ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 67, nº 1 (1997), p. 41-57

<http://www.numdam.org/item?id=AIHPA_1997__67_1_41_0>

Ann. Inst. Henri Poincaré,

Vol. 67, n° 1, 1997, p. 41-57. *Physique théorique*

Existence of Rayleigh resonances exponentially close to the real axis

by

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ABSTRACT. – The resonances associated to the Neumann problem in linear elasticity outside a compact obstacle with analytic boundary are studied. When the space dimension is odd we prove that there exits an infinite sequence of resonances tending to the real axis exponentially fast thus extending the result of [7] in the C^∞ case. Moreover, we get a large region free of resonances under the same assumptions as in [4].

RÉSUMÉ. – Nous étudions les résonances associées au problème de Neumann en élasticité linéaire dans l'extérieur d'un obstacle compact à bord analytique. Quand la dimension d'espace est impaire nous montrons qu'il existe une suite infinie de résonances convergeant exponentiellement vite vers l'axe réel, ce qui étend le résultat de [7] au cas C^∞. De plus, sous les mêmes hypothèses que celles de [4], nous montrons l'existence d'une grande région sans résonance.

1. INTRODUCTION AND STATEMENT OF RESULTS

The purpose of this work is to extend the results in [6], [7], concerning the distribution of the resonances associated to the Neumann problem in linear elasticity in the exterior of a bounded obstacle with C^∞- smooth boundary, to the case of analytic boundary. Let $\mathcal{O} \subset \mathbf{R}^n, n \geq 2$, be a compact set with C^∞-smooth boundary Γ and connected complement $\Omega = \mathbf{R}^n \setminus \mathcal{O}$. Denote by Δ_e the elasticity operator

$$\Delta_e v = \mu_0 \Delta v + (\lambda_0 + \mu_0)\nabla(\nabla \cdot v),$$

$v = {}^t(v_1, ..., v_n)$, where the Lamé constants λ_0 and μ_0 satisfy

$$\mu_0 > 0, \quad n\lambda_0 + 2\mu_0 > 0.$$

Consider Δ_e in Ω with Neumann boundary conditions on Γ

$$(Bv)_i := \sum_{j=1}^{n} \sigma_{ij}(v)\nu_j|_\Gamma = 0, \qquad i = 1, ..., n,$$

where $\sigma_{ij}(v) = \lambda_0 \nabla \cdot v\delta_{ij} + \mu_0(\partial_{x_j} v_i + \partial_{x_i} v_j)$ is the stress tensor, ν is the outer normal to Γ. Denote by Δ_e^N the self-adjoint realization of Δ_e in Ω with Neumann boundary conditions on Γ. Recall that the resonances associated to Δ_e^N are the poles of the meromorphic continuation of the cut-off resolvent $R_\chi(\lambda) = \chi(\Delta_e^N + \lambda^2)^{-1}\chi$ from $\operatorname{Im}\lambda < 0$ to the whole complex plane $\mathbf{C}, \chi \in C_0^\infty(\mathbf{R}^n)$ being a cut-off function equal to 1 near Γ. One defines the resonances associated to the Dirichlet realization, Δ_e^D, of Δ_e similarly.

Introduce the Dirichlet-to-Neumann map, $\mathcal{N}(\lambda)$, defined as follows:

$$\mathcal{N}(\lambda) : H^s(\Gamma) \ni f \mapsto Bv \in H^{s-1}(\Gamma),$$

where v solves the problem

$$\begin{cases} (\Delta_e + \lambda^2)v = 0 & \text{in } \Omega, \\ v = f & \text{on } \Gamma, \\ v - \lambda - \text{outgoing}. \end{cases} \tag{1.1}$$

Recall that the function v is said to be λ-outgoing if for some $\rho_0 \gg 1$ we have

$$v|_{|x|\geq\rho_0} = R_0(\lambda)g|_{|x|\geq\rho_0},$$

where $g \in L^2_{comp}(\Omega)$, with a support independent of λ, and $R_0(\lambda)$ is the free outgoing resolvent of Δ_e in $\mathbf{R}^n$. Here "outgoing" means that $R_0(\lambda) \in \mathcal{L}(L^2(\Omega), L^2(\Omega))$ for $\operatorname{Im}\lambda < 0$. Since $\mathcal{N}(\lambda)$ (resp. $\mathcal{N}(\lambda)^{-1}$) can be expressed in terms of the meromorphic continuation of the cutoff resolvent of Δ_e^D (resp. Δ_e^N) (e.g. see [5]), $\mathcal{N}(\lambda)$ (resp. $\mathcal{N}(\lambda)^{-1}$) is a meromorphic family with poles among the Dirichlet (resp. Neumann) resonances. Let $c_1 = \sqrt{\mu_0}, c_2 = \sqrt{\lambda_0 + 2\mu_0}$ be the two speeds of propagation of the elastic waves in Ω. Recall that in the elliptic region $\mathcal{E} = \{\zeta \in T^*\Gamma;\ c_1\|\zeta\| > 1\}$, $\mathcal{N}(\lambda)$ is a $\lambda - \Psi$DO with a characteristic variety $\Sigma = \{\zeta \in T^*\Gamma;\ c_R\|\zeta\| = 1\} \subset \mathcal{E}$ which is interpreted as existence of surface waves (called Rayleigh waves) on Γ moving with a speed $c_R, 0 < c_R < c_1$. When $n \geq 3$ is odd, it was proved in [7] that these waves generate an infinite sequence $\{\lambda_j\}$ of resonances of Δ_e^N with $\operatorname{Im}\lambda_j = O(|\lambda_j|^{-\infty})$. Moreover, it was shown in [6] that if the obstacle is strictly convex, there is a region of the form $C_N|\lambda|^{-N} < \operatorname{Im}\lambda < A\log|\lambda| - B_A, \forall A, N \gg 1$, with some $C_N, B_A > 0$, free of resonances. The same type of region free of resonances was obtained in [2] for obstacles which are nontrapping for Δ_e^D. A larger region free of resonances of the form $Ce^{-\gamma|\lambda|} < \operatorname{Im}\lambda < C_1|\lambda|^{1/3} - C_2$, with some positive constants C, C_1, C_2 and γ, was obtained in [5] in the case when $\mathcal{O}$ is a ball. In [4], an asymptotic with a first term of the counting function of the resonances generated by the Rayleigh waves is obtained when $n \neq 4$ and under the following two assumptions fulfiled for the class of obstacles studied in [2] and [6] (for example, strictly convex ones).

(H.1) There exist some constants $C_0 \geq 1, \delta_0, k_0 > 0$ such that there are no Dirichlet resonances in $\Lambda = \{\lambda \in \mathbf{C} : |\operatorname{Im}\lambda| \leq |\lambda|^{-\delta_0}, \operatorname{Re}\lambda \geq C_0\}$, and

$$\|\mathcal{N}(\lambda)\|_{\mathcal{L}(H^{3/2}(\Gamma), H^{1/2}(\Gamma))} \leq |\lambda|^{k_0}, \quad \lambda \in \Lambda. \tag{1.2}$$

Note that when n is odd, it suffices only to require that there are no Dirichlet resonances in some polynomial neighbourhood of the real axis, as this implies, in view of Proposition 1 in [7], that (1.2) holds in a smaller polynomial neighbourhood of the real axis.

(H.2) Let $\chi \in C_0^\infty(T^*\Gamma)$ be supported in $\mathcal{E}$, $\chi = 1$ in a neighbourhood of Σ. There exist constants $C, k_1 > 0$ such that

$$\|f\|_{H^{3/2}(\Gamma)} \leq C|\lambda|^{k_1}\left(\|\mathcal{N}(\lambda)f\|_{H^{1/2}(\Gamma)} + \|\operatorname{Op}_\lambda(\chi)f\|_{H^{3/2}(\Gamma)}\right), \tag{1.3}$$

$\forall f \in H^{3/2}(\Gamma), \forall \lambda \in \Lambda$.

It also follows from the analysis in [4] that under the assumptions (H.1) and (H.2) there are no Neumann resonances in $\{\lambda \in \Lambda : \operatorname{Im}\lambda > C_N|\lambda|^{-N}\}, \forall N \gg 1$. One of our goals in the present work is to improve this in the case of analytic boundary. Our first result is the following

THEOREM 1.1. – *If the boundary Γ is analytic, under the assumptions (H.1) and (H.2), there are no resonances associated to Δ_e^N in the region $\{\lambda \in \Lambda : \operatorname{Im}\lambda > Ce^{-\gamma|\lambda|}\}$ for some constants $C, \gamma > 0$.*

It follows from this theorem that the resonances of Δ_e^N in Λ must be exponentially close to the real axis, provided that the conditions of the theorem are fulfiled. In the next theorem we do not assume (H.1) and (H.2).

THEOREM 1.2. – *If n is odd and if at least one of the connected components of the obstacle $\mathcal{O}$ is of analytic boundary, then there exists a positive constant γ so that in the region $\{0 < \operatorname{Im}\lambda < e^{-\gamma|\lambda|}\}$ there are infinitely many resonances of Δ_e^N.*

Remark 1. – It is easy to see from the proof that if the analyticity is replaced by Gevrey class $G^s, s \geq 1$, the same type of results as in the above two theorems hold with $e^{-\gamma|\lambda|}$ replaced by $e^{-\gamma_s|\lambda|^{1/s}}$.

To prove Theorem 1.2 we combine some ideas from the proof in the C^∞ case (see [7]) with some results from [4]. The proof is based on the following analogue of Proposition 1 of [7].

PROPOSITION 1.3. – *Let $n \geq 3$ be odd. If, for some constants $C_1, \gamma > 0$, $\mathcal{N}(\lambda)^{-1}$ is holomorphic in $\{0 < \operatorname{Im}\lambda < e^{-\gamma|\lambda|}, \operatorname{Re}\lambda > C_1\}$, then*

$$\|\mathcal{N}(\lambda)^{-1}\|_{\mathcal{L}(H^{1/2}(\Gamma), H^{3/2}(\Gamma))} \leq C_\varepsilon e^{(\gamma+2\varepsilon)|\lambda|}, \tag{1.4}$$

for $|\operatorname{Im}\lambda| < e^{-(\gamma+\varepsilon)|\lambda|}, \operatorname{Re}\lambda > C_2, \forall\varepsilon > 0$.

The proof of this proposition is similar to the proof of Proposition 1 of [7] and we will give it for the sake of completeness in an appendix. Thus, to prove Theorem 1.2 it suffices to show that (1.4) fails for some choice of $\gamma > 0$.

Using the method developed to prove Theorems 1.1 and 1.2 we will extend the results in [2] on the rate of the local energy decay for the elastic wave equation with Neumann boundary conditions. Let $a > 0$ be such that $\mathcal{O}$ is contained in the ball $B_a = \{x \in \mathbf{R}^n : |x| \leq a\}$ and denote $\Omega_a = \Omega \cap B_a$. For any $m \geq 0$ set

$$p_{m,a}(t) = \sup\left\{ \frac{\|\nabla_x u(t,x)\|_{L^2(\Omega_a)} + \|\partial_t u(t,x)\|_{L^2(\Omega_a)}}{\|\nabla_x f_1\|_{H^m(\Omega_a)} + \|f_2\|_{H^m(\Omega_a)}}, \right.$$
$$\left. (0,0) \neq (f_1, f_2) \in C_0^\infty(\bar{\Omega}_a) \times C_0^\infty(\bar{\Omega}_a) \right\},$$

where $u(t,x)$ solves the equation

$$\begin{cases} (\partial_t^2 - \Delta_e)u = 0 & \text{in } \mathbf{R} \times \Omega, \\ Bu = 0 & \text{on } \mathbf{R} \times \Gamma, \\ u(0,x) = f_1, \quad \partial_t u(0,x) = f_2. & \end{cases} \tag{1.5}$$

Note that $p_{m,a}(t)$ measures the uniform behaviour as $t \to \infty$ of the local energy of the solutions of (1.5). When $m > 0$ it follows from [10] that $p_{m,a}(t) \to 0$ as $t \to +\infty$. On the other hand, when n is odd, it is proved in [1] that $p_{0,a}(t) \geq \alpha > 0$ for every obstacle with C^∞ boundary. We have the following

THEOREM 1.4. – *If $n \neq 4$ and if at least one of the connected components of the obstacle $\mathcal{O}$ is of analytic boundary, then for every $m \geq 0$,*

$$\limsup_{t \to +\infty} (\ln t)^m p_{m,a}(t) > 0. \tag{1.6}$$

Remark 2. – It is easy to see from the proof that if the analyticity is replaced by Gevrey class $G^s, s \geq 1$, the same result is true with $\ln t$ replaced by $(\ln t)^s$, while in the C^∞ case one should replace $\ln t$ by $t^\delta, \forall \delta > 0$.

In a way similar to that one in [2] we prove the following

PROPOSITION 1.5. – *If Theorem 1.4 is not true, then for any $\beta > 0$ there exist positive C_1 and C_2 depending on β so that $\mathcal{N}(\lambda)^{-1}$ is holomorphic in $\{|\mathrm{Im}\,\lambda| < e^{-\beta|\lambda|}, \mathrm{Re}\,\lambda > C_1\}$ and satisfies there the estimate*

$$\|\mathcal{N}(\lambda)^{-1}\|_{\mathcal{L}(H^{1/2}(\Gamma), H^{3/2}(\Gamma))} \leq C_2 e^{2\beta|\lambda|}. \tag{1.7}$$

Thus, to prove Theorem 1.4 it suffices to get a contradiction to (1.7) for a suitable choice of β. This in turn can be carried out in precisely the same way as in the proof of Theorem 1.2.

2. PROOF OF THEOREM 1.1

We begin by recalling the construction of the parametrix of $\mathcal{N}(\lambda)$ in the elliptic region (see also [6], [7]). Throughout this section λ will vary in the set Λ introduced in (H.1). Fix an integer $m \gg 1$. Choose a function $\chi_m \in C_0^\infty(T^*\Gamma)$, $\mathrm{supp}\,\chi_m \subset \mathcal{E}, \chi_m = 1$ in a neighbourhood of Σ, and

$$|\partial_{x,\xi}^\alpha \chi_m(x,\xi)| \leq (C|\alpha|)^{|\alpha|} \qquad \text{for} \quad |\alpha| \leq m, \tag{2.1}$$

with a constant $C > 0$ independent of α and m. As in [6], [7] one can construct an operator $H_m(\lambda) : C^\infty(\Gamma) \to C^\infty(\Omega)$ which solves the equation

$$\begin{cases} (\Delta_e + \lambda^2) H_m(\lambda) f = K_m(\lambda) f & \text{in } \tilde{\Omega}, \\ H_m(\lambda) f = \mathrm{Op}_\lambda(\chi_m) f & \text{on } \Gamma\,, \end{cases} \tag{2.2}$$

where $\widetilde{\Omega} \subset \Omega$ is a small neighbourhood near Γ, independent of m, $K_m(\lambda) : H^{3/2}(\Gamma) \to L^2(\widetilde{\Omega})$ is $O_m(|\lambda|^{-m})$. Moreover, when Γ is analytic, it follows from the analysis in [3] that one can controle the term O_m, namely one has

$$\|K_m(\lambda)\|_{\mathcal{L}(H^{3/2}(\Gamma),L^2(\widetilde{\Omega}))} \leq (Cm/|\lambda|)^m, \tag{2.3}$$

with a constant $C > 0$ independent of λ and m. To get the desired exponential bound we will take $m \sim \beta|\lambda|$ with $\beta > 0$ small enough. Recall next that $H_m(\lambda)$ is given as a finite sum of λ-FIO each of which in local coordonates has a kernel of the form

$$\left(\frac{\lambda}{2\pi}\right)^{n-1} \int e^{i(\varphi(x,\xi)-y\cdot\xi)} h_m(x,\xi,\lambda) d\xi,$$

where the local coordonates $x = (x_1, x')$ are taken such that Γ is defined locally by $x_1 = 0$ and the domain $x_1 > 0$ belongs to Ω. The amplitude h_m is of the form

$$h_m(x,\xi,\lambda) = \sum_{j=0}^{m} \lambda^{-j} b_j(x,\xi),$$

with $b_j \in C^\infty(\bar{\Omega} \times \mathbf{R}^{n-1})$. Moreover, when Γ is analytic, b_j satisfy (see [3]):

$$|\partial_{x,\xi}^\alpha b_j(x,\xi)| \leq (C|\alpha| + Cj)^{|\alpha|+j} \quad \text{for} \quad |\alpha| + j \leq m, \tag{2.4}$$

with a constant $C > 0$ independent of j, α and m. The phase φ satisfies $\varphi|_\Gamma = x' \cdot \xi$ and $\operatorname{Im} \varphi \geq cx_1$ with some $c > 0$ independent of m. Let $\phi(x) \in C^\infty(\Omega)$ be supported in $\widetilde{\Omega}$ and $\phi = 1$ in a neighbourhood of Γ. Set

$$\widetilde{H}_m(\lambda) = \phi H_m(\lambda) - R_0(\lambda)(\phi K_m(\lambda) + [\Delta_e, \phi] H_m(\lambda)),$$

where $R_0(\lambda)$ is the free outgoing resolvent. It is easy to see that $u = \widetilde{H}_m(\lambda) f$ solves the equation

$$\begin{cases} (\Delta_e + \lambda^2) u = 0 & \text{in } \Omega\ , \\ u = \operatorname{Op}_\lambda(\chi_m) f + \widetilde{K}_m(\lambda) f & \text{on } \Gamma, \\ u - \lambda - \text{outgoing}, \end{cases} \tag{2.5}$$

where

$$\widetilde{K}_m(\lambda) = \gamma R_0(\lambda)(\phi K_m(\lambda) + [\Delta_e, \phi] H_m(\lambda)),$$

$\gamma g := g|_\Gamma$. Now, since the coefficients of $[\Delta_e, \phi]$ vanish near Γ, it follows from the form of $H_m(\lambda)$, (2.4) and the fact $\operatorname{Im}\varphi \geq \rho > 0$ on $\operatorname{supp}[\Delta_e, \phi]$ that

$$\|[\Delta_e, \phi] H_m(\lambda)\|_{\mathcal{L}(H^{3/2}(\Gamma), L^2(\Omega))} \leq Ce^{-\rho|\lambda|},$$

if $\beta < (Ce)^{-1}$. This together with (2.3) imply

$$\|\widetilde{K}_m(\lambda)\|_{\mathcal{L}(H^{3/2}(\Gamma), H^{3/2}(\Gamma))} \leq (Cm/|\lambda|)^m + Ce^{-\rho|\lambda|}, \tag{2.6}$$

with a constant $C > 0$ independent of λ and m. Set

$$N_m(\lambda) f = BH_m(\lambda) f|_\Gamma.$$

It is easy to see that $N_m(\lambda)$ is a $\lambda - \Psi$DO on Γ with a symbol which has in local coordonates the form

$$n_m(x', \xi, \lambda) = \lambda \sum_{j=0}^{m} \lambda^{-j} a_j(x', \xi),$$

where $a_j \in C_0^\infty(T^*\Gamma)$ are supported in $\operatorname{supp}\chi_m$. Moreover, in view of (2.4), when Γ is analytic, a_j satisfy

$$|\partial_{x',\xi}^\alpha a_j(x', \xi)| \leq (C|\alpha| + Cj)^{|\alpha|+j} \qquad \text{for} \quad |\alpha| + j \leq m, \tag{2.7}$$

with a constant $C > 0$ independent of j, α and m. Denote by $R^D(\lambda) f$ the solution of (1.1). By (2.5) we have

$$\begin{aligned} R^D(\lambda)\mathrm{Op}_\lambda(\chi_m) f + R^D(\lambda)\widetilde{K}_m(\lambda) f \\ = \phi H_m(\lambda) f - R_0(\lambda)(\phi K_m(\lambda) f + [\Delta_e, \phi] H_m(\lambda) f). \end{aligned} \tag{2.8}$$

Hence

$$\mathcal{N}(\lambda)\mathrm{Op}_\lambda(\chi_m) f + \mathcal{N}(\lambda)\widetilde{K}_m(\lambda) f = N_m(\lambda) f + Q_m(\lambda) f, \tag{2.9}$$

where

$$Q_m(\lambda) f = -\gamma B R_0(\lambda)(\phi K_m(\lambda) f + [\Delta_e, \phi] H_m(\lambda) f).$$

As above,

$$\|Q_m(\lambda)\|_{\mathcal{L}(H^{3/2}(\Gamma), H^{1/2}(\Gamma))} \leq (Cm/|\lambda|)^m + Ce^{-\rho|\lambda|},$$

with a new constant $C > 0$ independent of λ and m. We also have, by (2.6) and under (H.1), that the same type of estimate holds for the operator

$\mathcal{N}(\lambda)\widetilde{K}_m(\lambda)$. Choose now a real-valued function $\eta_m \in C_0^\infty(T^*\Gamma)$ supported in a neighbourhood of Σ, satisfying (2.1) and such that $a_0+\eta_m$ is elliptic in a neighbourhood of Σ. It follows from Remark 3.2 in [4] that (H.2) implies that the operator $\widetilde{N}(\lambda) := \lambda^{-1}\mathcal{N}(\lambda) + \mathrm{Op}_\lambda(\eta_m) : H^{3/2}(\Gamma) \to H^{1/2}(\Gamma)$ is invertible for $\lambda \in \Lambda$. Moreover, it is easy to see that the norm of the inverse is upper bounded by a term of the form $O(|\lambda|^{k_1})$ involving only finite number (independent of m) of semi-norms $\sup|\partial_{x,\xi}\eta_m|$, and hence independent of m. We need now the following

LEMMA 2.1. – *Let $\psi_{1,m}, \psi_{2,m} \in C_0^\infty(T^*\Gamma)$ be supported in a neighbourhood of Σ, satisfying* (2.1), *equal to 1 near Σ, and $(1-\psi_{1,m})\psi_{2,m}$ is identically zero. Then we have*

$$\begin{aligned}&\|(I-\mathrm{Op}_\lambda(\psi_{1,m}))\widetilde{N}(\lambda)^{-1}\mathrm{Op}_\lambda(\psi_{2,m})\|_{\mathcal{L}(H^{3/2}(\Gamma),H^{3/2}(\Gamma))}\\&\quad\le (Cm/|\lambda|)^m + Ce^{-\rho|\lambda|},\end{aligned}\tag{2.10}$$

with a constant $C>0$ independent of λ and m.

Proof. – Choose a real-valued function $\varphi_m \in C_0^\infty(T^*\Gamma)$ supported in a neighbourhood of Σ, satisfying (2.1) and such that $|a_0+\eta_m+i(1-\varphi_m)| \ge c$ with some constant $c>0$ independent of m. Set

$$M(\lambda) = \lambda^{-1}N_m(\lambda) + \mathrm{Op}_\lambda(\eta_m) + i(I-\mathrm{Op}_\lambda(\varphi_m)).$$

Clearly, $M(\lambda)$ is elliptic and moreover, by (2.7) and the calculas with analytic symbols (see [3]), we have

$$M(\lambda)^{-1} = \sum_{j=0}^{m} \lambda^{-j}\mathrm{Op}_\lambda(d_j) + S_m(\lambda),\tag{2.11}$$

where $d_j \in C^\infty(T^*\Gamma)$ satisfy

$$|\partial_{x',\xi}^\alpha d_j(x',\xi)| \le (C|\alpha|+Cj)^{|\alpha|+j} \quad \text{for} \quad |\alpha|+j \le m,\tag{2.12}$$

and

$$\|S_m(\lambda)\|_{\mathcal{L}(H^{3/2}(\Gamma),H^{3/2}(\Gamma))} \le (Cm/|\lambda|)^m.\tag{2.13}$$

If $\psi_{1,m}$ and $\psi_{2,m}$ are as above, by (2.12) one deduces

$$\begin{aligned}&\|(I-\mathrm{Op}_\lambda(\psi_{1,m}))\mathrm{Op}_\lambda(d_j)\mathrm{Op}_\lambda(\psi_{2,m})\|_{\mathcal{L}(H^{3/2}(\Gamma),H^{3/2}(\Gamma))}\\&\quad\le (Cm)^m|\lambda|^{-(m-j)}.\end{aligned}\tag{2.14}$$

By (2.11)-(2.14),

$$\begin{aligned}&\|(I-\mathrm{Op}_\lambda(\psi_{1,m}))M(\lambda)^{-1}\mathrm{Op}_\lambda(\psi_{2,m})\|_{\mathcal{L}(H^{3/2}(\Gamma),H^{3/2}(\Gamma))}\\&\quad\le (Cm/|\lambda|)^m.\end{aligned}\tag{2.15}$$

We have

$$\begin{aligned}&(I-\mathrm{Op}_\lambda(\psi_{1,m}))\widetilde{N}(\lambda)^{-1}\mathrm{Op}_\lambda(\psi_{2,m})\\&\quad=(I-\mathrm{Op}_\lambda(\psi_{1,m}))M(\lambda)^{-1}\mathrm{Op}_\lambda(\psi_{2,m})\\&\quad-(I-\mathrm{Op}_\lambda(\psi_{1,m}))\widetilde{N}(\lambda)^{-1}(\widetilde{N}(\lambda)-M(\lambda))\\&\quad\times(I-\mathrm{Op}_\lambda(\widetilde{\chi}_m))M(\lambda)^{-1}\mathrm{Op}_\lambda(\psi_{2,m})\\&\quad-(I-\mathrm{Op}_\lambda(\psi_{1,m}))\widetilde{N}(\lambda)^{-1}(\widetilde{N}(\lambda)-M(\lambda))\\&\quad\times\mathrm{Op}_\lambda(\widetilde{\chi}_m)M(\lambda)^{-1}\mathrm{Op}_\lambda(\psi_{2,m}),\end{aligned}\tag{2.16}$$

where $\widetilde{\chi}_m\in C_0^\infty(T^*\Gamma)$ satisfies (2.1), $\widetilde{\chi}_m=1$ on the supports of $\psi_{2,m}$ and χ_m. It follows from the definitions of $\widetilde{N}$ and M that

$$\begin{aligned}&\|(\widetilde{N}(\lambda)-M(\lambda))\mathrm{Op}_\lambda(\widetilde{\chi}_m)\|_{\mathcal{L}(H^{3/2}(\Gamma),H^{1/2}(\Gamma))}\\&\quad\le(Cm/|\lambda|)^m+Ce^{-\rho|\lambda|}.\end{aligned}\tag{2.17}$$

Now (2.10) follows from (2.15)-(2.17). This completes the proof of Lemma 2.1.

To finish the proof of the theorem we will proceed as in [6]. Let $\lambda\in\Lambda$ be a resonance of Δ_e^N. Then, there exists a $f\in C^\infty(\Gamma)$, $\|f\|_{H^{3/2}(\Gamma)}=1$, such that $\mathcal{N}(\lambda)f=0$. Hence,

$$f=\mathrm{Op}_\lambda(\chi_m)f+(I-\mathrm{Op}_\lambda(\chi_m))\widetilde{N}(\lambda)^{-1}\mathrm{Op}_\lambda(\eta_m)f,\tag{2.18}$$

where $\chi_m\in C_0^\infty(T^*\Gamma)$ satisfies (2.1), $\chi_m=1$ on the support of η_m. Let $v=R^D(\lambda)f$ and let ϕ be as above. As in [6] we have

$$\operatorname{Im}\lambda^2\le\frac{\|[\Delta_e,\phi]v\|_{L^2}}{\|\phi v\|_{L^2}}\le C|\lambda|^2\|[\Delta_e,\phi]v\|_{L^2}.\tag{2.19}$$

Note that, since $R^D(\lambda)$ can be expressed in terms of $\mathcal{N}(\lambda)$, by (H.1), $R^D(\lambda):H^{3/2}(\Gamma)\to L^2_{loc}(\Omega)$ is holomorphic in Λ with norm $O(|\lambda|^{k_2})$. Thus, by (2.8), (2.10) and (2.18) we conclude that

$$\|[\Delta_e,\phi]v\|_{L^2}\le(Cm/|\lambda|)^m+Ce^{-\rho|\lambda|}.\tag{2.20}$$

By (2.19) and (2.20),

$$\operatorname{Im}\lambda^2\le(Cm/|\lambda|)^m+Ce^{-\rho|\lambda|}.\tag{2.21}$$

Recalling that $m\sim\beta|\lambda|$ and taking $\beta<(Ce)^{-1},\gamma=\min\{\beta,\rho\}$, completes the proof of Theorem 1.1.

3. PROOF OF THEOREM 1.2

Let $\mathcal{O}_1$ be a connected component of $\mathcal{O}$ with analytic boundary which will be denoted by Γ. Clearly, one can construct a parametrix of the Dirichlet problem in a neighbourhood of $\mathcal{O}_1$ in the same way as in the previous section. We keep the same notations. The key point of our proof of Theorem 1.2 is the following

LEMMA 3.1. – *For every integer $m \gg 1$, there exist an infinite sequence $\{\widetilde{\lambda}_j\}$ and $f_j \in C^\infty(\Gamma)$, $\|f_j\|_{H^{3/2}(\Gamma)} = 1$, depending on m, such that*

(i) $$\|N_m(\widetilde{\lambda}_j)f_j\|_{H^{1/2}(\Gamma)} \leq (Cm/|\widetilde{\lambda}_j|)^m + Ce^{-\rho|\widetilde{\lambda}_j|};$$

(ii) $$|\mathrm{Im}\,\widetilde{\lambda}_j| \leq (Cm/|\widetilde{\lambda}_j|)^m + Ce^{-\rho|\widetilde{\lambda}_j|};$$

(iii) $$r_j \leq |\widetilde{\lambda}_j| \leq r_j + C',$$

for some sequence of real numbers $r_j \to +\infty$, independent of m, and some constant $C' > 0$ independent of m and j.

Proof. – We are going to take advantage of the results in [4]. Note first that when λ is real, $N_m(\lambda)$ is self-adjoint for every m. It is shown in [4] that, when $n \neq 4$, $\lambda^{-1}N_{2n+1}(\lambda)$ (independent of m) can be extended to a $\lambda - \Psi$DO, $P_{2n+1}(\lambda) \in S^{0,0}_{cl}(\Gamma)$, self-adjoint for real λ, with a principal symbol vanishing only on Σ in $T^*\Gamma$. Moreover, $P_{2n+1}(\lambda) : L^2(\Gamma) \to L^2(\Gamma)$ is invertible for $\mathrm{Im}\,\lambda \neq 0$ (see Lemma 5.1 of [4]). It follows from Propositions 5.2 and 5.3 of [4] that there exist infinitely many closed curves $\{\gamma_j\}$, without self-intersections and symmetric with respect to the real axis, satisfying $\gamma_j \cap \gamma_k = \emptyset$ if $j \neq k$, $\mathrm{diam}\,\gamma_j \leq C'$ with some $C' > 0$ independent of j, so that in the interior of each γ_j there exists at least one pole of $P_{2n+1}(\lambda)^{-1}$ and the distance from any such a pole z to γ_j is $\geq |z|^{-n}$. Moreover,

$$\|P_{2n+1}(\lambda)^{-1}\|_{\mathcal{L}(L^2(\Gamma),L^2(\Gamma))} \leq C|\lambda|^n, \quad \lambda \in \gamma_j, \tag{3.1}$$

with some constant C independent of j and λ. Denote by r_j the distance from γ_j to the origine.

Let now $m \gg 2n+1$. Choose a real-valued function $\eta_m \in C_0^\infty(T^*\Gamma)$ supported in a neighbourhood of Σ, satisfying (2.1) and such that $a_0 + i(1-\eta_m)$ is elliptic outside Σ, where a_0 is the principal symbol of N_m. Set

$$P_m(\lambda) = \lambda^{-1}N_m(\lambda) + i(I - \mathrm{Op}_\lambda(\eta_m)).$$

Clearly, $P_m(\lambda)^{-1}$ forms a meromorphic Fredholm family on $\mathbf{C}$. We also have that if $\chi \in C_0^\infty(T^*\Gamma)$ is equal to 1 in a small neighbourhood of Σ, then

$$\|P_m(\lambda)\mathrm{Op}_\lambda(\chi) - P_{2n+1}(\lambda)\mathrm{Op}_\lambda(\chi)\|_{\mathcal{L}(L^2(\Gamma),L^2(\Gamma))} \leq C|\lambda|^{-2n-1}, \quad (3.2)$$

for $|\mathrm{Im}\,\lambda| \leq C_1$ with a constant $C > 0$ independent of m and λ. Thus $P_m(\lambda)$ has the same properties as the Neumann operator $\mathcal{N}(\lambda)$ does under the assumptions (H.1) and (H.2) as long as $\lambda \in \Lambda$ (and in fact has a simpler structure as it is a $\lambda - \Psi$DO). Hence the results in [4] apply to $P_m(\lambda)$. Now, comparing the poles of $P_{2n+1}(\lambda)^{-1}$ and $P_m(\lambda)^{-1}$, by (3.1), (3.2) and the fact that P_m is elliptic outside Σ, as in Proposition 6.4 of [4], it follows that inside γ_j there is at least one pole, $\widetilde{\lambda}_j$, of $P_m(\lambda)^{-1}$. By the Fredholm alternative, there is a nontrivial $f_j \in L^2(\Gamma)$ so that $P_m(\widetilde{\lambda}_j)f_j = 0$. It is a standard fact now that this equation implies $f_j \in C^\infty(\Gamma)$ and we normalize f_j so that $\|f_j\|_{H^{3/2}(\Gamma)} = 1$. Choose now a real-valued function $\varphi_m \in C_0^\infty(T^*\Gamma)$ satisfying (2.1), $\varphi_m = 1$ in a neighbourhood of Σ, and such that $|a_0+\varphi_m+i(1-\eta_m)| \geq c$ with some constant $c > 0$ independent of m. Then the operator $\widetilde{P}_m(\lambda) = P_m(\lambda) + \mathrm{Op}_\lambda(\varphi_m)$ is elliptic and we have

$$f_j = \mathrm{Op}_{\widetilde{\lambda}_j}(\psi_m)f_j + (I - \mathrm{Op}_{\widetilde{\lambda}_j}(\psi_m))\widetilde{P}_m(\widetilde{\lambda}_j)^{-1}\mathrm{Op}_{\widetilde{\lambda}_j}(\varphi_m)f_j, \quad (3.3)$$

where $\psi_m \in C_0^\infty(T^*\Gamma)$ satisfies (2.1), $\psi_m = 1$ in a neighbourhood of Σ, and $(1 - \psi_m)\varphi_m$ is identically zero. Using an analogue of (2.15), with M replaced by $\widetilde{P}_m$, we get (i) from (3.3). Furthermore, one can treat P_m in the same way as $\mathcal{N}$ in the previous section, using (2.2) instead of (1.1) and (3.3) instead of (2.18), to obtain (ii). Note finally that (iii) follows from the fact that each $\widetilde{\lambda}_j$ is in the interior of γ_j. This completes the proof of Lemma 3.1.

Suppose that there are no resonances in $\{0 < \mathrm{Im}\,\lambda < \widetilde{C}e^{-\gamma|\lambda|}\}$ with some $\widetilde{C}, \gamma > 0$ to be specified later on. Choose $m \sim \beta r_j$ with a parameter $\beta > 0$ to be fixed below. By Lemma 3.1, if $\beta < \min\{(Ce)^{-1}, \rho\}$, we get

$$|\mathrm{Im}\,\widetilde{\lambda}_j| \leq C_1 e^{-\beta|\widetilde{\lambda}_j|},$$

so taking $\gamma < \beta$ we can make all $\widetilde{\lambda}_j$ belong to the region near the real axis where (1.4) holds. Thus, by Proposition 1.3, Lemma 3.1, (2.9) combined with (3.3) we conclude

$$1 \leq C_\varepsilon e^{(\gamma+2\varepsilon)|\widetilde{\lambda}_j|}\Big((Cm/|\widetilde{\lambda}_j|)^m + C_2 e^{-\rho|\widetilde{\lambda}_j|}\Big)$$
$$\leq C'_\varepsilon e^{(\gamma+2\varepsilon)r_j}\Big(e^{-\beta(\ln\frac{1}{C\beta})r_j} + C_3 e^{-\rho r_j}\Big), \quad (3.4)$$

with a new constant $C > 0$. Now it suffices to choose $\beta < (Ce)^{-1}$, $\gamma < \beta$, $\varepsilon > 0$ small enough to get a contradiction in (3.4) as $r_j \to +\infty$. This completes the proof of Theorem 1.2.

4. PROOF OF THEOREM 1.4

We begin by the proof of Proposition 1.5. Denoting $R(\lambda) = (\Delta_e^N + \lambda^2)^{-1} : L^2(\Omega) \to L^2(\Omega)$ for $\operatorname{Im}\lambda < 0$, we have

$$\lambda R(\lambda) f = i \int_0^\infty e^{-it\lambda} \partial_t U(t) f dt, \quad \operatorname{Im}\lambda < 0, \quad f \in L^2(\Omega),$$

where $u(t,x) = U(t)f$ solves (1.5) with $f_1 = 0, f_2 = f$. Assume that (1.6) does not hold. Then for any $b > 0$ there exists $C = C(b) > 2$ so that, for $\operatorname{Im}\lambda < 0$, $|\operatorname{Im}\lambda| \ll 1$, we have

$$\begin{aligned} |\lambda| \|R(\lambda) f\|_{L^2(\Omega_a)} &\le \int_0^\infty e^{-t|\operatorname{Im}\lambda|} \|\partial_t U(t) f\|_{L^2(\Omega_a)} dt \\ &\le \left(\widetilde{C}(b) + b \int_C^\infty e^{-t|\operatorname{Im}\lambda|} (\ln t)^{-m} dt \right) \|f\|_{H^m(\Omega_a)}. \end{aligned} \tag{4.1}$$

Let us now see that

$$I = \int_2^\infty e^{-t/x} (\ln t)^{-m} dt \le C_1 x (\ln x)^{-m}, \quad x \gg 1. \tag{4.2}$$

Take $y = x(\ln x)^{-m}$. Clearly, $y(\ln y)^m \sim x$ as $x \to +\infty$. Hence,

$$\begin{aligned} I &= \int_2^y e^{-t/x} (\ln t)^{-m} dt + \int_y^\infty e^{-t/x} (\ln t)^{-m} dt \\ &\le (\ln 2)^{-1} \int_2^y e^{-t/x} dt + (\ln y)^{-m} \int_y^\infty e^{-t/x} dt \\ &\le (\ln 2)^{-1} y + x (\ln y)^{-m} \le C_1 y. \end{aligned}$$

Thus, by (4.1) and (4.2) we deduce

$$\|R(\lambda)\|_{\mathcal{L}(H^m(\Omega_a), L^2(\Omega_a))} \le C_1 b |\lambda|^{-1} |\operatorname{Im}\lambda|^{-1} \left(\ln \frac{1}{|\operatorname{Im}\lambda|} \right)^{-m} + \widetilde{C}(b) |\lambda|^{-1}. \tag{4.3}$$

Set $\gamma_- = \{-\operatorname{Im}\lambda = e^{-\beta|\lambda|}, \operatorname{Re}\lambda \ge A\}$, where $A > 1, \beta > 0$ will be fixed later on. By (4.3),

$$\|R(\lambda)\|_{\mathcal{L}(H^m(\Omega_a), L^2(\Omega_a))} \le C_1 b \beta^{-m} |\lambda|^{-m-1} e^{\beta|\lambda|}, \quad \lambda \in \gamma_-, \tag{4.4}$$

if $A = A(\beta)$ is large enough. On the other hand, it follows easily from the ellipticity of Δ_e^N and the resolvent identity

$$R(\lambda) = R(\lambda_0) + (\lambda^2 - \lambda_0^2) R(\lambda) R(\lambda_0),$$

with $\lambda_0 = -i$, that

$$\|R(\lambda)\|_{\mathcal{L}(L^2(\Omega_a),L^2(\Omega_a))} \leq C_2|\lambda|^m \|R(\lambda)\|_{\mathcal{L}(H^m(\Omega_a),L^2(\Omega_a))}, \tag{4.5}$$

for $\operatorname{Im}\lambda < 0$ and every $m \geq 0$. By (4.4) and (4.5) we conclude

$$\|R(\lambda)\|_{\mathcal{L}(L^2(\Omega_a),L^2(\Omega_a))} \leq C_3 b\beta^{-m}|\lambda|^{-1}e^{\beta|\lambda|}, \quad \lambda \in \gamma_-. \tag{4.6}$$

Let $\chi \in C_0^\infty(\mathbf{R}^n)$ be supported in B_a, $\chi = 1$ near the boundary. We are going to derive from (4.6) that the cut-off resolvent $R_\chi(z)$ extends holomorphically through the real axis. To this end we will use some results from [9]. Let $\chi_1, \chi_2 \in C_0^\infty(\mathbf{R}^n)$, $\chi_1 = 1$ near Γ, $\chi_2 = 1$ on $\operatorname{supp}\chi_1$, $\chi = 1$ on $\operatorname{supp}\chi_2$. As in [9], we have the following representation

$$R_\chi(z)(1 - K(z)) = K_1(z), \tag{4.7}$$

where

$$K(z) = ([\chi_1, \Delta_e]R_0(z)\eta - [\chi_1, \Delta_e]R_0(\lambda)\eta)\widetilde{K}(\lambda) + (z^2 - \lambda^2)\chi_2 R(\lambda)\chi,$$
$$K_1(z) = R_\chi(\lambda) + (1 - \chi_1)(\chi R_0(z)\eta - \chi R_0(\lambda)\eta)\widetilde{K}(\lambda),$$
$$\widetilde{K}(\lambda) = (1 - \chi_2)\chi + [\chi_2, \Delta_e]R_0(\lambda)\eta\chi + [\chi_2, \Delta_e]R_0(\lambda)[\eta, \Delta_e]R(\lambda)\chi,$$

where $\lambda \in \mathbf{C}, \operatorname{Im}\lambda < 0$, $R_0(z)$ is the outgoing free resolvent, $\eta \in C_0^\infty(\mathbf{R}^n)$, $\eta = 1$ on $\operatorname{supp}(1 - \chi_2)\chi$, $\eta = 0$ on $\operatorname{supp}\chi_1$. Clearly, $K(z)$ and $K_1(z)$ are analytic on $\mathbf{C}$ when n is odd and on the Riemann surface Λ of $\log z$ when n is even. Take now $z \in \mathbf{C}$ with $\operatorname{Re} z \geq A, 0 \leq \operatorname{Im} z < 1/2$, and let $\lambda \in \gamma_-$ be such that $\operatorname{Re}\lambda = \operatorname{Re} z$. In view of (4.6) and of some well known bounds on the free resolvent, we get

$$\|\widetilde{K}(\lambda)\| \leq C_4 + C_5 b\beta^{-m}e^{\beta|\lambda|}, \tag{4.8}$$

where $\|\cdot\|$ stands for the norm in $\mathcal{L}(L^2(\Omega), L^2(\Omega))$. On the other hand,

$$\|[\chi_1, \Delta_e]R_0(z)\eta - [\chi_1, \Delta_e]R_0(\lambda)\eta\| \leq |z - \lambda|\|Q(\tau)\|, \tag{4.9}$$

for some $\tau = \delta\lambda + (1 - \delta)z, \delta \in [0, 1]$, where

$$Q(w) = \frac{d}{dw}[\chi_1, \Delta_e]R_0(w)\eta.$$

Now we are going to show

$$\|Q(w)\| \leq C \qquad \text{for} \quad |\operatorname{Im} w| \leq 1, \tag{4.10}$$

with a constant $C > 0$ indepnedent of w. On the line $\operatorname{Im} w = -1$ we have

$$\begin{aligned} \|Q(w)\| &= 2|w| \, \|[\chi_1, \Delta_e](\Delta_e + w^2)^{-2}\eta\| \\ &\leq C_1|w| \, \|(\Delta_e + w^2)^{-1}\|_{\mathcal{L}(L^2(\Omega), H^1(\Omega))} \\ &\times \|(\Delta_e + w^2)^{-1}\| \leq C_2, \end{aligned} \tag{4.11}$$

with a constant $C_2 > 0$ independent of w. Let now w lie on $\operatorname{Im} w = 1$. By the identity

$$\begin{aligned} &[R_0(w) - R_0(-w)](x, y) \\ &= (i/2)(2\pi)^{-n+1} w^{n-2} \sum_{j=1}^{2} c_j^{-n+2} \int_{\mathbf{S}^{n-1}} e^{ic_j^{-1} w \langle x-y, \sigma \rangle} P_j(\sigma) d\sigma, \end{aligned}$$

where $P_1(\sigma) + P_2(\sigma) = I, P_1(\sigma)g := \langle g, \sigma \rangle \sigma$, one easily finds

$$\left\| \frac{d}{dw}([\chi_1, \Delta_e](R_0(w) - R_0(-w))\eta) \right\| \leq C_3. \tag{4.12}$$

By (4.11) and (4.12) we get

$$\|Q(w)\| \leq C_4$$

on $\operatorname{Im} w = \pm 1$, and hence (4.10) follows from Phragmen-Lindelöf principle. By (4.6)-(4.10) we deduce

$$\|K(z)\| \leq C b \beta^{-m} (\operatorname{Im} z + e^{-\beta|z|}) e^{\beta|z|}. \tag{4.13}$$

Taking $b = (4C)^{-1}\beta^m$, we conclude that $\|K(z)\| \leq 1/2$ for $\operatorname{Im} z \leq e^{-\beta|z|}, \operatorname{Re} z \gg 1$, and hence, $R_\chi(z)$ extends holomorphically in this region with norm $O(e^{\beta|z|})$. Since one can easily express $\mathcal{N}(z)^{-1}$ in terms of $R_\chi(z)$ (see [5], [7]), this completes the proof of Proposition 1.5.

Now, in precisely the same way as in Section 3 we get that (1.7) leads to a contradiction if β is properly chosen.

APPENDIX

In this appendix we will give a proof of Proposition 1.3 following [7]. The proof is based on the following two lemmae.

LEMMA A.1. – *Assume that the function* $f(z)$ *is analytic in* $\{z \in \mathbf{C} : \operatorname{Im} z \le C_1 e^{-\gamma|z|}, \operatorname{Re} z \ge C_2\}, \gamma, C_1, C_2 > 0$, *and satisfies there the estimate* $|f(z)| \le C_3 e^{C_3|z|^p}, C_3, p > 0$. *Assume moreover that* $|f(z)| \le C_4|z|^m/|\operatorname{Im} z|$ *for* $-1 < \operatorname{Im} z < 0, |z| > 1$, *with some* $C_4 > 0$ *and an integer* $m \ge 0$. *Then*

$$|f(z)| \le C_\varepsilon e^{(\gamma+2\varepsilon)|z|}$$

for $|\operatorname{Im} z| \le e^{-(\gamma+\varepsilon)|z|}, \operatorname{Re} z \ge C_\varepsilon, \forall \varepsilon > 0$.

Proof. – Set

$$u(z) = \exp\left(ie^{\delta z}\right) = \exp\left(-e^{\delta \operatorname{Re} z}\sin(\delta \operatorname{Im} z) + ie^{\delta \operatorname{Re} z}\cos(\delta \operatorname{Im} z)\right),$$

where $\delta > 0$ will be chosen later on. On $\gamma_+ := \{z \in \mathbf{C} : \operatorname{Im} z = e^{-\gamma|z|}, \operatorname{Re} z \ge C_2\}$ we have for $g := fu$,

$$\begin{aligned} |g(z)| &\le C_3 \exp\left(C_3|z|^p - \frac{\delta}{2}\operatorname{Im} z e^{\delta \operatorname{Re} z}\right) \\ &= C_3 \exp\left(C_3|z|^p - \frac{\delta}{2}e^{-\gamma|z|}e^{\delta \operatorname{Re} z}\right) \le C, \end{aligned} \tag{A1}$$

if $\delta > \gamma$ and $C_2 = C_2(\delta) > 0$ large enough. On $\gamma_- := \{z \in \mathbf{C} : -\operatorname{Im} z = e^{-\delta|z|}, \operatorname{Re} z \ge C_2\}$ we have

$$|g(z)| \le C_4'|z|^m e^{\delta|z|}. \tag{A2}$$

On the other hand, for $e^{-\gamma|z|} \ge \operatorname{Im} z \ge -e^{-\delta|z|}$ we have

$$|g(z)| \le C_3' e^{C_3|z|^p}. \tag{A3}$$

Hence, by (A.1)-(A.3) and the Phragmén-Lindelöf principle we conclude that between the curves γ_+, γ_- and $\operatorname{Re} z = C_2$ the function $g(z)$ satisfies the estimate

$$|g(z)| \le C_4'|z|^m e^{\delta|z|}. \tag{A4}$$

Since for $|\operatorname{Im} z| \le e^{-\delta|z|}$ we have

$$|u(z)^{-1}| \le C',$$

the desired bound follows from (A.4) taking $\delta = \gamma + \varepsilon, \forall \varepsilon > 0$. This completes the proof of the lemma.

LEMMA A.1. – *Assume that* $R_\chi(z)$ *is analytic in* $D_{C_1,C_2} := \{z \in \mathbf{C} : \operatorname{Im} z < C_1 e^{-\gamma|z|}, \operatorname{Re} z \geq C_2\}, \gamma, C_1, C_2 > 0$. *Then, for any* $C_1' < C_1, C_2' > C_2$, *we have*

$$\|R_\chi(z)\|_{\mathcal{L}(L^2,H^2)} \leq Ce^{C|z|^{n+1}} \quad \text{for} \quad z \in D_{C_1',C_2'}.$$

Proof. – We are going to take advantage of (4.7). Since $n \geq 3$ is odd, $K(z)^{(n+1)/2}$ is an entire family of trace class operators, and hence we can define the entire function

$$h(z) = \det(I - K(z)^{(n+1)/2}).$$

Moreover, it follows from the analysis in [9] that

$$|h(z)| \leq \prod_{j=1}^{\infty} \left(1 + \mu_j(K(z)^{(n+1)/2})\right) \leq Ce^{C|z|^n}, \quad \forall z \in \mathbf{C},$$

where $\mu_j(A)$ denote the characteristic values of A. Let $\{z_j\}_{j=1}^{\infty}$ be the zeros of $h(z)$ and set $V = \mathbf{C} \setminus \cup\{z \in \mathbf{C} : |z - z_j| < |z_j|^{-n-2}e^{-\gamma|z_j|}\}$. Let us first see that

$$\|R_\chi(z)\|_{\mathcal{L}(L^2,H^2)} \leq Ce^{C|z|^{n+1}} \quad \text{for} \quad z \in V. \tag{A5}$$

Since the function $h(z)$ is entire we can use the theorem of the minimum of the modul which gives

$$|h(z)^{-1}| \leq Ce^{C|z|^{n+1}} \quad \text{for} \quad z \in V. \tag{A6}$$

On the other hand, we have

$$|\det(I - K(z)^{(n+1)/2})| \|(I - K(z)^{(n+1)/2})^{-1}\|$$

$$\leq \prod_{j=1}^{\infty} \left(1 + \mu_j(K(z)^{(n+1)/2})\right) \leq Ce^{C|z|^n},$$

which combined with (4.7) and (A.6) implies (A.5).

We have that $\mathbf{C} \setminus V = \cup_{k=1}^{\infty} U_k$, where U_k are disjoint connected sets, each U_k is a union of a finite number of disks, and clearly $\operatorname{diam} U_k$ is upper bounded by a constant independent of k. Denote by r_k the distance from the origine to U_k. Then

$$\operatorname{diam} U_k \leq 2 \sum_{|z_j| \geq r_k} |z_j|^{-n-2}e^{-\gamma|z_j|} \leq Mr_k^{-1}e^{-\gamma r_k}. \tag{A7}$$

Set $K = \{k : D_{C'_1,C'_2} \cap U_k \neq \emptyset\}$. Because of (A.7) we have $U_k \subset D_{C_1,C_2}$ for large $k \in K$. Since (A.5) holds on ∂U_k, by the maximum principle (A.5) holds in U_k for large $k \in K$ with possibly new constant. Thus (A.5) holds in the whole $D_{C'_1,C'_2}$, which completes the proof of Lemma A.2.

Using that

$$\|R_\chi(z)\|_{\mathcal{L}(L^2,H^2)} \leq \frac{C|z|}{|\operatorname{Im} z|} \quad \text{for} \quad \operatorname{Im} z < 0,$$

by Lemmas A.1 and A.2 we conclude that if $R_\chi(z)$ is analytic in $\{z \in \mathbf{C} : \operatorname{Im} z < C_1 e^{-\gamma|z|}, \operatorname{Re} z \geq C_2\}, \gamma, C_1, C_2 > 0$, then

$$\|R_\chi(z)\|_{\mathcal{L}(L^2,H^2)} \leq C_\varepsilon e^{(\gamma+2\varepsilon)|z|}$$

for $\operatorname{Im} z < e^{-(\gamma+\varepsilon)|z|}, \operatorname{Re} z \geq C_\varepsilon$. Since we can express $\mathcal{N}(z)^{-1}$ in terms of $R_\chi(z)$ (and vice-versa), Proposition 3.1 follows.

ACKNOWLEDGEMENTS

I would like to thank the referee for the useful remarks and suggestions.

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(Manuscript received on February 2nd, 1996;
Revised version received on June 25th, 1996.)