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Dynamic pressure in relativistic thermodynamics

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by

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ABSTRACT. – While in classical monatomic ideal gases the dynamic pressure–*i.e.* the non-equilibrium contribution to the pressure–is equal to zero, this is not true in relativistic thermodynamics. Indeed, thermodynamics and the kinetic theory of relativistic gases indicate that there is a non-vanishing bulk viscosity (e.g. see [1], [2], [3]). However the bulk viscosity is small, of $\mathcal{O}(\frac{1}{c^4})$ in a non-degenerate gas. We confirm this and proceed to show that there is also an $\mathcal{O}(\frac{1}{c^2})$ contribution to the dynamic pressure due to a non-homogeneous temperature field or, equivalently, to heating. In the non-relativistic limit that new contribution is much bigger than the one due to bulk viscosity.

Key words: Bulk viscosity, dynamic pressure, relativistic thermodynamics.

RÉSUMÉ. – Alors que dans un gaz monoatomique la pression dynamique, c'est-à-dire la partie de la pression due au non équilibre, est nulle, elle ne l'est plus en thermodynamique relativiste. Aussi bien la thermodynamique que la théorie cinétique des gaz relativistes indiquent qu'il existe une viscosité de volume non nulle (voir par exemple [1], [2], [3]). Cependant cette viscosité est petite, de l'ordre de $\frac{1}{c^4}$ dans un gaz non dégénéré. Nous en donnons la confirmation et nous montrons qu'il se trouve aussi une contribution d'ordre $\frac{1}{c^2}$ à la pression dynamique, contribution due

à un champ de température non homogène ou, ce qui est équivalent, au chauffage. À la limite non relativiste cette nouvelle contribution est beaucoup plus importante que celle due a la viscosité de volume.

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1. PREVIEW AND DISCUSSION

It has been said,–e.g. see [2], [4]–that the dynamic pressure and the associated bulk viscosity play an important role in the thermodynamics of the early universe, even though the bulk viscosity is relativistically small, of order $\frac{1}{c^4}$. Indeed, the kinetic theory of gases and extended thermodynamics provide the result–for $mc^2 >> kT$–

$$\pi = \frac{1}{B_1^\pi} \frac{5}{6} nkT \left(\frac{kT}{mc^2}\right)^2 U^A_{,A}, \tag{1.1}$$

as first approximations either of the Chapman-Enskog procedure or of the Maxwell iteration. $U^A_{,A}$ is the divergence of the velocity field.

We show in this paper that the second approximation provides the result

$$\pi = \frac{1}{B_1^\pi} \frac{5}{6} nkT \left(\frac{kT}{mc^2}\right)^2 U^A_{,A} - \frac{5}{2} \frac{1}{(B_1^\pi)^2} \left(\frac{kT}{mc^2}\right)^2 \frac{dU^A_{,A}}{d\tau}$$
$$- \frac{2}{3} \frac{1}{B_1^\pi} \left(\frac{kT}{mc^2}\right) \left(1 - \frac{1}{2}\frac{kT}{mc^2}\right) q^A_{,A}. \tag{1.2}$$

$q^A_{,A}$ is the divergence of the heat flux.

Thus the second line–although part of the second approximation–is the leading term of relativistic order; it is $\mathcal{O}(\frac{1}{c^2})$, while the other terms are $\mathcal{O}(\frac{1}{c^4})$. B_1^π depends on the atomic interaction; it has been calculated for some interaction potentials (e.g. see [2]). In any case it does not change the relativistic order.

We conclude from (1.2) that the dynamic pressure π is due to expansion $U^A_{,A}$ and heating $q^A_{,A}$ in $\mathcal{O}(\frac{1}{c^4})$ and $\mathcal{O}(\frac{1}{c^2})$ respectively. Thus the heating provides the leading contribution to π. The strategists of the big bang might wish to know this.

2. A REMINDER OF RELATIVISTIC
EXTENDED THERMODYNAMICS

The principal objective of relativistic extended thermodynamics is the determination of the fields ([1])

$$\begin{cases} \text{particle flux vector } A^A, \\ \text{energy-momentum tensor } A^{AB}, \quad (A^{AB} = A^{BA}) \end{cases} \quad (2.1)$$

The necessary field equations are based upon the conservation laws of particle number and energy-momentum, viz.

$$A^A_{,A} = 0, \tag{2.2$_1$}$$

$$A^{AB}_{,B} = 0, \tag{2.2$_2$}$$

and on the flux balance

$$A^{ABC}_{,C} = I^{AB}, \qquad (I^A_A = 0, A^{AB}_B = c^2 A^A). \tag{2.2$_3$}$$

In order to close the system we need constitutive equations which in extended thermodynamics have the forms

$$A^{ABC} = \hat{A}^{ABC}(A^M, A^{MN}),$$

$$I^{AB} = \hat{I}^{AB}(A^M, A^{MN}), \tag{2.3}$$

where $\hat{A}$ and $\hat{I}$ are constitutive functions depending on the nature of the gas.

If the constitutive functions are known, the set (2.2) represents a set of 14 field equations for the 14 fields A^A, A^{AB}. Every solution of this set is a *thermodynamic process*.

The constitutive functions $\hat{A}$ and $\hat{I}$ are restricted by:

• the entropy principle,

• the principle of relativity, and

• the principle of causality and convexity.

For a systematic exploitation of these principles we refer the reader to [5], or [3], Chapter 4. Here we only cite the results of those memoirs, which both refer to a *linear* theory; non-linear combinations of quantities that vanish in equilibrium are ignored.

([1]) The indices $A, B, C, \ldots$ denote flat space-time indices and the metric tensor g^{AB} has signature $(1, -1, -1, -1)$.

In order to specify linearity we give up the elegant synthetic notation used in (2.1) through (2.3) and write A^A and A^{AB} as

$$A^A = n(\alpha, T)mU^A,$$

$$A^{AB} = t^{\langle AB \rangle} + (p(\alpha, T) + \pi)h^{AB} + \frac{1}{c^2}(U^A q^B + U^B q^A) + \frac{e(\alpha, T)}{c^2}U^A U^B. \quad (2.4)$$

U^A is the 4-velocity and h^{AB} is defined by $\frac{1}{c^2}U^A U^B - g^{AB}$. This is a common decomposition in relativistic thermodynamics with

$$\begin{cases} n - \text{particle density,} \\ t^{\langle AB \rangle} - \text{stress deviator,} \\ p - \text{equilibrium pressure,} \\ \pi - \text{dynamic pressure,} \\ q^A - \text{heat flux,} \\ e - \text{energy density.} \end{cases}$$

T is the absolute temperature and α is the chemical potential to within a factor $-\frac{1}{T}$ (²). The functions $n(\alpha, T)$, $p(\alpha, T)$ and $e(\alpha, T)$ represent the thermal and caloric equations of state of a gas, of which only $p(\alpha, T)$ is independent, because of the thermodynamic relations

$$nmT = -\left(\frac{\partial p}{\partial \alpha}\right)_T, \quad \text{and} \quad \frac{e}{T^2} = \left(\frac{\partial}{\partial T}\left(\frac{p}{T}\right)\right)_\alpha. \quad (2.5)$$

The state function $p(\alpha, T)$ for relativistic gases is given by the equation (e.g. see [3], p. 87)

$$p = e^{-\frac{m}{k}\alpha} 4\pi y m^2 c(kT)^2 K_2\left(\frac{mc^2}{kT}\right)$$

$$\xrightarrow{mc^2 \gg kT} y m^3 kT \left(2\pi \frac{k}{m}T\right)^{\frac{3}{2}} e^{-\frac{m}{k}\alpha - \frac{mc^2}{kT}}\left[1 + \frac{15}{8}\frac{kT}{mc^2} + \cdots\right]. \quad (2.6)$$

m is the rest mass of an atom, k the Boltzmann constant and y is related to Planck constant.

The equations (2.4) imply a change of variables

$$\text{from} \quad A^A, A^{AB} \quad \text{to} \quad \alpha, T, U^A, t^{\langle AB \rangle}, \pi, q^A.$$

Both sets contains 14 independent variables. The latter set is introduced because the last three of its entries, viz. $t^{\langle AB \rangle}$, π, and q^A vanish in equilibrium; thus non-equilibrium terms are easy to recognize.

(²) α was called fugacity in [3] and [5].

In terms of the new variables the flux tensor A^{ABC} and the flux production I^{AB} are given by the linear relations

$$A^{ABC} = (C_1^0 + C_1^\pi \pi)U^A U^B U^C$$
$$+ \frac{c^2}{6}(nm - C_1^0 - C_1^\pi \pi)(g^{AB}U^C + g^{BC}U^A + g^{CA}U^B)$$
$$+ C_3(g^{AB}q^C + g^{BC}q^A + g^{CA}q^B)$$
$$- \frac{6}{c^2}C_3(U^A U^B q^C + U^B U^C q^A + U^C U^A q^B)$$
$$+ C_5(t^{\langle AB\rangle}U^C + t^{\langle BC\rangle}U^A + t^{\langle CA\rangle}U^B), \tag{2.7}$$

$$I^{AB} = B_1^\pi \pi g^{AB} - \frac{4}{c^2}B_1^\pi \pi U^A U^B + B_3 t^{\langle AB\rangle} + \frac{1}{c^2}\hat{B}_4(q^A U^B + q^B U^A). \tag{2.8}$$

The coefficients C in (2.7) are fully determined in terms of $p(\alpha, T)$ by the equations

$$C_1^0 = \frac{1}{2c^2}\frac{\Gamma_1'}{T},$$

$$C_1^\pi = -\frac{3}{c^2}\frac{1}{T}\frac{\det\begin{bmatrix} -\ddot{p} & \dot{p} - \dot{p}' & \dot{\Gamma}_1 \\ \dot{p} - \dot{p}' & p' - p'' & \Gamma_1' - \Gamma_1 \\ \dot{\Gamma}_1 & \Gamma_1' - \Gamma_1 & \frac{5}{3}\Gamma_2 \end{bmatrix}}{\det\begin{bmatrix} -\ddot{p} & \dot{p} - \dot{p}' & \dot{\Gamma}_1 \\ \dot{p} - \dot{p}' & p' - p'' & \Gamma_1' - \Gamma_1 \\ -\dot{p} & -p' & \frac{5}{3}\Gamma_1 \end{bmatrix}}, \tag{2.9}$$

$$C_3 = -\frac{1}{2}\frac{1}{T}\frac{\det\begin{bmatrix} \dot{p} & -\dot{\Gamma}_1 \\ \Gamma_1 & \Gamma_2 \end{bmatrix}}{\det\begin{bmatrix} \dot{p} & -\dot{\Gamma}_1 \\ p' & \Gamma_1 - \Gamma_1' \end{bmatrix}}, \qquad C_5 = -\frac{1}{T}\frac{1}{2}\frac{\Gamma_2}{\Gamma_1},$$

where $\Gamma_1(\alpha, T)$, $\Gamma_2(\alpha, T)$ are given in terms of $p(\alpha, T)$ by [3]

$$\Gamma_1 = T^6\left(-2c^2\int\frac{\dot{p}}{T^7}dT\right), \quad \Gamma_2 = T^8\left[2c^2\int\frac{1}{T^3}\left(-2c^2\int\frac{\ddot{p}}{T^7}dT\right)dT\right]. \tag{2.10}$$

The simbols $\bullet$ and $'$ represent derivatives with respect to α and $\ln T$ respectively.

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[3] Constants of integration have been set equal to zero, since nobody this far has found them to have physical significance.

The coefficients B in (2.8) are the only unknown coefficients of relativistic extended thermodynamics. They will presently be related to the transport coefficients of ordinary thermodynamics, *i.e.* shear viscosity, bulk viscosity and thermal conductivity.

This is the full picture as far as extended thermodynamics goes: Given the transport coefficients we have 14 explicit field equations from which the 14 fields

$$\alpha, T, U^A, t^{\langle AB\rangle}, q^A, \pi$$

may be determined as solutions of initial and boundary value problems.

However, this is not the way in which thermodynamicists are used to think. Thermodynamicists like to think of stress, heat flux and dynamic pressure as constitutive quantities for which they write "phenomenological equations", typically the equations of Fourier and Navier-Stokes for the heat flux and the stress respectively. Such phenomenological relations are contained in the equations of extended thermodynamics in an approximate way. We proceed to describe that.

3. THE PHENOMENOLOGICAL RELATIONS OF ORDINARY RELATIVISTIC THERMODYNAMICS

The transition from extended to ordinary relativistic thermodynamics proceeds by an iteration scheme that is akin to the Maxwell iteration of the kinetic theory of gases. We calculate the first iterate $I^{AB^{(1)}}$ from the set of equtions (2.2) by inserting equilibrium values on their left hand sides: [4]

$$A^A_{,A} = 0, \qquad A^{AB}|_{E,B} = 0, \qquad A^{ABC}|_{E,C} = I^{AB^{(1)}}. \qquad (3.1)$$

Thus follows $t^{\langle AB\rangle^{(1)}}$, $\pi^{(1)}$, and $q^{A^{(1)}}$ (cf. (2.8)). The second iterate $I^{AB^{(2)}}$ results from insertion of the first iterates on the left hand sides of (2.2). Thus

$$A^A_{,A} = 0, \qquad A^{AB^{(1)}}_{,B} = 0, \qquad A^{ABC^{(1)}}_{,C} = I^{AB^{(2)}}. \qquad (3.2)$$

This scheme might be repeated, but we break it off here. Infact we are following the scheme to the second order only for the dynamic pressure which is the focal point of this paper.

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[4] A^A does not need the index E–for equilibrium–, since $A^A = n(\alpha, T)mU^A$ consists only of variables. Similary in (3.2) the index (1) need not appear on A^A.

With A^A, A^{AB}, A^{ABC} and I^{AB} given by (2.4), (2.7) and (2.8) the equations (3.1) read

$$0 = -\dot{p}\frac{d\alpha}{d\tau} - (\dot{p}' - \dot{p})\frac{d\ln T}{d\tau} - \dot{p}U^A_{,A},$$

$$0 = (\dot{p}' - \dot{p})\frac{d\alpha}{d\tau} + (p'' - p')\frac{d\ln T}{d\tau} + p'U^A_{,A},$$

$$0 = \dot{p}h^A_M\alpha_{,A} + p'h^A_M(\ln T)_{,A} - \frac{p'}{c^2}\frac{dU_M}{d\tau}, \qquad (3.3)$$

$$-3B^\pi_1\overset{(1)}{\pi} = \frac{\dot{\Gamma}'_1 - 3\dot{\Gamma}_1}{2T}\frac{d\alpha}{d\tau} + \frac{\Gamma''_1 - 4\Gamma'_1 + 3\Gamma_1}{2T}\frac{d\ln T}{d\tau} + \frac{\Gamma'_1 - \Gamma_1}{2T}U^A_{,A},$$

$$-\frac{1}{c^2}\hat{B}_4\overset{(1)}{q_M} = -\frac{\dot{\Gamma}_1}{2T}h^A_M\alpha_{,A} - \frac{\Gamma'_1 - \Gamma_1}{2T}h^A_M(\ln T)_{,A} - \frac{1}{c^2}\frac{\Gamma'_1 - \Gamma_1}{2T}\frac{dU_M}{d\tau},$$

$$B_3\overset{(1)}{t_{\langle MN\rangle}} = -\frac{\Gamma_1}{T}h^B_M h^C_N U_{\langle B,C\rangle}.$$

The first three of these equations are used to eliminate $\frac{d\alpha}{d\tau} = U^A\alpha_{,A}$, $\frac{d\ln T}{d\tau} = U^A(\ln T)_{,A}$ and $h^A_M\alpha_{,A}$ from the last three equations. Thus we obtain

$$\overset{(1)}{\pi} = -\frac{1}{2T}\frac{1}{B^\pi_1}\frac{\det\begin{bmatrix} -\ddot{p} & \dot{p} - \dot{p}' & \dot{\Gamma}_1 \\ \dot{p} - \dot{p}' & p' - p'' & \Gamma'_1 - \Gamma_1 \\ -\dot{p} & -p' & \frac{5}{3}\Gamma_1 \end{bmatrix}}{\det\begin{bmatrix} -\ddot{p} & \dot{p} - \dot{p}' \\ \dot{p} - \dot{p}' & p' - p'' \end{bmatrix}}[U^A_{,A}],$$

$$\overset{(1)}{q_M} = -\frac{1}{2T}\frac{c^2}{\hat{B}_4}\frac{\det\begin{bmatrix} \dot{p} & -\dot{\Gamma}_1 \\ p' & \Gamma_1 - \Gamma'_1 \end{bmatrix}}{\dot{p}}\left[h^A_M\left((\ln T)_{,A} - \frac{1}{c^2}\frac{dU_A}{d\tau}\right)\right], \quad (3.4)$$

$$\overset{(1)}{t_{\langle MN\rangle}} = -\frac{1}{T}\frac{1}{B_3}\Gamma_1\left[h^B_M h^C_N U_{\langle B,C\rangle}\right].$$

These equations are the relativistic analogues to the phenomenological equations of Fourier and Navier-Stokes. In the jargon of ordinary irreversible thermodynamics we may call the quantities

$$\pi.q_M.t_{\langle MN\rangle} \quad \text{and} \quad [U^A_{,A}], \left[h^A_M\left((\ln T)_{,A} - \frac{1}{c^2}\frac{dU_A}{d\tau}\right)\right], \left[h^B_M h^C_N U_{\langle B,C\rangle}\right].$$

$$(3.5)$$

thermodynamic fluxes and forces respectively. The factors of proportionality between these forces and fluxes are called transport coefficients. There are

three of them, viz. bulk viscosity η, thermal conductivity κ and shear viscosity μ. We have

$$\eta = \frac{1}{2T}\frac{1}{B_1^\pi}\frac{\det\begin{bmatrix} -\ddot{p} & \dot{p}-\dot{p}' & \dot{\Gamma}_1 \\ \dot{p}-\dot{p}' & p'-p'' & \Gamma_1'-\Gamma_1 \\ -\dot{p} & -p' & \frac{5}{3}\Gamma_1 \end{bmatrix}}{\det\begin{bmatrix} -\ddot{p} & \dot{p}-\dot{p}' \\ \dot{p}-\dot{p}' & p'-p'' \end{bmatrix}} = -\frac{1}{B_1^\pi}\frac{5}{6}\frac{1}{c^4}\frac{nk^3T^3}{m^2}.$$

$$\kappa = \frac{1}{2T^2}\frac{c^2}{\hat{B}_4}\frac{\det\begin{bmatrix} \dot{p} & -\dot{\Gamma}_1 \\ p' & \Gamma_1-\Gamma_1' \end{bmatrix}}{\dot{p}} = -\frac{1}{\hat{B}_4}\frac{5}{2}\frac{nk^2T^2}{m}, \qquad (3.6)$$

$$\mu = -\frac{1}{2T}\frac{1}{B_3}\Gamma_1 = -\frac{1}{B_3}nkT,$$

where the second expressions on the right hand sides result from the equation of state (2.6) in the limit of the non-relativistic gas, *i.e.* for $mc^2 >> kT$.

Thus we confirm that the bulk viscosity is of $\mathcal{O}(\frac{1}{c^4})$. There is no reason to assume that the B's in (2.8) and (3.4), (3.6) are anything else but $\mathcal{O}(1)$ and,– indeed– the kinetic theory of gases confirms that, e.g. see [2]. The equations (3.4), (3.6) can also be derived in the kinetic theory by either the Chapman-Enskog approximation or by the Grad approximation with subsequent application of the Maxwell iteration.

4. SECOND ITERATE FOR THE DYNAMIC PRESSURE

In order to calculate the second iterate of π we need only $(3.2)_1$ and the time-like components of $(3.2)_{2,3}$, viz.

$$0 = A^A{}_{,A},$$
$$0 = A^{AB^{(1)}}{}_{,B}\,U_A, \qquad (4.1)$$
$$-3B_1^\pi c^2\pi^{(2)} = A^{ABC^{(1)}}{}_{,C}U_A U_B.$$

We focus the attention on linear terms and drop all products of derivatives of α, T and U^A. Thus (4.1) reads

$$0 = A^A{}_{,A}.$$
$$0 = (A^{AB^{(1)}}U_A)_{,B} - A^{AB}|_E U_{B,A}, \qquad (4.2)$$
$$-3B_1^\pi c^2\pi^{(2)} = (A^{ABC^{(1)}}U_A U_B)_{,C} - A^{ABC}|_E(U_{A,C}U_B + U_{B,C}U_A).$$

or else, by (2.4), (2.7)

$$0 = \ddot{p}\frac{d\alpha}{d\tau} + (\dot{p}' - \dot{p})\frac{d\ln T}{d\tau} + \dot{p}U^C_{,C},$$

$$-q^{C^{(1)}}_{,C} = (\dot{p}' - \dot{p})\frac{d\alpha}{d\tau} + (p'' - p')\frac{d\ln T}{d\tau} + p'U^A_{,A},$$

$$-2TB^\pi_1\pi^{(2)} - \frac{c^2T}{3}C^\pi_1\frac{d\pi^{(1)}}{d\tau} + \frac{10T}{3}C_3q^{C^{(1)}}_{,C}$$

$$= \dot{\Gamma}_1\frac{d\alpha}{d\tau} + (\Gamma'_1 - \Gamma_1)\frac{d\ln T}{d\tau} + \frac{5}{3}\Gamma_1U^C_{,C}.$$

(4.3)

Hence follows by elimination of $\frac{d\alpha}{d\tau}$ and $\frac{d\ln T}{d\tau}$

$$\pi^{(2)} = -\frac{1}{2T}\frac{1}{B^\pi_1}\frac{\det\begin{bmatrix} -\ddot{p} & \dot{p} - \dot{p}' & \dot{\Gamma}_1 \\ \dot{p} - \dot{p}' & p' - p'' & \Gamma'_1 - \Gamma_1 \\ -\dot{p} & -p' & \frac{5}{3}\Gamma_1 \end{bmatrix}}{\det\begin{bmatrix} -\ddot{p} & \dot{p} - \dot{p}' \\ \dot{p} - \dot{p}' & p' - p'' \end{bmatrix}}[U^C_{,C}] -$$

$$-\left(\frac{\det\begin{bmatrix} \ddot{p} & \dot{p}' - \dot{p} \\ \dot{\Gamma}_1 & \Gamma'_1 - \Gamma_1 \end{bmatrix}}{\det\begin{bmatrix} \ddot{p} & \dot{p}' - \dot{p} \\ \dot{p}' - \dot{p} & p'' - p' \end{bmatrix}} - \frac{5}{3}\frac{\det\begin{bmatrix} \dot{p} & -\dot{\Gamma}_1 \\ \Gamma_1 & \Gamma_2 \end{bmatrix}}{\det\begin{bmatrix} \dot{p} & -\dot{\Gamma}_1 \\ p' & \Gamma_1 - \Gamma'_1 \end{bmatrix}}\right)\times$$

$$\times\frac{c^2}{4T^2B^\pi_1\hat{B}_4}\det\begin{bmatrix} \dot{p} & -\dot{\Gamma}_1 \\ p' & \Gamma_1 - \Gamma'_1 \end{bmatrix}\frac{1}{\dot{p}}h^{AB}\left((\ln T)_{,A} - \frac{1}{c^2}\frac{dU_A}{d\tau}\right)_{,B} -$$

$$-\frac{1}{4T^2}\frac{1}{(B^\pi_1)^2}\frac{\det\begin{bmatrix} -\ddot{p} & \dot{p} - \dot{p}' & \dot{\Gamma}_1 \\ \dot{p} - \dot{p}' & p' - p'' & \Gamma'_1 - \Gamma_1 \\ \dot{\Gamma}_1 & \Gamma'_1 - \Gamma_1 & \frac{5}{3}\Gamma_2 \end{bmatrix}}{\det\begin{bmatrix} -\ddot{p} & \dot{p} - \dot{p}' \\ \dot{p} - \dot{p}' & p' - p'' \end{bmatrix}}\frac{dU^A_{,A}}{d\tau}.$$

(4.4)

The first line reproduces the first iterate while the other lines represent the contributions from the second iteration. We see that the new terms are proportional to second derivatives of T and U^A and their coefficients contain squares of the B's.

In the kinetic theory the B's are proportional to the collision frequency of the atoms and therefore the Maxwell iteration furnishes a formal expansion in terms of powers of the flight time. This fact makes the new terms "small" compared to the first one.

However, the first term is of $\mathcal{O}(\frac{1}{c^4})$, cf. $(3.6)_1$ and the new terms– the second iterates– are only of $\mathcal{O}(\frac{1}{c^2})$, or at least this is true for the terms with the gradient of temperature. We proceed to show this.

The relativistic order, in terms of power of $\frac{1}{c^2}$, of the various terms in (4.4) may be determined by calculating the values of the determinants with the state equation (2.6) for a relativistic gas. So as to avoid complexity we show only the result for the non-relativistic gas $mc^2 \gg kT$. It reads

$$
\pi^{(2)} = \frac{1}{B_1^\pi} \frac{5}{6} \frac{nk^3T^3}{m^2} \frac{1}{c^4} U^A_{,A} - \frac{1}{(B_1^\pi)^2} \frac{5}{2} \frac{nk^3T^3}{m^2} \frac{1}{c^4} \frac{dU^A_{,A}}{d\tau}
$$
$$
- \frac{1}{B_1^\pi} \frac{1}{\hat{B}_4} \frac{5}{3} \frac{nk^3T^3}{m^2} \frac{1}{c^2} \left(1 + \frac{kT}{mc^2}\right) h^{AB}
$$
$$
\times \left((\ln T)_{,A} - \frac{1}{c^2} \frac{dU_A}{d\tau} \right)_{,B}. \tag{4.5}
$$

Therefore the leading term in the power series of $\frac{1}{c^2}$ is given by– in the rest frame–

$$
\pi^{(2)} = - \frac{1}{B_1^\pi} \frac{1}{\hat{B}_4} \frac{5}{3} \frac{nk^3T^3}{m^2} \frac{1}{c^2} \Delta T = - \frac{1}{B_1^\pi} \frac{2}{3} \left(\frac{kT}{mc^2} \right) q^{C^{(1)}}_{,C}, \tag{4.6}
$$

it is proportional to the Laplacian of temperature or to the heating $q^{C^{(1)}}_{,C}$. And while this dynamic pressure is still a relativistic effect it is not as small as it is considered to be in the literature. Indeed, it is $\mathcal{O}(\frac{1}{c^2})$ rather than $\mathcal{O}(\frac{1}{c^4})$!

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