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Stability of shock waves in relativistic radiation hydrodynamics

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ABSTRACT. – The linear stability of shock waves in a radiating relativistic gas is analyzed. According to the ratio between the sound speed and gas speed behind and ahead the front of discontinuity, the radiative shocks are classified as *slow* and *fast* shock waves respectively. With the help of the technique of dissipative energy integrals, it is proved that the fast shock waves are stable. An ill-posedness example for stability problem for slow shocks is constructed.

Key words: Relativistic radiation hydrodynamics, variable Eddington factor, stability of shoch waves.

RÉSUMÉ. – Nous analysons la stabilité linéaire des ondes de choc dans un gaz relativiste radiatif. Les chocs radiatifs sont classifiés comme chocs *lents* et *rapides* selon le rapport entre la vitesse du son et la vitesse du gaz à l'arrière et à l'avant du front de discontinuité. A l'aide de la technique des intégrales d'énergie dissipative on démontre que les ondes de choc rapides sont stables. On construit aussi un exemple mal posé pour le problème de la stabilité des chocs lents.

1. INTRODUCTION

Radiation represents the more efficient mechanism for dissipative effects occuring in many problems of relativistic astrophysics and cosmology.

Radiation can be well described by means of kinetic theory [1]. but such an approach is a formidable task even for numerical simulations. We shall follow a macroscopic description in the framework of radiation hydrodynamics [2]: radiation is considered as a dissipative fluid which obeys the usual balance law of continuum mechanics. We will adopt the mathematical model for radiation by Anile-Pennisi-Sammartino [3], [4] (hereafter APS), which is based on a generalization to the relativistic case of the variable Eddington factor given by Levermore [5]. The APM model is formulated in the framework of the Extended-Thermodynamics of irreversible processes [6], [7] and leads to equations of hyperbolic type (for other theories which give rise to nonlinear parabolic equations see Refs 8, 9).

As well-known, the solutions of hyperbolic systems can develope discontinuities in a finite time [10], [11] (shock waves, contact discontinuities, rarefaction waves). In particular, shock waves in radiating gases are of great importance in models of gravitational collapse, supernova explosions, formation of quark-gluon plasma.

Stability of relativistic shock waves in simple fluids was investigated by several authors (see Ref. 12 and references therein). We will follow the method previously adopted in [13], [14], [15] for relativistic simple gas, magnetofluid dynamics and superfluid. It consists in the application of the so-called *technique of dissipative energy integrals* to the investigation of the well-posedness of mixed linear problems (Cauchy and boundary value problem). If the mixed problem for the pertubation of the basic shock solution is well-posed, then the shock wave is stable; othewise, it is unstable.

We classify the shock waves as *fast* and *slow* ones. It is proved that fast radiative shocks are stable while for the slow ones a ill-posed problem is constructed. This imples in turn the instabilility of the second type solutions.

One of the most important stages of our approach is the symmetrization of initial quasilinear equations which describe the motion of radiating gas in order to apply, with suitable developments, the theory of mixed problems for *symmetric t-hyperbolic systems (by Friedrichs).*

The problem of stability of strong discontinuities in continuum mechanics has recently assumed a special importance in view of the wide application of computational methods to find approximate solutions to the problems of continuum mechanics with strong discontinuities (e.g. see ref. 16 for a recent numerical code in radiation hydrodynamics). Clearly, a positive answer on the question of stability of discontinuities in a given model is an essential preliminary step for the application of computational methods.

The paper is organized as follows. In section 2 we set forth the complete system of Radiation Hydrodynamics equations and in sections 3 the problem of their symmetrization is discussed. In section 4 the problem of shock wave stability is presented. Section 5 is devoted to prove the well-posedness of the mixed problem for fast shock waves. The case of slow shocks is investigated in section 6, where the fulfilment of the Lopatinsky conditions is discussed and an ill-posed problem is presented.

We shall use units such that $c = \hbar = 1$, with c being the speed of light in the vacuum. Repeated indeces are to consider summed and in local coordinates the Greek indices run from 0 to 3 and Latin ones from 1 to 3, except where stated otherwise. Moreover the simbol * indicates transposition for matrix.

2. THE COMPLETE SYSTEM OF RADIATION HYDRODYNAMICS EQUATIONS

The relativistic radiation hydrodynamics equations of motion are represented by the continuity equation and the balance equations for the energy momentum-tensor of matter and radiation,

$$\nabla_\mu(\rho u^\mu) = 0, \tag{1}$$

$$\nabla_\mu T^{\mu\nu}_{\rm mat} = f^\nu, \tag{2}$$

$$\nabla_\mu T^{\mu\nu}_{\rm rad} = -f^\mu, \tag{3}$$

where $T^{\mu\nu}_{\rm mat}$ is the energy momentum tensor of matter and $T^{\mu\nu}_{\rm rad}$ that of radiation. ∇_μ is the covariant derivative with respect to the metric $g_{\mu\nu}$ of the space-time, u^μ is the four-velocity of an observer comoving with the fluid.

In usual problems in which radiation is important the main dissipative processes occur due to the transport of photons, therefore we neglect viscosity and heat conduction of gas. In general matter and radiation behave like two distinct fluids whose interaction is expressed through the source terms f^ν. In thermodynamics equilibrium radiation becomes thermalized and it is completely determined by means of the local temperature of the gas (black-body radiation).

With respect to the normalized four-velocity u^μ, the energy-momentum tensors can be decomposed as

$$T^{\mu\nu}_{\rm mat} = (e+p)u^\mu u^\nu + pg^{\mu\nu},$$
$$T^{\mu\nu}_{\rm rad} = Ju^\mu u^\nu + H^\mu u^\nu + H^\nu u^\mu + K^{\mu\nu}.$$

e and p are the total energy density and pressure of gas. By writing $e = \rho(1+e_0)$ with e_0 the specific internal energy and ρ the density measured in the frame determined by u^μ and by introducing the specific entropy s and the absolute temperature T of the gas, one can relate e_0, ρ and p by means of an equation of state according to the Gibbs relation

$$Tds = de_0 + pd\frac{1}{\rho}, \tag{4}$$

on the basis of kinetic theory or statistical mechanics. For the moment we do not restrict ourselves to a particular equation of state.

J, H^μ and $K^{\mu\nu}$ are the radiative energy, density, flux and shear tensor. They satisfy

$$H^\mu u_\mu = K^{\mu\nu}u_\mu = 0,$$

For thermalized radiation the expression of J is obtained by the distribution of Planck

$$J = B = \sigma T^4,$$

where B is the black-body energy density and σ the Stefan-Boltzmann constant.

In order to close the system (1)-(3) one has to specify a relation between J, H^μ and $K^{\mu\nu}$ (closure problem), by taking into account

$$K^\mu_\mu = J.$$

If the radiation field is isotropic, $K^{\mu\nu}$ is given by the classical Eddington approximation

$$K^{\mu\nu} = \frac{1}{3}Jh^{\mu\nu},$$

where $h^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ is the projection tensor orthogonal to u^μ.

Levermore [5] in the case of static medium has obtained on the basis of kinematic considerations a variable Eddington factor,

$$K^{\mu\nu} = \frac{1}{3}J\varphi_1(u^\mu u^\nu + g^{\mu\nu}) + \frac{\varphi_2}{J}H^\mu H^\nu,$$

where $\varphi_1 = 1 - \varphi$, $\varphi_2 = \frac{\varphi}{\lambda^2}$, $\varphi = \varphi(\lambda^2) = 2 - \sqrt{4 - 3\lambda^2}$ $\lambda^2 = \frac{H^\mu H_\mu}{J^2}$. We observe that the physical range of values for λ is $0 \leq \lambda^2 << \leq 1$, the limiting case being isotropic radiation and free streaming respectively.

Lately APS [2], [3] have cast the previous closure in covariant form and showed that it can be derived by means of the entropy principle as formulated in the framework of extended thermodynamics [6], [7].

The source terms in (2)-(3), which model the interaction matter-radiation, are the moments of the collision kernel of the transport equation for photons. Except Thomson scattering, in general it is not possible to express them as explicit functions of J, H^μ and $K^{\mu\nu}$ [17], [18]. We will adopt a phenomenological description by means of a sort of mean absorption κ,

$$f^\nu = \rho\kappa\{H^\nu + u^\nu(J - B)\}. \tag{2.7}$$

We note that by using the continuity equation, the balance equation and the Gibbs relation, the following additional consevation law holds

$$\rho u^\alpha \nabla_\alpha s = \nabla_\alpha(\rho s u^\alpha) = \frac{\rho\kappa}{T}(J - B). \tag{5}$$

The fluid flow, in contrast with a simple ideal gas, is not adiabatic on account of the dissipative effects of radiation. The presence of the additional relation (5) will play an important role in the next section for the symmetrization of balance equations for matter.

Since the thickness of the shock is almost always negligible compared with the characteristic length for which curvature is relevant, we will work in Minkowski space-time $\mathcal{M}$. Only for special problems in very early universe we expect the inclusion of curvature effects to change significantly the results.

Let (t, x^i) be local coordinates in $\mathcal{M}$. Then

$$g_{\mu\nu} = \text{diagonal} \quad (-1, 1, 1, 1)$$

and the four-velocity of the fluid has components

$$u^\mu = \Gamma(1, v^i),$$

where $\Gamma = (1 - v^2)^{-1/2}$ is the Lorentz factor, with $v^2 = v^i v_i$.

Explicitly eqs (1)-(3) now read

$$\frac{\partial}{\partial t}(\rho\Gamma) + \frac{\partial}{\partial x^k}(\rho u^k) = 0, \tag{6}$$

$$\frac{\partial}{\partial t}(\rho h\Gamma^2 - p) + \frac{\partial}{\partial x^k}(\rho h\Gamma u^k) = \rho\kappa\left[H^i v_i + \Gamma(J - B)\right], \tag{7}$$

$$\frac{\partial}{\partial t}[(e + p)\Gamma u^i] + \frac{\partial}{\partial x^k}(\rho h u^i u^k + p\delta^{ik}) = \rho\kappa[H^i + u^i(J - B)]. \tag{8}$$

$$\frac{\partial}{\partial t}(J\Gamma^2 + 2u^i H_i + K^{00}) + \frac{\partial}{\partial x^i}(J\Gamma u^i + u^i v^k H_k + \Gamma H^i + K^{i0})$$
$$= -\rho\kappa[v^i H_i + \Gamma(J - B)], \tag{9}$$

$$\frac{\partial}{\partial t}(J\Gamma u^i + \Gamma H^i + u^i v^k H_k + K^{i0}) + \frac{\partial}{\partial x^j}(J u^i u^j + 2u^j H^i + K^{ij})$$
$$= -\rho\kappa[H^i + u^i(J - B)], \tag{10}$$

with $h = 1 + e_0 + p/\rho$ the relativistic hentalpy.

If the state equation is in the form

$$e_0 = e_0(\rho, s),$$

in view of (4) we can write

$$T = (e_0)_s(\rho, s), \quad p = -(e_0)_V(\rho, s) \tag{2.8}$$

and consider system (6)-(10) in the unknown variables vector

$$\mathbf{W} = \begin{pmatrix} \mathbf{U} \\ \Lambda \end{pmatrix},$$

where

$$\mathbf{U} = \begin{pmatrix} p \\ s \\ \mathbf{u} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} J \\ \mathbf{H} \end{pmatrix}.$$

3. SYMMETRIZATION OF BALANCE EQUATION

Our aim is to investigate the stability of shock waves arising from the system (1)-(3) by applying the energy method. One of the most fruitfull way to employ that method is to rewrite the balance equations in a symmetric form. We recall that a system of the type

$$\mathcal{B}^0 \frac{\partial U}{\partial t} + \mathcal{B}^i \frac{\partial U}{\partial x^i} = \mathcal{G}(U)$$

is said symmetric if $\mathcal{B}^0$ and $\mathcal{B}^i$ are symmetric matrices. Further if $\mathcal{B}^0$ is positive definite then the system is said symmetric t-hyperbolic by Friedrichs. We shall symmetrize the balance equation for matter and radiation saparately with different methods.

3.1. Symmetrization of the balance equations for medium

Since for gas the additional law

$$\frac{\partial}{\partial t}(\rho \Gamma s) + \frac{\partial}{\partial x^i}(\rho s u^i) = \frac{\rho \kappa}{J}(J - B),$$

holds we can use the procedure suggested in Refs 19, 20, 21, 22, 23. The basic ideas are the following.

If

$$\frac{\mathcal{P}(U)}{\partial t} + \frac{\partial}{\partial x^i}\mathcal{P}^i(U) = \mathcal{G}(U) \tag{11}$$

is a system of balance equations such that the supplementary relation

$$\frac{\partial \Phi^0(U)}{\partial t} + \frac{\partial \Phi^i}{\partial x^i} = g(U)$$

is satisfied by each solution of (11), then a new set of dependent variables $\mathbf{Q} = (q_1, ..., q_n)$ can be introduced by means of the relation

$$\mathbf{Q} \cdot d\mathcal{P}^0 = d\Phi^0,$$

with $\cdot$ standard scalar product in $\mathbf{R}^3$. Setting

$$\mathcal{L} = \mathcal{L}(\mathbf{Q}) = \mathbf{Q} \cdot \mathcal{P}^0 - \Phi^0 \qquad \text{and} \qquad \mathcal{M}^k = \mathbf{Q} \cdot \mathcal{P}^k - \Phi^k,$$

the system (2) can be rewritten in a symmetrical form as

$$\mathcal{A}^{(0)} \frac{\partial Q}{\partial t} + \mathcal{A}^{(k)} \frac{\partial Q}{\partial x^k} = \tilde{\mathcal{G}}(U),$$

where

$$\mathcal{A}^{(0)} = (a^0_{ij}) \qquad \text{and} \qquad \mathcal{A}^{(k)} = (a^k_{ij}),$$

with $a^0_{ij} = \mathcal{L}_{q_i q_j}$ and $a^k_{ij} = \mathcal{M}^k_{q_i q_j}$.

The calculation can be simplified by introducing the matrices $I^{(0)}$ and $I^{(k)}$ with the aid of the relations [13], [14], [15]

$$d\mathbf{Q} = Id\mathbf{U},$$
$$d\mathbf{L}_q = I^{(0)}d\mathbf{U},$$
$$d\mathbf{M}_q^{(k)} = I^{(k)}d\mathbf{U}, \quad k = 1, 2, 3,$$

where

$$d\mathbf{L}_q = (L_{q_1}, \dots, L_{q_5})^*, \quad d\mathbf{M}_q^{(k)} = (M_{q_1}^{(k)}, \dots, M_{q_5}^{(k)})^*, \quad k = 1, 2, 3.$$

Then

$$A^{(0)} = \left(L_{q_i q_j}\right) = I^{(0)} I^{-1},$$
$$A^{(k)} = \left(M_{q_i q_j}^{(k)}\right) = I^{(k)} I^{-1}, \quad k = 1, 2, 3, \ j, j = \overline{1, 5}.$$

In the present case we find the following canonical variables

$$\mathbf{Q} = (q_1, q_2, q_3, q_4, q_5)^* = \left(s - \frac{h}{T}, -\frac{u^1}{T}, -\frac{u^2}{T}, -\frac{u^3}{T}, \frac{\Gamma}{T}\right)^*$$

and productive function

$$L = -\frac{\Gamma}{T} p \qquad \text{and} \qquad M^{(k)} = L v^k.$$

It is easy to verify in the particular case $u^2 = u^3 = 0$ (which will be considered in the following, even if the same result holds in general) that $-A^{(0)} > 0$ if the following inequalities take place

$$\begin{gathered} m_0 > 0, \\ h m_0 - (u^1)^2 (m_0 m_2 h + m_1^2) > 0, \\ (u^1)^2 [m_2 - h m_0 - 2 m_1 + \Gamma^2 (m_0 m_2 h + m_1^2)] \\ -\Gamma^2 h m_0 (3 + \Gamma^2 m_2) - (1 + m_1 \Gamma^2)^2 > 0, \end{gathered} \tag{12}$$

where

$$m_0 = \frac{(e_0)_{ss}}{\Delta},$$

$$m_1 = -1 + \frac{\rho T (e_0)_{\rho s} - h (e_0)_{ss}}{\Delta},$$

$$m_2 = -3 - \frac{h}{c^2} - \frac{(c^2 T - h \rho (e_0)_{\rho s})^2}{h^2 c^2 \Delta},$$

$$\Delta = c^2 (e_0)_{ss} - \rho^2 (e_0)_{\rho s}^2, \quad c^2 = \left(\rho^2 (e_0)_\rho\right)_\rho.$$

It is easy to show that, as in classical Gas Dynamics, inequalities (12) are fulfilled if the state equation $e_0 = e_0(\rho, s)$ satisfies the following inequalities which express causality (sound speed smaller than the velocity of light) and convessity of the relativistic free entalpy:

$$\begin{gathered}(e_0)_V < 0, \quad (e_0)_s > 0,\\ (e_0)_{VV} > 0, \quad (e_0)_{VV}(e_0)_{ss} - (e_0)^2{}_{Vs} > 0.\end{gathered} \tag{13}$$

Note (see Ref. 15) that system (2) can be rewritten as

$$B^{(0)}\frac{\partial \mathbf{U}}{\partial t} + \sum_{k=1}^{3} B^{(k)}\frac{\partial \mathbf{U}}{\partial x^k} = \mathbf{F}, \tag{14}$$

where in the particular case $u^2 = u^3 = 0$

$$B^{(0)} = -I^*I^{(0)} = \begin{pmatrix} \frac{\Gamma}{\rho T c^2} & 0 & \frac{v^1}{T} & 0 & 0\\ 0 & \rho T m^0 & 0 & 0 & 0\\ \frac{v^1}{T} & 0 & \frac{\rho h}{T\Gamma} & 0 & 0\\ 0 & 0 & 0 & \frac{\rho h\Gamma}{T} & 0\\ 0 & 0 & 0 & 0 & \frac{\rho h\Gamma}{T}\end{pmatrix},$$

$$B^{(1)} = -I^*I^{(1)} = \begin{pmatrix} \frac{a_1 u^1}{c^2} & 0 & \frac{1}{T} & 0 & 0\\ 0 & -a_2\frac{u^1(e_0)_{Vs}}{c^2} & 0 & 0 & 0\\ \frac{1}{T} & 0 & \frac{\rho h u^1}{T\Gamma^2} & 0 & 0\\ 0 & 0 & 0 & \frac{\rho h u^1}{T} & 0\\ 0 & 0 & 0 & 0 & \frac{\rho h u^1}{T}\end{pmatrix},$$

$$B^{(2)} = -I^*I^{(2)} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{T} & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ \frac{1}{T} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix},$$

$$B^{(3)} = -I^*I^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1}{T}\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ \frac{1}{T} & 0 & 0 & 0 & 0\end{pmatrix},$$

$\mathbf{F} = -I^*\tilde{\mathbf{F}}$. It is easy to see that the matrices

$$B^{(\alpha)} = -I^*I^{(\alpha)} = -I^*A^{(\alpha)}I, \quad \alpha = \overline{0,3}$$

are symmetric and $B^{(0)} > 0$ since $-A^{(0)} > 0$ and inequalities (13) fulfilled. Consequently system (14) is also symmetric t-hyperbolic (by Friedrichs).

3.2. Symmetrization of radiation equation

In this subsection we discuss the topic of symmetrization of radiation equations (3). With this purpose we adopt the same idea used in refs 3,4 and based on the procedure of extended thermodynamics [6]. We want to impose the entropy principle on system (3), *i.e.* we assume that there are functions $\Phi^\alpha = \Phi^\alpha(\mathbf{W})$, $g = g(\mathbf{W})$ and *canonic variables* (or *Lagrange Multipliers*) $r_\alpha = r_\alpha(\mathbf{W})$ such that the relation

$$\begin{aligned} r_\alpha & \left\{ \frac{\partial}{\partial t}(\bar{T}^{0\alpha}_{\mathrm{rad}}) + \sum_{k=1}^{3} \frac{\partial}{\partial x^k}(\bar{T}^{k\alpha}_{\mathrm{rad}}) \right\} \\ &= r_0 \left\{ \frac{\partial}{\partial t}(\bar{T}^{00}_{\mathrm{rad}}) + \sum_{k=1}^{3} \frac{\partial}{\partial x^k}(\bar{T}^{k0}_{\mathrm{rad}}) \right\} \\ &\quad + \sum_{j=1}^{3} r_j \left\{ \frac{\partial}{\partial t}(\bar{T}^{0j}_{\mathrm{rad}}) + \sum_{k=1}^{3} \frac{\partial}{\partial x^k}(\bar{T}^{kj}_{\mathrm{rad}}) \right\} \\ &= -r_\alpha f^\alpha = g = \frac{\partial \Phi^{(0)}}{\partial t} + \mathrm{div}\boldsymbol{\Phi} \end{aligned} \tag{15}$$

holds for every smooth solution of system (1)-(3). Here

$$\boldsymbol{\Phi} = (\Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)})^*.$$

We define *the productive functions* L, $M^{(k)}$, $k = 1, 2, 3$:

$$L = r_\alpha \bar{T}^{0\alpha}_{\mathrm{rad}} - \Phi^{(0)} = r_0 \bar{T}^{00}_{\mathrm{rad}} + \sum_{k=1}^{3} r_k \bar{T}^{0k}_{\mathrm{rad}} - \Phi^{(0)},$$

$$M^{(k)} = r_\alpha \bar{T}^{k\alpha}_{\mathrm{rad}} - \Phi^{(k)}, \quad k = 1, 2, 3.$$

By using the theorem of representation formulas for tensor-valued isotropic function [24], we seek *the productive functions* in the form

$$L = \mathcal{L}(G) r^0,$$

$$M^{(k)} = \mathcal{L}(G) r^k, \qquad k = 1, 2, 3,$$

where

$$G = r_\alpha r^\alpha = r_0 r^0 + \sum_{k=1}^{3} r_k r^k = r_1^2 + r_2^2 + r_3^2 - r_0^2.$$

From (15) derive

$$\begin{aligned}
\bar{T}^{00}_{\mathrm{rad}} &= \frac{\partial L}{\partial r_0} = 2r_0^2\mathcal{L}' - \mathcal{L},\\
\bar{T}^{0k}_{\mathrm{rad}} &= \frac{\partial L}{\partial r_k} = -2r_k r_0 \mathcal{L}', \quad k = 1, 2, 3,\\
\bar{T}^{k0}_{\mathrm{rad}} &= \frac{\partial M^{(k)}}{\partial r_0} = -2r_0 r_k \mathcal{L}', \quad k = 1, 2, 3,\\
\bar{T}^{kj}_{\mathrm{rad}} &= \frac{\partial M^{(k)}}{\partial r_0} = 2r_k r_j \mathcal{L}' + \mathcal{L}\delta_{kj}, \quad j, k = 1, 2, 3.
\end{aligned}$$

To compare expressions (15) with expressions (3), we introduce the representation for H^α

$$H^\mu = c_0 u^\mu + d r^\mu,$$

where c_0, d are related by

$$\begin{aligned}
\Gamma r_0 + \sum_{k=1}^{3} r_k u^k &= \frac{c_0}{d},\\
J^2\lambda^2 &= d^2 G + c_0^2
\end{aligned} \tag{16}$$

and satisfy the conditions

$$\Gamma(c_0\Gamma - d r_0) = u_k H^k,$$

$$J^2\lambda^2 = H^\alpha H_\alpha.$$

Substituting the expressions (16) into (3) and comparing the obtained expression with (15), one has

$$\begin{aligned}
2\mathcal{L}' &= \frac{\varphi_2}{J} d^2,\\
\mathcal{L} &=, \frac{1}{3} J\varphi_1,\\
c_0 &= -\frac{J}{\varphi_2},\\
J\Big(1 + \frac{\varphi_1}{3}\Big) + 2c_0 &+ \frac{\varphi_2}{J} c_0^2 = 0.
\end{aligned} \tag{17}$$

actually, the last expression is an identity (for the given expression of c_0 and φ). Using relation $(16)_2$ and $(17)_{1,2}$, we have the differential equation for unknown function $\mathcal{L}$:

$$G\mathcal{L}' = -2\mathcal{L},$$

its solution is

$$\mathcal{L} = \frac{1}{G^2}.$$

Since

$$r_\alpha = \frac{1}{d}\left(H_\alpha + + \frac{J}{\varphi_2} u_\alpha\right),$$

we have that

$$G = r_1^2 + r_2^2 + r_3^2 - r_0^2 = -\frac{4}{3}\frac{J^2\varphi_1}{d^2\varphi_2} < 0,$$

wherefrom

$$G = -\sqrt{\frac{3}{J\varphi_1}},$$
$$d = \sqrt{\frac{4J}{\varphi_2}\left(\frac{J\varphi_1}{3}\right)^{3/2}}.$$

By means of the previous relations, we obtain

$$\Phi^{(0)} = \frac{4}{3}Jr_0\varphi_1,$$
$$\Phi^{(k)} = -\frac{2}{3}Jr_k\varphi_1, \quad k = 1, 2, 3,$$
$$g = \frac{\rho\kappa J}{d\varphi_2}(\varphi_1 J - B).$$

Consequently system (3) can be rewritten in the canonic form

$$A_{\rm rad}^{(0)}\frac{\partial \mathbf{R}}{\partial t} + \sum_{k=1}^{3} A_{\rm rad}^{(k)}\frac{\partial \mathbf{R}}{\partial x^k} = \tilde{\mathbf{\Phi}}. \tag{18}$$

Here

$$A_{\rm rad}^{(0)} = \left(L_{r_\alpha r_\beta}\right), \quad A_{\rm rad}^{(k)} = \left(M_{r_\alpha r_\beta}^{(k)}\right), \quad k = 1, 2, 3, \quad \alpha, \beta = \overline{0, 3}.$$

$$\tilde{\mathbf{\Phi}} = -(f^0, f^1, f^2, f^3)^*,$$

$$A_{\rm rad}^{(0)} = \frac{4}{G^4}\begin{pmatrix} -3r_0(G + 2r_0^2) & r_1(G + 6r_0^2) & r_2(G + 6r_0^2) & r_3(G + 6r_0^2) \\ r_1(G + 6r_0^2) & r_0(G - 6r_1^2) & -6r_0r_1r_2 & -6r_0r_1r_3 \\ r_2(G + 6r_0^2) & -6r_0r_1r_2 & r_0(G - 6r_2^2) & -6r_0r_2r_3 \\ r_3(G + 6r_0^2) & -6r_0r_1r_3 & -6r_0r_2r_3 & r_0(G - 6r_3^2) \end{pmatrix}.$$

$$A^{(k)}_{\mathrm{rad}} = \frac{4}{G^4}\left(a^{(k)}_{\alpha\beta}\right), \quad \alpha, \beta = \overline{0, 3};$$

$$a^{(k)}_{00} = r_k(G + 6r_0^2), \quad a^{(k)}_{0\alpha} = a^{(k)}_{\alpha 0} = r_k(G\delta_{\alpha k} - 6r_0 r_\alpha), \quad \alpha = \overline{1, 3}.$$

$$a^{(k)}_{\alpha 3} = a^{(k)}_{3\alpha} = 6r_k r_\alpha r_\beta - G(r_k\delta_{\alpha\beta} + r_\beta\delta_{\alpha k} + r_k\delta_{\alpha\beta}\delta_{\alpha k}), \quad \alpha, \beta = \overline{1, 3}.$$

Considering the case

$$u^2 = u^3 = 0, \quad H^k = 0, \quad k = 1, 2, 3,$$

we have

$$A^{(0)}_{\mathrm{rad}} = 4\Gamma n_0^5 \begin{pmatrix} 3(2\Gamma^2 - 1) & v^1(6\Gamma^2 - 1) & 0 & 0 \\ v^1(6\Gamma^2 - 1) & 1 + 6(u^1)^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A^{(1)}_{\mathrm{rad}} = 4\Gamma n_0^5 \begin{pmatrix} v^1(6\Gamma^2 - 1) & 1 + 6(u^1)^2 & 0 & 0 \\ 1 + 6(u^1)^2 & 3v^1(2\Gamma^2 - 1) & 0 & 0 \\ 0 & 0 & v^1 & 0 \\ 0 & 0 & 0 & v^1 \end{pmatrix},$$

$$A^{(2)}_{\mathrm{rad}} = 4\Gamma n_0^5 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & v^1 & 0 \\ 1 & v^1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A^{(3)}_{\mathrm{rad}} = 4\Gamma n_0^5 \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & v^1 \\ 0 & 0 & 0 & 0 \\ 1 & v^1 & 0 & 0 \end{pmatrix},$$

where $n_0 = \left(\frac{J}{3}\right)^{1/4}$.

It is easy to verify that in the particular case $u^2 = u^3 = 0$, $H^k = 0$, $k = 1, 2, 3$ the matrix $A^{(0)}_{\mathrm{rad}} > 0$, *i.e.*, system (17) is symmetric t-hyperbolic (by Friedrichs).

Moreover, we have for system (6)-(10) the entropy relation

$$\frac{\partial}{\partial t}\{\rho\Gamma s + \Phi^{(0)}\} + \mathrm{div}\{\rho s\mathbf{u} + \mathbf{\Phi}\} = \rho\kappa\left\{\frac{4\pi}{T}(J - B) + \frac{J}{d\varphi_2}(\varphi_1 J - B)\right\},$$

which holds for every smooth solution of system (6)-(10).

4. SHOCK WAVES IN RELATIVISTIC RADIATION HYDRODYNAMICS: THE STABILITY PROBLEM

A shock wave is an oriented hypersurface Σ in space-time, across which the field variables suffer a jump discontinuity, *i.e.* the tensor fields governing the motion are regularly discontinuous [12].

We will be deal with shocks between two equilibrium states. In such a case radiation is completely thermalized and radiative stress-energy tensor is, then, given in the smoothness regions by

$$T_{\mathrm{rad}}^{0\mu\nu} = \frac{4}{3}B(T)u^{\mu}u^{\nu} + \frac{1}{3}B(T)g^{\mu\nu}. \tag{19}$$

The mathematical properties of shocks for relativistic radiating gas were discussed in Ref. 25, where it was proved that under the assumption (13) on the equation of state the jump conditions admit a unique solution satisfying the entropy inequality. Moreover, it was shown that the ipersurface Σ is space-like. Therefore, if Σ is described by the equation

$$\varphi(x^{\alpha}) = 0, \quad \mathrm{d}\varphi \,|_{\Sigma} \neq 0,$$

we can introduce the unitary space-like four vector

$$l^{\mu} = g^{\mu\nu}\nabla_{\nu}\phi / g^{\alpha\beta}\nabla_{\alpha}\phi\nabla_{\beta}\phi$$

and the conditions

$$[\rho u^{\mu}\varphi_{\mu}] = 0, \tag{20}$$

$$[T_{\mathrm{mat}}^{\mu\nu}\varphi_{\mu}] = 0, \tag{21}$$

$$[T_{\mathrm{rad}}^{0\mu\nu}\varphi_{\mu}] = 0, \tag{22}$$

must hold on Σ for the system (1)-(3) where for every regularly discontinuous function f is $[f] = f_{-} - f_{+}$, the subscripts $+$, $-$ denote the value of the function ahead and behind the shock front, respectively. We assume the undisturbed shock to be stationary. Then, one can choose the local coordinates (t, x^{i}) adapted to the shock front. In particular, the local chart can be taken such that the equation $x^1 = 0$ represents locally the hypersuface Σ. The shock wave will be given by a step-wise function with different constant value for for each t in the two region $x^1 < 0$ and $x^1 > 0$.

We will investigate the stability of the solution under small disturbances (linear stability).

The indisturbed state has the following form:
when $x^1 < 0$:

$$p = \hat{p}_\infty = \text{const}, \quad s = \hat{s}_\infty = \text{const},$$

$$v^1 = \hat{v}^1_\infty = \text{const} > 0, \quad v^2 = v^3 = 0,$$

$$\rho = \hat{\rho}_\infty = \text{const}, \quad \hat{p}_\infty = -(e_0)_V(\hat{\rho}_\infty, \hat{s}_\infty),$$

$$\hat{c}^2_\infty = \hat{V}^2_\infty(e_0)_{VV}(\hat{\rho}_\infty, \hat{s}_\infty) = \left(\rho^2(e_0)_\rho\right)_\rho(\hat{\rho}_\infty, \hat{s}_\infty), \quad \hat{V}_\infty = 1/\hat{\rho}_\infty,$$

$$\hat{c}^2_{s\infty} = \frac{\hat{c}^2_\infty}{\hat{h}_\infty}, \quad \hat{h}_\infty = 1 + e_0(\hat{\rho}_\infty, \hat{s}_\infty) + \hat{p}_\infty \hat{V}_\infty, \quad \hat{\Gamma}_\infty = \frac{1}{\sqrt{1 - (\hat{v}^1_\infty)^2}},$$

$$J = \hat{J}_\infty = \sigma \hat{T}^4_\infty, \quad \hat{T}_\infty = (e_0)_s(\hat{\rho}_\infty, \hat{s}_\infty), \quad H^1 = H^2 = H^3 = 0;$$

when $x^1 > 0$:

$$p = \hat{p} = \text{const}, \quad s = \hat{s} = \text{const}, \quad v^1 = \hat{v}^1 = \text{const} > 0, \quad v^2 = v^3 = 0,$$

$$\rho = \hat{\rho} = \text{const}, \quad \hat{p} = -(e_0)_V(\hat{\rho}, \hat{s}),$$

$$\hat{c}^2 = \hat{V}^2(e_0)_{VV}(\hat{\rho}, \hat{s}) = \left(\rho^2(e_0)_\rho\right)_\rho(\hat{\rho}, \hat{s}),$$

$$\hat{V} = 1/\hat{\rho}, \quad \hat{c}^2_s = \frac{\hat{c}^2}{\hat{h}}, \quad \hat{h} = 1 + e_0(\hat{\rho}, \hat{s}) + \hat{p}\hat{V}, \quad \hat{\Gamma} = \frac{1}{\sqrt{1 - (\hat{v}^1)^2}},$$

$$J = \hat{J} = \sigma \hat{T}^4, \quad \hat{T} = (e_0)_s(\hat{\rho}, \hat{s}), \quad H^1 = H^2 = H^3 = 0.$$

We assume that at $x^1 = 0$ conditions (20)-(21) hold on the discontinuity surface

$$\hat{\rho}\hat{u}^1 = \hat{\rho}_\infty \hat{u}^1_\infty = \hat{j} \neq 0, \tag{23}$$

$$\hat{\rho}\hat{h}(\hat{u}^1)^2 + \hat{p} = \hat{\rho}_\infty \hat{h}_\infty (\hat{u}^1_\infty)^2 + \hat{p}_\infty, \tag{24}$$

$$\hat{h}\hat{\Gamma} = \hat{h}_\infty \hat{\Gamma}_\infty = \hat{l} \neq 0 \tag{25}$$

$$\left[\hat{h}^2\right] = [\hat{p}](\hat{h}\hat{V} + \hat{h}_\infty \hat{V}_\infty), \tag{26}$$

$$\left[\hat{J}\hat{\Gamma}\hat{u}^1\right] = 0, \tag{27}$$

$$\left[\hat{J}(1 + 4(\hat{u}^1)^2)\right] = 0 \tag{28}$$

($[\hat{h}^2] = \hat{h}^2 - \hat{h}^2_\infty$, $[\hat{p}] = \hat{p} - \hat{p}_\infty$)) and require the fulfilment of the necessary conditions:

$$\begin{gathered}\hat{p} > \hat{p}_\infty, \quad \hat{\rho} > \hat{\rho}_\infty, \quad \hat{s} > \hat{s}_\infty, \\ (e_0)_{VV}(\hat{\rho}_\infty, \hat{s}_\infty) > 0, \quad (e_0)_{VV}(\hat{\rho}, \hat{s}) > 0, \\ \{(e_0)_{VV}(e_0)_{ss} - (e_0)^2_{Vs}\}(\hat{\rho}_\infty, \hat{s}_\infty) > 0, \\ \{(e_0)_{VV}(e_0)_{ss} - (e_0)^2_{Vs}\}(\hat{\rho}, \hat{s}) > 0, \\ \hat{v}^1_\infty > \hat{c}_{s\infty}, \quad \hat{v}^1 < \hat{c}_s \quad (\hat{v}^1, \hat{v}^1_\infty, \hat{c}_{s\infty}, \hat{c}_s < 1!).\end{gathered} \tag{29}$$

Let us consider a *small* pertubation of the previous solution and denote by $\mathcal{B}^\alpha(\hat{U})$, $A^{(0)}_{\text{rad}}(\hat{W})$, $\mathcal{B}^\alpha(\hat{U}_\infty)$, $A^{(0)}_{\text{rad}}(\hat{W}_\infty)$, the matrices $\mathcal{B}^{(\alpha)}$ and $A^{(0)}_{\text{rad}}$ for $U = \hat{U}$ and so on. Let us linearize the system (14)-(18) around the indisturbed state. In order to simplify the notation we indicate again the pertubation of the matter and radiative variables with the vectors U and W.

The stability is investigated by studying the well-posedness of the following mixed problem with moving boundary conditions.

MAIN PROBLEM I. – To find piece-wise smooth solution which satisfies

$$B^{(0)}(\hat{\mathbf{U}})\frac{\partial \mathbf{U}}{\partial t} + \sum_{k=1}^{3} B^{(k)}(\hat{\mathbf{U}})\frac{\partial \mathbf{U}}{\partial x^k} = -I^*(\hat{\mathbf{U}})Q_{\text{mat}}(\hat{\mathbf{U}})\mathbf{W}, \tag{30}$$

$$A^{(0)}_{\text{rad}}(\hat{\mathbf{W}})\frac{\partial \mathbf{R}}{\partial t} + \sum_{k=1}^{3} A^{(k)}_{\text{rad}}(\hat{\mathbf{W}})\frac{\partial \mathbf{R}}{\partial x^k} = -Q_{\text{rad}}(\hat{\mathbf{U}})\mathbf{W}, \tag{31}$$

for $t > 0$ and $\mathbf{x} \in R^3_+$;

$$B^{(0)}(\hat{\mathbf{U}}_\infty)\frac{\partial \mathbf{U}}{\partial t} + \sum_{k=1}^{3} B^{(k)}(\hat{\mathbf{U}}_\infty)\frac{\partial \mathbf{U}}{\partial x^k} = -I^*(\hat{\mathbf{U}}_\infty)Q_{\text{mat}}(\hat{\mathbf{U}}_\infty)\mathbf{W}, \tag{32}$$

$$A^{(0)}_{\text{rad}}(\hat{\mathbf{W}}_\infty)\frac{\partial \mathbf{R}}{\partial t} + \sum_{k=1}^{3} A^{(k)}_{\text{rad}}(\hat{\mathbf{W}}_\infty)\frac{\partial \mathbf{R}}{\partial x^k} = -Q_{\text{rad}}(\hat{\mathbf{U}}_\infty)\mathbf{W}, \tag{33}$$

for $t > 0$ and $\mathbf{x} \in R^3_-$;

the boundary conditions:

$$\begin{aligned} F_t &= \mu p + \mu_1 p_\infty + \mu_2 s_\infty + \mu_3 u^1_\infty, \\ u^1 + dp &= d_1 p_\infty + d_2 s_\infty + d_3 u^1_\infty, \\ u^2 - \frac{\lambda}{\mu}F_{x^2} &= \lambda_1 u^2_\infty, \\ u^3 - \frac{\lambda}{\mu}F_{x^3} &= \lambda_1 u^3_\infty, \\ s &= \nu p + \nu_1 p_\infty + \nu_2 s_\infty; \end{aligned} \tag{34}$$

$$[A^{(1)}_{\rm rad}(\hat{\mathbf{W}})\mathbf{R}] = \begin{pmatrix} F_t[(4\hat\Gamma^2-1)\hat n_0^4] \\ 0 \\ -F_{x^2}[16(\hat u^1)^2\hat n_0^4] \\ -F_{x^3}[16(\hat u^1)^2\hat n_0^4] \end{pmatrix} \quad (35)$$

for $t > 0$ and $\mathbf{x}' \in R^2$, $x^1 = 0$; and the initial data

$$\mathbf{W}(0,\mathbf{x}) = \mathbf{W}_0(\mathbf{x}), \quad \mathbf{x} \in R^3_\pm, \quad F(0,\mathbf{x}') = F_0(\mathbf{x}'), \quad \mathbf{x}' \in R^2 \quad (36)$$

for $t = 0$.

Here $x^1 = F(t,\mathbf{x}')$ is a small displacement of the discontinuity front,

$$\mu = -\frac{\lambda}{\hat\Gamma[\hat v^1]}, \quad d = \frac{\hat\Gamma^2}{\hat v^1}\cdot\frac{a-\hat\beta^2}{a}, \quad \lambda = \frac{\hat v^1_\infty}{c_1\hat v^1\hat\Gamma^2}(d\hat v^1 - \hat\Gamma^2) = -\frac{\hat v^1_\infty\hat\beta^2}{\hat v^1 c_1 a},$$

$$\nu = -\frac{\hat h\hat\Gamma\hat\beta^2[\hat v^1]}{\hat T\hat v^1 c_1 a}, \quad a = \hat c_s^2 a_1, \quad a_1 = 2 - \frac{\hat v^1[\hat v^1]}{c_1}\left(1 + \frac{(e_0)_{Vs}(\hat\rho,\hat s)}{\hat T\hat c_s^2\hat\rho}\right),$$

$$c_1 = 1 - \hat v^1\hat v^1_\infty, \quad \mu_1 = -\frac{\hat v^1_\infty}{\hat\rho\hat\Gamma[\hat v^1]}\lambda_\infty, \quad d_1 = \frac{\hat\Gamma^2}{\hat v^1_\infty}\frac{a_1(\hat v^1_\infty)^2 + \hat\beta^2_\infty}{a}\frac{\hat c^2}{\hat c^2_\infty},$$

$$\lambda_\infty = -\frac{\hat u^1_\infty}{\hat h_\infty\hat c^2_{s\infty}}(e_0)_{Vs}(\hat\rho_\infty,\hat s_\infty)\nu_1 - \frac{\hat\rho_\infty\hat v^1_\infty\hat\Gamma^2_\infty}{\hat c^2_{s\infty}} - \hat\rho_\infty d_1,$$

$$\nu_1 = -\frac{\hat c^2}{\hat c^2_\infty}\frac{\hat h\hat\Gamma[\hat v^1]\hat\beta^2_\infty}{\hat T\hat v^1_\infty c_1 a}, \quad \hat\beta^2_\infty = \hat c^2_{s\infty} - (\hat v^1_\infty)^2, \quad \lambda_1 = d_3 = \frac{\hat\Gamma}{\hat\Gamma_\infty},$$

$$\tilde\mu_2 = \frac{\hat u^1}{\hat h\hat c_s^2}(e_0)_{Vs}(\hat\rho,\hat s)\nu_2 - \frac{\hat u^1_\infty}{\hat h_\infty\hat c^2_{s\infty}}(e_0)_{Vs}(\hat\rho_\infty,\hat s_\infty) + \hat\rho d_2,$$

$$\mu_2 = -\frac{\hat v^1_\infty}{\hat\rho\hat\Gamma[\hat v^1]}\tilde\mu_2, \quad a_{1\infty} = 2 + \frac{\hat v^1_\infty[\hat v^1]}{c_1}\left(1 + \frac{(e_0)_{Vs}(\hat\rho_\infty,\hat s_\infty)}{\hat T_\infty\hat c^2_{s\infty}\hat\rho_\infty}\right),$$

$$\nu_2 = \frac{\hat\Gamma_\infty\hat T_\infty a_{1\infty}}{\hat\Gamma\hat T a_1}, \quad \mu_3 = \frac{1}{\hat\Gamma_\infty}, \quad d_2 = \frac{c_1\hat\Gamma_\infty\hat T_\infty}{\hat h[\hat v^1]}\left(a_{1\infty} - \frac{a_{1\infty}}{a_1} - 1\right).$$

By performing the following change of coordinates, which is regular for sufficiently small values of $F(t,\mathbf{x}')$,

$$(t,x^i) \longmapsto (t, x^1 - F(t,\mathbf{x}'), x^2, x^3)$$

we reduce the problem to a problem with fixed boundary conditions.

Necessary conditions for the stability of shock waves are the geometrical Lax conditions (see Ref. 10 for a review), which assure that the problem is well formulated with respect to the number of boundary conditions. Shock solutions satisfying the Lax conditions are said evolutionary.

We observe that the matrix

$$B_1 = I^{(0)}(\hat{\mathbf{U}}))^{-1}I^{(1)}(\hat{\mathbf{U}}) = \begin{pmatrix} b_1 & 0 & b_2 & 0 & 0 \\ 0 & \hat{v}^1 & 0 & 0 & 0 \\ b_3 & 0 & b_1 & 0 & 0 \\ 0 & 0 & 0 & \hat{v}^1 & 0 \\ 0 & 0 & 0 & 0 & \hat{v}^1 \end{pmatrix},$$

where $b_1 = \dfrac{\hat{v}^1(1-\hat{c}_s^2)}{\hat{\Delta}}$, $b_2 = \dfrac{\hat{\rho}\hat{h}\hat{c}_s^2}{\hat{\Gamma}^3\hat{\Delta}}$, $b_3 = \dfrac{1}{\hat{\rho}\hat{h}\hat{\Gamma}\hat{\Delta}}$, has the following eigen-values:

$$\lambda_{1,2,3} = \hat{v}^1, \quad \lambda_{4,5} = \frac{\hat{v}^1(1-\hat{c}_s^2) \pm \frac{\hat{c}_s}{\hat{\Gamma}^2}}{\hat{\Delta}},$$

and if $\hat{v}_1 > \hat{c}_s$, then $\lambda_j > 0,\ j = \overline{1,5}$; if $\hat{v}^1 < \hat{c}_s$, then $\lambda_{1,2,3,4} > 0,\ \lambda_5 < 0$.

The matrix $A^{(1)}_{\text{rad}}(\hat{\mathbf{W}})$ has the following eigen-values:

$$\lambda_{1,2} = 4\hat{\Gamma}\hat{n}_0^5\left\{2\hat{v}^1(3\hat{\Gamma}^2-1) \pm \sqrt{4(\hat{v}^1)^2(3\hat{\Gamma}^2-1)^2 - (3(\hat{v}^1)^2-1)}\right\},$$

$$\lambda_{3,4} = 4\hat{v}_1\hat{\Gamma}\hat{n}_0^5,$$

and $\lambda_{1,3,4} > 0$, $\lambda_2 < 0$ if $3(\hat{v}^1)^2 < 1$, and $\lambda_{1,2,3,4} > 0$ if $3(\hat{v}^1)^2 > 1$. Therefore, one has evolutionary shocks when

$$\hat{v}^1_\infty > \max\left(\frac{1}{\sqrt{3}}, \hat{c}_{s\infty}\right), \tag{37}$$

$$\frac{1}{\sqrt{3}} < \hat{v}^1 < \hat{c}_s. \tag{38}$$

or when

$$\frac{1}{\sqrt{3}} > \hat{v}^1_\infty > \hat{c}_{s\infty}, \tag{39}$$

$$\hat{v}^1 < \min\left(\frac{1}{\sqrt{3}}, \hat{c}_s\right). \tag{40}$$

We call the last two type of discontinuous solutions as *fast* and *slow* shock waves respectively. In the next sections the stability properties will be studied for both the type of evolutionary shocks.

We observe that fast shocks can occour only in very special situations, as in early universe, because it is required that the sound speed of gas is greater than that of radiation. An example can be a barotropic fluid with equation of state $e = \gamma\rho$ with $\gamma = 1$ (stiff matter), considered in several cosmologial models.

The slow shock waves include almost all the cases of interest for radiative phenomena in astrophysics or laboratory experiments.

5. WELL-POSEDNESS OF STABILITY PROBLEM FOR FAST SHOCKS

By virtue of the first inequality from (37), the matrix $B_1(\hat{\mathbf{U}}_\infty)$ has five positive eigen-values, and the matrix $A^{(1)}_{\mathrm{rad}}(\hat{\mathbf{W}}_\infty)$ has four positive eigenvalues. Consequently system (32), (33) does not need the boundary conditions for $x^1 = 0$ and its solutions in this case is completely determined by initial data for $x^1 < 0$. Without loss of generality we presume that $\mathbf{W}(t, \mathbf{x}) \equiv 0$ for $x^1 < 0$ (see also Refs 13, 14).

We can turn the stability problem for fast shock waves into the proof of the well-posedness of the following problem.

MAIN PROBLEM II. – We seek a solution to the system of equations (30), (31) which satisfies the boundary conditions

$$\begin{aligned} F_t &= \mu p, \\ u^1 + dp &= 0, \\ u^{2,3} - \frac{\lambda}{\mu} F_{x^{2,3}} &= 0, \\ s &= \nu p; \end{aligned} \tag{41}$$

$$[A^{(1)}_{\mathrm{rad}}(\hat{\mathbf{W}})\mathbf{R}] = \begin{pmatrix} F_t[(4\hat{\Gamma}^2 - 1)\hat{n}_0^4] \\ 0 \\ -F_{x^2}[16(\hat{u}^1)^2\hat{n}_0^4] \\ -F_{x^3}[16(\hat{u}^1)^2\hat{n}_0^4] \end{pmatrix} \tag{42}$$

for $t > 0$ and $\mathbf{x}' \in R^2$, $x^1 = 0$ and the initial data

$$\mathbf{W}(0, \mathbf{x}) = \mathbf{W}_0(\mathbf{x}), \quad \mathbf{x} \in R^3_+, \quad F(0, \mathbf{x}') = F_0(\mathbf{x}'), \quad \mathbf{x}' \in R^2 \tag{43}$$

for $t = 0$. The coefficients μ, d, λ, ν are described above.

As in Gas Dynamics (see Ref. 14), we can derive the following boundary condition at $x^1 = 0$ for Main problem II

$$m(\tau')^2\mathcal{P} + n(\xi_1')^2\mathcal{P} - \frac{\hat{c}_s^2}{\hat{v}^1}\tau'\xi_1'\mathcal{P} + \mathcal{F}_0 = 0, \tag{44}$$

where

$$n = -\left(\frac{\hat{c}_s}{\hat{\Gamma}\hat{\beta}}\right)^2\lambda, \quad m = \left(\frac{\hat{c}_s}{\hat{\Gamma}\hat{\beta}}\right)^2\lambda + \frac{\hat{c}_s^2}{\hat{v}^1\hat{\Gamma}^2}d, \quad \mathcal{P} = \frac{p}{\hat{\rho}\hat{h}\hat{\Gamma}},$$

$$\mathcal{F}_0 = a_1\tau\mathcal{P} + a_2\xi_2\mathcal{P} + a_3\xi_3\mathcal{P},$$

$$\tau = \frac{\partial}{\partial t}, \quad \xi_k = \frac{\partial}{\partial x^k}, \quad k = 1, 2, 3,$$

$$\tau = \hat{p}_0\tau', \quad \xi_1 = \hat{p}_1\xi_1' + \hat{p}_2\tau',$$

with $\hat{p}_0 = \hat{\Gamma}\hat{\beta}$, $\hat{p}_1 = \frac{\hat{c}_s}{\hat{p}_0}$, $\hat{p}_2 = \hat{v}^1\hat{\Gamma}\frac{1-\hat{c}_s^2}{\hat{\beta}}$, $\hat{\beta}^2 = \hat{c}_s^2 - (\hat{v}^1)^2$ while the coefficients a_k, $k = 1, 2, 3$ can be determined from boundary conditions (41), (42).

We will obtain the so-called *a priori estimation without loss of smoothness* for Main problem II. This allows us, by using standard techniques (see for example Ref. 27), to prove the well-posedness of Main Problem II and in turn to achieve the fast shock waves stability in Radiation Hydrodynamics.

The basic idea used is to consider besides the eqs (30)-(31) also another problem with dissipative boundary condition at $x^1 = 0$. By taking a suitable linear combination of the two systems we obtain an expanded system which has also dissipative boundary conditions. This allows us to get the desidered *a priori* estimate for Main problem II. The construction process of the expanded system consists of two stages. Firstly from the system

$$A_0\mathbf{V}_t + \sum_{k=1}^{3} A_k\mathbf{V}_{x^k} + \Omega\mathbf{V} = 0$$

we construct the following symmetric t-hyperbolic (by Friedrichs) system:

$$A_{0p}(\mathbf{V}_p)_t + \sum_{k=1}^{3} A_{kp}(\mathbf{V}_p)_{x^k} + \Omega_p\mathbf{V}_p = 0. \tag{45}$$

Here

$$\mathbf{V} = \begin{pmatrix}\mathbf{U}\\ \mathbf{R}\end{pmatrix}, \quad A_\alpha = \begin{pmatrix} B^{(\alpha)}(\hat{\mathbf{U}}) & 0 \\ 0 & A_{\text{rad}}^{(\alpha)}(\hat{\mathbf{W}})\end{pmatrix},$$

Ω is the square matrix such that

$$\Omega \mathbf{V} = \begin{pmatrix} I^*(\hat{\mathbf{U}})\tilde{\mathbf{F}} \\ -\tilde{\mathbf{\Phi}} \end{pmatrix};$$

$$\mathbf{V}_p = (\mathbf{V}^*, \tau\mathbf{V}^*, \xi_1\mathbf{V}^*, \xi_2\mathbf{V}^*, \xi_3\mathbf{V}^*, \tau^2\mathbf{V}^*, \tau\xi_1\mathbf{V}^*, \tau\xi_2\mathbf{V}^*, \tau\xi_3\mathbf{V}^*, \xi_1^2\mathbf{V}^*, \xi_1\xi_2\mathbf{V}^*, \xi_1\xi_3\mathbf{V}^*, \xi_2^2\mathbf{V}^*, \xi_2\xi_3\mathbf{V}^*, \xi_3^2\mathbf{V}^*)^*;$$

$A_{\alpha p} = \mathrm{diag}(I_5 \times A, \epsilon(I_{10} \times A_\alpha))$, $\alpha = \overline{0,3}$, $\Omega_p = \mathrm{diag}(I_5 \times \Omega, \epsilon(I_{10} \times \Omega))$ are block-diagonal matrices, $I_5 \times A_\alpha$ is the Kronecker product of the matrices I_5 and A_α, I_5 is the unit matrix of order 5 etc.; ϵ is a positive constant.

Writing the energy integral in differential form for the symmetric system (45) and integrating it over the domain R^3_+, we obtain

$$\frac{d}{dt}J_0(t) - \iint_{R^2} (A_{1p}\mathbf{V}_p, \mathbf{V}_p)\Big|_{x^1=0} d\mathbf{x}' + \iiint_{R^3_+} ((\Omega_p + \Omega_p^*)\mathbf{V}_p, \mathbf{V}_p) d\mathbf{x} = 0, \tag{46}$$

where

$$J_0(t) = \iiint_{R^3_+} (A_{0p}\mathbf{V}_p, \mathbf{V}_p) d\mathbf{x},$$

$$(A_{0p}\mathbf{V}_p, \mathbf{V}_p) = (A_0\mathbf{V}, \mathbf{V}) + (A_0\mathbf{V}_t, \mathbf{V}_t) + \ldots + (A_0\mathbf{V}_{x^3}, \mathbf{V}_{x^3}) + \epsilon\{(A\mathbf{V}_{tt}, \mathbf{V}_{tt}) + \ldots + (A_0\mathbf{V}_{x^3x^3}, \mathbf{V}_{x^3x^3})\}$$

etc. In order to deduce eq. (46), we have assumed $\mathbf{V}_p$ to be square integrable in R^3_+. Therefore $|\mathbf{V}_p| \to 0$ when $x^1 \to \infty$ or $|x^{2,3}| \to \infty$.

Estimating the second and the third term in equality (46) with the help of boundary conditions (41), (42) and system (14), we obtain the inequality

$$\frac{d}{dt}J_0(t) - M_1 \iint_{R^2} \{\mathcal{P}^2 + (u^2)^2 + (u^3)^2 + \mathcal{P}_t^2 + \mathcal{P}_{x^1}^2 + \mathcal{P}_{x^2}^2 + \mathcal{P}_{x^3}^2 + \epsilon(P + Q)\}|_{x^1=0} d\mathbf{x}' \leq M_2 J_0(t), \tag{47}$$

where $M_1, M_2 > 0$ are constants;

$$P = \mathcal{P}_{tt}^2 + \mathcal{P}_{tx^1}^2 + \mathcal{P}_{tx^2}^2 + \mathcal{P}_{tx^3}^2 + \sum_{i,j=1}^{3} \mathcal{P}_{x^ix^j}^2, \quad Q = \sum_{i=2}^{3} \sum_{j,k=2}^{3} (u^i_{x^jx^k})^2.$$

From boundary conditions (42) and system (30) (31) at $x^1 = 0$ we deduce

$$(\xi_2^2 + \xi_3^2)u^i = (\beta_1\tau + \beta_2\xi_1)\xi_i\mathcal{P} + \left(\gamma_0\tau + \sum_{k=1}^{3} \gamma_k\xi_k\right)\mathcal{P}, \quad i = 2, 3, \tag{48}$$

where $\beta_{1,2}$, $\gamma_{0,1,2}$ are constants. Using the inequality [29] that

$$\iint_{R^2} Q|_{x^1=0} d\mathbf{x}' \leq \text{const} \iint_{R^2} \sum_{i=1}^{2} (u^i_{x^2x^2} + u^i_{x^3x^3})^2|_{x^1=0} d\mathbf{x}'$$
$$\leq \text{const} \iint_{R^2} \sum_{i=1}^{2} ((\beta_1\tau + \beta_2\xi_1)\xi_i\mathcal{P})^2|_{x^1=0} d\mathbf{x}',$$

and the property of the trace of a function in $W_2^1(R^3_+)$ on the plane $x^1 = 0$ (see Ref. 29), we transform inequality (47) to the following form:

$$\frac{d}{dt} J_0(t) - \epsilon \tilde{M}_1 \iint_{R^2} P|_{x^1=0} d\mathbf{x}' \leq \tilde{M}_2 J_0(t), \tag{49}$$

where $\tilde{M}_1$, $\tilde{M}_2 > 0$ are constants.

Now we proceed to the second more complicated stage consisting in the construction of the expanded system. Since $\hat{v}^1 < \hat{c}_s$, we observe that $\mathcal{P}$ satisfies the wave equation

$$\{(\tau')^2 - (\xi_1')^2 - \xi_2^2 - \xi_3^2\}\mathcal{P} = \tilde{h}.$$

with

$$\tilde{h} = \frac{\hat{\Gamma}^2\hat{\Delta}}{\hat{c}_s^2} L h^0 - \frac{1}{\hat{\Gamma}^2}\xi_1 h^1 - \xi_2 h^2 - \xi_3 h^3.$$

Then the vector

$$\mathbf{Y} = (\tau'\mathcal{P}, \xi_1'\mathcal{P}, \xi_2\mathcal{P}, \xi_3\mathcal{P})^*$$

satisfies the symmetric system [13]

$$(E\tau' + Q\xi_1' + R_2\xi_2 + R_3\xi_3)\mathbf{Y} = \mathbf{h},$$

$$E = E(m_1, l_2, l_3) = \begin{pmatrix} 1 & -m_1 & -l_2 & -l_3 \\ -m_1 & 1 & 0 & 0 \\ -l_2 & 0 & 1 & 0 \\ -l_3 & 0 & 0 & 1 \end{pmatrix},$$

$$Q = Q(m_1, l_2, l_3) = \begin{pmatrix} m_1 & -1 & 0 & 0 \\ -1 & m_1 & l_2 & l_3 \\ 0 & l_2 & -m_1 & 0 \\ 0 & l_3 & 0 & -m_1 \end{pmatrix},$$

$$R_2 = R_2(m_1, l_2, l_3) = \begin{pmatrix} l_2 & 0 & -1 & 0 \\ 0 & -l_2 & m_1 & 0 \\ -1 & m_1 & l_2 & l_3 \\ 0 & 0 & l_3 & -l_2 \end{pmatrix},$$

$$R_3 = R_3(m_1, l_2, l_3) = \begin{pmatrix} l_3 & 0 & 0 & -1 \\ 0 & -l_3 & 0 & m_1 \\ 0 & 0 & -l_3 & l_2 \\ -1 & m_1 & l_2 & l_3 \end{pmatrix},$$

where m_1, l_2, l_3 are some constants, and $E > 0$ if $1 > m_1^2 + l_2^2 + l_3^2$;

$$\mathbf{h} = \mathbf{h}(m_1, l_2, l_3) = (\tilde{h}, -m_1\tilde{h}, -l_2\tilde{h}, -l_3\tilde{h})^*.$$

We rewrite boundary conditions (44) in the form:

$$(\tau' - a\xi_1')\hat{L}\mathcal{P} + \mathcal{F}_0 = 0, \quad x^1 = 0,$$

where

$$\hat{L} = a_1\tau + a_2\xi_1'.$$

The constants a, a_1, a_2 are derived from the system

$$a_1 = m, \quad aa_2 = -n, \quad am - a_2 = \gamma = \frac{\hat{c}_s}{\hat{v}^1}.$$

Solving this system, we choose, for example, the number a as follows:

$$a = \frac{\gamma + \sqrt{\gamma^2 - 4mn}}{2m}.$$

In general, the number a is complex (if $\gamma^2 - 4mn > 0$, then a is real). Therefore, the function $\hat{L}\mathcal{P}$ is a complex function and the vector

$$\mathbf{Y}_p = (\tau'\mathbf{Y}^*, \xi_1'\mathbf{Y}^*, \xi_2\mathbf{Y}^*, \xi_3\mathbf{Y}^*, \hat{L}\mathbf{Y}^*)^*$$

is a complex vector which satisfies a symmetric t-hyperbolic (by Friedrichs) system:

$$\{E_p\tau' + Q_p\xi_1' + R_{2p}\xi_2 + R_{3p}\xi_3\}\mathbf{Y}_p = \mathbf{h}_p, \tag{50}$$

where E_p, Q_p, R_{2p}, R_{3p} are block-diagonal matrices of order 20,

$$E_p = \text{diag}(\sigma_1 E_1, \sigma_2 E_2, \sigma_3 E_3, \sigma_4 E_4, \sigma_5 E_5), \quad E_i = E(m_{1i}, l_{2i}, l_{3i})$$

etc.; $\sigma_i > 0$, m_{1i}, l_{2i}, l_{3i} are constants, $m_{1i}^2 + l_{2i}^2 + m_{3i}^2 < 1$,

$$\mathbf{h}_p = (\sigma_1\mathbf{h}_1^*, \sigma_2\mathbf{h}_2^*, \sigma_3\mathbf{h}_3^*, \sigma_4\mathbf{h}_4^*, \sigma_5\mathbf{h}_5^*)^*, \quad \mathbf{h}_i = \mathbf{h}(m_{1i}, l_{2i}, l_{3i}), \quad i = \overline{1, 5}.$$

We choose

$$l_{2i} = l_{3i} = 0, \quad i = \overline{1,5}, \quad m_{11} = 0, \quad m_{12} = -1/2,$$

$$m_{13} = m_{14} = b > 0, \quad b = \frac{1}{2}\min\left\{\frac{m|a|^2}{n\text{Re}a}, \frac{n\text{Re}a}{m|a|^2}\right\},$$

$$m_{15} = \frac{2\text{Re}a}{1+|a|^2}, \qquad \sigma_1 = \frac{m}{n}\sigma_2,$$

$$\sigma_3 = \sigma_4 = \frac{mn}{1+|a|^2}\frac{(\text{Re}a)^2}{|a|^2}\sigma_5, \quad \sigma_2 = \left(\frac{n\text{Re}a}{m|a|^2} - b\right)\sigma_4,$$

σ_5 is an arbitrary positive number. Thanks to this choice, we obtain

$$\begin{aligned} &-(Q_pY_p, Y_p)|_{x^1=0} \\ &\leq \left(\sum_{i=2}^{3}\{k_{1i}|\hat{L}\xi_i\mathcal{P}|^2 + k_{2i}(\tau'\xi_i\mathcal{P})^2 + k_{3i}(\xi_1'\xi_i\mathcal{P})^2 + k_{4i}(\xi_2\xi_i\mathcal{P})^2\}\right. \\ &+ k_5(\xi_3^2\mathcal{P})^2 + \left(\frac{2\gamma}{n} - \frac{1}{2}\right)(\tau'\xi_1'\mathcal{P})^2 + \frac{1}{2}\sigma_2((\xi_1')^2\mathcal{P})^2 \\ &\left.- \epsilon_1|\hat{L}\tau'\mathcal{P}|^2 - \frac{1}{\epsilon_1}\mathcal{F}_0^2\right)\Bigg|_{x^1=0} \leq (M_3 - \epsilon_1 M_4)\mathcal{P}|_{x^1=0} - \frac{1}{\epsilon_1}\mathcal{F}_0^2|_{x^1=0}, \end{aligned} \tag{51}$$

where $k_{ij}, i = \overline{1,4}, \ j = 2,3, \ k_5, \ M_{3,4}$ are positive constants; $\epsilon_1 > 0$ is a constant such that

$$\epsilon_1 < \frac{M_3}{M_4}.$$

We write out the energy integral in the differential form for system (50):

$$\begin{aligned} (D_p\mathbf{Y}_p, \mathbf{Y}_p)_t + \frac{1}{p_1}(Q_p\mathbf{Y}_p, \mathbf{Y}_p)_{x^1} + (R_{2p}\mathbf{Y}_p, \mathbf{Y}_p)_{x^2} \\ + (R_{2p}\mathbf{Y}_p, \mathbf{Y}_p)_{x^3} + 2(\mathbf{Y}_p, \mathbf{h}_p) = 0. \end{aligned} \tag{52}$$

Here $D_p = \frac{1}{p_0}E_p - \frac{p_2}{p_1}Q_p$. It is easy to verify that $D_p > 0$. We integrate equality (52) over the domain R_+^3 assuming again that $|\mathbf{Y}_p|$ is square integrable in R_+^3 and therefore $|\mathbf{Y}_p| \to 0$ whether $x^1 \to \infty$ or $|x^{2,3}| \to \infty$. As a result, in view of (50), with the help of the property of function trace from $W_2^1(R_+^3)$ on the plane $x^1 = 0$, we obtain the following inequality:

$$\frac{d}{dt}J_1(t) + M_5\iint_{R^2} P|_{x^1=0}d\mathbf{x}' \leq M_6(J_1(t) + J_0(t)), \tag{53}$$

where

$$J_1(t) = \iiint_{R^3_+} (D_p \mathbf{Y}_p, \mathbf{Y}_p) d\mathbf{x},$$

$M_5 = \frac{1}{p_1}(M_3 - M_4) > 0$, $M_6 > 0$ are constants dependent on ϵ_1.

Adding inequality (49) to inequality (53) and considering that under an appropriate choice of the constant ϵ the quadratic form

$$\mathcal{A} = (M_5 - \epsilon \tilde{M}_1) P|_{x_1=0}$$

is positive definite, we obtain the inequality:

$$\frac{d}{dt} J_2(t) \leq M_7 J_2(t), \quad t > 0,$$

where $J_2(t) = J_0(t) + J_1(t)$; $M_7 > 0$ is a constant. The last inequality implies the desired *a priori* estimation for Main problem II:

$$J_2(t) \leq e^{M_7 t} J_2(0), \quad t > 0, \tag{54}$$

Thus, from (54) we conclude that the estimation

$$\|\mathbf{W}(t)\|_{W_2^2(R^3_+)} \leq M_8, \quad 0 < t \leq \tilde{T} < \infty \tag{55}$$

holds for Main problem II. Here $M_8 < \infty$ is a positive constant dependent on $\tilde{T}$.

Note that for the function $F(t, \mathbf{x}')$ the estimation

$$\|F\|_{W_2^3((0,T) \times R^2)} \leq M_9$$

may be obtained ($M_9 < \infty$ is a positive constant dependent on $\tilde{T}$).

From estimations (54), (55) it follows (see 27) that Main problem II is well-posed for $m, n > 0$ and this implies *the fast shock waves are stable.*

6. INSTABILITY OF SLOW SHOCK WAVES

Now the slow shocks will be studied.

By virtue of (39), the matrix $B_1(\hat{\mathbf{U}}_\infty)$ has five positive eigen-values, and the matrix $A^{(1)}_{\text{rad}}(\hat{\mathbf{W}}_\infty)$ has three positive eigen-values and *one negative eigen-value*. At the same time, as a consequence of (40), the matrix $B_1(\hat{\mathbf{U}})$ has four positive eigen-values and the matrix $A^{(1)}_{\text{rad}}(\hat{\mathbf{W}})$ has three positive eigen-values, *i.e.*, at $x^1 = 0$ it is necessary to set up eight boundary conditions for systems (30), (31) and (32), (33) plus a condition to determine the function $F(t, \mathbf{x}')$.

The study of stability for slow radiative shock corresponds to the analysis of the well-posedness of Main Problem I under the conditions (39)-(40). Applying the same technique to the case of slow shock waves, omitting details, we conclude that if the inequality

$$-[(A^{(1)}_{\text{rad}}(\hat{\mathbf{W}}))^{-1}] = (A^{(1)}_{\text{rad}}(\hat{\mathbf{W}}_\infty))^{-1} - (A^{(1)}_{\text{rad}}(\hat{\mathbf{W}}))^{-1} > 0 \tag{56}$$

holds, then the boundary conditions of the corresponding expanded system are dissipative. However, by virtue of (39), (40) it is not difficult to see that condition (56) is not fulfilled. Indeed, we shall show, by following the Ref. 30, that it is possible to construct ill-posed examples of Hadamard-type for slow shock waves, that is a sequence of solutions of the form

$$\mathbf{Y}_n = \begin{cases} \Lambda e^{-\sqrt{n}+n(\eta_0 t + i\omega_2^0 x^2)} \left\{ \begin{pmatrix} 0 \\ c_2 \\ 0 \\ 0 \end{pmatrix} e^{-n\hat{\nu} x^1} + \begin{pmatrix} 0 \\ 0 \\ c_3 \\ c_4 \end{pmatrix} e^{-n\frac{\eta_0}{\hat{v}^1} x^1} \right\}, & x^1 > 0, \\ \Lambda_\infty \begin{pmatrix} c_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-\sqrt{n}+n(\eta_0 t + \hat{\nu}_\infty x^1 + i\omega_2^0 x^2)}, & x^1 < 0; \end{cases}$$

where the matrices Λ and Λ_∞ and the constants $\hat{\nu}$, $\hat{\nu}_\infty$, η_0, ω_2^0 will be described below and the constants c_i will not be all zero and determined by the boundary conditions.

In order to simplify the calculations we rewrite system (2)-(3) as

$$B^{(0)}_{\text{mat}}(\hat{\mathbf{U}})\frac{\partial \mathbf{U}}{\partial t} + \sum_{k=1}^{3} B^{(k)}_{\text{mat}}(\hat{\mathbf{U}})\frac{\partial \mathbf{U}}{\partial x^k} = -Q_{\text{mat}}(\hat{\mathbf{W}})\mathbf{U} - D_{\text{mat}}(\hat{\mathbf{W}})\mathbf{Y}, \tag{57}$$

$$B^{(0)}_{\text{rad}}(\hat{\mathbf{U}})\frac{\partial \mathbf{Y}}{\partial t} + \sum_{k=1}^{3} B^{(k)}_{\text{rad}}(\hat{\mathbf{U}})\frac{\partial \mathbf{Y}}{\partial x^k} = -Q_{\text{rad}}(\hat{\mathbf{W}})\mathbf{Y} - D_{\text{rad}}(\hat{\mathbf{W}})\mathbf{U}, \tag{58}$$

where

$$\mathbf{Y} = 12\hat{n}_0^5 \mathcal{A}^{-1}\mathbf{R} = \begin{pmatrix} J \\ \mathbf{H} + \frac{4}{3}\hat{J}\mathbf{u} \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} \hat{\Gamma} & -3\hat{v}^1 & 0 & 0 \\ -\hat{u}^1 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

$$B^{(\alpha)}_{\rm rad}(\hat{\mathbf{U}}) = \frac{1}{12\hat{n}_0^5}\mathcal{A}^* A^{(\alpha)}_{\rm rad}(\hat{\mathbf{W}})\mathcal{A}, \quad \alpha = \overline{0,3},$$

$$B^{(0)}_{\rm rad}(\hat{\mathbf{U}}) = \begin{pmatrix} \hat{\Gamma} & \hat{v}^1 & 0 & 0 \\ \hat{v}^1 & \frac{3}{\hat{\Gamma}} & 0 & 0 \\ 0 & 0 & 3\hat{\Gamma} & 0 \\ 0 & 0 & 0 & 3\hat{\Gamma} \end{pmatrix},$$

$$B^{(1)}_{\rm rad}(\hat{\mathbf{U}}) = \begin{pmatrix} \hat{u}^1 & 1 & 0 & 0 \\ 1 & \frac{3\hat{v}^1}{\hat{\Gamma}} & 0 & 0 \\ 0 & 0 & 3\hat{u}^1 & 0 \\ 0 & 0 & 0 & 3\hat{u}^1 \end{pmatrix},$$

$$B^{(2)}_{\rm rad}(\hat{\mathbf{U}}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B^{(3)}_{\rm rad}(\hat{\mathbf{U}}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix};$$

$Q_{\rm mat}$, $Q_{\rm rad}$ are quadratic matrices and $D_{\rm mat}$, $D_{\rm rad}$ are rectangular matrices such that

$$Q\mathbf{V} = \begin{pmatrix} -I^*(\hat{\mathbf{U}})\tilde{\mathbf{F}} \\ \mathcal{A}^*\tilde{\mathbf{\Phi}} \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} \mathbf{U} \\ \mathbf{Y} \end{pmatrix}, \quad Q = \begin{pmatrix} Q_{\rm mat} & D_{\rm mat} \\ D_{\rm rad} & Q_{\rm rad} \end{pmatrix}.$$

Applying the Fourier-Laplace transform to the Main Problem I, we obtain the following boundary value problem for the system of ordinary differential equations:

$$-\frac{d\hat{\mathbf{V}}}{dx^1} + \mathcal{M}(s,\omega)\hat{\mathbf{V}} = 0, \quad x^1 > 0, \tag{59}$$

$$-\frac{d\hat{\mathbf{V}}}{dx^1} + \mathcal{M}_\infty(s,\omega)\hat{\mathbf{V}} = 0, \quad x^1 < 0, \tag{60}$$

$$S_0\hat{\mathbf{V}} - S_1\hat{\mathbf{V}}_\infty = 0 \quad \text{for } x^1 = 0. \tag{61}$$

Here

$$\hat{\mathbf{V}} = \hat{\mathbf{V}}(s, x^1, \omega) = (2\pi)^{-3/2}\iiint_{R^3} e^{-st-i(\omega,\mathbf{x}')}\mathbf{V}(t,\mathbf{x})dtd\mathbf{x}',$$

$$s = \eta + i\xi, \quad \eta > 0, \quad \omega = (\omega_2, \omega_3), \quad |\xi|, |\omega| < \infty;$$

$$\mathcal{M}(s,\omega) = \mathcal{M}(\hat{\mathbf{W}}) = \begin{pmatrix} M_{\rm mat}(\hat{\mathbf{W}}) & -(B^{(1)}_{\rm mat}(\hat{\mathbf{U}}))^{-1}D_{\rm mat}(\hat{\mathbf{W}}) \\ -(B^{(1)}_{\rm rad}(\hat{\mathbf{U}}))^{-1}D_{\rm rad}(\hat{\mathbf{W}}) & M_{\rm rad}(\hat{\mathbf{W}}) \end{pmatrix},$$

$$M_{\rm mat}(\hat{\mathbf{W}}) = M^{(0)}_{\rm mat}(\hat{\mathbf{U}}) - (B^{(1)}_{\rm mat}(\hat{\mathbf{U}}))^{-1}Q_{\rm mat}(\hat{\mathbf{W}}),$$

$$M_{\rm mat}^{(0)}(\hat{\mathbf{U}}) = M_{\rm mat}^{(0)}(s,\omega) = -(B_{\rm mat}^{(1)}(\hat{\mathbf{U}}))^{-1}\Big\{sB_{\rm mat}^{(0)}(\hat{\mathbf{U}}) + i\sum_{k=2}^{3}\omega_k B_{\rm mat}^{(k)}(\hat{\mathbf{U}})\Big\},$$

$$M_{\rm rad}(\hat{\mathbf{W}}) = M_{\rm rad}^{(0)}(\hat{\mathbf{U}}) - (B_{\rm rad}^{(1)}(\hat{\mathbf{U}}))^{-1}Q_{\rm rad}(\hat{\mathbf{W}}),$$

$$M_{\rm rad}^{(0)}(\hat{\mathbf{U}}) = -(B_{\rm rad}^{(1)}(\hat{\mathbf{U}}))^{-1}\Big\{sB_{\rm rad}^{(0)}(\hat{\mathbf{U}}) + i\sum_{k=2}^{3}\omega_k B_{\rm rad}^{(k)}(\hat{\mathbf{U}})\Big\},$$

$$\mathcal{M}_\infty(s,\omega) = \mathcal{M}(\hat{\mathbf{W}}_\infty), \quad M_{\rm rad}^{(0)}(\hat{\mathbf{U}}_\infty) = M_{\rm rad}^{(0)}(\hat{\mathbf{U}}_\infty),$$

$$\mathcal{B}^{(\alpha)} = \mathcal{B}^{(\alpha)}(\hat{\mathbf{U}}) = \begin{pmatrix} B_{\rm mat}^{(\alpha)}(\hat{\mathbf{U}}) & 0 \\ 0 & B_{\rm rad}^{(\alpha)}(\hat{\mathbf{U}}) \end{pmatrix}, \alpha = \overline{0,3},$$

$$\hat{\mathbf{V}}_\infty = \hat{\mathbf{V}}|_{x^1 \to -0},$$

$$S_0 = \begin{pmatrix} S_{\rm mat}^0 & 0 \\ S_{\rm rad}^0 & G_{\rm rad}(\hat{\mathbf{W}}) \end{pmatrix}, \quad S_1 = \begin{pmatrix} S_{\rm mat}^1 & 0 \\ S_{\rm rad}^1 & G_{\rm rad}(\hat{\mathbf{W}}_\infty) \end{pmatrix},$$

$$S_{\rm mat}^0 = S_{\rm mat}^0(s,\omega) = \begin{pmatrix} -i\omega_2\tilde{\mu} & 0 & 0 & s\frac{\mu}{\lambda} & 0 \\ -i\omega_3\tilde{\mu} & 0 & 0 & 0 & s\frac{\mu}{\lambda} \\ \tilde{d} & 0 & 1 & 0 & 0 \\ -\tilde{\nu} & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$S_{\rm mat}^1 = S_{\rm mat}^1(s,\omega) = \begin{pmatrix} i\omega_2\tilde{\mu}_1 & i\omega_2\mu_2 & i\omega_2\mu_3 & s\frac{\mu\lambda_1}{\lambda} & 0 \\ i\omega_3\tilde{\mu}_1 & i\omega_3\mu_2 & i\omega_3\mu_3 & 0 & s\frac{\mu\lambda_1}{\lambda} \\ \tilde{d}_1 & d_2 & d_3 & 0 & 0 \\ \tilde{\nu}_1 & \nu_2 & 0 & 0 & 0 \end{pmatrix},$$

$$S_{\rm rad}^0 = \begin{pmatrix} -[(4\hat{\Gamma}^2-1)\hat{J}]\tilde{\mu} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & [16(\hat{u}^1)^2\hat{J}]\frac{\mu}{\lambda} & 0 \\ 0 & 0 & 0 & 0 & [16(\hat{u}^1)^2\hat{J}]\frac{\mu}{\lambda} \end{pmatrix},$$

$$S_{\rm rad}^1 = \begin{pmatrix} [(4\hat{\Gamma}^2-1)\hat{J}]\tilde{\mu}_1 & [(4\hat{\Gamma}^2-1)\hat{J}]\mu_2 & [(4\hat{\Gamma}^2-1)\hat{J}]\mu_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & [16(\hat{u}^1)^2\hat{J}]\frac{\mu\lambda_1}{\lambda} & 0 \\ 0 & 0 & 0 & 0 & [16(\hat{u}^1)^2\hat{J}]\frac{\mu\lambda_1}{\lambda} \end{pmatrix}$$

While applying the Fourier-Laplace transform, we assume that $\mathbf{V}(t,\mathbf{x}) = 0$ for $t \leq 0$.

Let

$$s = n\hat{s} = n(\hat{\eta}+i\hat{\xi}) = \eta+i\xi, \quad \eta > 0, \quad \omega = n\hat{\omega} = n(\hat{\omega}_2, \hat{\omega}_3), \quad n = 1, 2, 3, \dots$$

Then

$$\mathcal{M}(s,\omega) = n\hat{\mathcal{M}}(\hat{s},\hat{\omega}), \quad \mathcal{M}_\infty(s,\omega) = n\hat{\mathcal{M}}_\infty(\hat{s},\hat{\omega}).$$

Further let $n \gg 1$. We can expand the values $\hat{\eta}$, $\hat{\xi}$, $\hat{\omega}_{2,3}$, the elements of matrices etc. as a power series in the small parameter $\frac{1}{n}$ because the element of the matrices $\hat{\mathcal{M}}$, $\hat{\mathcal{M}}_\infty$ are analytic functions:

$$\hat{\eta} = \hat{\eta}^{(0)} + \frac{1}{n}\hat{\eta}^{(1)} + \frac{1}{n^2}\hat{\eta}^{(2)} + \dots,$$

$$\hat{\xi} = \hat{\xi}^{(0)} + \frac{1}{n}\hat{\xi}^{(1)} + \frac{1}{n^2}\hat{\xi}^{(2)} + \dots,$$

$$\hat{\omega}_{2,3} = \hat{\omega}_{2,3}^{(0)} + \frac{1}{n}\hat{\omega}_{2,3}^{(1)} + \frac{1}{n^2}\hat{\omega}_{2,3}^{(2)} + \dots$$

etc. Then we have:

$$\hat{\mathcal{M}}(\hat{s},\hat{\omega}) = T\begin{pmatrix} B_m & 0 & 0 & 0 \\ 0 & Q_m & 0 & 0 \\ 0 & 0 & B_r & 0 \\ 0 & 0 & 0 & Q_r \end{pmatrix} T^{-1}$$

$$= T\begin{pmatrix} B_0(\hat{\mathbf{U}}) + N_1 & 0 & 0 & 0 \\ 0 & Q_0(\hat{\mathbf{U}}) + N_2 & 0 & 0 \\ 0 & 0 & B(\hat{\mathbf{U}}) + N_3 & 0 \\ 0 & 0 & 0 & Q(\hat{\mathbf{U}}) + N_4 \end{pmatrix} T^{-1},$$

$$T = \begin{pmatrix} \Lambda_0(\hat{\mathbf{U}}) + N_5 & 0 \\ 0 & \Lambda(\hat{\mathbf{U}}) + N_6 \end{pmatrix},$$

where

$$B = \frac{1}{D^2\hat{\Gamma}^2}(2\hat{u}^1\hat{\Gamma}s + \sigma), \quad Q = \begin{pmatrix} \frac{2\hat{u}^1\hat{\Gamma}s-\sigma}{D^2\hat{\Gamma}^2} & 0 & 0 \\ 0 & -\frac{s}{\hat{v}^1} & 0 \\ 0 & 0 & -\frac{s}{\hat{v}^1} \end{pmatrix},$$

$$\Lambda = \frac{1}{\hat{\Gamma}^2}\begin{pmatrix} 3\hat{v}^1\sigma + 3s & -3\hat{v}^1\sigma + 3s & 0 & 0 \\ -3\hat{u}^1 s - \hat{\Gamma}\sigma & -3\hat{u}^1 s + \hat{\Gamma}\sigma & \hat{u}^1\hat{\Gamma}i\omega_2 & \hat{u}^1\hat{\Gamma}i\omega_3 \\ -i\omega_2 D^2\hat{\Gamma} & -i\omega_2 D^2\hat{\Gamma} & s & 0 \\ -i\omega_3 D^2\hat{\Gamma} & -i\omega_3 D^2\hat{\Gamma} & 0 & s \end{pmatrix}$$

for $\eta \neq \hat{u}^1|\omega|$ ($\eta^2 + \xi^2 + \omega_k^2 \neq 0$, $k = 2, 3$);

$$B = \frac{1}{D^2\hat{\Gamma}}|\omega|(2(\hat{u}^1)^2 + 1), \quad Q = \begin{pmatrix} -|\omega|\hat{\Gamma} & 1 & 0 \\ 0 & -|\omega|\hat{\Gamma} & 1 \\ 0 & 0 & -|\omega|\hat{\Gamma} \end{pmatrix}$$

for $\eta = \hat{u}^1|\omega|$, $\xi = 0$ (it is also not difficult to write out the matrix Λ for this case and for the case when $\eta^2+\xi^2+\omega_2^2 = 0$ or $\eta^2+\xi^2+\omega_3^2 = 0$). Here

$$D = \sqrt{1 - 3(\hat{v}^1)^2}, \quad \sigma = \sqrt{3s^2 + |\omega|^2 D^2 \hat{\Gamma}^2};$$

Re$B > 0$ and the eigen-values of the matrix Q lie strictly in the left semi-plane (Re$\lambda_j(Q) < 0$).

The explicit form of the matrices Q_0, Λ_0 and the parameter B_0 are not relevant. We report only that Re$B_0 > 0$ and the eigen-values of the matrix Q_0 lie strictly in the left semi-plane. N_k, $k = \overline{1,6}$ are the matrices with elements of order $O(\frac{1}{n})$. It is not difficult to write out the analogous representation for the matrix $\hat{\mathcal{M}}_\infty$:

$$\hat{\mathcal{M}}_\infty(\hat{s}, \hat{\omega}) = T_\infty \begin{pmatrix} Q_{m\infty} & 0 & 0 \\ 0 & B_{r\infty} & 0 \\ 0 & 0 & Q_{r\infty} \end{pmatrix} T_\infty^{-1}$$

$$= T_\infty \begin{pmatrix} M_{\text{mat}}^{(0)}(\hat{\mathbf{U}}_\infty) + N_7 & 0 & 0 \\ 0 & B(\hat{\mathbf{U}}_\infty) + N_8 & 0 \\ 0 & 0 & Q(\hat{\mathbf{U}}_\infty) + N_9 \end{pmatrix} T_\infty^{-1},$$

$$T_\infty = \begin{pmatrix} I_4 & 0 \\ 0 & \Lambda(\hat{\mathbf{U}}_\infty) + N_{10} \end{pmatrix},$$

$$Q_{m\infty} = M_{\text{mat}}^{(0)}(\hat{\mathbf{U}}_\infty) - \frac{1}{n}(B_{\text{mat}}^{(1)}(\hat{\mathbf{U}}_\infty))^{-1} Q_{\text{mat}}(\hat{\mathbf{W}}_\infty).$$

The eigen-values of the matrix $M_{\text{mat}}^{(0)}(\hat{\mathbf{U}}_\infty)$ lie strictly in the left semi-plane; I_4 is the unit matrix of order 5; N_k, $k = \overline{7,10}$ are the matrices with elements of order $O(\frac{1}{n})$.

We have the following relations for constants c_k, $k = \overline{1,9}$:

$$S_0 T \begin{pmatrix} \mathbf{C}_{\text{mat}} \\ 0 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} - S_1 T_\infty \begin{pmatrix} 0 \\ \vdots \\ 0 \\ c_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0}, \quad \mathbf{C}_{\text{mat}} = \begin{pmatrix} c_5 \\ c_6 \\ c_7 \\ c_8 \\ c_9 \end{pmatrix}.$$

The last equation has only the trivial solutions if

$$\det \mathcal{L} \neq 0, \tag{62}$$

where

$$\det \mathcal{L} = l^{(0)} + \frac{1}{n} l^{(1)} + \ldots + \frac{1}{n^k} l^{(k)} + O(\frac{1}{n^k}), \quad l^0 = \det \mathcal{L}^{(0)},$$

$$\mathcal{L}^{(0)} = \mathcal{L}^{(0)}(\hat{s}^{(0)}, \hat{\omega}^{(0)}) = \begin{pmatrix} S^0_{\mathrm{mat}}(\hat{s}^{(0)}, \hat{\omega}^{(0)})\Lambda_0(\hat{s}^{(0)}, \hat{\omega}^{(0)}) & 0 \\ S_{\mathrm{rad}}\Lambda_0(\hat{s}^{(0)}, \hat{\omega}^{(0)}) & L_{\mathrm{rad}}(\hat{s}^{(0)}, \hat{\omega}^{(0)}) \end{pmatrix},$$

$$\hat{s}^{(0)} = \hat{\eta}^{(0)} + i\hat{\xi}^{(0)}, \quad \hat{\omega}^{(0)} = (\hat{\omega}_2^{(0)}, \hat{\omega}_3^{(0)}),$$

$$\det \mathcal{L}^{(0)} = \det \left\{ S^0_{\mathrm{mat}}(\hat{s}^{(0)}, \hat{\omega}^{(0)})\Lambda_0(\hat{s}^{(0)}, \hat{\omega}^{(0)}) \right\} \det L_{\mathrm{rad}}(\hat{s}^{(0)}, \hat{\omega}^{(0)}),$$

$$\det L_{\mathrm{rad}}(\hat{s}^{(0)}, \hat{\omega}^{(0)}) = 3(\hat{v}^1)^2 D^2 D^2_\infty \det L(\hat{s}^{(0)}, \hat{\omega}^{(0)}).$$

Here

$$L = \begin{pmatrix} \sigma_\infty - \hat{v}^1_\infty s & \hat{v}^1 s + \sigma & (2\hat{\Gamma}^2 - 1)\omega_2 \hat{\Gamma} & (2\hat{\Gamma}^2 - 1)\omega_3 \hat{\Gamma} \\ \hat{v}^1_\infty \sigma_\infty - s & s + \hat{v}^1 \sigma & 2\hat{u}^1 \omega_2 & 2\hat{u}^1 \omega_3 \\ -\hat{v}^1_\infty \omega_2 & \hat{v}^1 \omega_2 & \hat{\Gamma} s & 0 \\ -\hat{v}^1_\infty \omega_3 & \hat{v}^1 \omega_3 & 0 & \hat{\Gamma} s \end{pmatrix},$$

$$\sigma_\infty = \sqrt{3s^2 + |\omega|^2 D^2_\infty \hat{\Gamma}^2_\infty}, \quad D_\infty = \sqrt{1 - 3(\hat{v}^1_\infty)^2},$$

$$\eta \neq \hat{u}^1 |\omega|, \quad \eta \neq \hat{u}^1_\infty |\omega|.$$

It is also not difficult to write out the matrix L for the cases: 1). $\eta = \hat{u}^1|\omega|$, $\xi = 0$; 2). $\eta = \hat{u}^1_\infty|\omega|$, $\xi = 0$.

One says that the boundary conditions (34), (35) of Main Problem satisfy the Lopatinsky condition if requirement (62) is fulfilled for all $\hat{\eta} > 0$, $(\hat{\xi}, \hat{\omega}) \in R^3$. This means that exponentially growing solutions do not exist for the problem under consideration. Let us prove that the Lopatinsky condition is not fulfilled.

We consider the equality

$$\det \mathcal{L} = 0 \tag{63}$$

as a relation which must be satisfied by the values $\hat{\eta}$, $\hat{\xi}$, $\hat{\omega}_{2,3}$. We show that there exist $\hat{\eta} > 0$, $(\hat{\xi}, \hat{\omega}) \in R^3$ such that equality (6.5) is fulfilled. As proved in the Appendix, there exist $\eta_0 > 0$, ξ_0, $\omega_0 = (\omega_2^0, \omega_3^0)$ $(\eta_0^2 + \xi_0^2 + |\omega_0|^2 = 1$, $\omega^0_{2,3} \geq 0)$ such that $\det L(\hat{s}^{(0)}, \hat{\omega}^{(0)}) = 0$, *i.e.*, $l^0 = \det \mathcal{L}^{(0)} = 0$.

We choose $\hat{\eta}^{(0)} = \eta_0$, $\hat{\xi}^{(0)} = 0$, $\hat{\omega}^{(0)} = (\omega_2^0, 0)$ $(q(\eta_0, 0, \omega_2^0, 0) = 0)$ and the values $\hat{s}^{(k)}$, $\hat{\omega}_{2,3}^{(k)}$, $k = 1, 2, 3 \ldots$ as roots of the corresponding relations:

$$l^{(k)} = 0, \quad k = 1, 2, 3 \ldots$$

Thus, for

$$\hat{\eta} = \eta_0 + \frac{1}{n}\hat{\eta}^{(1)} + O\left(\frac{1}{n^2}\right),$$

$$\hat{\xi} = \frac{1}{n}\hat{\xi}^{(1)} + O\left(\frac{1}{n^2}\right),$$

$$\hat{\omega}_2 = \omega_2^0 + \frac{1}{n}\hat{\omega}_2^{(1)} + O\left(\frac{1}{n^2}\right),$$

$$\hat{\omega}_3 = \frac{1}{n}\hat{\omega}_3^{(1)} + O\left(\frac{1}{n^2}\right)$$

the Lopatinsky determinant $\det \mathcal{L}$ is equal to zero. Consequently, boundary conditions (34), (35) do not satisfy the Lopatinsky condition.

The last fact allows us to write out the ill-posedness example of Hadamard type for Main problem:

$$\mathbf{V}_n = \begin{cases} T\begin{pmatrix} e^{nQ_m x^1}\mathbf{C}_{\text{mat}} \\ e^{nQ_r x^1}\begin{pmatrix} 0 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \end{pmatrix} e^{-\sqrt{n}+n(\hat{\eta}t+i(\hat{\omega},\mathbf{x}'))}, & x^1 > 0, \\ T_\infty\begin{pmatrix} 0 \\ \vdots \\ 0 \\ e^{nB_{r\infty}x^1}\begin{pmatrix} c_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix} e^{-\sqrt{n}+n(\hat{\eta}t+i(\hat{\omega},\mathbf{x}'))}, & x^1 < 0. \end{cases}$$

$$F_n = \frac{1}{n}F^{(0)}e^{-\sqrt{n}+n(\hat{\eta}t+i(\hat{\omega},\mathbf{x}'))}, \quad F^{(0)} = \text{const} \neq 0,$$

with the eigenvalues of the matrices Q_m, Q_r in the left semi-plane and $\mathrm{Re}B_{r\infty} > 0$ $(B_{r\infty} = \hat{\nu}_\infty + O(\frac{1}{n}))$. For the previous solutions it is not possible to determine a uniform bound for $t > 0$ and therefore we have proved *the instability of slow shock waves.*

7. CONCLUSIONS

The present analysis reveals substantial differences between fast and slow shocks in radiation hydrodynamics. However, the obtained results have a rather clear physical meaning.

Even if the shocks are treated as discontinuities, they are really represented by thin region where the field variables have strong gradients. The thickness of those regions is of the order of the mean free path of the fluid (a mixture in our case). For very high temperatures (e.g. ultrarelativistic limit) the sound speed of matter tends to radiative one. As a conseguence, the characteristic velocities of matter and radiation are of the same order.

For slow gas at relatively low temperature matter and radiation behave quite differently. For example if the temperature is $10^4 - 10^5$ K the mean free path of photons is of the order of $10^{-2} - 10^{-1}$ cm, while that of gas particles is of the order of 10^{-5} cm. Since the viscous exchange zones are determined by the longer mean free path, one can consider pratically the gas shock embedded in the radiative one. In fact in newtonian problems, in the analysis of shock strucure for radiating gas continuity is required for the radiative variables and jumps are allowed only for the matter fields [31].

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APPENDIX

PROOF OF THE EXISTENCE OF THE ROOT OF THE LOPATINSKY DETERMINANT

Here we prove that there exist $\eta_0 > 0$, ξ_0, $\omega_0 = (\omega_2^0, \omega_3^0)$ $(\eta_0^2 + \xi_0^2 + |\omega_0|^2 = 1$, $\omega_{2.3}^0 \geq 0)$ such that $q(\eta, \xi, \omega) = \det L(\hat{s}^{(0)}, \hat{\omega}^{(0)}) = 0$. Let

$$\eta \neq \hat{u}^1 |\omega|, \quad \eta \neq \hat{u}_\infty^1 |\omega|, \quad \xi = 0, \quad \omega_3 = 0.$$

Then we have

$$q(\eta, 0, \omega_2, 0) = \hat{\Gamma}\eta q_1(\eta, \omega_2),$$

$$q_1(\eta, \omega_2) = \det\begin{pmatrix} \sigma_\infty - \hat{v}^1_\infty\eta & \hat{v}^1\eta + \sigma & (2\hat{\Gamma}^2 - 1)\omega_2\hat{\Gamma} \\ \hat{v}^1_\infty\sigma_\infty - \eta & \eta + \hat{v}^1\sigma & 2\hat{u}^1\omega_2 \\ -\hat{v}^1_\infty\omega_2 & \hat{v}^1\omega_2 & \hat{\Gamma}\eta \end{pmatrix},$$

$$\sigma = \sqrt{3\eta^2 + \omega_2^2 D^2\hat{\Gamma}^2}, \quad \sigma_\infty = \sqrt{3\eta^2 + \omega_2^2 D^2_\infty\hat{\Gamma}^2_\infty},$$

where $q_1(\eta, \omega_2)$ is a real-valued function.

Let us show that the equality

$$q_1(\eta, \omega_2) = 0$$

has roots such that

$$\eta > 0, \quad \omega_2 \geq 0, \quad \eta^2 + \omega_2^2 = 1.$$

On one hand we have

$$q_1|_{\omega_2=0} = q_1(1, 0) = \frac{2\{\hat{v}^1(2 - \sqrt{3}\hat{v}^1_\infty) + \sqrt{3} - 2\hat{v}^1_\infty\}}{\hat{\Gamma}^3\hat{\Gamma}^2_\infty} > 0$$

(see (39)-(40)). On the other hand

$$q_1|_{\omega_2=1} = q_1(0, 1) = -\frac{\hat{v}^1}{\hat{\Gamma}^3\hat{\Gamma}^2_\infty}\mathcal{B},$$

where

$$\mathcal{B} = D\hat{\Gamma}\hat{v}^1_\infty(3 - 2\hat{\Gamma}^2) + D_\infty\hat{\Gamma}_\infty(2\hat{v}^1 - (2\hat{\Gamma}^2 - 1)\hat{v}^1_\infty).$$

It is straightforward to verify

$$D\hat{\Gamma} > D_\infty\hat{\Gamma}_\infty.$$

Then

$$\mathcal{B} > 2D_\infty\hat{\Gamma}_\infty\hat{v}^1(1 - 2\hat{v}^1\hat{v}^1_\infty\hat{\Gamma}^2),$$

$$1 - 2\hat{v}^1\hat{v}^1_\infty\hat{\Gamma}^2 = (1 - (\hat{v}^1)^2 - 2\hat{v}^1\hat{v}^1_\infty)\hat{\Gamma}^2$$
$$> \hat{\Gamma}^2(1 - 3(\hat{v}^1_\infty)^2) > 0 \quad (\hat{v}^1_\infty < 1/\sqrt{3}\ !).$$

Thus $\mathcal{B} > 0$ and we have:

$$q_1|_{\omega_2=0} > 0, \quad q_1|_{\omega_2=1} < 0.$$

This implies that there exist $0 < \eta_0 < 1$, $\omega_0 = (\omega_2^0, 0)$, $0 < \omega_2^0 < 1$, $\xi_0 = 0$ such that

$$q(\eta_0, 0, \omega_2^0, 0) = 0.$$

It remains to show that

$$\eta_0 \neq \hat{u}^1 \omega_2^0, \quad \eta_0 \neq \hat{u}_\infty^1 \omega_2^0. \tag{A1}$$

It is easy to see that

$$q_1(\hat{u}^1 \omega_2^0, \omega_2^0) > 0, \quad q_1(\hat{u}_\infty^1 \omega_2^0, \omega_2^0) > 0.$$

Therefore conditions (A1) hold. This completes the proof.

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