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Standard generalized vectors for partial O*-algebras

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ABSTRACT. – Standard generalized vectors for an O*-algebra $\mathfrak{N}$ lead to a Tomita-Takesaki theory of modular automorphisms on $\mathfrak{N}$, and this is a key step in constructing KMS states on $\mathfrak{N}$ (which would represent equilibrium states if $\mathfrak{N}$ is the observable algebra of some physical system). In this paper, we extend to partial O*-algebras the notion of standard generalized vector and we show that they indeed satisfy the KMS condition. We also discuss several less restrictive classes of generalized vectors for a partial O*-algebra $\mathfrak{M}$, which all give rise to standard generalized vectors for a partial GW*-algebra canonically associated to $\mathfrak{M}$, either on the same domain or on a smaller dense domain. Finally we discuss the extension of standard generalized vectors from a von Neumann algebra $\mathfrak{A}$ to a suitable partial GW*-algebra containing $\mathfrak{A}$. Thus here also partial GW*-algebras play a distinguished role among all partial O*-algebras.

Key words: partial O*-algebras, partial GW*-algebras, generalized vectors, KMS condition.

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RÉSUMÉ. – Les vecteurs généralisés standard pour une O*-algèbre $\mathfrak{N}$ mènent à une théorie des automorphismes modulaires sur $\mathfrak{N}$, au sens de Tomita-Takesaki, et ceci est une étape cruciale dans la construction d'états KMS sur $\mathfrak{N}$ (qui représentent les états d'équilibre si $\mathfrak{N}$ est l'algèbre des observables d'un système physique). Dans le présent travail, on étend aux O*-algèbres partielles la notion de vecteurs généralisés standard et on montre qu'ils vérifient effectivement la condition KMS. On discute également différentes classes moins restrictives de vecteurs généralisés pour une O*-algèbre partielle $\mathfrak{M}$, qui toutes donnent lieu à des vecteurs généralisés standard pour une GW*-algèbre partielle canoniquement associée à $\mathfrak{M}$, tantôt sur le même domaine, tantôt sur un domaine dense plus petit. Enfin, étant donné une algèbre de von Neumann $\mathfrak{A}$, on discute le problème de l'extension des vecteurs généralisés standard pour $\mathfrak{A}$ à une GW*-algèbre partielle convenable contenant $\mathfrak{A}$. Il appert donc qu'ici aussi les GW*-algèbres partielles jouent un rôle privilégié parmi toutes les O*-algèbres partielles.

1. INTRODUCTION

The first concern of quantum statistical mechanics is to identify the equilibrium states of a given physical system. In the traditional algebraic formulation [1], the system is characterized by the algebra $\mathfrak{A}$ of its observables, usually taken as an algebra of bounded operators. The latter in turn may be obtained by applying the well-known GNS construction defined by a state on some abstract *-algebra. Then the standard treatment of the basic problem consists in applying to $\mathfrak{A}$ the Tomita-Takesaki theory of modular automorphisms, which yields states on $\mathfrak{A}$ that satisfy the KMS condition. The latter is a characteristic of equilibrium, indeed it is generally admitted ([1], [2]) that KMS states may be interpreted as equilibrium states in the Gibbs formulation, at least if the system is described as a C*- or W*-dynamical system.

In quantum field theory too, the Tomita-Takesaki theory plays an important role. In particular, the modular group of the von Neumann algebra associated to a wedge domain has in certain cases a nice geometric interpretation, in terms of Lorentz transformations or dilations. This line of thought has undergone substantial developments in the last years, as described in the recent review of Borchers [3].

However, there are systems for which the standard approach fails. On one hand, it is often more natural in physical applications to consider unbounded operators, e.g. generators of symmetry groups, such as position, momentum, energy, angular momentum, etc. In that case it is usually assumed that all the relevant operators have a common dense invariant domain. The standard example is that of the canonical variables, represented in the Hilbert space $L^2(\mathbb{R}^3)$ by the unbounded operators $\vec{Q}$ and $\vec{P}$. The natural dense domain for these operators is Schwartz space $\mathcal{S}(\mathbb{R}^3)$. They both leave it invariant and thus generate on this domain an algebra of unbounded operators or O*-algebra [4]. Similarly, (smeared) quantum boson fields are unbounded operators, with a natural invariant domain, either the Gårding domain or the domain obtained by applying polynomials in the fields to the vacuum vector [5].

On the other hand, there are systems, such as spin systems with long range interactions (e.g. the BCS-Bogoliubov model of superconductivity [6]), for which nonlocal observables are important and the thermodynamic limit does not exist in any C*-norm topology. However, it does exist in a suitable O*-algebra ([7, 8]). For these reasons, it seems reasonable to represent the observables of the system (either local or in the thermodynamical limit) by the elements of an O^*-algebra $\mathfrak{M}$.

However, it is sometimes inconvenient or unnatural, or even impossible, to demand a common *invariant* domain for all relevant operators in a given problem. To give a trivial example: in the simple case described above, $\mathcal{S}$ is invariant under $\vec{Q}$ and $\vec{P}$, but it is of course not invariant under any of their spectral projections. Another instance is a Wightman field theory, where the Gårding domain is not always invariant under the elements of (local) field algebras [9]. Still another one is the existence of systems (e.g. a particle on an interval) which require nonself-adjoint observables [10]. All this suggests that one should go one step further and drop the invariance property of the common domain. The result is that the observable algebra is replaced by a partial *-algebra of operators on some dense domain or, more concisely, a *partial O*-algebra*. This object, originally introduced by W. Karwowski and one of us [11], has been studied systematically in a series of papers (see [12] and the review [13]), to which we refer for further details and references to the original papers.

Now let us come back to our question: how does one construct KMS states on an O*-algebra or a partial O*-algebra? In the O* case, it was first shown by one of us [14]-[16] that a suitable Tomita-Takesaki theory may be derived for an O*-algebra $\mathfrak{M}$ if, among other conditions, $\mathfrak{M}$ possesses a strongly cyclic vector. In that case, one obtains states on $\mathfrak{M}$ (in the usual sense) that satisfy the KMS condition. However, the existence of the cyclic vector is a rather restrictive condition, that should be avoided.

A possible solution is to consider *generalized vectors* ([17], [18]). If $\mathfrak{M}$ is an O*-algebra on the dense invariant domain $\mathcal{D}$, a generalized vector for $\mathfrak{M}$ is a linear map λ from some left ideal $D(\lambda)$ of $\mathfrak{M}$ into $\mathcal{D}$, satisfying the relation

$$\lambda(XA) = X\lambda(A), \qquad \forall X \in \mathfrak{M}, \quad A \in D(\lambda). \tag{1.1}$$

Now generalized vectors are closely related to *weights* and *quasi-weights* on O*-algebras, which extend the notion of states (roughly speaking, a (quasi)-weight on a *-algebra $\mathfrak{A}$ is a linear functional that takes finite values only on certain positive elements of $\mathfrak{A}$.) Indeed, it was shown in [19] that, under suitable restrictions, a generalized vector for an O*-algebra $\mathfrak{M}$ defines a quasi-weight on $\mathfrak{M}$, which satisfies the KMS condition. For a system whose observable algebra is assumed to be an O*-algebra, these KMS quasi-weights may be interpreted as representing equilibrium states.

The next step is to extend the whole scheme to a partial O*-algebra, and this is the aim of the present paper, which may be seen as a sequel to [19]. As a matter of fact, the definition and main properties of generalized vectors are almost the same as in the O* case, provided due care is taken of the possible nonexistence of the product XA in (1.1). Here too, arbitrary generalized vectors are too general for obtaining a Tomita-Takesaki theory, only the subclass of *standard* generalized vectors will do, as shown in [17] in the O* case. However, their definition is rather restrictive and can be weakened to *essentially standard* and *quasi-standard*, and even further to *modular* generalized vectors, while still reaching the original aim, in a restricted sense at least (the definitions will be given in Sections 4 and 5).

A new feature of the present results is the particular role of the partial GW*-algebras, that is, partial O*-algebras that constitute a natural generalization of von Neumann algebras to the partial O*-algebra setting. Indeed, a partial GW*-algebra is basically a partial O*-algebra that coincides with its bicommutant (technically, its weak unbounded bicommutant $\mathfrak{M}''_{w\sigma}$, see Section 2 for the definitions). Equivalently, a partial GW*-algebra contains a (strong*) dense subset of bounded operators, which constitute a von Neumann algebra . Now, if $\mathfrak{M}$ is a fully closed partial O*-algebra on $\mathcal{D}$ and its weak bounded commutant $\mathfrak{M}'_{w}$ leaves the domain $\mathcal{D}$ invariant, then the bicommutant $\mathfrak{M}''_{w\sigma}$ is a partial GW*-algebra containing $\mathfrak{M}$ and it coincides with the strong* closure of the von Neumann algebra $(\mathfrak{M}'_{w})'$. The relevance of this in the present context is that, in this case, a generalized vector λ for $\mathfrak{M}$, satisfying a mild density condition, may be extended to a standard generalized vector $\overline{\lambda}$ for $\mathfrak{M}''_{w\sigma}$. A similar result holds if we require only λ to be modular, but in this case $\overline{\lambda}$ is a standard generalized vector for a partial GW*-algebra that lives on a smaller dense domain. All this points again to the natural role of partial GW*-algebras among all partial O*-algebras, especially for applications.

The paper is organized as follows. In Section 2, we begin by recalling the main definitions and properties of partial O*-algebras that will be needed. More details may be found in [12] and [13]. In Section 3, we define a generalized vector λ on a partial O*-algebra $\mathfrak{M}$, and its commutant λ^c, which is a generalized vector on the von Neumann algebra $(\mathfrak{M}'_w)'$. Sections 4 and 5 form the core of the paper. In Section 4 we consider the extension of a generalized vector λ to a generalized vector $\overline{\lambda}$ on the partial GW*-algebra $\mathfrak{M}''_{w\sigma}$. From this we infer the appropriate definition of standard generalized vector , and show that a standard generalized vector indeed satisfies the KMS condition. In Section 5, we discuss various weaker variants, namely modular and quasi-modular generalized vectors. Section 6 is devoted to several special cases and examples, whereas in Section 7 we come back to partial GW*-algebras, with the following result: Given a von Neumann algebra $\mathfrak{M}_o$ and a standard generalized vector λ_o on $\mathfrak{M}_o$, we show that one may construct a partial GW*-algebra $\mathfrak{M}$, with bounded part $\mathfrak{M}_o$, and a standard generalized vector λ on $\mathfrak{M}$ that extends λ_o.

2. NOTATIONS AND DEFINITIONS

For the sake of completeness, we recall first the main definitions and properties of partial O*-algebras that will be needed in the sequel. Further details and references to original work may be found in [12] and [13].

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{D}$ a dense subspace of $\mathcal{H}$. We denote by $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ the set of all (closable) linear operators X such that $\mathcal{D}(X) = \mathcal{D}$, $\mathcal{D}(X^*) \supseteq \mathcal{D}$. The set $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is a partial *-algebra with respect to the following operations : the usual sum $X_1 + X_2$, the scalar multiplication λX, the involution $X \mapsto X^\dagger = X^*\lceil\mathcal{D}$ and the *(weak)* partial multiplication $X_1 \square X_2 = {X_1}^{\dagger *} X_2$, defined whenever X_2 is a weak right multiplier of X_1 (equivalently, X_1 is a weak left multiplier of X_2), that is, iff $X_2\mathcal{D} \subset \mathcal{D}({X_1}^{\dagger *})$ and $X_1{}^*\mathcal{D} \subset \mathcal{D}(X_2{}^*)$ (we write $X_2 \in R^w(X_1)$ or $X_1 \in L^w(X_2)$). When we regard $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ as a partial *-algebra with those operations, we denote it by $\mathcal{L}^\dagger_w(\mathcal{D},\mathcal{H})$.

A partial O*-algebra on $\mathcal{D}$ is a *-subalgebra $\mathfrak{M}$ of $\mathcal{L}_{\mathrm{w}}^{\dagger}(\mathcal{D},\mathcal{H})$, that is, $\mathfrak{M}$ is a subspace of $\mathcal{L}_{\mathrm{w}}^{\dagger}(\mathcal{D},\mathcal{H})$, containing the identity and such that $X^{\dagger} \in \mathfrak{M}$ whenever $X \in \mathfrak{M}$ and $X_1 \square X_2 \in \mathfrak{M}$ for any $X_1, X_2 \in \mathfrak{M}$ such that $X_2 \in R^{\mathrm{w}}(X_1)$. Thus $\mathcal{L}_{\mathrm{w}}^{\dagger}(\mathcal{D},\mathcal{H})$ itself is the largest partial O*-algebra on the domain $\mathcal{D}$.

For a partial O*-algebra $\mathfrak{M}$, its (internal) universal right multipliers are the elements of the set:

$$\begin{aligned} R(\mathfrak{M}) &= \{Y \in \mathfrak{M};\ Y \in R^{\mathrm{w}}(X) \text{ for all } X \in \mathfrak{M}\} \\ &= \{Y \in \mathfrak{M};\ X \square Y \text{ is well-defined, } \forall X \in \mathfrak{M}\}. \end{aligned}$$

Similarly we define $R(X) = R^{\mathrm{w}}(X) \cap \mathfrak{M}$ and $L(X) = L^{\mathrm{w}}(X) \cap \mathfrak{M}$. The space $R(\mathfrak{M})$ will play a role in the definition of generalized vectors, to be discussed in Section 3.

A †-invariant subset $\mathfrak{N}$ of $\mathcal{L}^{\dagger}(\mathcal{D},\mathcal{H})$ is called *fully closed* if $\mathcal{D} = \widehat{\mathcal{D}}(\mathfrak{N}) \equiv \bigcap_{X\in\mathfrak{N}} \mathcal{D}(\overline{X})$. If $\mathfrak{N}$ is not fully closed, its full closure is the smallest fully closed set containing it, namely $\widehat{\mathfrak{N}} = \{\hat{\iota}(X) \equiv \overline{X} \upharpoonright \widehat{\mathcal{D}}(\mathfrak{N});\ X \in \mathfrak{N}\}$.

Let $\mathfrak{M}$ be a partial O*-algebra. If it is not fully closed, it may always be embeddded into its full closure $\widehat{\mathfrak{M}} = \hat{\iota}(\mathfrak{M})$, which is a fully closed partial O*-algebra on the domain $\widehat{\mathcal{D}}(\mathfrak{M})$, isomorphic to $\mathfrak{M}$. Thus one may always restrict the analysis to fully closed partial O*-algebras without loss of generality, and this we shall do in the sequel.

On the space $\mathcal{L}^{\dagger}(\mathcal{D},\mathcal{H})$ we will consider the *strong** topology t_{s^*}, which is generated by the family of seminorms : $p_{\xi}{}^*(X) = \|X\xi\| + \|X^{\dagger}\xi\|, \xi \in \mathcal{D}$. The space $\mathcal{L}^{\dagger}(\mathcal{D},\mathcal{H})$ is complete for t_{s^*}. For $\mathfrak{N} \subset \mathcal{L}^{\dagger}(\mathcal{D},\mathcal{H})$, we denote by $[\mathfrak{N}]^{s^*}$ the t_{s^*}-closure of $\mathfrak{N}$, and similarly by $[\mathfrak{N}]^{\tau}$ its closure in some topology τ.

Given a †-invariant subset $\mathfrak{N}$ of $\mathcal{L}^{\dagger}(\mathcal{D},\mathcal{H})$, we define, as usual, its weak unbounded commutant:

$$\begin{aligned} \mathfrak{N}'_{\sigma} = \{Y \in \mathcal{L}^{\dagger}(\mathcal{D},\mathcal{H}); (X\xi|Y\eta) = (Y^{\dagger}\xi|X^{\dagger}\eta) \\ \text{for each } \xi,\eta \in \mathcal{D} \text{ and } X \in \mathfrak{N}\} \end{aligned} \tag{2.1}$$

and its weak bounded commutant:

$$\begin{aligned} \mathfrak{N}'_{\mathrm{w}} = \{C \in \mathcal{B}(\mathcal{H}); (CX\xi|\eta) = (C\xi|X^{\dagger}\eta) \\ \text{for each } \xi,\eta \in \mathcal{D} \text{ and } X \in \mathfrak{N}\}. \end{aligned} \tag{2.2}$$

The restriction to $\mathcal{D}$ of $\mathfrak{N}'_{\mathrm{w}}$ is the bounded part of $\mathfrak{N}'_{\sigma}$. Both $\mathfrak{N}'_{\sigma}$ and $\mathfrak{N}'_{\mathrm{w}}$ are weakly closed, $\dagger$-invariant subspaces, but not necessarily algebras.

As for bicommutant, we consider the weak unbounded one, namely $\mathfrak{N}''_{\mathrm{w}\sigma} = (\mathfrak{N}'_{\mathrm{w}})'_{\sigma}$. Its bounded part is (the restriction to $\mathcal{D}$ of) $(\mathfrak{N}'_{\mathrm{w}})'$, where $\mathcal{B}'$ denotes the usual bounded commutant of a subset $\mathcal{B} \subset \mathcal{B}(\mathcal{H})$. We note the relation $(\mathfrak{N}''_{\mathrm{w}\sigma})''_{\mathrm{w}\sigma} = \mathfrak{N}''_{\mathrm{w}\sigma}$ and remark that $\mathfrak{N}''_{\mathrm{w}\sigma}$ is fully closed whenever $\mathfrak{N}$ is, because of the obvious inclusions $\mathcal{D} \subset \widehat{\mathcal{D}}(\mathfrak{N}''_{\mathrm{w}\sigma}) \subset \widehat{\mathcal{D}}(\mathfrak{N})$. The crucial fact is that, for any $\dagger$-invariant subset $\mathfrak{N}$ of $\mathcal{L}^{\dagger}(\mathcal{D}, \mathcal{H})$, $\mathfrak{N}'_{\mathrm{w}}$ is a von Neumann algebra if, and only if, $\mathfrak{N}''_{\mathrm{w}\sigma} = [(\mathfrak{N}'_{\mathrm{w}})' \lceil \mathcal{D}]^{s^*}$.

A partial O*-algebra $\mathfrak{M}$ on $\mathcal{D}$ is said to be a *partial GW*-algebra* if it is fully closed and satisfies the two conditions $\mathfrak{M}'_{\mathrm{w}}\mathcal{D} = \mathcal{D}$ and $\mathfrak{M}''_{\mathrm{w}\sigma} = \mathfrak{M}$. In that case, $\mathfrak{M}'_{\mathrm{w}}$ is a von Neumann algebra, the (closure of the) bounded part of $\mathfrak{M}$ is also a von Neumann algebra, namely $\mathfrak{M}_o \equiv (\mathfrak{M}'_{\mathrm{w}})'$, and $\mathfrak{M} = [(\mathfrak{M}'_{\mathrm{w}})' \lceil \mathcal{D}]^{s^*}$ (we usually say that $\mathfrak{M}$ is a partial GW*-algebra over $\mathfrak{M}_o$). The good properties of partial GW*-algebras stem precisely from the fact that they contain a t_{s^*}-dense subset of bounded operators.

The easiest way of constructing a partial GW*-algebra is to take a bicommutant. Indeed, if $\mathfrak{N}$ is a fully closed $\dagger$-invariant subset of $\mathcal{L}^{\dagger}(\mathcal{D}, \mathcal{H})$, then $\mathfrak{N}''_{\mathrm{w}\sigma}$ is a partial GW*-algebra on $\mathcal{D}$ iff $\mathfrak{N}'_{\mathrm{w}}\mathcal{D} = \mathcal{D}$. On the other hand, if $\mathfrak{M}$ is a partial O*-algebra on $\mathcal{D}$ (not necessarily fully closed), such that $\mathfrak{M}'_{\mathrm{w}}\mathcal{D} = \mathcal{D}$ and $\mathfrak{M}''_{\mathrm{w}\sigma} = \mathfrak{M}$, then $\widehat{\mathfrak{M}}$ is a partial GW*-algebra on $\widehat{\mathcal{D}}(\mathfrak{M})$.

Along the way we will use some standard tools from the theory of bounded operator algebras, for instance the notion of (achieved) left Hilbert algebra and the corresponding Tomita algebra. For all these, we refer to standard texts, such as [20].

3. GENERALIZED VECTORS FOR PARTIAL O*-ALGEBRAS

If $\mathfrak{N}$ is an O*-algebra on $\mathcal{D}$, so that $\mathfrak{N}\mathcal{D} \subset \mathcal{D}$, then a map of $\mathfrak{N}$ into $\mathcal{D}$ is called a generalized vector for $\mathfrak{N}$ if its domain $D(\lambda)$ is a left ideal of $\mathfrak{N}$, λ is a linear map from $D(\lambda)$ into $\mathcal{D}$ and $\lambda(XA) = X\lambda(A)$, for all $X \in \mathfrak{N}$ and $A \in D(\lambda)$. This definition does not extend immediately to a partial O*-algebra, since the product XA is not necessarily defined and, in addition, the partial multiplication is not associative, which creates difficulties with the notion of left ideal.

Let $\mathfrak{M}$ be a partial O*-algebra on $\mathcal{D} \subset \mathcal{H}$. Throughout the paper, we will assume that $\mathfrak{M}$ is fully closed (as stated in Section 2, this is not a real restriction), and that $\mathfrak{M}'_{\mathrm{w}}\mathcal{D} \subset \mathcal{D}$. This implies that $\mathfrak{M}'_{\mathrm{w}}$ is a von Neumann algebra and that $\mathfrak{M}''_{\mathrm{w}\sigma} = [(\mathfrak{M}'_{\mathrm{w}})' \lceil \mathcal{D}]^{s^*}$ is a partial GW*-algebra.

DEFINITION 3.1. – A map $\lambda : \mathfrak{M} \to \mathcal{H}$ is a *generalized vector* for the partial O*-algebra $\mathfrak{M}$ if there exists a subspace $B(\lambda)$ of $\mathfrak{M}$ such that:

(i) the domain $D(\lambda)$ is the linear span of $\{Y \square X; X \in B(\lambda), Y \in L(X)\}$ and the map $\lambda : D(\lambda) \to \mathcal{H}$ is linear;

(ii) $\lambda(B(\lambda)) \subset \mathcal{D}$;

(iii) $\lambda(Y \square X) = Y\lambda(X), \ \forall X \in B(\lambda), Y \in L(X)$.

The subspace $B(\lambda)$ is called a *core* for λ. By Zorn's lemma, λ possesses a maximal core containing $B(\lambda)$, denoted by $B_M(\lambda)$.

DEFINITION 3.2. – A generalized vector λ for $\mathfrak{M}$ is said to be *strongly cyclic* (resp. *cyclic*) if it possesses a core $B(\lambda)$ such that:

(i) $B(\lambda) \subset R(\mathfrak{M})$;

(ii) $\lambda(B(\lambda))$ is dense in $\mathcal{D}[t_{\mathfrak{M}}]$ (resp. $\mathcal{H}$).

Let λ be a strongly cyclic generalized vector for $\mathfrak{M}$. Since $B(\lambda) \subset R(\mathfrak{M})$, it follows that

$$D(\lambda) = \text{linear span of } \{Y \square X; Y \in \mathfrak{M}, \ X \in B(\lambda)\},$$

$$\lambda(Y \square X) = Y\lambda(X), \ \forall Y \in \mathfrak{M}, \ X \in B(\lambda).$$

Moreover, since $\lambda(B(\lambda))$ is dense in $\mathcal{D}[t_{\mathfrak{M}}]$, the full closure of the partial O*-algebra $\mathfrak{M} \lceil \lambda(B(\lambda))$ coincides with $\mathfrak{M}$.

Given a generalized vector λ for $\mathfrak{M}$, we proceed to define its *commutant*, noted λ^c. Suppose the following condition holds:

(S_1) $\lambda((B(\lambda) \cap B(\lambda)^\dagger)^2)$ is total in $\mathcal{H}$, for some core $B(\lambda)$ for λ,

where we have defined

$$(B(\lambda) \cap B(\lambda)^\dagger)^2 = \{X \square Y; \ X, Y \in B(\lambda) \cap B(\lambda)^\dagger, \ X \square Y \text{ is well-defined}\}.$$

We put

$$\begin{cases} D(\lambda^c) = \{K \in \mathfrak{M}'_{\mathrm{w}}; \ \exists \xi_K \in \mathcal{D} \text{ s.t. } K\lambda(X) = X\xi_K \text{ for all } X \in B(\lambda)\}, \\ \lambda^c(K) = \xi_K, \quad K \in D(\lambda^c), \end{cases}$$

so that $K\lambda(X) = X\lambda^c(K)$, which justifies the name 'commutant'. Then we have the following

PROPOSITION 3.3. – (1) *The vector* ξ_K *is uniquely determined for every* $K \in D(\lambda^c)$, *and* λ^c *is a generalized vector for the von Neumann algebra* $\mathfrak{M}'_w$.

(2) λ^c *is independent of the choice of the core* $B(\lambda)$ *for* λ *satisfying condition* (S_1).

Proof. – (1) This follows immediately from condition (S_1).

(2) Let $B_1(\lambda)$ and $B_2(\lambda)$ be two cores for λ, both satisfying condition (S_1), $\lambda^c_{B_1(\lambda)}$ and $\lambda^c_{B_2(\lambda)}$ the corresponding commutants of λ. Choose any $K \in D(\lambda^c_{B_1(\lambda)})$, that is

$$\lambda^c_{B_1(\lambda)}(K) \in \mathcal{D} \quad \text{and} \quad K\lambda(X) = X\lambda^c_{B_1(\lambda)}(K),\ \forall X \in B_1(\lambda).$$

Since

$$\begin{aligned} D(\lambda) &= \text{linear span of } \{Y \Box X; X \in B_1(\lambda),\ Y \in L(X)\} \\ &= \text{linear span of } \{Y \Box X; X \in B_2(\lambda),\ Y \in L(X)\}, \end{aligned}$$

it follows that every element X of $B_2(\lambda)$ may be represented as

$$X = \sum_k Y_k \Box X_k,\ X_k \in B_1(\lambda),\ Y_k \in L(X_k).$$

We have

$$\begin{aligned} K\lambda(X) &= K\sum_k Y_k\lambda(X_k) = \sum_k Y_k^{\dagger *} K\lambda(X_k) \\ &= \sum_k Y_k^{\dagger *} X_k \lambda^c_{B_1(\lambda)}(K) \\ &= X\lambda^c_{B_1(\lambda)}(K), \end{aligned}$$

where the first equality results from the fact that, for all $\xi \in \mathcal{D}$,

$$\begin{aligned} (\xi | KY_k\lambda(X_k)) &= (K^*\xi | Y_k\lambda(X_k)) = (K^*Y_k^\dagger \xi | \lambda(X_k)) \\ &= (Y_k^\dagger \xi | K\lambda(X_k)). \end{aligned}$$

Thus we get

$$K \in D(\lambda^c_{B_2(\lambda)}) \quad \text{and} \quad \lambda^c_{B_2(\lambda)}(K) = \lambda^c_{B_1(\lambda)}(K),$$

and therefore $\lambda^c_{B_1(\lambda)} \subset \lambda^c_{B_2(\lambda)}$. The reverse inclusion is proved in the same way. This completes the proof. □

In this section, we give only two simple, yet important, examples of generalized vectors for partial O*-algebras, namely those associated to vectors in $\mathcal{D}$ and $\mathcal{H}\backslash\mathcal{D}$, respectively. More sophisticated ones will be discussed in Section 6.

EXAMPLE 3.4. – (1) Let $\xi_o \in \mathcal{D}$. We put

$$B(\lambda_{\xi_o}) = \mathbb{C}I,$$

$$D(\lambda_{\xi_o}) = \mathfrak{M},$$

$$\lambda_{\xi_o}(X) = X\xi_o,\ X \in \mathfrak{M}.$$

Then λ_{ξ_o} is a generalized vector for $\mathfrak{M}$ and $B_M(\lambda_{\xi_o}) = \{X \in \mathfrak{M};\ X\xi_o \in \mathcal{D}\}$. Suppose that $B_M(\lambda_{\xi_o})\xi_o$ is dense in $\mathcal{H}$. Then

$$\begin{cases} D(\lambda^c_{\xi_o}) = \mathfrak{M}'_{\mathrm{w}}, \\ \lambda^c_{\xi_o}(K) = K\xi_o,\ K \in \mathfrak{M}'_{\mathrm{w}}. \end{cases}$$

Suppose, in addition, that $(B_M(\lambda_{\xi_o}) \cap R(\mathfrak{M}))\xi_o$ is dense in $\mathcal{D}[t_{\mathfrak{M}}]$. Then λ_{ξ_o} is strongly cyclic.

(2) Let $\xi_o \in \mathcal{H}\backslash\mathcal{D}$. Suppose that $\mathcal{C} \equiv \{K \in \mathfrak{M}'_{\mathrm{w}}; K\xi_o, K^*\xi_o \in \mathcal{D}\}$ is nondegenerate, that is, $\mathcal{CH}$ is dense in $\mathcal{H}$. We put

$$\begin{cases} B_e(\lambda_{\xi_o}) = \{X \in \mathfrak{M};\ \xi_o \in D(X^{\dagger *}) \text{ and } X^{\dagger *}\xi_o \in \mathcal{D}\}, \\ D(\lambda_{\xi_o}) = \text{linear span of } \{Y \square X;\ X \in B_e(\lambda_{\xi_o}),\ Y \in L(X)\}, \\ \lambda_{\xi_o}(\sum_k Y_k \square X_k) = \sum_k Y_k X_k^{\dagger *}\xi_o,\ \sum_k Y_k \square X_k \in D(\lambda_{\xi_o}). \end{cases}$$

Then λ_{ξ_o} is a generalized vector for $\mathfrak{M}$ and $B_e(\lambda_{\xi_o})$ is the largest core of λ_{ξ_o}. First, it is clear that $B_e(\lambda_{\xi_o})$ is a subspace of $\mathfrak{M}$. Next, suppose that $\sum_k Y_k \square X_k = 0$, with $X_k \in B_e(\lambda_{\xi_o})$, $Y_k \in L(X_k)$. Since

$$(XK^*\xi_o|\xi) = (\xi_o|KX^\dagger\xi) = (\xi_o|X^\dagger K\xi) = (K^*X^{\dagger *}\xi_o|\xi)$$

for all $X \in \mathfrak{M}$, $K \in \mathcal{C}$ and $\xi \in \mathcal{D}$, it follows that

$$XK^*\xi_o = K^*X^{\dagger *}\xi_o. \tag{3.1}$$

Therefore

$$\begin{aligned} (\sum_k Y_k X_k^{\dagger *}\xi_o|K\eta) &= \sum_k (K^*X_k^{\dagger *}\xi_o|Y_k^\dagger\eta) \\ &= \sum_k (X_k K^*\xi_o|Y_k^\dagger\eta) \\ &= ((\sum_k Y_k \square X_k)K^*\xi_o|\eta) \\ &= 0 \end{aligned}$$

for all $\eta \in \mathcal{D}$ and $K \in C$. Since $C\mathcal{D}$ is total in $\mathcal{H}$, this implies $\sum_k Y_k X_k^{\dagger *}\xi_o = 0$, and thus λ_{ξ_o} is indeed a generalized vector for $\mathfrak{M}$.

Take now an arbitrary $X \in \mathfrak{M}$ such that $X \in D(\lambda_{\xi_o})$ and $\lambda_{\xi_o}(X) \in \mathcal{D}$. Then,

$$X = \sum_k Y_k \square X_k \quad (X_k \in B_e(\lambda_{\xi_o}),\, Y_k \in L(X_k)),$$

$$\lambda_{\xi_o}(X) = \sum_k Y_k X_k^{\dagger *}\xi_o \in \mathcal{D}.$$

Hence, introducing a net $\{K_\alpha\}$ in $\mathcal{C}$ which converges strongly to I, we get, for all $\xi \in \mathcal{D}$,

$$\begin{aligned}(X^\dagger \xi|\xi_o) &= \sum_k (X_k^* Y_k^\dagger \xi|\xi_o)\\ &= \lim_\alpha \sum_k (X_k^* Y_k^\dagger \xi|K_\alpha \xi_o)\\ &= \lim_\alpha \sum_k (Y_k^\dagger \xi|K_\alpha X_k^{\dagger *}\xi_o), \qquad \text{by (3.1)}\\ &= \sum_k (Y_k^\dagger \xi|X_k^{\dagger *}\xi_o)\\ &= (\xi|\lambda_{\xi_o}(X)),\end{aligned}$$

which implies that $\xi_o \in D(X^{\dagger *})$ and $X^{\dagger *}\xi_o = \lambda_{\xi_o}(X) \in \mathcal{D}$. Hence $X \in B_e(\lambda_{\xi_o})$. This means that $B_e(\lambda_{\xi_o})$ is the largest core of λ_{ξ_o}.

If we assume now that $\lambda_{\xi_o}(B_e(\lambda_{\xi_o}) \cap B_e(\lambda_{\xi_o})^\dagger)^2)$ is total in $\mathcal{H}$, then we have

$$\begin{cases} D(\lambda_{\xi_o}^c) = \{K \in \mathfrak{M}'_{\mathrm{w}};\, K\xi_o \in \mathcal{D}\},\\ \lambda_{\xi_o}^c(K) = K\xi_o,\ K \in D(\lambda_{\xi_o}^c).\end{cases}$$

Suppose finally that $\lambda_{\xi_o}(B_e(\lambda_{\xi_o}) \cap R(\mathfrak{M}))$ is dense in $\mathcal{D}[t_\mathfrak{M}]$. Then λ_{ξ_o} is strongly cyclic.

(3) Let $\mathfrak{M}$ be a closed O^*-*algebra* on $\mathcal{D}$, such that $\mathfrak{M}'_{\mathrm{w}}\mathcal{D} \subset \mathcal{D}$, and λ a generalized vector for $\mathfrak{M}$. Suppose

(i) $\lambda\Big((D(\lambda) \cap (D(\lambda)^\dagger)^2\Big)$ is total in $\mathcal{H}$;

(ii) $\lambda^c\big((D(\lambda^c) \cap (D(\lambda^c)^*)^2\big)$ is total in $\mathcal{H}$.

Then we put

$$\begin{cases} B_e(\overline{\lambda}) = \{X \in [(\mathfrak{M}'_{\rm w})' \lceil \mathcal{D}]^{s^*} ; \exists \{A_\alpha\} \subset D(\lambda^{cc}) \\ \qquad\qquad \text{s.t. } A_\alpha \xrightarrow{t_{s^*}} X \text{ and } \lambda^{cc}(A_\alpha) \to \xi_X \in \mathcal{D}\}, \\ \overline{\lambda}(X) = \xi_X, \; X \in B_e(\overline{\lambda}), \end{cases}$$

and

$$\begin{cases} D(\overline{\lambda}) = \text{linear span of } \{Y \square X ; X \in B_e(\overline{\lambda}),\ Y \in L(X)\}, \\ \overline{\lambda}(\sum_k Y_k \square X_k) = \sum_k Y_k \overline{\lambda}(X_k), \quad \sum_k Y_k \square X_k \in D(\overline{\lambda}). \end{cases}$$

As shown in Theorem 4.3 below, $\overline{\lambda}$ is a generalized vector for the partial GW*-algebra $\mathfrak{M}''_{\rm w\sigma} = [(\mathfrak{M}'_{\rm w})' \lceil \mathcal{D}]^{s^*}$, $B_e(\overline{\lambda})$ is the largest core for $\overline{\lambda}$, $\lambda \subset \overline{\lambda}$ and $\lambda^c = \overline{\lambda}^c$.

4. STANDARD GENERALIZED VECTORS FOR PARTIAL O*-ALGEBRAS

Let again $\mathfrak{M}$ be a fully closed partial O*-algebra on $\mathcal{D} \subset \mathcal{H}$, such that $\mathfrak{M}'_{\rm w}\mathcal{D} \subset \mathcal{D}$, and λ a generalized vector for $\mathfrak{M}$. Suppose the following conditions hold:

(S_1) $\lambda((B(\lambda) \cap B(\lambda)^\dagger)^2)$ is total in $\mathcal{H}$, for some core $B(\lambda)$ for λ;

(S_2) $\lambda^c((D(\lambda^c) \cap D(\lambda^c)^*)^2)$ is total in $\mathcal{H}$.

Then we define the commutant λ^{cc} of λ^c as follows:

$$\begin{cases} D(\lambda^{cc}) = \{A \in (\mathfrak{M}'_{\rm w})'; \\ \qquad\qquad \exists \, \xi_A \in \mathcal{H} \text{ s.t. } A\lambda^c(K) = K\xi_A \text{ for all } K \in D(\lambda^c)\}, \\ \lambda^{cc}(A) = \xi_A, \quad A \in D(\lambda^{cc}). \end{cases}$$

The vector ξ_A is uniquely determined for each $A \in D(\lambda^{cc})$ and λ^{cc} is a generalized vector for the von Neumann algebra $(\mathfrak{M}'_{\rm w})'$. Then we have the following

LEMMA 4.1. – (1) *$\lambda^{cc}(D(\lambda^{cc}) \cap D(\lambda^{cc})^*)$ is a left Hilbert algebra in $\mathcal{H}$, whose left von Neumann algebra equals $(\mathfrak{M}'_{\rm w})'$.*

(2) *Consider the following involutions:*

$$S_\lambda : \ \lambda(X) \mapsto \lambda(X^\dagger), \qquad X \in B(\lambda) \cap B(\lambda)^*;$$
$$S_{\lambda^{cc}} : \ \lambda^{cc}(A) \mapsto \lambda^{cc}(A^*), \qquad A \in D(\lambda^{cc}) \cap D(\lambda^{cc})^*.$$

Then S_λ and $S_{\lambda^{cc}}$ are closable conjugate linear operators in $\mathcal{H}$, whose closures we denote again by S_λ and $S_{\lambda^{cc}}$, and $S_\lambda \subset S_{\lambda^{cc}}$.

Proof. – (1) It is sufficient to show that $\lambda^{cc}((D(\lambda^{cc}) \cap D(\lambda^{cc})^*)^2)$ is total in $\mathcal{H}$. Choose an arbitrary element $X \in B(\lambda)$. Let $\overline{X} = U|\overline{X}|$ be the polar decomposition of $\overline{X}$ and $|\overline{X}| = \int_0^\infty \lambda\, dE(\lambda)$ the spectral resolution of $|\overline{X}|$. We put

$$E_n = \int_0^n dE(\lambda) \text{ and } X_n = \overline{X}E_n, \ \forall n \in \mathbb{N}.$$

Then $U, E_n, X_n \in (\mathfrak{M}'_{\mathrm{w}})'$, $\forall n \in \mathbb{N}$, and

$$X_n\lambda^c(K) = \overline{X}E_n\lambda^c(K) = UE_nU^*X\lambda^c(K) = KUE_nU^*\lambda(X)$$

for all $K \in D(\lambda^c)$. Hence

$$X_n \in D(\lambda^{cc}) \text{ and } \lambda^{cc}(X_n) = UE_nU^*\lambda(X), \ \forall n \in \mathbb{N}. \tag{4.1}$$

Since we have, for all $K_1, K_2 \in D(\lambda^c) \cap D(\lambda^c)^*$,

$$\begin{aligned}(UU^*\lambda(X)|\lambda^c(K_1^*K_2)) &= (UU^*X\lambda^c(K_1)|\lambda^c(K_2)) \\ &= (X\lambda^c(K_1)|\lambda^c(K_2)) \\ &= (\lambda(X)|\lambda^c(K_1^*K_2)),\end{aligned}$$

it follows from (S_2) that

$$UU^*\lambda(X) = \lambda(X). \tag{4.2}$$

Take an arbitrary element $X \in B(\lambda) \cap B(\lambda)^\dagger$. Since

$$X_n^*\lambda^c(K) = E_nX^\dagger\lambda^c(K) = KE_n\lambda(X^\dagger)$$

for all $K \in D(\lambda^c)$, it follows that

$$X_n^* \in D(\lambda^{cc}) \text{ and } \lambda^{cc}(X_n^*) = E_n\lambda(X^\dagger), \ \forall n \in \mathbb{N}. \tag{4.3}$$

Then, putting together (4.1), (4.2) and (4.3), we obtain

$$\left.\begin{aligned}&\{X_n\} \subset D(\lambda^{cc}) \cap D(\lambda^{cc})^*, \quad X_n \longrightarrow^{t_{s^*}} X, \\ &\lambda^{cc}(X_n) \to \lambda(X), \quad \lambda^{cc}(X_n^*) \to \lambda(X^\dagger).\end{aligned}\right\} \tag{4.4}$$

Take now two arbitrary elements $X, Y \in B(\lambda) \cap B(\lambda)^\dagger$ such that $X \in L(Y)$. By (4.4) we have

$$\{X_m Y_n;\ m, n \in \mathbb{N}\} \subset (D(\lambda^{cc}) \cap D(\lambda^{cc})^*)^2$$

and

$$\begin{aligned}\lim_{m,n\to\infty} \lambda^{cc}(X_m Y_n) &= \lim_{m,n\to\infty} X_m \lambda^{cc}(Y_n)\\ &= \lim_{m\to\infty} X_m \lambda(Y)\\ &= X\lambda(Y)\\ &= \lambda(X \square Y).\end{aligned}$$

Since $\lambda((B(\lambda) \cap B(\lambda)^\dagger)^2)$ is total in $\mathcal{H}$, it follows that $\lambda^{cc}((D(\lambda^{cc}) \cap D(\lambda^{cc})^*)^2)$ is total in $\mathcal{H}$ too.

(2) It is easily shown that S_λ and $S_{\lambda^{cc}}$ are closable conjugate linear operators in $\mathcal{H}$. Now observe that

$$(S_\lambda \lambda(X) | \lambda^c(K_1^* K_2)) = (\lambda^c(K_2^* K_1) | \lambda(X))$$

for all $K_1, K_2 \in D(\lambda^c)$ and $X \in B(\lambda) \cap B(\lambda)^\dagger$. Since $\lambda^c((D(\lambda^c) \cap D(\lambda^c)^*)^2)$ is total in the Hilbert space $D(S^*_{\lambda^{cc}})$, it follows that

$$(S_\lambda \lambda(X) | \eta) = (S^*_{\lambda^{cc}} \eta | \lambda(X)),$$

for all $\eta \in D(S^*_{\lambda^{cc}})$ and $X \in B(\lambda) \cap B(\lambda)^\dagger$, which implies that

$$\eta \in D(S_\lambda^*) \ \text{ and } \ S_\lambda^* \eta = S^*_{\lambda^{cc}} \eta.$$

Hence, $S_\lambda^* \supset S^*_{\lambda^{cc}}$, and thus $S_\lambda \subset S_{\lambda^{cc}}$. This completes the proof. □

Notice that $S_\lambda \neq S_{\lambda^{cc}}$ in general. Let now $S_\lambda = J_\lambda \Delta_\lambda^{1/2}$ and $S_{\lambda^{cc}} = J_{\lambda^{cc}} \Delta_{\lambda^{cc}}^{1/2}$ be the polar decompositions of S_λ and $S_{\lambda^{cc}}$, respectively. From the Tomita fundamental theorem [21], we derive the following

LEMMA 4.2. – (1) *The strongly continuous one-parameter groups* $\{\sigma_t^{\lambda^{cc}}\}_{t\in\mathbb{R}}$ *and* $\{\sigma_t^{\lambda^{ccc}}\}_{t\in\mathbb{R}}$ *of the von Neumann algebras* $(\mathfrak{M}'_{\rm w})'$ *and* $\mathfrak{M}'_{\rm w}$ *are defined by*

$$\begin{aligned}\sigma_t^{\lambda^{cc}}(A) &= \Delta_{\lambda^{cc}}^{it} A \Delta_{\lambda^{cc}}^{-it}, \qquad A \in (\mathfrak{M}'_{\rm w})',\ t \in \mathbb{R};\\ \sigma_t^{\lambda^{ccc}}(C) &= \Delta_{\lambda^{cc}}^{it} C \Delta_{\lambda^{cc}}^{-it}, \qquad C \in \mathfrak{M}'_{\rm w},\ t \in \mathbb{R},\end{aligned}$$

and they satisfy the relations

$$\sigma_t^{\lambda^{cc}}(D(\lambda^{cc})) = D(\lambda^{cc})$$

and

$$\lambda^{cc}(\sigma_t^{\lambda^{cc}}(B)) = \Delta_{\lambda^{cc}}^{it}\lambda^{cc}(B),\ \forall B \in D(\lambda^{cc}),\ \forall t \in \mathbb{R};$$
$$\sigma_t^{\lambda^{ccc}}(D(\lambda^{ccc})) = D(\lambda^{ccc})$$

and

$$\lambda^{ccc}(\sigma_t^{\lambda^{ccc}}(K)) = \Delta_{\lambda^{cc}}^{it}\lambda^{ccc}(K),\ \forall K \in D(\lambda^{ccc}),\ \forall t \in \mathbb{R}.$$

(2) λ^{cc} *satisfies the* KMS *condition with respect to the modular automorphism group* $\{\sigma_t^{\lambda^{cc}}\}$, *that is, for any* $A, B \in D(\lambda^{cc}) \cap D(\lambda^{cc})^*$, *there exists an element* $f_{A,B}$ *of* $A(0,1)$ *such that*

$$f_{A,B}(t) = (\lambda^{cc}(\sigma_t^{\lambda^{cc}}(A))|\lambda^{cc}(B)),$$
$$f_{A,B}(t+i) = (\lambda^{cc}(B^*))|\lambda^{cc}(\sigma_t^{\lambda^{cc}}(A^*)))$$

for all $t \in \mathbb{R}$, *where* $A(0,1)$ *is the set of all complex-valued functions, bounded and continuous on* $0 \leq \mathrm{Im}\ z \leq 1$ *and analytic in the interior.*

The next step is to determine how the modular automorphism group $\{\sigma_t^{\lambda^{cc}}\}_{t\in\mathbb{R}}$ of the von Neumann algebra $(\mathfrak{M}'_{\mathrm{w}})'$ acts on the partial O*-algebra $\mathfrak{M}$. For that purpose we need the notion of full generalized vector, that we proceed to define.

First we show that the generalized vector λ extends to a generalized vector $\overline{\lambda}$ for the partial GW*-algebra $\mathfrak{M}''_{\mathrm{w}\sigma} = [(\mathfrak{M}'_{\mathrm{w}})' \lceil \mathcal{D}]^{s^*}$.

THEOREM 4.3. – *Let* λ *be a generalized vector for* $\mathfrak{M}$, *satisfying the conditions* (S_1) *and* (S_2). *We put*

$$\begin{cases} B_e(\overline{\lambda}) = \{X \in \mathfrak{M}''_{\mathrm{w}\sigma}; \exists\ \{A_\alpha\} \subset D(\lambda^{cc})\ s.t. \\ \qquad A_\alpha \xrightarrow{t_{s^*}} X \text{ and } \lambda^{cc}(A_\alpha) \to \xi_X \in \mathcal{D}\}, \\ \overline{\lambda}(X) = \xi_X, \quad X \in B_e(\overline{\lambda}), \end{cases}$$

and

$$\begin{cases} D(\overline{\lambda}) = \text{linear span of } \{Y \square X; X \in B_e(\overline{\lambda}),\ Y \in L(X)\}, \\ \overline{\lambda}(\sum_k Y_k \square X_k) = \sum_k Y_k \overline{\lambda}(X_k), \quad \sum_k Y_k \square X_k \in D(\overline{\lambda}). \end{cases}$$

Then $\overline{\lambda}$ *is a generalized vector for the partial GW*-algebra* $\mathfrak{M}''_{w\sigma} = [(\mathfrak{M}'_w)' \lceil \mathcal{D}]^{s^*}$*, and it has the following properties:*

(1) $\lambda \subset \overline{\lambda}$;

(2) $\lambda^c = \overline{\lambda}^c$ *and* $\lambda^{cc} = \overline{\lambda}^{cc}$;

(3) $\overline{\lambda}$ *is the largest among the generalized vectors* μ *for* $\mathfrak{M}''_{w\sigma}$ *that satisfy condition* (S_1) *and* $\mu^c = \lambda^c$;

(4) $B_e(\overline{\lambda}) = \{X \in \mathfrak{M}''_{w\sigma};\ \exists \xi_X \in \mathcal{D}$ *s.t.* $X\lambda^c(K) = K\xi_X$ *for all* $K \in D(\lambda^c)\}$*, and it is the largest core for* $\overline{\lambda}$.

Proof. – It is clear that $B_e(\overline{\lambda})$ is a subspace of $\mathfrak{M}''_{w\sigma} = [(\mathfrak{M}'_w)' \lceil \mathcal{D}]^{s^*}$. Let X be an arbitrary element of $B_e(\overline{\lambda})$ and $\{A_\alpha\}$, $\{B_\beta\}$ two nets in $D(\lambda^{cc})$ such that $A_\alpha \xrightarrow{t_{s^*}} X$ and $\lambda^{cc}(A_\alpha) \to \xi_X \in \mathcal{D}$, $B_\beta \xrightarrow{t_{s^*}} X$; and $\lambda^{cc}(B_\beta) \to \xi'_X \in \mathcal{D}$. Then we have, for all $K_1, K_2 \in D(\lambda^c) \cap D(\lambda^c)^*$,

$$\begin{aligned}(\xi_X - \xi'_X|\lambda^c(K_1^*K_2)) &= \lim_{\alpha,\beta}(\lambda^{cc}(A_\alpha - B_\beta)|\lambda^c(K_1^*K_2)) \\ &= \lim_{\alpha,\beta}((A_\alpha - B_\beta)\lambda^c(K_1)|\lambda^c(K_2)) \\ &= 0.\end{aligned}$$

Since $\lambda^c((D(\lambda^c) \cap D(\lambda^c)^*)^2)$ is total in $\mathcal{H}$, by (S_2), we have $\xi_X = \xi'_X$, so that $\overline{\lambda}$ is a well-defined map from $B_e(\overline{\lambda})$ into $\mathcal{D}$. Suppose that $Y \square X = 0$ $(X \in B_e(\overline{\lambda}),\ Y \in L(X))$. Then we get, for all $K_1, K_2 \in D(\lambda^c) \cap D(\lambda^c)^*$,

$$\begin{aligned}(Y\overline{\lambda}(X)|\lambda^c(K_1^*K_2)) &= (K_1\overline{\lambda}(X)|Y^\dagger\lambda^c(K_2)) \\ &= \lim_{\alpha}(\lambda^{cc}(A_\alpha)|Y^\dagger\lambda^c(K_2)) \\ &= \lim_{\alpha}(A_\alpha\lambda^c(K_1)|Y^\dagger\lambda^c(K_2)) \\ &= (X\lambda^c(K_1)|Y^\dagger\lambda^c(K_2)) \\ &= ((Y\square X)\lambda^c(K_1)|\lambda^c(K_2)) \\ &= 0,\end{aligned}$$

where $\{A_\alpha\}$ is a net in $D(\lambda^{cc})$ such that $A_\alpha \xrightarrow{t_{s^*}} X$ and $\lambda^{cc}(A_\alpha) \to \overline{\lambda}(X)$. By (S_2) again, we have $Y\overline{\lambda}(X) = 0$. Similarly, the condition $\sum_k Y_k \square X_k = 0$ $(X_k \in B_e(\overline{\lambda}),\ Y_k \in L(X_k))$ implies $\sum_k Y_k\overline{\lambda}(X_k) = 0$. Therefore $\overline{\lambda}$ is a generalized vector for $\mathfrak{M}''_{w\sigma}$. Furthermore, it follows from (4.1) and (4.2) that $B(\lambda) \subset B_e(\overline{\lambda})$ and $\lambda(X) = \overline{\lambda}(X)$, $\forall X \in B(\lambda)$, which implies that $\lambda \subset \overline{\lambda}$.

Since $\lambda \subset \overline{\lambda}$, it follows that $\overline{\lambda}$ satisfies condition (S_1) and $\overline{\lambda}^c \subset \lambda^c$. Take arbitrary elements $K \in D(\lambda^c)$ and $X \in D(\overline{\lambda})$. Then we have

$$K\overline{\lambda}(X) = \lim_{\alpha} K\lambda^{cc}(A_\alpha) = \underset{\alpha}{} A_\alpha \lambda^c(K) = X\lambda^c(K),$$

where again $\{A_\alpha\}$ is a net in $D(\lambda^{cc})$ such that $A_\alpha \xrightarrow{t_{s^*}} X$ and $\lambda^{cc}(A_\alpha) \to \overline{\lambda}(X)$. Hence $K \in D(\overline{\lambda}^c)$ and $\overline{\lambda}^c(K) = \lambda^c(K)$. Thus $\overline{\lambda}^c = \lambda^c$. Take now an arbitrary generalized vector μ for $\mathfrak{M}''_{w\sigma}$ satisfying condition (S_1) and $\mu^c = \lambda^c$. By the definition of $\overline{\lambda}$, we have $\mu \subset \overline{\mu} = \overline{\lambda}$. Hence statement (3) holds true. It remains to prove statement (4). We put

$$\begin{aligned} B(\nu) = &\{X \in \mathfrak{M}''_{w\sigma};\ \exists \xi_X \in \mathcal{D} \\ &\quad \text{s.t. } X\lambda^c(K) = K\xi_X \ \text{ for all } \ K \in D(\lambda^c)\}, \\ \nu(X) = &\ \xi_X,\ X \in B(\nu). \end{aligned}$$

Then it is easily shown that ν is a generalized vector for $\mathfrak{M}''_{w\sigma}$, with core $B(\nu)$, such that $B_e(\overline{\lambda}) \subset B(\nu)$ and $\overline{\lambda} \subset \nu$. Conversely, take an arbitrary element $X \in B(\nu)$. One can show, in the same way as for (4.1) and (4.2), that there exists a sequence $\{X_n\}$ in $D(\lambda^{cc})$ such that $X_n \xrightarrow{t_{s^*}} X$ and $\lambda^{cc}(X_n) \to \nu(X)$, which means that $X \in B_e(\overline{\lambda})$ and $\overline{\lambda}(X) = \nu(X)$. Therefore we have

$$B_e(\overline{\lambda}) = B(\nu) \ \text{ and } \ \overline{\lambda} = \nu. \tag{4.5}$$

Let now $B(\overline{\lambda})$ be an arbitrary core for $\overline{\lambda}$ and $Z \in B(\overline{\lambda})$. Then

$$Z = \sum_k Y_k \Box X_k \ \text{ and } \ \overline{\lambda}(Z) = \sum_k Y_k \overline{\lambda}(X_k),$$

where $X_k \in B_e(\overline{\lambda})$ and $Y_k \in L(X_k)$. For every $K \in D(\lambda^c)$ we have

$$\begin{aligned} K\overline{\lambda}(Z) &= \sum_k KY_k\overline{\lambda}(X_k) = \sum_k Y_k^{\dagger *} K\overline{\lambda}(X_k) \\ &= \sum_k (Y_k \Box X_k)\lambda^c(K) \\ &= Z\lambda^c(K), \end{aligned}$$

and so $Z \in B(\nu) = B_e(\overline{\lambda})$ by (4.5). Thus we conclude that $B_e(\overline{\lambda})$ is indeed the largest core for $\overline{\lambda}$. This completes the proof. □

We may remark that the extension from λ to $\overline{\lambda}$ is the analogue of a closure operation for generalized vectors.

If we now restrict the generalized vector $\overline{\lambda}$ from $\mathfrak{M}''_{w\sigma}$ to $\mathfrak{M}$, we get a new one, which is an extension of λ, as results from the following corollary of Theorem 4.3.

COROLLARY 4.4. – *Let us put*

$$B_e(\overline{\lambda}\lceil\mathfrak{M}) = \mathfrak{M} \cap B_e(\overline{\lambda}),$$

$$(\overline{\lambda}\lceil\mathfrak{M})(X) = \overline{\lambda}(X), \qquad X \in B_e(\overline{\lambda}\lceil\mathfrak{M}).$$

Then $\overline{\lambda}\lceil\mathfrak{M}$ is a generalized vector for $\mathfrak{M}$ such that

(1) $\lambda \subset \overline{\lambda}\lceil\mathfrak{M}$;

(2) $\lambda^c = (\overline{\lambda}\lceil\mathfrak{M})^c$ and $\lambda^{cc} = (\overline{\lambda}\lceil\mathfrak{M})^{cc}$;

(3) $\overline{\lambda}\lceil\mathfrak{M}$ *is the largest among the generalized vectors μ for $\mathfrak{M}$ that satisfy condition (S_1) and $\mu^c = \lambda^c$;*

(4) $B_e(\overline{\lambda}\lceil\mathfrak{M}) = \{X \in \mathfrak{M};\ \exists \xi_X \in \mathcal{D}$ *s.t.* $X\lambda^c(K) = K\xi_X$ *for all* $K \in D(\lambda^c)\}$, *and it is the largest core for $\overline{\lambda}\lceil\mathfrak{M}$.* □

Then, of course, requiring λ to coincide with its extension $\overline{\lambda}\lceil\mathfrak{M}$ leads to a useful class of generalized vectors, namely:

DEFINITION 4.5. – A generalized vector λ for $\mathfrak{M}$ satisfying the conditions (S_1) and (S_2) is said to be *full* if $\lambda = \overline{\lambda}\lceil\mathfrak{M}$.

When λ is full, we denote simply $B_e(\overline{\lambda}\lceil\mathfrak{M})$ by $B_e(\lambda)$. If $\mathfrak{M}$ itself is a partial GW*-algebra, so that $\mathfrak{M} = \mathfrak{M}''_{w\sigma} = [(\mathfrak{M}'_w)'\lceil\mathcal{D}]^{s^*}$, then $\overline{\lambda} = \overline{\lambda}\lceil\mathfrak{M}$, but we still have $\lambda \subset \overline{\lambda}$ in general.

Now we are in a position to define the central concept of this paper, namely standard generalized vectors, that will play the role of KMS states.

DEFINITION 4.6. – A generalized vector λ for $\mathfrak{M}$ is said to be *standard* if the following conditions are satisfied:

(S_1) $\lambda((B(\lambda) \cap B(\lambda)^\dagger)^2)$ is total in $\mathcal{H}$, for some core $B(\lambda)$ of λ.

(S_2) $\lambda^c((D(\lambda^c) \cap D(\lambda^c)^*)^2)$ is total in $\mathcal{H}$.

(S_3) $\Delta^{it}_{\lambda^{cc}}\mathcal{D} \subset \mathcal{D}$, $\forall t \in \mathbb{R}$.

(S_4) $\Delta^{it}_{\lambda^{cc}}\mathfrak{M}\Delta^{-it}_{\lambda^{cc}} = \mathfrak{M}$, $\forall t \in \mathbb{R}$.

(S_5) λ is full.

The generalized vector λ for $\mathfrak{M}$ is said to be *essentially standard* (resp. *quasi-standard*) if the conditions $(S_1)-(S_4)$ (resp. $(S_1)-(S_3)$) are satisfied.

THEOREM 4.7. – *Let λ be a standard generalized vector for $\mathfrak{M}$. Then the following statements hold:*

(1) $S_\lambda = S_{\lambda^{cc}}$, *and thus* $J_\lambda = J_{\lambda^{cc}}$ *and* $\Delta_\lambda = \Delta_{\lambda^{cc}}$.

(2) *We put*

$$\sigma_t^\lambda(X) = \Delta_\lambda^{it} X \Delta_\lambda^{-it} = \sigma_t^{\lambda^{cc}}(X), \ X \in \mathfrak{M}, \ t \in \mathbb{R}.$$

Then $\{\sigma_t^\lambda\}_{t\in\mathbb{R}}$ *is a one-parameter group of *-automorphisms of* $\mathfrak{M}$, *such that* $\sigma_t^\lambda(B_e(\lambda)) = B_e(\lambda)$ for every $t \in \mathbb{R}$.

(3) λ *satisfies the KMS condition with respect to* $\{\sigma_t^\lambda\}_{t\in\mathbb{R}}$, *that is, for each* $X, Y \in B_e(\lambda) \cap B_e(\lambda)^\dagger$ *there exists an element* $f_{X,Y}$ of $A(0,1)$ *such that*

$$f_{X,Y}(t) = (\lambda(\sigma_t^\lambda(X))|\lambda(Y))$$

and

$$f_{X,Y}(t+i) = (\lambda(Y^\dagger)|\lambda(\sigma_t^\lambda(X^\dagger)))$$

for all $t \in \mathbb{R}$.

Proof. – The proof is entirely analogous to that of [17, Theorem 5.5], that we simply follow. Take two arbitrary elements $X, Y \in B_e(\lambda) \cap B_e(\lambda)^\dagger$. By (4.4) there exist two sequences $\{X_n\}$ and $\{Y_n\}$ in $D(\lambda^{cc}) \cap D(\lambda^{cc})^*$ such that

$$\left.\begin{array}{ll} \lim_n \lambda^{cc}(X_n) = \lambda(X), & \lim_n \lambda^{cc}(X_n^*) = \lambda(X^\dagger), \\ \lim_n \lambda^{cc}(Y_n) = \lambda(Y), & \lim_n \lambda^{cc}(Y_n^*) = \lambda(Y^\dagger). \end{array}\right\} \tag{4.6}$$

By Lemma 4.2 (2), there exists an element f_n of $A(0,1)$ such that, for all $t \in \mathbb{R}$,

$$f_n(t) = (\lambda^{cc}(\sigma_t^{\lambda^{cc}}(X_n))|\lambda^{cc}(Y_n)) = (\Delta_{\lambda^{cc}}^{it}\lambda_t^{cc}(X_n)|\lambda^{cc}(Y_n)),$$
$$f_n(t+i) = (\lambda^{cc}(Y_n^*)|\lambda^{cc}(\sigma_t^{\lambda^{cc}}(X_n^*))) = (\lambda^{cc}(Y_n^*)|\Delta_{\lambda^{cc}}^{it}\lambda^{cc}(X_n^*)),$$

By (S_3) and (S_4), we may define a one-parameter group $\{\sigma_t^{\lambda^{cc}}\}_{t\in\mathbb{R}}$ of *-automorphisms of $\mathfrak{M}$ by the relation:

$$\sigma_t^{\lambda^{cc}}(X) = \Delta_{\lambda^{cc}}^{it} X \Delta_{\lambda^{cc}}^{-it}, \ X \in \mathfrak{M}, \ t \in \mathbb{R}.$$

Since λ is full, it follows from Corollary 4.4 (4) and Lemma 4.2 (1) that

$$\left.\begin{array}{c} \sigma_t^{\lambda^{cc}}(B_e(\lambda)) = B_e(\lambda) \text{ and } \lambda(\sigma_t^{\lambda^{cc}}(X)) = \Delta_{\lambda^{cc}}^{it}\lambda(X), \\ \forall X \in B_e(\lambda), \ t \in \mathbb{R}; \\ \sigma_t^{\lambda^{cc}}(B_e(\lambda) \cap B_e(\lambda)^\dagger) = B_e(\lambda) \cap B_e(\lambda)^\dagger \\ \text{and} \quad \lambda(\sigma_t^{\lambda^{cc}}(X)) = \Delta_{\lambda^{cc}}^{it}\lambda(X), \\ \forall X \in B_e(\lambda) \cap B_e(\lambda)^\dagger, \ t \in \mathbb{R}. \end{array}\right\} \tag{4.7}$$

By (4.6) and (4.7), this implies that

$$\lim_n \sup_{t\in\mathbb{R}} |f_n(t) - (\lambda(\sigma_t^{\lambda^{cc}}(X))|\lambda(Y))| = 0,$$

$$\lim_n \sup_{t\in\mathbb{R}} |f_n(t+i) - (\lambda(Y^\dagger)|\lambda(\sigma_t^{\lambda^{cc}}(X^\dagger)))| = 0.$$

Hence there exists an element $f_{X,Y}$ of $A(0,1)$ such that

$$\left.\begin{aligned} & f_{X,Y}(t) = (\lambda(\sigma_t^{\lambda^{cc}}(X))|\lambda(Y)) \\ \text{and}\quad & f_{X,Y}(t+i) = (\lambda(Y^\dagger)|\lambda(\sigma_t^{\lambda^{cc}}(X^\dagger))), \\ & \forall t \in \mathbb{R}. \end{aligned}\right\} \tag{4.8}$$

Next we show that $S_\lambda = S_{\lambda^{cc}}$. Let $\mathcal{K}$ be the closure in $\mathcal{H}$ of the set $\{\lambda(X);\ X^\dagger = X \in B_e(\lambda) \cap B_e(\lambda)^\dagger\}$. Then it is easy to see that $\mathcal{K}$ is a closed real subspace of $\mathcal{H}$ such that $\mathcal{K}+i\mathcal{K}$ is dense in $\mathcal{H}$ and $\mathcal{K}\cap i\mathcal{K} = \{0\}$. Thus, by [22], S_λ equals the closed operator defined by

$$S(\xi + i\eta) = \xi - i\eta, \quad \xi, \eta \in \mathcal{K}. \tag{4.9}$$

Furthermore, it follows from (4.7) and (4.8) that the one-parameter group $\{\Delta_{\lambda^{cc}}^{it}\}_{t\in\mathbb{R}}$ of unitary operators satisfies the KMS condition with respect to $\mathcal{K}$ in the sense of [22, Definition 3.4], and $\Delta_{\lambda^{cc}}^{it}\mathcal{K} \subset \mathcal{K}$ for all $t \in \mathbb{R}$. By [22, Theorem 3.8] and (4.9), this implies that $\Delta_{\lambda^{cc}}^{it} = \Delta_\lambda^{it}$ for all $t \in \mathbb{R}$. Therefore, it follows that $S_{\lambda^{cc}} = S_\lambda$, which implies by (4.7) and (4.8) that λ satisfies the KMS condition with respect to one-parameter group $\{\sigma_t^\lambda\}_{t\in\mathbb{R}}$ of *-automorphisms of $\mathfrak{M}$. This completes the proof. □

Combining all these results, we get in addition

THEOREM 4.8. – *Let λ be a generalized vector for $\mathfrak{M}$. Then the following statements hold true:*

(1) *If λ is essentially standard, then $\overline{\lambda}\lceil\mathfrak{M}$ is a standard generalized vector for $\mathfrak{M}$.*

(2) *If λ is quasi-standard, then $\overline{\lambda}$ is a standard generalized vector for the partial GW*-algebra $\mathfrak{M}''_{w\sigma} = [(\mathfrak{M}'_w)']^{s^*}$.*

Proof. – The statement (1) follows from Corollary 4.4 and Theorem 4.7, while (2) results from Theorems 4.3 and 4.7. □

COROLLARY 4.9. – *Let $\mathfrak{M}$ be a partial GW*-algebra and λ a generalized vector for $\mathfrak{M}$. If λ is quasi-standard,* a fortiori *if it is essentially standard, then $\overline{\lambda}$ is a standard generalized vector for $\mathfrak{M}$.*

This does not mean, however, that every quasi-standard generalized vector is essentially standard, for conditions (S_3) and (S_4) are not equivalent, even in the case of a partial GW*-algebra.

5. MODULAR GENERALIZED VECTORS

The notion of standard generalized vector developed at length in Section 4 is powerful, but restrictive. In this section, we will weaken our requirements on generalized vectors and introduce modular generalized vectors, as we did in the O*-case in [19] (but the two definitions are different). The result, here too, is that a modular generalized vector will give rise to a standard generalized vector for a partial GW*-algebra, but the latter will act on a dense domain smaller than the original one.

LEMMA 5.1. – *Let λ be a generalized vector for $\mathfrak{M}$. Suppose there exists a core $B(\lambda)$ for λ such that*

(M_1) $\lambda((B(\lambda) \cap B(\lambda)^\dagger)^2)$ *is total in* $\mathcal{H}$;

(M_2) $\lambda^c((D(\lambda^c) \cap D(\lambda^c)^*)^2)$ *is total in* $\mathcal{H}$.

(M_3) $\mathcal{D}_\lambda \equiv \{\xi \in \mathcal{D}; \Delta^{it}_{\lambda^{cc}}\xi \in \mathcal{D},\ \forall t \in \mathbb{R}\}$ *is total in* $\mathcal{H}$.

Then the following statements hold true.

(1) $[(\mathfrak{M}'_{\mathrm{w}})' \lceil \mathcal{D}_\lambda]^{s^*}$ *is a partial GW*-algebra on* $\mathcal{D}_\lambda$ over $(\mathfrak{M}'_{\mathrm{w}})'$.

(2) $\{\Delta^{it}_{\lambda^{cc}}\}_{t\in\mathbb{R}}$ *implements a one-parameter group* $\{\sigma^{\lambda^{cc}}_t\}_{t\in\mathbb{R}}$ *of *-automorphisms of the partial GW*-algebra* $[(\mathfrak{M}'_{\mathrm{w}})' \lceil \mathcal{D}_\lambda]^{s^*}$.

(3) *We put*

$$\begin{cases} B(\lambda_s) = \{X \in [(\mathfrak{M}'_{\mathrm{w}})' \lceil \mathcal{D}_\lambda]^{s^*}; \exists \{A_\alpha\} \subset D(\lambda^{cc}) \\ \qquad\qquad \text{s.t. } A_\alpha \xrightarrow{t_{s^*}} X \text{ and } \lambda^{cc}(A_\alpha) \to \xi_X \in \mathcal{D}_\lambda\}, \\ \lambda_s(X) = \xi_X,\ X \in B(\lambda_s). \end{cases}$$

Then λ_s is a generalized vector for $[(\mathfrak{M}'_{\mathrm{w}})' \lceil \mathcal{D}_\lambda]^{s^*}$.

Proof. – By Lemma 4.2 (1), we have $\mathfrak{M}'_{\mathrm{w}}\mathcal{D}_\lambda \subset \mathcal{D}_\lambda$, so that $\mathfrak{N} \equiv [(\mathfrak{M}'_{\mathrm{w}})' \lceil \mathcal{D}_\lambda]^{s^*}$ is a partial O*-algebra on $\mathcal{D}_\lambda$, such that $\mathfrak{N} \supset \mathfrak{M} \lceil \mathcal{D}_\lambda$ and $\Delta^{it}_{\lambda^{cc}}\mathfrak{N}\Delta^{it}_{\lambda^{cc}} = \mathfrak{N}$ for all $t \in \mathbb{R}$. Hence we have

$$\bigcap_{X\in\mathfrak{N}} D(\overline{X}) \subset \bigcap_{X\in\mathfrak{M}} D(\overline{X \lceil \mathcal{D}_\lambda}) \subset \bigcap_{X\in\mathfrak{M}} D(\overline{X}) = \mathcal{D},$$

$$\Delta^{it}_{\lambda^{cc}}\xi \in D(\Delta^{it}_{\lambda^{cc}}(\overline{\Delta^{-it}_{\lambda^{cc}}X\Delta^{it}_{\lambda^{cc}}})) = D(\overline{X}),$$

for every $X \in \mathfrak{N}$ and $\xi \in \bigcap_{X\in\mathfrak{N}} D(\overline{X})$, which implies that $\mathcal{D}_\lambda = \bigcap_{X\in\mathfrak{N}} D(\overline{X})$. Therefore, $\mathfrak{N}$ is a partial GW*-algebra on $\mathcal{D}_\lambda$ over $(\mathfrak{M}'_{\mathrm{w}})'$.

Then one can show as in the proof of Theorem 4.3 that λ_s is a generalized vector for $\mathfrak{N}$. □

It should be clear that the two sets $\{X \lceil \mathcal{D}_\lambda; X \in B(\lambda)$ and $\lambda(X) \in \mathcal{D}_\lambda\}$ and $\{A \lceil \mathcal{D}_\lambda;\ A \in D(\lambda^{cc})$ and $\lambda^{cc}(A) \in \mathcal{D}_\lambda\}$ are both contained in $\mathcal{D}_\lambda$, but we don't know whether $\lambda_s((B(\lambda_s) \cap B(\lambda_s)^\dagger)^2)$ is total in $\mathcal{H}$. Thus we have to restrict the generalized vector λ and introduce the following notion.

DEFINITION 5.2. – A generalized vector λ for $\mathfrak{M}$ is said to be *modular* if the conditions $(M_1), (M_2), (M_3)$ above and, in addition, the following conditions $(M_4), (M_5)$ are all satisfied:

(M_4) $\lambda_s((B(\lambda_s) \cap B(\lambda_s)^\dagger)^2)$ is total in $\mathcal{H}$.

(M_5) $\lambda_s^c((D(\lambda_s^c) \cap D(\lambda_s^c)^*)^2)$ is total in $\mathcal{H}$ and $\lambda_s^{cc} = \lambda^{cc}$.

The notion of modular generalized vector indeed answers our question, as the next theorem shows, but at the price of restricting ourselves to a smaller dense domain.

THEOREM 5.3. – *Let λ be a modular generalized vector for $\mathfrak{M}$. Then λ_s is a standard generalized vector for the partial GW*-algebra $[(\mathfrak{M}'_{\mathrm{w}})' \lceil \mathcal{D}_\lambda]^{s^*}$ and $\lambda_s^{cc} = \lambda^{cc}$.*

Proof. – It follows from the definition of $\mathcal{D}_\lambda$ and the assumption (M_5) that the generalized vector λ_s for $[(\mathfrak{M}'_{\mathrm{w}})' \lceil \mathcal{D}_\lambda]^{s^*}$ satisfies the conditions (S_3) and (S_4) in Definition 4.6. Furthermore, it follows from the assumption (M_5) and Theorem 4.3 that $\lambda_s = \overline{\lambda_s}$. Therefore λ_s is full. This completes the proof. □

Knowing that modular generalized vectors indeed will lead us to generalized KMS states, it remains to find criteria for a given generalized vector to be modular. We present two of them.

PROPOSITION 5.4. – *Let λ be a generalized vector for $\mathfrak{M}$. Suppose there exists a core $B(\lambda)$ for λ such that*

(i) $\lambda((B(\lambda) \cap B(\lambda)^\dagger)^2)$ *is total in $\mathcal{H}$ and* $\lambda(B(\lambda)) \subset \mathcal{D}_\lambda$;

(ii) $\{\lambda^c(K_1K_2); K_i \in D(\lambda^c) \cap D(\lambda^c)^*$ *and* $\lambda^c(K_i), \lambda^c(K_i^*) \in \mathcal{D}_\lambda\}$ *is total in $\mathcal{H}$;*

(iii) $\{\lambda^c(K); K \in D(\lambda^c) \cap D(\lambda^c)^*$ *and* $\lambda^c(K), \lambda^c(K^*) \in \mathcal{D}_\lambda\}$ *is dense in the Hilbert space $D(S^*_{\lambda^{cc}})$.*

Then λ is modular.

Proof. – Since $\lambda(B(\lambda)) \subset \mathcal{D}_\lambda$, it follows from (4.1) and (4.2) that $X \lceil \mathcal{D}_\lambda \in B(\lambda_s)$ and $\lambda_s(X \lceil \mathcal{D}_\lambda) = \lambda(X)$ for all $X \in B(\lambda)$. By (i) this implies that $\lambda_s((B(\lambda_s) \cap B(\lambda_s)^\dagger)^2)$ is total in $\mathcal{H}$. Furthermore, $D(\lambda_s^c) = \{K \in D(\lambda^c); \lambda^c(K) \in \mathcal{D}_\lambda\}$ and $\lambda_s^c(K) = \lambda^c(K)$, for all $K \in D(\lambda_s^c)$. Hence, by (ii), we conclude that $\lambda_s^c((D(\lambda_s^c) \cap D(\lambda_s^c)^*)^2)$ is total in $\mathcal{H}$. Furthermore, (iii) implies that $\lambda_s^c(D(\lambda_s^c) \cap D(\lambda_s^c)^*)$ is dense in the Hilbert space $D(S^*_{\lambda^{cc}})$. It follows that $\lambda_s^{ccc} = \lambda^{ccc}$ and so $\lambda_s^{cc} = \lambda^{cc}$. Thus λ is modular. □

PROPOSITION 5.5. – *Let λ be a generalized vector for $\mathfrak{M}$. Suppose there exists a core $B(\lambda)$ for λ such that*

(i) $\lambda((B(\lambda) \cap B(\lambda)^\dagger)^2)$ *is total in* $\mathcal{H}$;

(ii) $\lambda^c((D(\lambda^c) \cap D(\lambda^c)^*)^2)$ *is total in* $\mathcal{H}$;

(iii) $\mathcal{D}$ *contains the maximal Tomita algebra* $\mathcal{B}$ *of the achieved left Hilbert algebra* $\mathfrak{A} \equiv \lambda^{cc}(D(\lambda^{cc}) \cap D(\lambda^{cc})^*)$, *that is,* $\mathcal{B} = \{\xi \in \mathfrak{A}; \xi \in \bigcap_{t\in\mathbb{R}} D(\Delta^t_{\lambda^{cc}})\}$.

Then λ is modular.

Proof. – By (iii) we have $\mathcal{B} \subset \mathcal{D}_\lambda$, thus condition (M_3) in Definition 5.2 is satisfied and $\mathcal{B} \subset \lambda_s(B(\lambda_s) \cap B(\lambda_s)^\dagger)$. Hence condition (M_4) in Definition 5.2 is also satisfied. We show that $\lambda_s^{cc} = \lambda^{cc}$. It is easily shown that

$$\left.\begin{array}{r}\{K \in D(\lambda^c); \lambda^c(K) \in \mathcal{D}_\lambda\} \subset D(\lambda_s^c), \\ \lambda_s^c(K) = \lambda^c(K), \ \forall K \in \mathfrak{M}'_{\mathrm{w}} \ \text{s.t.} \ D(\lambda_s^c) \subset \mathcal{D}_\lambda.\end{array}\right\} \tag{5.1}$$

Take arbitrary elements $K \in D(\lambda_s^c)$ and $A \in D(\lambda^{cc}) \cap D(\lambda^{cc})^*$. According to [23, Lemma 1.3], there exists a sequence $\{B_n\}$ in $D(\lambda^{cc}) \cap D(\lambda^{cc})^*$ such that $\lambda^{cc}(B_n) \in \mathcal{B} \subset \mathcal{D}_\lambda$, $B_n \xrightarrow{t_{s^*}} A$, $\lambda^{cc}(B_n) \xrightarrow{t_{s^*}} \lambda^{cc}(A)$ and $\lambda^{cc}(B_n^*) \xrightarrow{t_{s^*}} \lambda^{cc}(A^*)$. From this it follows that $\{B_n\} \subset B(\lambda_s) \cap B(\lambda_s)^\dagger$, $\lambda_s(B_n) = \lambda^{cc}(B_n)$ for every $n \in \mathbb{N}$ and

$$K\lambda^{cc}(A) = \lim_n K\lambda^{cc}(B_n) = \lim_n K\lambda_s(B_n) = \lim_n B_n\lambda_s^c(K) = A\lambda_s^c(K),$$

which implies

$$K \in D(\lambda^{ccc}) \quad \text{and} \quad \lambda^{ccc}(K) = \lambda_s^c(K). \tag{5.2}$$

In addition, we have

$$\mathcal{B} \subset \mathcal{E} \equiv \{\lambda^c(K); K \in D(\lambda^c) \cap D(\lambda^c)^* \ \text{s.t.} \ \lambda^c(K), \lambda^c(K^*) \in \mathcal{D}_\lambda\}. \tag{5.3}$$

Take now any $\xi \in \mathcal{B}$. Since λ^{ccc} is full, we may write $\xi = \lambda^{ccc}(K)$ and $S^*_{\lambda^{cc}}\xi = \lambda^{ccc}(K^*)$ for some $K \in D(\lambda^{ccc}) \cap D(\lambda^{ccc})^*$. Since $\xi, S^*_{\lambda^{cc}}\xi \in \mathcal{D}_\lambda \subset \mathcal{D}$, we see that $K \in \mathcal{E}$. Since

$$\mathcal{B} \subset \mathcal{E} \subset \lambda_s^c(D(\lambda_s^c) \cap D(\lambda_s^c)^*) \subset D(\lambda^{ccc}) \cap D(\lambda^{ccc})^*$$

by (5.1), (5.2) and (5.3), it follows that $\lambda_s^c((D(\lambda_s^c) \cap D(\lambda_s^c)^*)^2)$ is total in $\mathcal{H}$. Furthermore, since $\mathcal{B}$ is dense in the Hilbert space $D(S^*_{\lambda^{cc}})$, it follows that $\lambda_s^{ccc} = \lambda^{ccc}$ and so $\lambda_s^{cc} = \lambda^{cc}$. Thus λ is modular. □

6. SPECIAL CASES AND EXAMPLES

In this section we examine some particular cases of standard or modular generalized vectors for partial O*-algebras.

6.1. Generalized vectors associated to individual vectors

We consider, as usual, a fully closed partial O*-algebra $\mathfrak{M}$ on $\mathcal{D}$ such that $\mathfrak{M}'_{\mathrm{w}}\mathcal{D} \subset \mathcal{D}$. Let λ_ξ be the generalized vector associated to $\xi \in \mathcal{H}$, as defined in Examples 3.4 (1), (2). When is it standard or modular?

PROPOSITION 6.1. – *Let $\xi \in \mathcal{D}$. Suppose that*

(i) *$\{X\xi;\ X \in \mathfrak{M}$ s.t. $X\xi \in \mathcal{D}\}$ is dense in $\mathcal{H}$;*

(ii) *$\mathfrak{M}'_{\mathrm{w}}\xi$ is dense in $\mathcal{H}$.*

Then λ_ξ is modular.

Let $\triangle_\xi$ denote the modular operator associated to the achieved left Hilbert algebra $(\mathfrak{M}'_{\mathrm{w}})'\xi$. If

(iii) *$\triangle_\xi^{it}\mathcal{D} \subset \mathcal{D}$, $\forall t \in \mathbb{R}$.*

then λ_ξ is quasi-standard. Finally, if

(iv) *$\triangle_\xi^{it}\mathfrak{M}\triangle_\xi^{-it} = \mathfrak{M}$, $\forall t \in \mathbb{R}$,*

then λ_ξ is standard.

Proof. – It is easily shown from (i) and (ii) that

$$D(\lambda_\xi^c) = \mathfrak{M}'_{\mathrm{w}} \text{ and } \lambda_\xi^c(K) = K\xi,\ \forall K \in \mathfrak{M}'_{\mathrm{w}};$$
$$D(\lambda_\xi^{cc}) = (\mathfrak{M}'_{\mathrm{w}})' \text{ and } \lambda_\xi^{cc}(A) = A\xi,\ \forall A \in (\mathfrak{M}'_{\mathrm{w}})',$$

and so $\lambda_\xi^{cc}(D(\lambda_\xi^{cc}) \cap D(\lambda_\xi^{cc})^*)$ is an achieved left Hilbert algebra in $\mathcal{H}$, which equals the left Hilbert algebra $(\mathfrak{M}'_{\mathrm{w}})'\xi$. Hence we have

$$\mathcal{B} \subset \mathfrak{M}'_{\mathrm{w}}\xi \subset \mathcal{D}_{\lambda_\xi} \subset \mathcal{D},$$

where $\mathcal{B}$ is the maximal Tomita algebra for the achieved left Hilbert algebra $(\mathfrak{M}'_{\mathrm{w}})'\xi$. Proposition 5.5 then implies that λ_ξ is modular. The rest is immediate. □

Let now $\xi \in \mathcal{H}\backslash\mathcal{D}$. Suppose

(i) $\{K_1^* K_2\xi;\ K_1, K_2 \in \mathfrak{M}'_{\mathrm{w}}$ s.t. $K_1\xi, K_2\xi \in \mathcal{D}\}$ *is total in* $\mathcal{H}$.

According to Example 3.4 (2), the generalized vector λ_ξ is defined as follows:

$$\begin{aligned} B_e(\lambda_\xi) &= \{X \in \mathfrak{M}; \xi \in D(X^{\dagger *}) \text{ and } X^{\dagger *}\xi \in \mathcal{D}\}, \\ D(\lambda_\xi) &= \text{linear span of } \{Y \square X; X \in B_e(\lambda_\xi),\ Y \in L(X)\}, \\ \lambda_\xi(\textstyle\sum_k Y_k \square X_k) &= \textstyle\sum_k Y_k X_k^{\dagger *}\xi,\ \sum_k Y_k \square X_k \in D(\lambda_\xi). \end{aligned}$$

Suppose in addition that

(ii) $(B_e(\lambda_\xi) \cap B_e(\lambda_\xi)^\dagger)^2$ *is total in* $\mathcal{H}$.

Then we have

$$\left.\begin{aligned} D(\lambda_\xi^c) &= \{K \in \mathfrak{M}'_{\mathrm{w}}; K\xi \in \mathcal{D}\}, \\ \lambda_\xi^c(K) &= K\xi,\ K \in D(\lambda_\xi^c). \end{aligned}\right\} \tag{6.1}$$

$$\left.\begin{aligned} D(\lambda_\xi^{cc}) &= (\mathfrak{M}'_{\mathrm{w}})', \\ \lambda_\xi^{cc}(A) &= A\xi,\ A \in (\mathfrak{M}'_{\mathrm{w}})', \end{aligned}\right\} \tag{6.2}$$

and so $\lambda_\xi^{cc}(D(\lambda_\xi^{cc}) \cap D(\lambda_\xi^{cc})^*)$ equals the achieved left Hilbert algebra $(\mathfrak{M}'_{\mathrm{w}})'\xi$. For simplicity, we denote by S_ξ and $\triangle_\xi$ the operators $S_{(\mathfrak{M}'_{\mathrm{w}})'\xi}$ and $\triangle_{(\mathfrak{M}'_{\mathrm{w}})'\xi}$, respectively. Then we have the following

PROPOSITION 6.2. – *Let $\xi \in \mathcal{H}\backslash\mathcal{D}$. Suppose that the conditions (i) and (ii) above are satisfied. If $\triangle_\xi^{it}\mathcal{D} \subset \mathcal{D}$, for every $t \in \mathbb{R}$, then λ_ξ is quasi-standard. Furthermore, if $\triangle_\xi^{it}\mathfrak{M}\triangle_\xi^{-it} = \mathfrak{M}$, for every $t \in \mathbb{R}$, then λ_ξ is standard.* □

Now we investigate the modularity of λ_ξ. Suppose the conditions (i) and (ii) in Proposition 6.2 hold. We define

$$\mathcal{C}_\triangle(\mathcal{D}) = \{K \in \mathfrak{M}'_{\mathrm{w}}; \triangle_\xi^{it}K\xi,\ \triangle_\xi^{it}K^*\xi \in \mathcal{D},\ \forall t \in \mathbb{R}\}.$$

Then $\mathcal{C}_\triangle(\mathcal{D})$ is a *-subalgebra of $\mathfrak{M}'_{\mathrm{w}}$ and $\mathcal{C}_\triangle(\mathcal{D})\xi \subset \mathcal{D}_{\lambda_\xi}$. Thus, if $\mathcal{C}_\triangle(\mathcal{D})\xi$ is dense in $\mathcal{H}$, then, by Lemma 5.1, the generalized vector $(\lambda_\xi)_s$ for the partial GW*-algebra $\left[(\mathfrak{M}'_{\mathrm{w}})' \lceil \mathcal{D}_{\lambda_\xi}\right]^{s^*}$ may be defined. We have the following

PROPOSITION 6.3. – *Let* $\xi \in \mathcal{H}\backslash\mathcal{D}$. *Suppose that the conditions* (i) *and* (ii) *in Proposition* 6.2 *hold, as well as the following conditions* (iii) *and* (iv):

(iii) $\{K_1K_2\xi;\ K_1, K_2 \in \mathcal{C}_\triangle(\mathcal{D})\}$ *is total in* $\mathcal{H}$.

(iv) $\{X_2X_1^{\dagger *}\xi; X_i \in \left[(\mathfrak{M}'_{\rm w})' \lceil \mathcal{D}_{\lambda_\xi}\right]^{s^*}$ s.t. $\xi \in D(X_i^{\dagger *})$ *and* $X_i^{\dagger *}\xi \in \mathcal{D}_{\lambda_\xi}$, $i = 1, 2\}$ *is total in* $\mathcal{H}$.

Then λ_ξ *is modular.*

Proof. – By (iii) $\mathcal{D}_{\lambda_\xi}$ is dense in $\mathcal{H}$ and thus, by Lemma 5.1, the generalized vector $(\lambda_\xi)_s$ for the partial GW*-algebra $\left[(\mathfrak{M}'_{\rm w})' \lceil \mathcal{D}_{\lambda_\xi}\right]^{s^*}$ may be defined. It satisfies the following relations

$$\left.\begin{aligned} B((\lambda_\xi)_s) = \{X \in \left[(\mathfrak{M}'_{\rm w})' \lceil \mathcal{D}_{\lambda_\xi}\right]^{s^*};\\ \xi \in D(X^{\dagger *}) \text{ and } X^{\dagger *}\xi \in \mathcal{D}_{\lambda_\xi}\},\\ (\lambda_\xi)_s(X) = X^{\dagger *}\xi,\ X \in B((\lambda_\xi)_s). \end{aligned}\right\} \quad (6.3)$$

Take indeed an arbitrary element $X \in B((\lambda_\xi)_s)$. Then there exists a net $\{A_\alpha\}$ in $(\mathfrak{M}'_{\rm w})'$ such that $A_\alpha \xrightarrow{t_{s^*}} X$ and $A_\alpha\xi \to (\lambda_\xi)_s(X)$. Thus we get, for all $\eta \in \mathcal{D}_{\lambda_\xi}$,

$$(X^\dagger\eta|\xi) = \lim_\alpha (A_\alpha^*\eta|\xi) = \lim_\alpha (\eta|A_\alpha\xi) = (\eta|(\lambda_\xi)_s(X)).$$

Hence $X^{\dagger *}\xi \in \mathcal{D}_{\lambda_\xi}$ and $X^{\dagger *}\xi = (\lambda_\xi)_s(X) \in \mathcal{D}_{\lambda_\xi}$. Conversely, let X be an arbitrary element of $\left[(\mathfrak{M}'_{\rm w})' \lceil \mathcal{D}_{\lambda_\xi}\right]^{s^*}$ such that $\xi \in D(X^{\dagger *})$ and $X^{\dagger *}\xi \in \mathcal{D}_{\lambda_\xi}$. From the spectral resolution of $|X^{\dagger *}|$, we see there exists a sequence $\{X_n\}$ in $(\mathfrak{M}'_{\rm w})'$ such that $X_n \xrightarrow{t_{s^*}} X^{\dagger *}$ and $X_n\xi \to X^{\dagger *}\xi$, which implies that $X \in B((\lambda_\xi)_s)$.

By (6.3) and (iv), the condition (M_4) in Definition 5.2 is satisfied. Let us show the following relations.

$$\left.\begin{aligned} D((\lambda_\xi)_s^c) = \{K \in \mathfrak{M}'_{\rm w};\ K\xi \in \mathcal{D}_{\lambda_\xi}\},\\ (\lambda_\xi)_s^c(K) = K\xi,\ K \in D((\lambda_\xi)_s^c). \end{aligned}\right\} \quad (6.4)$$

Indeed it is easy to show from (6.3) that $\{K \in \mathfrak{M}'_{\rm w};\ K\xi \in \mathcal{D}_{\lambda_\xi}\} \subset D((\lambda_\xi)_s^c)$. Conversely, let K be an arbitrary element of $D((\lambda_\xi)_s^c)$. Then, for all $X, Y \in B((\lambda_\xi)_s) \cap B((\lambda_\xi)_s)^\dagger$ such that $Y \in L(X)$, we have

$$\begin{aligned} (K\xi|(\lambda_\xi)_s(Y\Box X)) &= (K\xi|YX^{\dagger *}\xi) = (KY^*\xi|X^{\dagger *}\xi)\\ &= (Y^\dagger(\lambda_\xi)_s^c(K)|X^{\dagger *}\xi)\\ &= ((\lambda_\xi)_s^c(K)|YX^{\dagger *}\xi)\\ &= ((\lambda_\xi)_s^c(K)|(\lambda_\xi)_s(Y\Box X)). \end{aligned}$$

Hence it follows from (6.1) and (iv) that $K\xi = (\lambda_\xi)_s^c(K) \in \mathcal{D}_{\lambda_\xi}$.

On the other hand, (6.4) and (iii) imply that $(\lambda_\xi)_s^c((D((\lambda_\xi)_s^c) \cap D((\lambda_\xi)_s^c)^*)^2)$ is total in $\mathcal{H}$. Finally we show that $S_{(\lambda_\xi)_s^{cc}} = S_{\lambda_\xi^{cc}}$. It is clear that $\mathcal{K} \equiv \mathcal{C}_\triangle(\mathcal{D})\xi$ is a dense *-subalgebra of the right Hilbert algebra $\mathfrak{M}'_{\mathrm{w}}\xi$, whose commutant $\mathcal{K}'$ contains $(\mathfrak{M}'_{\mathrm{w}})'\xi$. Conversely, let η be an element of $\mathcal{K}'$, *i.e.* there exists an element $\eta^\#$ of $\mathcal{H}$ and an element A of $\mathcal{B}(\mathcal{H})$ such that $(\eta|K\xi) = (K^*\xi|\eta^\#)$ and $AK\xi = K\eta$ for all $K \in \mathcal{C}_\triangle(\mathcal{D})$. Since $\mathcal{C}_\triangle(\mathcal{D})$ is a nondegenerate *-subalgebra of $\mathfrak{M}'_{\mathrm{w}}$ by (iii), it follows that $AK\xi = K\eta$ for every $K \in \mathfrak{M}'_{\mathrm{w}}$, which implies that $\eta = A\xi \in (\mathfrak{M}'_{\mathrm{w}})'\xi$. Thus we have $\mathcal{K}' = (\mathfrak{M}'_{\mathrm{w}})'\xi$. Hence it follows from (6.4) that $S_{(\lambda_\xi)_s^{cc}} = S_{\lambda_\xi^{cc}}$. Therefore λ_ξ is modular. □

6.2. Standard generalized vectors constructed from Hilbert-Schmidt operators

Let $\mathfrak{M}$ be a fully closed (resp. self-adjoint) partial O*-algebra on $\mathcal{D} \subset \mathcal{H}$. We denote by $\mathcal{H} \otimes \overline{\mathcal{H}}$ the Hilbert space of Hilbert-Schmidt operators on $\mathcal{H}$, and put:

$$\begin{aligned}\mathcal{D} \otimes \overline{\mathcal{H}} &= \{T \in \mathcal{H} \otimes \overline{\mathcal{H}};\ T\mathcal{H} \subset \mathcal{D}\},\\ \mathfrak{S}_2(\mathfrak{M}) &= \{T \in \mathcal{D} \otimes \overline{\mathcal{H}}; XT \in \mathcal{H} \otimes \overline{\mathcal{H}},\ \forall X \in \mathfrak{M}\},\\ \pi(X)T &= XT,\ X \in \mathrm{M}, T \in \mathfrak{S}_2(\mathfrak{M}).\end{aligned}$$

Then it is easily shown [24] that, for any X_1, $X_2 \in \mathfrak{M}$, $\pi(X_1) \square \pi(X_2)$ is well-defined iff $X_1 \square X_2$ is, and, moreover, that $\pi(\mathfrak{M})$ is fully closed (resp. self-adjoint) iff $\mathfrak{M}$ is fully closed (resp. self-adjoint). Hence $\pi(\mathfrak{M})$ is a fully closed (resp. self-adjoint) partial O*-algebra on $\mathfrak{S}_2(\mathfrak{M}) \subset \mathcal{H} \otimes \overline{\mathcal{H}}$. Furthermore, we can show as in [25, Lemma 2.4] that, if $\mathfrak{M}'_{\mathrm{w}} = \mathbb{C}I$, then $\pi(\mathfrak{M})'_{\mathrm{w}} = \pi'(\mathcal{B}(\mathcal{H}))$ and $(\pi(\mathfrak{M})'_{\mathrm{w}})' = \pi''(\mathcal{B}(\mathcal{H}))$, where

$$\begin{aligned}\pi'(A)T &= TA,\\ \pi''(A)T &= AT,\ A \in \mathcal{B}(\mathcal{H}), T \in \mathcal{H} \otimes \overline{\mathcal{H}}.\end{aligned}$$

Let $\mathfrak{M}$ be a fully closed partial O*-algebra on $\mathcal{D} \subset \mathcal{H}$ such that $\mathfrak{M}'_{\mathrm{w}} = \mathbb{C}I$ and Ω a nonsingular, positive Hilbert-Schmidt operator on $\mathcal{H}$. Suppose that

(i) $\Omega\mathcal{E} \subset \mathcal{D}$, for some dense subspace $\mathcal{E}$ in $\mathcal{H}$, contained in $\mathcal{D}$.

Then it follows that $\{\pi'(A);\ A \in \mathcal{B}(\mathcal{H}) \text{ s.t. } \Omega A, \Omega A^* \in \mathfrak{S}_2(\mathfrak{M})\}$ is nondegenerate, so that we can define the generalized vector λ_Ω for $\pi(\mathfrak{M})$

as follows:

$$\left.\begin{aligned} B_e(\lambda_\Omega) &= \{\pi(X);\ X \in \mathfrak{M} \text{ and } X^{\dagger *}\Omega \in \mathfrak{S}_2(\mathfrak{M})\},\\ D(\lambda_\Omega) &= \text{linear span of } \{\pi(Y \square X); \pi(X) \in B_e(\lambda_\Omega) \\ &\qquad\qquad \text{and } Y \in L(X)\},\\ \lambda_\Omega(\textstyle\sum_k \pi(Y_k \square X_k)) &= \textstyle\sum_k Y_k X_k^{\dagger *}\Omega,\ \sum_k \pi(Y_k \square X_k) \in D(\lambda_\Omega). \end{aligned}\right\} \quad (6.5)$$

Concerning the standardness of λ_Ω, we have the following criterion.

PROPOSITION 6.4. – *Suppose*

(i) $\Omega\mathcal{E} \subset \mathcal{D}$, *for some dense subspace* $\mathcal{E}$ *in* $\mathcal{H}$, *contained in* $\mathcal{D}$;

(ii) $\{X_2 X_1^{\dagger *}\Omega;\ X_i \in \mathfrak{M}$ *s.t.* $X_i^{\dagger *} \in \mathfrak{S}_2(\mathfrak{M}),\ i = 1, 2\}$ *is total in* $\mathcal{H} \otimes \overline{\mathcal{H}}$;

(iii) $\Omega^{it}\mathcal{D} \subset \mathcal{D},\ \forall t \in \mathbb{R}$.

Then λ_Ω *is quasi-standard. Moreover, if*

(iv) $\Omega^{it}\mathfrak{M}\,\Omega^{-it} = \mathfrak{M},\ \forall t \in \mathbb{R}$,

then λ_Ω *is standard.*

Proof. – From (6.5) and (ii), we see that $\lambda_\Omega((B_e(\lambda_\Omega) \cap B_e(\lambda_\Omega)^\dagger)^2)$ is total in $\mathcal{H} \otimes \overline{\mathcal{H}}$, so that, by (6.1),

$$\begin{cases} D(\lambda_\Omega^c) = \{\pi'(K);\ K \in \mathcal{B}(\mathcal{H}) \text{ s.t. } \Omega K \in \mathfrak{S}_2(\mathfrak{M})\},\\ \qquad \lambda_\Omega^c(\pi'(K)) = \Omega K,\ K \in D(\lambda_\Omega^c). \end{cases}$$

Hence, by (i), we have $\pi'(\xi \otimes \overline{\eta}) \in (D(\lambda_\Omega^c) \cap D(\lambda_\Omega^c)^*)^2$, for every $\xi, \eta \in \mathcal{E}$. Since Ω is nonsingular, this implies that $\lambda_\Omega^c((D(\lambda_\Omega^c) \cap D(\lambda_\Omega^c)^*)^2)$ is total in $\mathcal{H} \otimes \overline{\mathcal{H}}$, and therefore, by (6.2),

$$\begin{cases} \qquad D(\lambda_\Omega^{cc}) = \pi''(\mathcal{B}(\mathcal{H})),\\ \lambda_\Omega^{cc}(\pi''(A)) = A\Omega,\ A \in \mathcal{B}(\mathcal{H}). \end{cases}$$

Then it follows from [14, Lemma 5.2] that $\lambda_\Omega^{cc}(D(\lambda_\Omega^{cc}) \cap D(\lambda_\Omega^{cc})^*)$ $(= \pi''(\mathcal{B}(\mathcal{H}))\Omega)$ is an achieved left Hilbert algebra in $\mathcal{H} \otimes \overline{\mathcal{H}}$, such that

$$\Delta_{\lambda_\Omega^{cc}}^{it} = \pi'(\Omega^{-2it})\pi''(\Omega^{2it}),\ \forall t \in \mathbb{R}. \qquad (6.6)$$

All our assertions follow from this relation. □

As for the modularity of λ_Ω, we may state:

PROPOSITION 6.5. – *Suppose that* $\Omega \in \mathcal{D} \otimes \overline{\mathcal{H}}$. *Then the following statements hold:*

(1) *If the condition (ii) in Proposition 6.4 is satisfied, then* λ_Ω *is modular;*

(2) *If* $\mathcal{D}$ *contains an orthonormal basis* $\{\xi_n\}$ *of* $\mathcal{H}$ *such that* $\xi_n \otimes \overline{\xi_m} \in \mathfrak{M}$ *for all* $n, m \in \mathbb{N}$, *then* λ_Ω *is modular.*

Proof. – (1) Since Ω is a nonsingular, positive Hilbert-Schmidt operator on $\mathcal{H}$ such that $\Omega\mathcal{D} \in \mathcal{H}$, it may be represented as $\Omega = \sum_{n=1}^{\infty} \omega_n f_n \otimes \overline{f_n}$, where $\omega_n > 0$ for every $n \in \mathbb{N}$ and $\{f_n\}$ is an orthonormal basis of $\mathcal{H}$ contained in $\mathcal{D}$. Since the linear span of $\{f_n \otimes \overline{f_m},\ n,m \in \mathbb{N}\}$ is contained in $\mathcal{D}_{\lambda_\Omega}$ and is dense in $\mathcal{H} \otimes \overline{\mathcal{H}}$, it follows that $\mathcal{D}_{\lambda_\Omega}$ is also dense in $\mathcal{H} \otimes \overline{\mathcal{H}}$. Thus the conditions (M_1), (M_2) and (M_3) in Lemma 5.1 are satisfied, and so one may define the generalized vector $(\lambda_\Omega)_s$ for the partial GW*-algebra $[(\pi''(\mathcal{B}(\mathcal{H})) \lceil \mathcal{D}_{\lambda_\Omega}]^{s^*}$. Since the set $\{(f_n \otimes \overline{f_n})A(f_n \otimes \overline{f_n}),\ A \in \mathcal{B}(\mathcal{H}),\ n \in \mathbb{N}\}$ is contained in $(\lambda_\Omega)_s((B((\lambda_\Omega)_s) \cap B((\lambda_\Omega)_s)^\dagger)^2)$, and is total in $\mathcal{H} \otimes \overline{\mathcal{H}}$, the condition (M_4) in Definition 5.2 is satisfied. Moreover, (6.5) implies that $\{\pi'(f_n \otimes \overline{f_n})K(f_n \otimes \overline{f_n}),\ K \in \mathcal{B}(\mathcal{H}),\ n \in \mathbb{N}\}$ is contained in $\mathcal{C}_\triangle(\mathfrak{S}_2(\mathfrak{M}))^2$ and is total in $\mathcal{H} \otimes \overline{\mathcal{H}}$. Thus the condition (ii) in Proposition 6.3 holds and λ_Ω is modular.

(2) Since

$$\{\xi_n \otimes \overline{\xi_m};\ n,m \in \mathbb{N}\} \subset (B_e(\lambda_\Omega) \cap B_e(\lambda_\Omega)^\dagger)^2$$

and $\{\lambda_\Omega(\xi_n \otimes \overline{\xi_m})\ (= (\xi_n \otimes \overline{\xi_m})\Omega);\ n,m \in \mathbb{N}\}$ is total in the dense subspace of $\mathcal{H} \otimes \overline{\mathcal{H}}$ generated by $(f_n \otimes \overline{f_n})\Omega;\ n,m \in \mathbb{N}\}$, it follows that $\lambda_\Omega((B_e(\lambda_\Omega) \cap B_e(\lambda_\Omega)^\dagger)^2)$ is total in $\mathcal{H} \otimes \overline{\mathcal{H}}$. Therefore, by (1), λ_Ω is modular. □

6.3. Partially modular generalized vectors for O*-algebras

In this subsection we consider generalized vectors for *O*-algebras*. For such objects, a notion of modularity has been defined in [17, Definition 5.10] in the framework of O*-algebras. Roughly speaking, a generalized vector λ for an O*-algebra $\mathfrak{M}$ is said to be modular if there exists a suitable dense domain $\mathcal{E}$ such that (i) $\triangle_{\lambda^{cc}}^{it}\mathcal{E} \subset \mathcal{E},\ \forall t \in \mathbb{R}$; and (ii) $\mathfrak{M}\mathcal{E} \subset \mathcal{E}$. Now, since an O*-algebra is *a fortiori* a partial *-algebra, we can also define a notion of modularity for the generalized vector λ in the framework of partial *-algebras. In a nutshell, we may omit the condition (ii) above. Let us state this precisely.

Let $\mathfrak{M}$ be a closed *O*-algebra* on $\mathcal{D} \subset \mathcal{H}$ such that $\mathfrak{M}'_{\mathrm{w}}\mathcal{D} \subset \mathcal{D}$. A generalized vector λ for $\mathfrak{M}$ is said to be *partially modular* if it is modular when regarded as a generalized vector for the partial O*-algebra $\mathfrak{M}$, that is, the following conditions hold:

(PM_1) $\lambda((D(\lambda) \cap D(\lambda)^\dagger)^2)$ is total in $\mathcal{H}$;

(PM_2) $\lambda^c((D(\lambda^c) \cap D(\lambda^c)^*)^2)$ is total in $\mathcal{H}$.

(PM_3) $\mathcal{D}_\lambda \equiv \{\xi \in \mathcal{D}; \triangle_{\lambda^{cc}}^{it}\xi \in \mathcal{D},\ \forall t \in \mathbb{R}\}$ is total in $\mathcal{H}$.

(PM_4) $\lambda_s((B(\lambda_s) \cap B(\lambda_s)^\dagger)^2)$ is total in $\mathcal{H}$.

(PM_5) $\lambda_s^c((D(\lambda_s^c) \cap D(\lambda_s^c)^*)^2)$ is total in $\mathcal{H}$ and $\lambda_s^{cc} = \lambda^{cc}$.

If λ is a partially modular generalized vector for $\mathfrak{M}$, then, by Theorem 5.3, λ_s is a standard generalized vector for the partial GW*-algebra $[(\mathfrak{M}'_{\mathrm{w}})' \lceil \mathcal{D}_\lambda]^{s^*}$, and $\lambda_s^{cc} = \lambda^{cc}$. Thus all the results of Propositions 6.1 to 6.5 apply to the partial modularity of generalized vectors for O*-algebras. In addition, we have:

PROPOSITION 6.6. – *Let $\mathfrak{M}$ be a closed O*-algebra on $\mathcal{D} \subset \mathcal{H}$ such that $\mathfrak{M}'_{\mathrm{w}}\mathcal{D} \subset \mathcal{D}$. Then the following statements hold:*

(1) *Suppose $\xi \in \mathcal{D}$ satisfies the following two conditions:*

(i) *$\mathfrak{M}\xi$ is dense in $\mathcal{H}$,*

(ii) *$\mathfrak{M}'_{\mathrm{w}}\xi$ is dense in $\mathcal{H}$.*

Then λ_ξ is partially modular.

(2) *Suppose $\mathfrak{M}'_{\mathrm{w}} = \mathbb{C}I$. If Ω is a nonsingular, positive Hilbert-Schmidt operator in $\mathfrak{S}_2(\mathfrak{M})$ such that $\pi(\mathfrak{M})\Omega$ is dense in $\mathcal{H} \otimes \overline{\mathcal{H}}$, then λ_Ω is partially modular.* □

REMARK 6.7. – Let again λ be a generalized vector for the O*-algebra $\mathfrak{M}$, and suppose that it is modular. Then λ is partially modular. Indeed, one can show, exactly as in the proof of [17, Theorem 5.11], that $\{X \lceil \mathcal{D}_\lambda$ $X \in D(\lambda) \cap D(\lambda)^\dagger\} \subset B(\lambda_s) \cap B(\lambda_s)^\dagger$ and $\lambda_s(X \lceil \mathcal{D}_\lambda) = \lambda(X)$, for every $X \in D(\lambda) \cap D(\lambda)^\dagger$. Thus the condition (PM_4) above is satisfied. Moreover, since $K \in \mathfrak{M}'_{\mathrm{w}}; \lambda^c(K) \in \mathcal{D}_\lambda\} \subset D(\lambda_s^c)$, the condition (PM_5) holds also, which proves the assertion.

7. STANDARD GENERALIZED VECTORS CONSTRUCTED FROM STANDARD VON NEUMANN ALGEBRAS

In this last section, we pursue the study of generalized vectors on partial GW*-algebras, reverting to the usual approach of extending a given result from the dense subset of bounded operators to the full partial GW*-algebra. More precisely, we address the following question:

Let $\mathfrak{M}_o$ be a von Neumann algebra on $\mathcal{H}$ with a standard generalized vector λ_o. Can one construct a partial GW*-algebra $\mathfrak{M}$ on a dense subspace $\mathcal{D} \subset \mathcal{H}$ over $\mathfrak{M}_o$ and a standard generalized vector λ for $\mathfrak{M}$ such that $\lambda^c = \lambda_o^c$ and $\lambda^{cc} = \lambda_o$?

We will give a positive answer to that question, using arguments similar to [14, Theorem 4.1]. Let $\mathfrak{M}_o$ be a von Neumann algebra on $\mathcal{H}$ with a standard generalized vector λ_o and $\mathfrak{T}(\lambda_o)$ a *-subalgebra of $\mathfrak{M}_o$ such that $\lambda_o(\mathfrak{T}(\lambda_o))$ is a maximal Tomita algebra equivalent to the achieved left Hilbert algebra $\lambda_o(D(\lambda_o) \cap D(\lambda_o)^*)$, with involution #. Let $\triangle_{\lambda_o}$ and J_{λ_o} be, respectively, the modular operator and the modular conjugation operator for the achieved left Hilbert algebra $\lambda_o(D(\lambda_o) \cap D(\lambda_o)^*)$.

THEOREM 7.1. – *Let τ be a locally convex topology on $\lambda_o(\mathfrak{T}(\lambda_o))$ such that:*

(i) $\lambda_o(\mathfrak{T}(\lambda_o))[\tau]$ *is a locally convex *-algebra with the involution #,*

(ii) τ *is finer than the norm topology of $\mathcal{H}$.*

Then there exist a dense domain $\mathcal{D}_\tau$, a partial GW-algebra $\mathfrak{M}_\tau$ on $\mathcal{D}_\tau$ over $\mathfrak{M}_o$ and a strongly cyclic full modular generalized vector λ_τ for $\mathfrak{M}_\tau$ such that $\lambda_\tau^{cc} = \lambda_o$. Moreover, if*

(iii) $\triangle_{\lambda_o}^{it}$ *is a continuous operator on $\lambda_o(\mathfrak{T}(\lambda_o))[\tau]$ for every $t \in \mathbb{R}$,*

then λ_τ is standard.

Proof. – We denote by $\mathcal{B}_\tau$ the completion $[\lambda_o(\mathfrak{T}(\lambda_o))]^\tau$ of $\lambda_o(\mathfrak{T}(\lambda_o))[\tau]$. For every $\xi \in \mathcal{B}_\tau$ we put

$$\begin{aligned} L_\xi \lambda_o(B) &\equiv \lim_\alpha \lambda_o(B_\alpha)\lambda_o(B), \ B \in \mathfrak{T}(\lambda_o), \\ &= \lim_\alpha B_\alpha \lambda_o(B), \end{aligned}$$

where $\{B_\alpha\}$ is a net in $\mathfrak{T}(\lambda_o)$ such that $\lambda_o(B_\alpha) \xrightarrow{\tau} \xi$. Since $\lambda_o(\mathfrak{T}(\lambda_o))[\tau]$ is a locally convex *-algebra, L_ξ is a well-defined linear operator from $\lambda_o(\mathfrak{T}(\lambda_o))$ to $\mathcal{B}_\tau$. First we have $L_{\xi^\#} \subset L_\xi^*$. Indeed, by (i) and (ii), we may write, for any $B_1, B_2 \in \mathfrak{T}(\lambda_o)$,

$$\begin{aligned} (L_\xi \lambda_o(B_1)|\lambda_o(B_2)) &= \lim_\alpha (B_\alpha \lambda_o(B_1)|\lambda_o(B_2)) \\ &= \lim_\alpha (\lambda_o(B_1)|B_\alpha^* \lambda_o(B_2)) \\ &= (\lambda_o(B_1)|L_{\xi^\#} \lambda_o(B_2)), \end{aligned}$$

which implies the statement. The crucial property of L_ξ is that

$$\overline{L_\xi} \text{ is affiliated with } \mathfrak{M}_o. \tag{7.1}$$

To show this, take any $\xi \in \mathcal{B}_\tau$ and $B, B_1, B_2 \in \mathfrak{T}(\lambda_o)$. Then we have:

$$\begin{aligned} &(L_\xi \lambda_o(B_1) \mid J_{\lambda_o} \triangle_{\lambda_o}^{1/2} B \triangle_{\lambda_o}^{-1/2} J_{\lambda_o} \lambda_o(B_2)) \\ &= \lim_\alpha (B_\alpha \lambda_o(B_1) \mid J_{\lambda_o} \triangle_{\lambda_o}^{1/2} B \triangle_{\lambda_o}^{-1/2} J_{\lambda_o} \lambda_o(B_2)) \\ &= \lim_\alpha \ (B_\alpha \lambda_o(B_1 B) \mid \lambda_o(B_2)) \\ &= (L_\xi (J_{\lambda_o} \triangle_{\lambda_o}^{1/2} B^* \triangle_{\lambda_o}^{-1/2} J_{\lambda_o}) \lambda_o(B_1) \mid \lambda_o(B_2)), \end{aligned}$$

so that

$$(J_{\lambda_o}\Delta_{\lambda_o}^{1/2}B^*\Delta_{\lambda_o}^{-1/2}J_{\lambda_o})\overline{L_\xi} \subset \overline{L_\xi}(J_{\lambda_o}\Delta_{\lambda_o}^{1/2}B^*\Delta_{\lambda_o}^{-1/2}J_{\lambda_o}).$$

Since $\{J_{\lambda_o}\Delta_{\lambda_o}^{1/2}B^*\Delta_{\lambda_o}^{-1/2}J_{\lambda_o}; B \in \mathfrak{T}(\lambda_o)\}'' = \mathfrak{M}_o'$, it follows that $C\overline{L_\xi} \subset \overline{L_\xi}C$ for all $C \in \mathfrak{M}_o'$, that is, $\overline{L_\xi}$ is affiliated with $\mathfrak{M}_o$. Define now the domain

$$\mathcal{D}_\tau = \bigcap_{\xi \in \mathcal{B}_\tau} D(\overline{L_\xi}).$$

By (7.1) we have

$$\mathfrak{M}_o'\mathcal{D}_\tau \subset \mathcal{D}_\tau. \tag{7.2}$$

We put

$$\mathfrak{M}_\tau = \{X \in \mathcal{L}^\dagger(\mathcal{D}_\tau, \mathcal{H}); \overline{X} \text{ is affiliated with } \mathfrak{M}_o\}.$$

Then it follows from (7.1) and (7.2) that $\mathfrak{M}_\tau$ is a partial GW*-algebra on $\mathcal{D}_\tau$ over $\mathfrak{M}_o$, which contains $\{\overline{L_\xi}\lceil\mathcal{D}_\tau; \xi \in \mathcal{B}_\tau\}$.

We proceed to define a strongly cyclic generalized vector λ for $\mathfrak{M}_\tau$, which extends λ_o. First let us show that

$$\mathfrak{T}(\lambda_o)\lceil\mathcal{D}_\tau \subset R(\mathfrak{M}_\tau). \tag{7.3}$$

It follows from the definition of $\mathcal{B}_\tau$ that $L_\xi\lambda_o(B) \in \mathcal{B}_\tau$ and $L_{L_\xi\lambda_o(B)} \subset L_\xi B$, for every $B \in \mathfrak{T}(\lambda_o)$ and $\xi \in \mathcal{B}_\tau$, and this implies

$$(L_\xi^*\zeta|B\eta) = (B^*L_\xi^*\zeta|\eta) = ((L_\xi B)^*\zeta|\eta) = (\zeta|\overline{L_{L_\xi\lambda_o(B)}}\eta),$$

for every $\eta \in \mathcal{D}_\tau$ and $\zeta \in D(L_\xi^*)$. Thus $B\eta \in D(\overline{L_\xi})$ and $\overline{L_\xi}B\eta = \overline{L_{L_\xi\lambda_o(B)}}\eta$. Hence $B\eta \in \mathcal{D}_\tau$, so that indeed $\mathfrak{T}(\lambda_o)\lceil\mathcal{D}_\tau \subset R(\mathfrak{M}_\tau)$.

We put now

$$\begin{aligned} B(\lambda) &= \mathfrak{T}(\lambda_o)\lceil\mathcal{D}_\tau, \\ D(\lambda) &= \text{linear span of } \{X\Box B\lceil\mathcal{D}_\tau; X \in \mathfrak{M}_\tau,\ B \in B(\lambda)\}, \\ \lambda(\textstyle\sum_k Y_k\Box X_k) &= \textstyle\sum_k X_k\lambda_o(B_k),\ \sum_k X_k\Box B_k \in D(\lambda). \end{aligned}$$

Then λ is a generalized vector for $\mathfrak{M}_\tau$ and $\lambda((B(\lambda) \cap B(\lambda)^\dagger)^2)$ is total in $\mathcal{H}$. We show that

$$\lambda^{cc} = \lambda_o. \tag{7.4}$$

On one hand, since $\lambda_o \lceil \mathfrak{T}(\lambda_o) \subset \lambda$, we have $(\lambda_o \lceil \mathfrak{T}(\lambda_o))^c \supset \lambda^c$. Conversely, suppose that $K \in (\lambda_o \lceil \mathfrak{T}(\lambda_o))^c$, that is, $K \in \mathfrak{M}_o'$ and $\exists \xi_K \in \mathcal{D}_\tau$ s.t. $K\lambda_o(B) = B\xi_K$, $\forall B \in \mathfrak{T}(\lambda_o)$. For every $X \in \mathfrak{M}_\tau$, $B \in \mathfrak{T}(\lambda_o)$, we have

$$\begin{aligned} K\lambda(X \square B) &= KX\lambda_o(B) = XK\lambda_o(B) \\ &= XB\,(\lambda_o \lceil \mathfrak{T}(\lambda_o))^c(K) = (X \square B)\,(\lambda_o \lceil \mathfrak{T}(\lambda_o))^c(K), \end{aligned}$$

and so $K \in D(\lambda^c)$. Thus we get $(\lambda_o \lceil \mathfrak{T}(\lambda_o))^c = \lambda^c$. Since

$$\{\lambda_o(B);\ B \in \mathfrak{T}(\lambda_o)\} \subset \lambda_o^c(D((\lambda_o \lceil \mathfrak{T}(\lambda_o))^c)) = \lambda^c(D(\lambda^c)) \subset \lambda_o'(D(\lambda_o'))$$

and $\{\lambda_o(B);\ B \in \mathfrak{T}(\lambda_o)\}$ is dense in the Hilbert space $D(S_{\lambda_o}^*)$, it follows that $(\lambda^c)^{cc} \equiv (\lambda^c)'' = \lambda_o'$. Hence $\lambda^{cc} = \lambda_o$.

Moreover it is easily shown that $\Delta_{\lambda_o}^{it}\lambda_o(\mathfrak{T}(\lambda_o)) \subset \lambda_o(\mathfrak{T}(\lambda_o))$, for all $t \in \mathbb{R}$, and that $\lambda_o(\mathfrak{T}(\lambda_o))$ is dense in $\mathcal{D}[\,t_{\mathfrak{M}_\tau}]$. This implies that λ is a strongly cyclic modular generalized vector for $\mathfrak{M}_\tau$. Then it follows from Theorem 4.3 that $\lambda_\tau \equiv \overline{\lambda}$ is a strongly cyclic full modular generalized vector for $\mathfrak{M}_\tau$. Suppose, in addition, that Δ_{λ_o} is τ-continuous for every $t \in \mathbb{R}$. Then it is easily shown that

$$\Delta_{\lambda_o}^{it}\xi \in \mathcal{B}_\tau \quad\text{and}\quad L_{\Delta_{\lambda_o}^{it}\xi} = \Delta_{\lambda_o}^{it} L_\xi \Delta_{\lambda_o}^{-it},$$

for every $\xi \in \mathcal{B}_\tau$ and $t \in \mathbb{R}$, which implies that $\Delta_{\lambda_o}^{it}\mathcal{D}_\tau \subset \mathcal{D}_\tau$ for every $t \in \mathbb{R}$. Finally, it is easily shown that $\Delta_{\lambda_o}^{it}\mathfrak{M}_\tau\Delta_{\lambda_o}^{-it} = \mathfrak{M}_\tau$, $\forall t \in \mathbb{R}$. Therefore, λ is standard. This completes the proof. □

It remains to discuss the choice of the topologies τ. Denote by $\mathcal{T}$(resp. $\mathcal{T}_o$) the set of all locally convex topologies on $\lambda_o(\mathfrak{T}(\lambda_o))$ satisfying the conditions (i) and (ii) (resp. (i), (ii) and (iii)) of Theorem 7.1. The latter implies

PROPOSITION 7.2. – *Let* $\tau_1, \tau_2 \in \mathcal{T}$. *If* $\tau_1 \prec \tau_2$, *then* $(\mathfrak{M}_{\tau_1}, \mathcal{D}_{\tau_1}, \lambda_{\tau_1}) \succ (\mathfrak{M}_{\tau_2}, \mathcal{D}_{\tau_2}, \lambda_{\tau_2})$, *that is,*

(i) $\mathcal{D}_{\tau_1} \subset \mathcal{D}_{\tau_2}$ *and* $\mathfrak{M}_{\tau_2} \lceil \mathcal{D}_{\tau_1}$ *is a *-subalgebra of* $\mathfrak{M}_{\tau_1}$ (*in* [12], *this situation was denoted* $(\mathfrak{M}_{\tau_1}, \mathcal{D}_{\tau_1})) \underset{\leftarrow}{\supset} (\mathfrak{M}_{\tau_2}, \mathcal{D}_{\tau_2})$;

(ii) $\{X \lceil \mathcal{D}_{\tau_1};\ X \in D(\lambda_{\tau_2})\} \subset D(\lambda_{\tau_1})$ *and* $\lambda_{\tau_1}(X \lceil \mathcal{D}_{\tau_1}) = \lambda_{\tau_2}(X)$, $X \in D(\lambda_{\tau_2})$. □

Let us give some examples of suitable topologies.

EXAMPLES 7.3. – (1) *Norm topology* $\tau[\Delta_{\lambda_o}^{1/2}]$:

This topology is defined by the norm

$$\|\lambda_o(B)\|_{\Delta_{\lambda_o}^{1/2}} = \|\lambda_o(B)\| + \|\Delta_{\lambda_o}^{1/2}\lambda_o(B)\|,\ B \in \mathfrak{T}(\lambda_o).$$

It is easily shown that $\tau[\Delta_{\lambda_o}^{1/2}]$ is the weakest topology in $\mathcal{T}_o$ and that the $\tau[\Delta_{\lambda_o}^{1/2}]$-completion of $\lambda_o(\mathfrak{T}(\lambda_o))$ is simply $D(\Delta_{\lambda_o}^{1/2})$. In particular, if λ_o is tracial, that is, $(\lambda_o(A)|\lambda_o(B)) = (\lambda_o(B^*)|\lambda_o(A^*)), \forall A, B \in D(\lambda_o) \cap D(\lambda_o)^*$, then the usual Hilbert norm topology belongs to $\mathcal{T}_o$.

(2) *Strong* topology* $\tau[s^*]$:

This is the l.c. topology on $\lambda_o(\mathfrak{T}(\lambda_o))$ defined by the family of norms $\{||\cdot||_B;\ B \in \mathfrak{T}(\lambda_o)\}$, where

$$||\lambda_o(A)||_B = ||\lambda_o(A)|| + ||\lambda_o(A^*)|| + ||\lambda_o(AB)|| + ||\lambda_o(A^*B)||,\ A \in \mathfrak{T}(\lambda_o).$$

It is easy to see that $\tau[s^*] \in \mathcal{T}_o$.

(3) Δ-*topology* $\tau[\Delta]$:

This is the l.c. topology on $\lambda_o(\mathfrak{T}(\lambda_o))$ defined by the family of norms $\{||\cdot||_{\Delta_{\lambda_o}^{\alpha}};\ \alpha \in \mathbb{C}\}$, where

$$||\lambda_o(B)||_{\Delta_{\lambda_o}^{\alpha}} = ||\lambda_o(B)|| + ||\Delta_{\lambda_o}^{\alpha}\lambda_o(B)||,\ B \in \mathfrak{T}(\lambda_o),\ \alpha \in \mathbb{C}.$$

It is easily shown that $\tau[\Delta] \in \mathcal{T}_o$ and that $[(\lambda_o(\mathfrak{T}(\lambda_o))]^{\tau[\Delta]} = \bigcap_{\alpha \in \mathbb{C}} D(\Delta_{\lambda_o}^{\alpha})$.

8. OUTCOME: PHYSICAL APPLICATIONS

The aim of the theory developed throughout this paper, as already in [19], is to generalize the Tomita-Takesaki to partial O*-algebras, as a tool for constructing KMS states or appropriate substitutes thereof. It turns out that standard generalized vectors are an efficient answer to this quest. Can one hope to find physically relevant applications of the results presented here?

As described in the introduction, there are several instances where the set of observables of a physical system could (or even should) be taken as a partial *-algebra. One of them is the possible occurrence of non-self-adjoint observables [10]. In that case, the natural candidate is the partial GW*-algebra associated to a complete set $\mathcal{S} = \{S_1, \ldots, S_n\}$ of compatible observables, constructed as follows. Each observable S_j is a maximal symmetric operator, which, if not self-adjoint, generates a semigroup of isometries $\{V_j(t),\ t \geq 0\}$. Let $\mathfrak{N}$ be the von Neumann algebra generated by the isometry semigroups $\{V_j(t),\ j = 1, \ldots, n\}$. Then the partial GW*-algebra of observables associated to $\mathcal{S}$ is taken as the bicommutant $\mathfrak{N}''_{w\sigma}$ of $\mathfrak{N}$.

An even simpler case is that of a particle on an interval [26], where the observables are the elements of the quasi*-algebra $\mathcal{L}(\mathcal{D}, \mathcal{D}')$, where

$\mathcal{D} = \mathcal{D}^\infty(H)$, $H = P^2$ is the hamiltonian and $\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}'$ is the corresponding rigged Hilbert space.

Another example yet is quantum field theory. In the situation described by Horuzhy and Voronin [9], the natural setup would be a partial O*-algebra of field operators on the Gårding domain. Going one step further, one may consider in a very natural fashion [27], unsmeared field operators as elements of the quasi*-algebra $\mathcal{L}(\mathcal{D}, \mathcal{D}')$, where again $\mathcal{D} = \mathcal{D}^\infty(H)$ and H is the hamiltonian, that is, the infinitesimal generator of time translations. More precisely, these field operators may be taken as operators in the Hilbert scale generated by H, which constitute a CQ*-algebra [28]. Since both quasi*-algebras and CQ*-algebras are simple types of partial *-algebras, this case suggests the extension of the notion of generalized vector to other types of partial *-algebras than partial O*-algebras.

In all these instances, vectors in the Hilbert space, and in particular vectors $\xi \in \mathcal{H} \backslash \mathcal{D}$, where $\mathcal{D}$ is the relevant domain, describe states of the system. So one may conjecture that the corresponding generalized vectors λ_ξ (when properly defined for a general partial *-algebra) would play a significant role in the definition of appropriate KMS states of the system. Also, in a Wightman field theory, if $\mathfrak{N}$ is the von Neumann algebra associated to a given wedge region, it might be interesting to study the corresponding partial GW*-algebra $\mathfrak{N}''_{w\sigma}$ and its modular group.

At this stage, of course, this is only a list of questions, but in our opinion, these problems are worth studying and could yield some physically relevant results.

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