1	0	0
(4)	t	0	q
(5)	q	$2t$	0
(6)	qt	t^2	$\frac{1}{2}(q^2 - \hbar t)$

The Lie algebra of the infinitesimal symmetries of the one-dimensional free heat equation (3.12) is spanned by the 6 vector fields (3.11) associated with those generators. The symmetries (2) and (3) are, respectively, particular instances of b) and a) above.

d) For the one dimensional harmonic oscillator $V(q) = \frac{\omega^2}{2} q^2$, the symmetry group is also six-dimensional and, actually, locally isomorphic to the one of the case c). The generators are the following:

	$X(q, t)$	$T(t)$	$\Phi(q, t)$
(1)	0	0	-1
(2)	0	1	0
(3)	$\cosh \omega t$	0	$\omega q \sinh \omega t$
(4)	$\frac{\sinh \omega t}{\omega}$	0	$q \cosh \omega t$
(5)	$q \cosh(2\omega t)$	$\frac{\sinh(2\omega t)}{\omega}$	$\omega q^2 \sinh(2\omega t) - \frac{\hbar}{2}(\cosh(2\omega t) - 1)$
(6)	$q \frac{\sinh(2\omega t)}{2\omega}$	$\frac{\cosh(2\omega t) - 1}{2\omega^2}$	$\frac{q^2 \cosh(2\omega t)}{2} - \frac{\hbar}{4\omega} \sinh(2\omega t)$

For some purposes, other basis may be more natural for this Lie algebra.

It is clear that the Lie group component of our Theorem of Noether is crucial. As a matter of fact, we are going to show that a pure Lie group proof of its conclusion is available.

Let us consider the following differential operator associated with the transformation (3.10):

$$N = X^i(q, t)\frac{\partial}{\partial q^i} + T(t)\frac{\partial}{\partial t} - \frac{\Phi}{\hbar}(q, t) \tag{3.30}$$

A solution η of the heat equation is said to satisfy the invariance surface condition if $(N\eta)(q, t) = 0$. When this is not the case, Lie group theory shows that the invariance of Eq. (3.12) under (3.10) implies that the exponential map

$$\begin{aligned}\eta_\alpha(q, t) =& (e^{-\frac{\alpha}{\hbar}N}\eta)(q, t)\\ \equiv& \Sigma_{k=0}^{\infty}\frac{(-\alpha/\hbar)^k}{k!}(N^k\eta)(q, t), \text{for } \alpha \text{ close enough from } 0,\end{aligned} \tag{3.31}$$

defines a one parameter family of solutions of the same heat equation [20]. One shows easily that, in these conditions, for any real constants γ_j, $j = 0, 1, \ldots n$

$$\Sigma_{j=0}^{n}\gamma_j(N^j\eta)(q, t) \text{ is also solution of Eq. (3.12).}$$

In particular, if η is a solution of (3.12), $N\eta$ is also a solution so that, introducing the heat operator for Eq. (3.12):

$$\aleph = \hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2}\Delta - V \equiv \hbar\frac{\partial}{\partial t} - H, \tag{3.32}$$

if $\aleph\eta = 0$ then $\aleph N\eta = 0$.

Now define a real-valued function n on $\mathbb{R}^3 \times I$ by

$$n(q, t)\,\eta(q, t) = (N\eta)(q, t) \tag{3.33}$$

Equivalently, from the definition (2.17), $n(q, t)$ is the (forward) random variable associated with the differential operator N. Suppose that $(N\eta) \in \mathcal{D}_H$, where $\mathcal{D}_H$ is the domain of the Hamiltonian H.

By Eqs. (2.22)-(3.33), along a critical diffusion z_t, $t \in I$,

$$D n(z_t, t) = \frac{1}{\eta(z_t, t)}\Big(\frac{\partial}{\partial t} - \frac{1}{\hbar}H\Big)(N\eta)$$

is zero, almost surely, since $N\eta$ is solution of the heat equation (3.12) when η is. This is equivalent to say that $n(z_t, t)$ is a $\mathcal{P}_t$-martingale. In other words, using the definitions (3.30) and (3.33), as well as (2.10) (1) and (2.19),

$$\begin{aligned} n(q, t) &= X^i(q, t)\hbar\frac{\partial}{\partial q^i}\log\eta(q, t) + T(t)\hbar\frac{\partial}{\partial t}\log\eta(q, t) - \Phi(q, t) \\ &= B^i(q, t)\,X^i(q, t) + \varepsilon(q, t)\,T(t) - \Phi(q, t), \quad i = 1, 2, 3 \end{aligned}$$

is a $\mathcal{P}_t$ martingale, as found in (3.17). We have proved the

PROPOSITION 3.12. – *The real valued function n on $\mathbb{R}^3 \times I$ defined by $n(q, t)\,\eta(q, t) = (N\eta)(q, t)$, where N is the differential operator* (3.30) *and η is a positive solution of Eq.* (3.12) *in $\mathcal{D}_N$ which does not satisfy the invariance surface condition $(N\eta) = 0$, is a $\mathcal{P}_t$-martingale when evaluated along the corresponding Bernstein diffusion z_t indexed by $t \in I$.*

The proof of the last proposition involves implicitly, besides the starting Bernstein diffusion z_t associated with the solution η of Eq. (3.12) another one, $\tilde{z}_t$, associated with the new solution $(N\eta)$. What is the probabilistic relationship between these diffusions? The answer is the

PROPOSITION 3.13. – *The diffusion $\tilde{z}_t$, $t \in \tilde{I}$, is a Doob's h-transform of the original one. In particular, when the drift vector field of z_t is given by Eq.* (2.10) (1), *the drift of $\tilde{z}_t$ (starting from the same point as z_t) is*

$$\tilde{B}(q, t) = B(q, t) + \nabla\log n(q, t) \tag{3.34}$$

where $n(q, t)$ was defined in Prop. 3.12 *and assumed to be strictly positive.*

Proof. – Let P the probability measure of z_t on the measurable space $(\Omega, \mathcal{P}_t)$, $t \in I$, and for which the transition function is $P(s, x, t, A) = \int_A p(s, x, t, y)\, dy$, where $s \leq t \in I$, $x, y \in \mathbb{R}^3$ and A is a Borelian of $\mathbb{R}^3$. According to Prop. 3.12, the function n solves, a.s, $Dn(z_t, t) = 0$ *i.e.* $n(z_t, t)$ is a $\mathcal{P}_t$-martingale. Then a measure $\tilde{P}$ exists, is absolutely continuous with respect to P, and with Radon-Nikodym density

$$\frac{\mathrm{d}\tilde{P}}{\mathrm{d}P} \,|_{\mathcal{P}_t} = n(z_t, t)$$

Under $\tilde{P}$ the process has the transition density with respect to $\mathrm{d}y$

$$\tilde{p}(s, x, t, y) = p(s, x, t, y)\, \frac{n(y, t)}{n(x, s)}, \quad s \leq t,\ x, y \in \mathbb{R}^3$$

and therefore, by definition of the drift vector field

$$\tilde{B}(q, t) = B(q, t) + \nabla \log n(q, t)$$

This is the definition of Doob's h-transform [21].

As shown by the Euclidean Born interpretation (2.16), a fundamental property of Bernstein diffusions is their time symmetry, if appropriate boundary conditions for (2.11) (1) and (2.11) (2) are chosen.

The Noether's Theorem proved above, as well as all the results of this chapter, involve exclusively the heat equation (2.11) (1), *i.e.* the nondecreasing filtration $\mathcal{P}_t$. In order to restore the time symmetry needed for quantum physics, one shows that these results admit a counterpart with respect to the nonincreasing filtration $\mathcal{F}_t$, *i.e.* in terms of the associated positive solution of (2.11) (2). Starting from an action functional which is formally a time reversal of (3.1) (if the scalar potential V is time independent) namely

$$S_*[z(\cdot)] = E_{t_1} \int_{t_0}^{t_1} L(z_s, D_* z_s, s)\, \mathrm{d}s, \quad t_0 < t_1 \text{ in } I \tag{3.35}$$

where a conditional expectation giving the future has been used, as well as the associated (backward) derivative (2.9), one may introduce an invariance up to a divergence term as in the Definition 3.3. The effect of any space-time transformation, denoted now by

$$\begin{cases} Q = q + \alpha X_*(q, t) + o(\alpha) \\ \tau = t + \alpha T_*(t) + o(\alpha) \end{cases} \tag{3.36}$$

where $X_* : \mathbb{R}^3 \times I \to \mathbb{R}^3$ and $T_* : I \to \mathbb{R}$ are as regular as X and T, on the stochastic differential equation (2.8) (2) with respect to the filtration $\mathcal{F}_t$, is analyzed as in Prop. 3.4. After addition of a smooth scalar generator Φ_*:

$$\tilde{\eta}_* = \eta_* - \frac{\alpha}{\hbar} \Phi_*(q,\, t)\, \eta_* + o(\alpha) \tag{3.37}$$

the study of the invariance of (3.35) becomes the study of the symmetry group of the heat equation (2.11) (2). The determining equations of this group are slightly different of the ones of Lemma 3.5:

$$\begin{array}{ll}
(1) & \dfrac{\mathrm{d}T_*}{\mathrm{d}t} = 2\, \dfrac{\partial X_*^i}{\partial q^i} \\
(2) & \dfrac{\partial X_*^i}{\partial t} = -\dfrac{\partial \Phi_*}{\partial q^i} \\
(3) & \dfrac{\partial \Phi_*}{\partial t} - \dfrac{\hbar}{2} \Delta \Phi_* = -\dfrac{\mathrm{d}T_*}{\mathrm{d}t} V - X_*^i \dfrac{\partial V}{\partial q^i} - T_* \dfrac{\partial V}{\partial t} \\
(4) & \dfrac{\partial X_*^i}{\partial q^j} + \dfrac{\partial X_*^j}{\partial q^i} = 0, \quad \text{for } i = 1,\, 2,\, 3,\ j \neq i
\end{array} \tag{3.38}$$

Comparing (3.38) with the lemma 3.5, it is clear that we may always choose $X = X_*$ and $T = T_*$. But Φ will differ, in general, from Φ_*.

Then, the invariance of the action functional (3.35) under the above Lie group of transformations imply that the process

$$B_*^i(z_t,\, t)\, X_*^i(z_t,\, t) + \varepsilon_*(z_t,\, t)\, T_*(t) - \Phi_*(z_t,\, t) \tag{3.39}$$

(where the summation convention is used, $i = 1,\, 2,\, 3$) is an $\mathcal{F}_t$-(2.20) and ε_* the backward energy (2.21) of the diffusion z_t, $t \in I$.

One shows as in Lemma 3.7 that the form of the backward version of the Euler-Lagrange equation (2.15), namely, here,

$$D_* D_* z_t = \nabla V\,(z_t, t) \text{ a.s., } t \in [t_0,\, t_1] \tag{3.40}$$

is preserved by the change of variables (3.36).

The necessary condition for the invariance of the action is as (3.25), for the $*$ generators and the backward derivative D_*. The right hand side of (3.25) is replaced by $-D_* \Phi_*$.

In analogy with (3.30) the differential operator

$$N_* = X_*^i(q,\, t) \frac{\partial}{\partial q^i} + T_*(t) \frac{\partial}{\partial t} + \frac{\Phi_*}{\hbar}\,(q,\, t) \tag{3.41}$$

is associated with the transformations (3.36-3.37) so that one defines a real valued random variable n_* by

$$n_*(q, t)\eta_*(q, t) = (N_*\eta_*)(q, t) \tag{3.42}$$

For η_* any positive solution of the heat equation (2.11) (2) such that the r.h.s. of (3.42) is nonzero. Using Eq. (2.23), as well as the definitions (2.20) and (2.21), one verifies that $n_*(z_t, t)$ coincides with (minus) the $\mathcal{F}_t$-martingale (3.39).

Finally we can illustrate, once again, the central role of time symmetry in our construction by considering the $\mathcal{F}_t$ analogue of Proposition 3.13. As noticed there, $n(z_t, t)$ is a $\mathcal{P}_t$-martingale used to define a new law $\tilde{P}$ under which the process z_t, $t \in I$, is a diffusion $\tilde{z}_t$ with drift (3.34). The analogue is true if the Bernstein diffusion z_t is regarded as defined on $(\Omega, \mathcal{F}_t)$ and solving (2.8) (2). Then $n_*(z_t, t)$, for n_* defined by (3.42), is an $\mathcal{F}_t$-martingale used to define a measure $\tilde{P}_*$ under which the process has the backward drift

$$\tilde{B}_*(q, t) = B_*(q, t) - \nabla \log n_*(q, t) \tag{3.43}$$

when ended at the same point as for (2.8) (2).

Each of those 2 descriptions uses only one filtration and then choose definitely a direction of time. Now, suppose that $(N\eta)$ and $(N_*\eta_*)$ are two positive solutions of the heat equations (2.11) (1) and (2) if η and η_* are. From them, a Bernstein process $\tilde{z}_t$ can be constructed, as in Chapt. 2. According to Eq. (2.16), its probability law at time t has density

$$(N\eta)(N_*\eta^*)(q, t)\,\mathrm{d}q = n(q, t)\,n_*(q, t)\,\mu_t^B(\mathrm{d}q) \tag{3.44}$$

where the density (2.16) of the starting diffusion z_t (2.16) has been introduced. So $(n\,n_*)(q, t)$ is the Radon-Nykodim density of the distribution of $\tilde{z}_t$ with respect to the distribution of z_t.

The relation (3.44) is time symmetric. As for any Bernstein diffusion associated with Eqs. (2.11) (1) and (2.11) (2),

$$\int_{\mathbb{R}^3} (N\eta)(N_*\eta^*)(q, t)\,\mathrm{d}q \tag{3.45}$$

is time independent (and assumed to be normalizable to 1) since the Hamiltonian H is self-adjoint in $L^2(\mathbb{R}^3)$). The r.h.s. of Eq. (3.44) can also be read, after integration, as

$$E[n\,n_*(z_t, t)] \tag{3.46}$$

Indeed, using the Theorem 11.12 of [18] and the fact that n (respectively n_*) is a $\mathcal{P}_t$ (respectively $\mathcal{F}_t$) martingale, assumed to be in $\mathcal{D}_D$ and $\mathcal{D}_{D_*}$ respectively,

$$\frac{\mathrm{d}}{\mathrm{d}t} E[n\, n_*(z_t,\, t)] = E\left[D\, n\,(z_t,\, t) \cdot n_* + n\, D_* n_*(z_t,\, t)\right] = 0 \qquad (3.47)$$

This is another illustration of the time symmetry of this probabilistic framework.

4. ANALYTICAL CONTINUATION IN TIME AND QUANTUM SYMMETRIES

Schrödinger's Euclidean quantum mechanics uses some unfamiliar tools of stochastic analysis. Its probability structure is, however, as close as possible to the one of regular quantum mechanics.

After analytical continuation in time, the Euclidean Born interpretation (2.16) reduces to the usual probabilistic interpretation of the wave function ψ solving the Schrödinger equation for the same Hamiltonian (2.7). But the construction of the Bernstein diffusions summarized in paragraph 2 collapses; the associated probability measures require to be built from positive solutions of the heat equation and not the Schrödinger one. In particular, the positivity of the integral kernel of the heat equation is crucial to this construction (*cf.* [9 b)]) while the corresponding quantum kernel is not even real.

Still, our diffusion z_t was built in such a way that its role is played, in real time, by Heisenberg's position observable $Q_H(t)$ (*cf.* Eq. (2.17)). It follows, in particular, that the structure of the quantum symmetries is quite similar to the Euclidean ones described above.

The determining equations of the "real time" symmetry group are very similar (except for a few factors i) to the ones of lemma 3.5, namely:

LEMMA 3.5. – *In real time.*

(1) $$\frac{\mathrm{d}T}{\mathrm{d}t} = 2\frac{\partial X^i}{\partial q^i}$$

(2) $$\frac{\partial X^j}{\partial t} = -i\frac{\partial \Phi}{\partial q^j}$$

(3) $$i\frac{\partial \Phi}{\partial t} + \frac{\hbar}{2}\Delta\Phi = \frac{\mathrm{d}T}{\mathrm{d}t}V + X^i\frac{\partial V}{\partial q^i} + T\frac{\partial V}{\partial t}$$

(4) $$\frac{\partial X^i}{\partial q^j} + \frac{\partial X^j}{\partial q^i} = 0 \textit{ for } i = 1, 2, 3, j \neq i.$$

Their solutions are essentially the same, evaluated on (it) instead of t. Reading backward Eq. (2.17) and the relation (2.22) between probability theory and analysis in L^2 space, one sees that the real time version of the conclusion (3.17) of our Theorem of Noether is that the collection of time dependent observables (before symmetrization)

$$N(t) = X^i(Q_H(t), t)P^i - T(t)H + i\Phi(Q_H(t), t),\ i = 1, 2, 3 \qquad (4.1)$$

are quantum constants of the motion, for P and H, respectively, the momentum and Hamiltonian observables of the system, and X, T, Φ the solutions of the real time determining equations.

Of course, the rigorous proof involves the familiar tools of quantum theory in Hilbert space, with the determination of the dense domains of operators etc. This is the content of [36], together with various applications in quantum physics. However, the verification that the observables (4.1) are constants of motion is easily done, using the Heisenberg equations of motion of the system, the real time determining equations and formal calculations with commutators.

As an illustration (*cf.* [36] for more) let us mention the real time version of the one dimensional free case. The table of generators becomes

	$X(q, t)$	$T(t)$	$\Phi(q, t)$
(1)	0	0	$-i$
(2)	0	1	0
(3)	1	0	0
(4)	$-t$	0	$-iq$
(5)	q	$2t$	$\frac{-\hbar}{2}$
(6)	$-qt$	$-t^2$	$\frac{1}{2}(\hbar t - iq^2)$

After substitution in (4.1), we observe that the first conservation laws of the free quantum particle are familiar: energy, momentum, pure Galilean transformations. But the two last ones are unfamiliar, time dependent, constants of motion of the free quantum particle.

The possible $\hbar$-dependence of the Φ factor, in (4.1), is interesting. It means that Φ does not, in general, coincide with its classical counterpart. In fact, denoting by φ this classical (real) coefficient, one shows that

$\Phi = -i\varphi - \frac{\hbar}{2} \frac{\partial X^i}{\partial q^i}$. This relation is a necessary condition for the symmetry of the operator $\tilde{N}(t)$ defined formally by (4.1) (*cf.* [36]).

More generally, all the familiar quantum conservation laws will always be included in the family of constant observables (4.1) associated with a given Hamiltonian (2.7), but new ones will appear as well. This is due to the fact that our Euclidean Noether Theorem (3.16)-(3.17) includes a divergence term Φ, rarely considered even at the classical limit $\hbar = 0$ and also to the, generally, time dependent nature of the constant observables (4.1). It does not seem that time dependent constants of motion in quantum mechanics have been thoroughly investigated.

5. DISCUSSION AND PROSPECTS

As observed in Lemma 3.7, the form of the regularized Euler-Lagrange equations is preserved under the space-time transformations (3.10). In the classical case $\hbar = 0$, this property characterizes a canonical transformation [1-19b]. This means that all the transformations such that the invariance condition (3.16) holds correspond to probabilistic canonical transformations. In particular, those transformations preserve as well Feynman's commutation relations $\widetilde{(1.14)}$,

$$E\left[Q_l(\tau)\, D_* Q_m(\tau) - D Q_m(\tau) Q_l(\tau)\right] = \hbar \delta_{lm}, \quad l, m = 1, 2, 3$$

It will be interesting to study systematically such transformations, especially in the perspective of (Euclidean) quantum field theory.

Let us call R the system with space-time coordinates (q^i, t), $i = 1, 2, 3$ and $\tilde{R}$ the one with coordinates (Q^i, τ). If we allow, in the basic action functional $S[z(\cdot)]$, time transformations of the form $\tau = t + \alpha T(t, q)$, then two simultaneous events with respect to the system of reference R will not be, in general, simultaneous anymore with respect to $\tilde{R}$. However, if we limit ourselves (motivated by the parameter invariance of the action, cf. Lemma 3.2) to transformations of the form $\tau = t + \alpha T(t)$, the simultaneity will be preserved. Consequently, our class of time transformations was just a basic requirement for our non relativistic framework.

The family of Lagrangians L for which our Noether Theorem is valid seems much more restricted than in classical mechanics. It is, however, not limited to the elementary case (2.14) of a classical particle in a scalar potential V. For example, it holds as well for the Lagrangians (or action) considered in [12], *i.e.* for the (Euclidean) classical particle in an electro-magnetic field. Such limitations have nothing to do with our Noether Theorem itself, but are due to the structure of Schrödinger's Euclidean Quantum Mechanics. It seems that this strategy works for classical Lagrangians quadratic in the velocities. This limitation is already present in Feynman's path integral approach [3]; the coincidence is not accidental.

However, and in striking contrast with Feynman's approach, our Euclidean strategy is not limited to (Markovian) diffusion processes. Specialists of stochastic control theory know, for example, how to deal with Jump Markov processes [14b)]. The role of the Hamilton-Jacobi Bellman equation is played by another nonlinear partial differential equation ("Dynamic programming equation") and the analogue of our starting action functional (2.13) is known. Since Schrödinger's strategy (*cf.* (2.11) and (2.16)) is, essentially, independent on the form of the quantum Hamiltonian H (*cf.* [22]), most of what we did here for diffusions will hold for other Markov processes. The case of jump processes is especially attractive since Feynman complained often that his path integrals do not permit a simple incorporation of spin degrees of freedom (*cf.* Conclusion of [3]).

There are other directions of generalization of the present Theorem of Noether.

From the strictly probabilistic point of view, our strategy may present some interest since it forces us to introduce there some geometrical structures of classical mechanics, which are not at all familiar in this context, but may have a more general range of validity than for Euclidean Quantum Mechanics.

For example, the martingale constructed in Eq. (3.17) can be regarded as the value of the stochastic Poincaré-Cartan differential form (involving the Stratonovich product $\circ$ [5])

$$\omega^1 = B \circ \mathrm{d}z - \varepsilon\, \mathrm{d}t$$

on the tangent vector field (X, T) of the family of transformations $U_\alpha(q, t) = (Q, \tau)$ defined, when $\alpha \to 0$, as in Prop. 3.4. A stochastic geometry of the phase space can be constructed, for each filtration, from ω^1, in analogy with the classical case [1].

Let us mention another exciting direction of investigation regarding the Hamilton-Jacobi-Bellman equation.

At the classical limit $\hbar = 0$, it is well known that one of the most powerful method of integration of the classical equations of motion is the one of Jacobi, involving a "complete integral" of the Hamilton-Jacobi equation.

This is a solution of this equation depending on n parameters (where n is the dimension of the configuration space of the system, here $n = 3$). The n parameters are constants of motion of the equation of motion. If such a complete integral of Hamilton-Jacobi equation is known, the Theorem of Jacobi says that the classical equations of motion are integrable "by quadrature" (*cf.* [1]). Our Theorem of Noether suggests that a probabilistic concept of complete integrability for the Hamilton-Jacobi-Bellman equation is accessible, and that the knowledge of n martingales could enable us to solve explicitly this equation.

From the geometrical point of view, the method advocated here gives access to various interesting problems in the probabilistic approaches to Quantum Physics. The above mentioned study of canonical transformations is just one of them; Djehiche and Kolsrud [23] consider those transformations as harmonic morphisms. Also, as mentioned in Chapt. 1, in the case of diffusions, very weak regularity conditions are required for the critical processes, using the method of viscosity solutions for the Hamilton Jacobi-Bellman equation [14]. The regularity imposed here can, therefore, be weakened without modification of our strategy. This is true even in infinite dimensions and therefore should be relevant for Euclidean Quantum Field Theory [24] since it has been shown recently that Schrödinger's strategy still holds in this case [30].

Regarding the consequence of our Noether Theorem in regular quantum mechanics, the new family of constant observables (4.1) is often more natural than the Hamiltonian itself for the investigation of a quantum system. This is true, for example, when the potential V depends on time, i.e. when the system is not conservative. Various physical applications along this lines will be considered in [36].

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