ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 66, nº 4 (1997), p. 373-410
<http://www.numdam.org/item?id=AIHPA_1997__66_4_373_0>

Ann. Inst. Henri Poincaré,

Vol. 66, n° 4, 1997, p. 373-410. *Physique théorique*

Reaction-diffusion systems with prescribed large time behaviour

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ABSTRACT. – A special class of two-component reaction-diffusion systems is considered. These systems have remarkable properties. They possess inertial manifolds and moreover one can control the motion in this manifold. For any given structurally stable system of ordinary differential equations one can find such system from this class that the dynamics in the inertial manifold will coincide with the given one (up to orbital topological equivalency).

RÉSUMÉ. – Nous considérons une classe des systèmes de réaction-diffusion avec deux variables. Nous prouvons que ces systèmes ont des variétes invariantes inertielles. Pour chaque système d'équations différentielles définissant une dynamique structurellement stable, il existe dans sa classe, un système dont la dynamique inertielle est topologiquement équivalente à la dynamique initiale.

1. INTRODUCTION

1.1. Inertial manifolds, attractors and inertial forms

Reaction-diffusion systems

$$\frac{\partial u_i}{\partial t} = d_i \Delta u_i + f_i(x, u), \quad x \in \Omega \subset \mathbf{R}^p, \quad u \in \mathbf{R}^k \tag{1.1}$$

play a key role in a number of applications for example in biology, physics and chemistry [1], [2]. In this field the large time behavior of systems (1.1) is an extremely important problem. It has been known for some time that many nonlinear dissipative systems possess global attractors which have finite Hausdorff dimension [3]-[8]. Under some restrictions (in particular, excluding blow-up effects) one can prove for (1.1) the existence of the finite dimensional attractors [4]-[6] or even inertial manifolds [9]-[12]. The inertial manifold is finite dimensional and invariant under dynamics. Thus, by restricting the evolutionary equations to this manifold one obtains a finite system of ordinary differential equations (ODE). The inertial manifold is globally attracting with asymptotic phase, and thus, every solution of evolution equations converges to a solution of the mentioned ODE on the inertial manifold as $t \to \infty$. In particular, global attractor, all invariant sets and local attractors lie in the inertial manifold.

These ODE controlling the dynamics on the inertial manifold were called inertial forms [11].

The main aim of this work is to give an analytical proof of the fact that the dynamics in the inertial manifold can be unboundedly complicated even if we investigate systems (1.1) with only two components. To make it, one constructs a special class of systems which have remarkable properties. They possess inertial manifolds and in addition one can control the motion in these manifolds. For any given structurally stable dynamics (defined by ordinary differential equations) one can find such system from this class that the dynamics in the inertial manifold will coincide with the given one (up to orbital topological equivalency).

1.2. Parabolic nonlinear equations. Complicated dynamics

First let us consider the nonlinear parabolic equation (*i.e.* $k = 1$) in a bounded domain Ω

$$\frac{\partial u}{\partial t} = d\Delta u + f(x, u) + g(x), \quad x \in \Omega \subset \mathbf{R}^p, \quad u \in \mathbf{R} \tag{1.2}$$

under the Neumann or Dirichlet boundary conditions. Then at least for "generic" g the global attractor structure can be studied [4] [8]. For such g this attractor is a union of hyperbolic rest points together with the corresponding unstable manifolds.

If nonlinearity f can depend on ∇u and on time t periodically one obtains (under some regularity conditions) that "almost all" trajectories converge to cycles ([15], [16], the method uses the general approaches from the theory of monotone flows and maps in Banach spaces [13], [14]). In details, the dynamics (1.2) is studied in one-dimensional ($p = 1$) case [17], [18].

Finally for (1.2) there exist no interesting complicated large time regimes, at least for generic initial data. If we suppose that f in (1.2) can depend on ∇u then for especial initial states complicated behaviour is possible. For instance when $p \geq 2$ (multi-dimensional case) P. Polacik has proved some results on the existence of complicated dynamics for (1.2) [19], [20]. These dynamics are embedded in unstable invariant manifolds.

First the complicated dynamics existence in PDE was obtained analytically in the pioneering work [21] for Korteweg de Vries (KdV) equation with small perturbation which is a complicated nonlinear functional (analytic in some special norm). For such perturbed KdV the Smale-Ruelle-Takens tori breakdown mechanism arises and dimensions of these tori can be arbitrary (depending on these exotic perturbations).

A more simple nonlocal equation

$$\begin{gathered}\frac{\partial u}{\partial t} = u_{xx} + f(x,u) + g(x)\int_0^1 \beta(x)u dx,\\ x \in [0,1], \quad u(0,t) = 0, \quad u(1,t) = 0.\end{gathered} \tag{1.3}$$

was investigated by the beautiful paper [22]. In this case one also can expect the Smale-Ruelle-Takens tori breakdown however a complete proof is absent. Some results for system (1.1) were obtained by the author [23]-[25]. Unfortunately in these papers the dimensions of strange attractors are bounded by number $k+1$ (k is the number of components).

1.3. Aim of the investigation. Main results

The aim of this paper is to study a new special class $\mathcal{V}$ of system of two nonlinear parabolic equations with two unknown functions $u_1 = u,\ u_2 = v$. The equations contain the two parameters ϵ and λ. These systems define dissipative dynamics and have inertial manifolds $\mathcal{M}$.

Interesting property of these systems is that in a sense the corresponding inertial forms can be prescribed. More precisely it can be described as follows.

In the two-dimensional box $\Omega = [0,1]\times[0,1]$ let us consider the following system

$$\frac{\partial u}{\partial t} = A_1 u + \lambda\epsilon^{\mu} a(x,y)v - 2\lambda^2\epsilon^{\mu+1/2}u, \quad u(x,y,0) = u_0(x,y), \tag{1.4}$$

$$\frac{\partial v}{\partial t} = A_2 v + \lambda\epsilon^{\mu} g(x,y,u,\epsilon) - \epsilon^{\mu+1/2}v, \quad v(x,y,0) = v_0(x,y) \tag{1.5}$$

where $\epsilon, \lambda \in [0,1]$, $\mu > 1$ and $A_i = A_i(\epsilon)$ are special linear operators

$$A_1 u = \Delta u - \epsilon^{-2}(V_m(x) - \epsilon)u, \quad A_2 v = \Delta v - \epsilon^{-2}(\tilde{V}_n(y) - \epsilon)v \tag{1.6}$$

where Δ is the two-dimensional Laplacian.

Let us set the Neumann boundary conditions

$$\frac{\partial}{\partial \nu} u(x,y,t) \equiv 0, \quad \frac{\partial}{\partial \nu} v(x,y,t) \equiv 0 \qquad (x,y) \in \partial\Omega \tag{1.7}$$

where ν is the normal (similarly one also can consider the case of periodic boundary conditions). Potentials $V_m(z)$ and $\tilde{V}_n(z)$ have m and n identical potential wells respectively. (All these wells have the forms $(z - z_i)^2$ *i.e.* coincide with the potential of the quantum harmonic oscillator.)

Denote $\mathcal{Q}_K = \{|Q_i| \le K, \ i = 1, ..., n\}$ n-dimensional cube in $\mathbf{R}^n$.

THEOREM 1.1. – *Let us consider in the n-dimensional unit cube $\mathcal{Q}_1$ an arbitrary prescribed C^1-vector field $F^{pr}(Q)$ such that*

$$\max_i \sum_j |\frac{\partial F_i^{pr}}{\partial Q_j}| < 1. \tag{1.8}$$

Suppose this field is directed inside the cube at the boundary $\partial\mathcal{Q}_1$ i.e. *for every i one has:*

$$Q_i F_i^{pr}(Q) < 0 \quad for\ all\ vectors \quad Q \in \mathcal{Q}_1 \quad such\ that \quad |Q_i| = 1. \tag{1.9}$$

Let us consider the ordinary differential equations in the cube $\mathcal{Q}_1$ induced by F^{pr}

$$\frac{dQ}{d\tau} = F^{pr}(Q). \tag{1.10}$$

Moreover, let δ be some positive number.

Then there exist potentials V_m, $\tilde{V}_n$, a function $g(x, y, u, \epsilon) \in C^2$ *and a function $a(x,y) \in C^1$ such that*

I. *System* (1.4) *and* (1.5) *defines a global semiflow S^t, $t > 0$ in an ambient phase space H;*

II. *For sufficiently small positive $\epsilon < \epsilon_0(\lambda)$ and sufficiently small positive $\lambda < \lambda_0$ this dynamics is dissipative* i.e. *there exists an absorbing set $\mathcal{A}$. In addition, the dynamics S^t has a n-dimensional inertial manifold $\mathcal{M}$. This manifold is globally attracting in H and locally invariant in the absorbing set $\mathcal{A}$;*

III. *There are special coordinates Q in $\mathcal{M}$ such that in the intersection $\mathcal{A} \bigcap \mathcal{M}$ the inertial forms of global semiflow S^t are defined by ODE*

$$\frac{dQ}{d\tau} = F(Q, \epsilon, \lambda), \quad F \in C^1(\mathcal{Q}_1) \tag{1.11}$$

where τ is a rescaling time.

The field $F(\cdot, \epsilon, \lambda)$ is close to the prescribed i.e.

$$|F(\cdot, \epsilon, \lambda) - F^{pr}(\cdot)|_{C^1(\mathcal{Q}_1)} < \delta \tag{1.12}$$

for any $\epsilon \in (0, \epsilon_0(\lambda))$, $\lambda \in (0, \lambda_0)$.

The third assertion has corollaries describing the existence of complicated dynamics. They hold only if the prescribed fields F^{pr} satisfy some "structural stability" conditions.

First let us recall that a reasoning definition of structural stability for finite dimensional dynamical systems is connected with the orbitally topological equivalence of trajectories (see [26], [29], [32], [33]). Systems $dQ/dt = F(Q)$ and $dQ/dt = \tilde{F}(Q)$ are topologically equivalent if there exists a homeomorphism h connecting trajectories of these systems.

System (1.10) is structurally stable if all the dynamical systems from a small C^1-neighborhood of (1.10) are orbitally topologically equivalent to original and corresponding homeomorphisms h are close to the identity.

Also some results can obtained if prescribed system (1.10) has local hyperbolic attractors or more generally hyperbolic sets Γ.

Trivial examples of such sets are given by saddle rest points Q_* (where the linearization F at Q_* gives a linear operator having no zero eigenvalues) or saddle periodical cycle.

However, beginning with the famous works [27]-[31] we know a number of nontrivial examples of hyperbolic sets Γ where the dynamics restricted to Γ can be chaotic for instance the Smale horseshoe, Anosov systems and systems of the Lorentz type [29]. From this list, the Smale horseshoe and homoclinic invariant sets are not local attracting. Local hyperbolic attractors occur in the Smale-Ruelle-Takens models of tori breakdown mentioned above. This breakdown gives structurally stable systems on p-dimensional tori and dynamics can be chaotic for $p > 2$.

Due to classic results [28], [29], [32], [33], this chaotic dynamics is stable *i.e.* holds under small C^1-perturbations. For instance, let us suppose system (1.10) has hyperbolic invariant set Γ and (1.9), (1.12) hold. Due to the Persistence Hyperbolic Set Theorem [32], [33], for sufficiently small δ system (1.11) also has hyperbolic sets $\tilde{\Gamma}$ and the dynamics restricted to these sets $\Gamma, \tilde{\Gamma}$ are orbitally topologically equivalent.

Finally, one has the following key corollaries of Theorem.

COROLLARY 1.2. – *Suppose the field F^{pr} satisfy* (1.9) *and* (1.12) *and defines structurally stable dynamics* (1.10). *Then there exist potentials V_m,*

$\tilde{V}_n$, a function $g(x, y, u, \epsilon)$ and a function $a(x, y)$ such that for sufficiently small positive ϵ, $\epsilon < \epsilon_0(\lambda)$ and small $\lambda < \lambda_0$ inertial form (1.11) *defines in the cube $\mathcal{Q}_1$ the dynamics which are orbitally topologically equivalent to prescribed dynamics* (1.10).

COROLLARY 1.3. – *Suppose the field F^{pr} (satisfying* (1.9), (1.12)) *has some hyperbolic set Γ. Then there exist potentials V_m, $\tilde{V}_n$, a function $g(x, y, u, \epsilon)$ and a function $a(x, y)$ such that for sufficiently small positive ϵ, $\epsilon < \epsilon_0(\lambda)$ and λ inertial forms* (1.11) *also have hyperbolic sets $\tilde{\Gamma}$ and $\tilde{\Gamma} = h_{\epsilon,\lambda}\Gamma$ where $h_{\epsilon,\lambda}$ are homeomorphisms.*

If the prescribed set Γ is a local attractor, then $\tilde{\Gamma}$ also is a local attractor for the global semiflow S^t.

Briefly, system (1.4), (1.5) can have arbitrary complicated chaotic dynamics for appropriate coefficients $a, V, \tilde{V}$ and nonlinearity g. We shall show that fractal dimensions (which is less than n) of these invariant sets and attractors can be unbounded as $\epsilon \to 0$ since for small ϵ one can take large n. When the number n increases then generally the thresholds ϵ_0, λ_0 decrease.

Finally, to conclude this subsection, let us note that using of classic Differential Dynamics results allows (by this construction) to show analytically the existence of arbitrary prescribed complicated dynamics. In the excellent book [34] D. Henry has developed the geometric approach to dissipative dynamics in PDE and has generalized results of the theory of finite dimensional dynamical systems on infinite dimensional systems (see also [7]). In particular, this extension contains theory of invariant manifolds, averaging methods and theory of gradient-like systems.

However most interesting and nontrivial achievements of Differential Dynamics (connected with works by S. Smale, Yu. Sinai, D. V. Anosov, D. Ruelle and others) are analytical approaches to different models of the chaotic motion.

These classic results were in the main outside geometric theory of nonlinear parabolic systems and in particular the reaction-diffusion ones. This paper allows to construct some map from the set of ordinary differential equations in the class $\mathcal{V}$ of special reaction diffusion system (1.4), (1.5). In general this map can glue different but equivalent dynamics however for any prescribed topological class of dynamics one can find system (1.4), (1.5) with analogous dynamics. Finally roughly speaking the main result is that in a sense all Differential Dynamics can be contained in the geometric theory for semilinear parabolic systems. In addition it is sufficient to consider only two-components systems *i.e.* this "embedding" constructed in the given paper uses an almost minimal class (see subsection 1.2).

Moreover it is shown that at least for "generic" reaction diffusion systems we have "unbounded jump of complexity". Generic nonlinear parabolic equation (1.3) has a relatively simple structure of attractor nonetheless systems of two such equations can define unboundedly complicated dynamics.

To conclude this subsection let us note that (as it described below) systems (1.4), (1.5) have interesting connections with classical systems of coupled oscillators for example with equations suggested A. N. Kolmogorov [35] for a description of hydrodynamic turbulence.

1.4. Outline of proof and key ideas

In general, the strategy is quite traditional however some new ideas are used. First let us recall some previous methods from [22]. Consider equation (1.3).

A centre manifold reduction to a finite dimensional ODE is the basic approach to obtain complicated dynamics near $u = 0$. To describe this reduction, authors [22] considered operator $Au = -u_{xx} - a(x)u - c(x)u\int_0^1 \nu(x)u(x)dx$. Let $X = L_2[0,1]$. It is shown in [22] that, for any m, one can adjust the functions $\nu(x), a(x)$ and $c(x)$ such that the spectrum of A contains m arbitrary prescribed imaginary pairs of simple eigenvalues. All the other eigenvalues of A are positive. Thus, there arises a natural factorization (splitting) $X = X_1 \times X_2$ of phase space where $X_1 = span\{\psi_1, ..., \psi_m\}$, $A\psi_j = \lambda_j\psi_j$, $Re(\lambda_j) = 0$.

The proof of these facts is quite nontrivial.

After of this investigation of A one can, in a standard way, obtain a centre manifold W^c for (1.3) which has the form $u_2 = w(u_1)$, $u_i \in X_i$. This manifold carries all important information about dynamics of (1.3) near $u = 0$. As a result, one has [22] the following assertion (which is stated in a brief form):

Let V be any polynomial vector fields on $\mathbf{R}^n$ of degree $N \geq 2$ such that $V(0) = 0$ and $V'(0) = 0$. Let W be reduced vector field representing the flow on a C^N-center manifold at $u = 0$ for equation (1.3).

Then one can choose $f(x,u), a(x), c(x), \nu(x) \in C^\infty$ so that, in real coordinates, the Taylor expansion of W at 0 *coincides with V, for orders* 2 *through N.*

This result affirms to a certain extent the existence of complicated dynamics in (1.3) connected with tori breakdown.

The approach of this paper is similar. We are going to move towards complicated dynamics via linear mode interactions however we use another linear operator $A = (A_1(\epsilon), A_2(\epsilon))^t$ given by (1.6).

For operators A_k a factorization of the phase space $L_2(S^1 \times S^1)$ can be obtained in a relatively simple way with using of properties of V_m, $\tilde{V}_n$.

One obtains that

$$A_1 u = O(\epsilon^N) \quad \text{for } u \in X_u = span\{\psi_1(x,\epsilon), ..., \psi_m(x,\epsilon)\}$$

and

$$A_2 v = O(\epsilon^N) \quad \text{for } v \in X_v = span\{\phi_1(y,\epsilon), ..., \phi_n(y,\epsilon)\}$$

where N is any positive integer and ψ_i, ϕ_j are some "almost" eigenfunctions.

If u and v are orthogonal to principal modes ψ_i, ϕ_j respectively one has $< A_1 u, u > \leq -c||u||^2$, $< A_2 v, v > \leq -c||v||^2$. This means that if one restricts A to the orthogonal addition to $X_u \times X_v$ the spectrum of this operator will be shifted in the negative half-plane and will be separated from the imaginary axis by a spectral barrier. The width of this barrier does not depend on small parameters ϵ, λ as $\epsilon, \lambda \to 0$.

This phase space splitting allows to use the standard approach described above.

To obtain the globally attracting manifold it is sufficiently to take the exponent $\mu > 1$ *i.e.* take small nonlinearity $\epsilon^\mu g$.

The three additional key ideas are essential. The first point is that one uses the Persistence Hyperbolic Set Theorem.

The second nontrivial point can be described as follows.

The flavor of construction (1.4), (1.5) is that the potential V_n depends on x only and $\tilde{V}_m$ depends on y. Namely such choice allows to obtain a complicated interaction of the linear modes. If we denote q_i and p_j amplitudes of these modes one obtains (for any $\lambda \in [0,1]$ and for small $\epsilon < \epsilon_0(\lambda)$) that there exists a globally attracting invariant manifold $\mathcal{M}_1$ (let us note that $\mathcal{M} \subset \mathcal{M}_1$). The corresponding embedded dynamics (induced by global semiflow in this manifold) is defined by the equations

$$\frac{dq_i}{dT} = \sum_{j=1}^{n} \lambda Q_{ij}(q_i, p_j, a, g, \lambda) - 2\lambda^2 q_i + \tilde{Q}_i(q, p, \epsilon, \lambda), \tag{1.13a}$$

$$\frac{dp_j}{dT} = \lambda \sum_{i=1}^{m} P_{ij}(q_i, p_j, a, g, \lambda) - p_j + \tilde{P}_j(q, p, \epsilon, \lambda) \tag{1.13b}$$

where in (1.13) a, f and λ play a role of parameters and T is a rescaling time. Contributions $\tilde{Q}_i$ and $\tilde{P}_j$ are small (in C^1 norm) corrections vanishing as $\epsilon \longrightarrow 0$.

The following important property holds for the right hands Q and P. By changing a and g in (1.4) and (1.5) one can adjust Q, P.

In (1.13) the parameter λ is in $[0, 1]$. To simplify (1.13) let us suppose λ is small enough. It allows to construct a new invariant (inertial for (1.13)) manifold $\mathcal{M}_2$ that is embedded in $\mathcal{M}_1$. The corresponding inertial forms arising in this second manifold are quite remarkable. These forms are similar to the classical model [33]

$$\frac{dq_i}{d\tau} = \sum_j \sum_k \alpha^i_{jk} q_j q_k - b_i q_i + f_i, \quad b > 0. \tag{1.14}$$

These equations were first suggested by A. N. Kolmogorov to describe the turbulence for the Navier-Stokes equations (NSE). In (1.14) the first term (under some restrictions on matrix α) describes an inertial contribution connected with the Euler terms in NSE, the second term $-b_i q_i$ describes a dissipation connected with viscosity and the third term is an external force.

The investigation of (1.14) is a formidable task [35], [36]. However one can show (and it is the third key idea of our construction) that there exists a simple modification of (1.14). This system has the form

$$\frac{dq_i}{d\tau} = \sum_{j=1}^{n} \alpha_{ij} \Phi_j(q) - b q_i, \quad \tau = \lambda T. \tag{1.15}$$

It can arise as an inertial form of (1.13) for appropriate Q, P (and thus, for original equations (1.4) and (1.5) for appropriate coefficients g, a). In opposite to the A. N. Kolmogorov case, equations (1.15) can be investigated successfully for some special nonlinearities Φ.

The last step is a simple proof of the following facts.

Under some restrictions the main dynamics in (1.15) is captured by the first n components $(q_1, q_2, ... q_n)$ of vector $q \in \mathbf{R}^m$. In addition, roughly speaking one can choose parameters a, g in (1.4) and (1.5) so that system (1.15) has an inertial manifold $\mathcal{M}$ and the corresponding inertial form can be prescribed *i.e.* for any dynamics (1.10) one can adjust Φ_j and α in (1.15) so that the inertial form of (1.15) coincides with equations (1.11) and estimate (1.12) holds.

Let us note that terms in (1.15) have the same physical meaning that in (1.14). Namely if $b = 0$ shorted equations (1.15) have a number of conservation laws (more exactly under some restrictions if $m > n$ one has $m - n$ integrals) thus it can be considered as an inertial term. We know from V. I. Arnol'd investigations that the Euler equations have infinite number of integrales [37].

In model (1.4) and (1.5) one can choose any m changing $\tilde{V}_m$ and thus one can have any number of the integrals in shorted equations (1.15). Finally system (1.15) can be considered as some rough simulation of the Navier-Stokes system.

1.5. Organization of the paper

The following section 2 contains need technical assertions about linear operators. It allows to establish the splitting of phase space. In sec. 3, one shows that equations (1.4) and (1.5) define global semiflow $S^t, t > 0$. Furthermore, in sec. 4, one rewrites equations (1.4), (1.5) so that one can effectively to use the phase separation on fast and slow modes.

In sec. 5, one investigates dissipative properties of this transformed system. It will be shown that dynamics S^t induced by system (1.4) and (1.5) is dissipative (there exists an absorbing set $\mathcal{A}_1$).

After that, in sec. 6-10 one studies dynamics inside this absorbing set. In sec. 6 one obtains the locally invariant and globally attracting manifold $\mathcal{M}_1$ mentioned in previous subsection. Equations (1.13) are obtained in section 7 as an inertial form of (1.4), (1.5) in $\mathcal{M}_1$. Section 8 contain an investigation of these equations. In this section we use that λ is also a small parameter and simplify (1.13). It gives equation (1.15).

Choosing a, g in (1.4), (1.5) one can obtain in (1.15) nonlinearities Φ_j in a special form. It allows, in section 9, to investigate (1.15) and to obtain the need inertial manifold $\mathcal{M} \subset \mathcal{M}_1$.

Section 10 contains the final part of the proof of main Theorem. For given F^{pr}, this section describes an algorithm of finding of the coefficients $V_n, \tilde{V}_m, a$ and nonlinearity g which give the prescribed dynamics in the inertial manifold $\mathcal{M}$.

2. SOME PRELIMINARIES

One begins with the construction of operators A_i. Also in this section one obtains some estimates playing an important role below.

Let us consider the interval $I = [0, 1]$ and a subset X_m consisting of m different points x_i such that $x_i < x_{i+1}$. Let us denote d_m the minimum of distances between of these points and boundaries $0, 1$ *i.e.* $d_m = \min\{x_{i+1} - x_i, \ x_1, 1 - x_m\}$ where $= 1, 2, ..., m-1$. Let us set $\delta_m = d_m/4$.

Let us define a special potential $V_m(x)$ consisting of m identical potential wells at the points x_i. Namely, one assumes

$$V_m(x) \equiv (x - x_i)^2, \quad \text{for} \quad x \in \Omega_i \tag{2.1}$$

where Ω_i are intervals centered at x_i, $\Omega_i = (x_i - \delta_m, x_i + \delta_m)$. Moreover, let us suppose

$$V_m \in C^2, \quad V_m > c > 0 \ \text{ for any } x \notin \bigcup_{i=1}^{m} \Omega_i. \tag{2.2}$$

Let us define the linear operator in $L_2([0, 1]) = L_2(S)$

$$B_m(\epsilon) = \frac{\partial^2}{\partial x^2} - \epsilon^{-2} V_m(x) + \epsilon^{-1}. \tag{2.3}$$

With the above standing assumptions this operator has the following "almost" eigenfunctions $\psi_i(x, \epsilon)$

$$\psi_i(x, \epsilon) = a_\epsilon \epsilon^{-1/4} \exp(-(x - x_i)^2/2\epsilon) \xi((x - x_i)/\delta_m) \tag{2.4}$$

where ξ is a C^∞ cut-off functions such that $0 \leq \xi \leq 1$, $\xi(z) \equiv 1$ for $z < 1/2$, $supp\ \xi = (-1, 1)$. Constants $a_\epsilon = \sqrt{2\pi} + O(\exp(-c\epsilon^{-1/2}))$ are chosen so that

$$||\psi_i||_{L_2(I)} = 1. \tag{2.5}$$

Using the inequality $dist(x_i, \Omega_j) > \delta_m > 0$ that holds for $i \neq j$ one obtains the estimates

$$\sup |B_m \psi_i| < C_N \epsilon^N, \quad \text{for any } N > 0. \tag{2.6}$$

(To simplify denotations, throughout one uses the following convention. One denotes C, c sufficiently large positive constants which do not depend on small parameters ϵ, λ as $\epsilon, \lambda \to 0$ and sometimes one omits index. These constants can vary from line to line. Similarly, small positive constants (which do not depend on ϵ, λ) are denoted by δ.)

Let us prove the following technical but essential Lemma.

LEMMA 2.1. – *Let $H^{(m)}$ is the subspace in $L_2(I)$ consisting of functions u which are orthogonal to all ψ_i*

$$H^{(m)} = \{u \in L_2(I) : \ < u, \psi_i >_{L_2(I)} = 0\}. \tag{2.7}$$

Then for sufficiently small ϵ the operator B_m is negatively defined in this subspace

$$< B_m u, u >_{L_2(I)} \leq -c_m \epsilon^{-1} ||u||^2_{L_2(I)}, \quad u \in H^{(m)} \bigcap Dom B_m. \tag{2.8}$$

Proof. – Clearly this Lemma holds for the functions $u(x)$ with supports in the intervals $I_i = (x_i - \frac{1}{2}\delta_m, \ x_i + \frac{1}{2}\delta_m)$.

It follows immediately from definitions (2.1) and (2.3) and a spectral decomposition for linear operator $\frac{\partial^2}{\partial x^2} + x^2$ (the quantum harmonic oscillator). One sets

$$u_i(x) = \xi_i(x) u(x), \quad \xi_i = xi(2(x - x_i)/\delta_m), \quad \bar{u} = \sum_{i=1}^{m} u_i, \quad \tilde{u} = u - \bar{u}. \tag{2.9}$$

Thus one has $< B_m u_i, u_i > \leq -c\epsilon^{-1} ||u_i||^2$. Since supports of functions u_i do not intersect, immediately one concludes

$$< B_m \bar{u}, \bar{u} > \leq -c\epsilon^{-1} ||\bar{u}||^2. \tag{2.10}$$

The similar estimate holds for $\tilde{u}$

$$< B_m \tilde{u}, \tilde{u} > \leq -c\epsilon^{-2} ||\tilde{u}||^2. \tag{2.11}$$

holds due to (2.2). In fact, the support of $\tilde{u}$ lies in a set where the potential $V_m(x) > c > 0$.

Finally to prove the assertion one remains to estimate $I_m = < B_m \bar{u}, \tilde{u} >$. Taking into account that, if $u_i(x) \neq 0$ and $\tilde{u}(x) \neq 0$ then $u_i(x) = \xi_i(x) u(x)$, $\tilde{u}_i(x) = (1 - xi_i) u$, one obtains

$$I_m = - \sum_{i=1}^{m} \int_0^1 [((1 - \xi_i) u)_x (\xi_i u)_x - \epsilon^{-2} V_m(x)(1 - \xi_i)\xi_i u^2] dx.$$

Integrating by parts and using that $V_m \geq 0$ one finds

$$I_m \leq - \sum_{i=1}^{m} \int_0^1 (1 - \xi_i)\xi_i u_x^2 + \int_0^1 \bar{p}(x) u^2 dx$$

where the function $\bar{p}$ is bounded and does not depend on ϵ. Thus,

$$I_m \leq c_2 ||u||^2. \tag{2.12}$$

Combining (2.9), (2.10), (2.11) and (2.12), one obtains for small ϵ

$$< B_m u, u > \leq -c\epsilon^{-1}||\bar{u}||^2 - c_1\epsilon^{-2}||u - \bar{u}||^2 + c_2||u||^2 \leq -c_3||u||^2.$$

This completes the proof.

In a similar way, let us take a set $Y_n = \{y_1, y_2, ..., y_n\}$ of points in S where $n \leq m$ and intervals $\tilde{\Omega}_i$. Similarly, let us construct a potential $\tilde{V}_n(y)$ with n potential wells and take the linear operator

$$\tilde{B}_n(\epsilon) = \frac{\partial^2}{\partial y^2} - \epsilon^{-2}\tilde{V}_n(y) + \epsilon^{-1}. \tag{2.13}$$

Observe that the same estimate (2.8) holds for $\tilde{B}_n$:

$$< \tilde{B}_n u, u >_{L_2(I)} \leq c_n \epsilon^{-1}||u||^2_{L_2(I)}, \quad u \in \tilde{H}^{(n)} \bigcap Dom\tilde{B}_n \tag{2.14}$$

where

$$\tilde{H}^{(n)} = \{u \in L_2(I) : < u, \phi_j >_{L_2(I)} = 0\}, \quad j = 1, ..., n \tag{2.15}$$

and $\phi_j(y, \epsilon)$ are "almost" eigenfunctions of $\tilde{B}_n$

$$\phi_j(y, \epsilon) = b_\epsilon \epsilon^{-1/4} \exp(-(y - y_j)^2/2\epsilon)\tilde{\xi}_i(y/\delta_n). \tag{2.16}$$

Constants b_ϵ are defined so that

$$||\phi_j||_{L_2(I)} = 1. \tag{2.17}$$

Notice that due to the choice of the cut-off functions ξ_i and $\tilde{\xi}_j$ one has

$$< \psi_i, \psi_j >_{L_2(I)} = \delta_{ij}, \quad < \phi_i, \phi_j >_{L_2(I)} = \delta_{ij} \tag{2.18}$$

where δ_{ij} is the Kronecker symbol.

Beginning with this moment, $< \, , \, >$ denotes the inner scalar product in $H = L_2(\Omega)$ and $|| \; ||$ is the corresponding norm.

Let us define the operators $A_k(\epsilon)$ by (1.6). Definition domains $DomA_i$ of these operators are dense in the Sobolev spaces $W_2^2(\Omega) = \{u : \Delta u \in L_2(\Omega)\}$. To investigate the spectrum location of these operators, it is sufficiently to apply the standard variable separation. It allows to use the information on the spectra given by estimates (2.8) and (2.14).

Now one can prove the key assertion of this section.

PROPOSITION 2.2. – *Let us define subspaces in* $L_2(\Omega)$*:*

$$H^m = \{u \in L_2(\Omega) : < u, \psi_i >= 0, \ i = 1, ..., m\},$$

$$\tilde{H}^n = \ \{v \in L_2(\Omega) : < v, \phi_j >= 0, \ j = 1, ..., n\}.$$

If $u \in H^m \bigcap DomA_1$ *and* $v \in \tilde{H}^n \bigcap DomA_2$ *one has*

$$< A_1 u, u > \le -c||u||^2, \quad < A_2 v, v > \le -c||v||^2. \tag{2.19}$$

The Proposition can be proved easily by Lemma 2.1.

In fact, one has the decomposition $u = \bar{u}(x) + \tilde{u}(x, y)$ where $\bar{u} = \int_0^1 u(x, y)dy$. For $\bar{u}$ inequality (2.19) holds due to Lemma 2.1 and for $\tilde{u}$ it holds due to the Poincare inequality.

Estimates (2.19) means that if we restrict the self-adjoint operators A_k on corresponding subspaces H^m and $\tilde{H}^n$ respectively the corresponding spectra lie in negative half-plane and are separated from the imaginary axis by some "spectral barrier". The existence of such barrier (which does not vanish as $\epsilon \longrightarrow 0$) helps us to construct explicitly an inertial manifold for system (1.4) and (1.5).

3. GLOBAL EXISTENCE OF SOLUTIONS

Let us consider system (1.4) and (1.5) where the functions $a, \ g \in C^2$ and the positive exponent μ satisfy conditions

$$|D^k g(\cdot, \cdot, u, \epsilon)|_k < C\epsilon^{-k/2}, \quad k = 0, 1, 2 \tag{3.1}$$

uniformly respectively u and for all D^k where D^k denote arbitrary operators $(\frac{\partial}{\partial x})^{k_1}(\frac{\partial}{\partial y})^{k_2}$ with $k_1 + k_2 = k, k_i \ge 0$ and

$$|a|_2 < C, \quad \mu > 1. \tag{3.2}$$

Here $| \ |_s$ denotes C^s-Holder norm.

To investigate the dynamics generated by (1.4) and (1.5) one applies the well known approach [34]. Let us define scale of the Hilbert spaces (spaces of fractional powers)

$$H_\alpha = \{w = (u, v) : ||(-\Delta + I)^\alpha u|| < \infty, \quad ||(-\Delta + I)^\alpha v|| < \infty\} \tag{3.3}$$

where $0 \le \alpha < 1$ and $H_0 = H = L_2(\Omega) \times L_2(\Omega)$. The corresponding norms $|| \ ||_\alpha$ are defined by

$$||w||^2 = ||u||_\alpha^2 + ||v||_\alpha^2, \quad ||u||_\alpha^2 = ||(-\Delta + I)^\alpha u||. \tag{3.4}$$

Rewrite system (1.4) and (1.5) as

$$\frac{dw}{dt} = A_\epsilon w + F(w, \epsilon, \lambda), \quad w(0) = w_0 \tag{3.5}$$

where $w = (u, v)^t$, $A = (A_1, A_2)^t$, $G = (G_1, G_2)^t$, the maps G_k are defined by

$$G_1(u, v, \epsilon, \lambda) = \lambda \epsilon^\mu a(x, y) v - \lambda^2 \epsilon^{\mu+1/2} u \tag{3.6a}$$

$$G_2(u, v, \epsilon, \lambda) = \epsilon^\mu g(x, y, u, \epsilon) - \epsilon^{\mu+1/2} v. \tag{3.6b}$$

Let us take $\alpha \in (3/4, 1)$ and suppose that the initial data w_0 lie in H_α.

Due to results [34] to prove local (for some minimal time interval) existence and uniqueness of solutions of (3.5), it is sufficient to check that the map $(u, v) \to G(u, v)$ from H_α to H (induced by the function G) is in the Lipschitz class.

Let us show that this map $F : H_\alpha \to H$ really lies at least in $C^{1+\kappa}$-Holder class where $0 < \kappa < 1$. (This is useful below.) To prove it, one uses so called Converse Taylor Theorem [34]. Suppose $h, w \in H_\alpha$. Then, since $g \in C^2$, one has

$$||G(w + h) - G(w) - L(w)h|| < C_G||h^2|| < C|h|_0||h|| \tag{3.7}$$

where $L(w)$ is some linear bounded operator $L : H \to H$. This operator is the Frechet derivative of G and is defined by

$$L(w) = (L_1, L_2)^t, \quad L_1(w)h = \lambda \epsilon^\mu a(x, y) h_2 - \lambda^2 \epsilon^{\mu+1/2} h_1,$$

$$L_2(w)h = g'_u(x, y, \epsilon, u) h_1 - \epsilon^{\mu+1/2} h_2.$$

Using the well known embeddings [34]

$$|u|_0 < c||u||_\alpha, \quad |v|_0 < c||v||_\alpha$$

one has from (3.7)

$$||G(w + h) - G(w) - L(w)h|| < C||h||_\alpha^2$$

that immediately yields the need assertion. Finally the local existence of solutions in some minimal time interval is shown.

Let us turn into the global existence. To continue solutions in $(0, \infty)$, it is sufficient to obtain some *a priori* estimates.

First let us note that rough estimates of $|u|$ and $|v|$ can be obtained trivially as a consequence of assumptions (3.1) and (3.2). More delicate estimates are obtained in section 5 under additional suppositions on a and g.

It gives, due to the well known results [34], that the norms $||w||_\alpha$ also are bounded. The global existence is proved.

4. TRANSFORMATION OF ORIGINAL SYSTEM. FAST AND SLOW VARIABLES

The key idea in showing that (1.4), (1.5) can be effectively investigated is a splitting in fast and slow variables. We shall show that slow modes capture essential dynamics and define the inertial form of the system in the inertial manifold $\mathcal{M}_1$ for small ϵ.

Taking into account Proposition 2.2 it is natural to make the following transformation. Namely one sets $u \to (q, w_1)$ and $v \to (p, w_2)$ where $q = (q_1, q_2, ..., q_m)$ and $p = (p_1, p_2, ..., p_n)$. These new variables are defined by

$$u = \epsilon^{1/4} \sum_{i=1}^{m} q_i \psi_i(x, \epsilon) + w_1, \quad < \psi_i, w_1 > \equiv 0, \tag{4.1}$$

$$v = \epsilon^{1/4} \sum_{j=1}^{n} p_j \phi_j(y, \epsilon) + w_2, \quad < \phi_j, w_2 > \equiv 0 \tag{4.2}$$

where ψ_i and ϕ_j are defined in sec. 2. Clearly

$$q_i = \epsilon^{-1/4} < u, \psi_i >, \quad p_j = \epsilon^{-1/4} < v, \phi_j > . \tag{4.3}$$

By substituting (4.1) and (4.2) in (1.4) and (1.5) one obtains the following system

$$\begin{aligned} \frac{dq_i}{dt} = & - 2\lambda^2 \epsilon^{\mu+1/2} q_i + R_i(w_1, q, \epsilon) \\ & + \lambda \epsilon^{\mu-1/4} < a(x, y)(\epsilon^{1/4} \sum_k p_k \phi_k + w_2), \psi_i >, \end{aligned} \tag{4.4}$$

$$\frac{dp_j}{dt} = -\,\epsilon^{\mu+1/2}p_j + \tilde{R}_j(w_2,p,\epsilon) + \lambda\epsilon^{\mu-1/4} < g(x,y,\epsilon^{1/4}\sum_l q_l\psi_l + w_1,\epsilon),\phi_j >, \tag{4.5}$$

$$\frac{\partial w_1}{\partial t} = A_1^\epsilon w_1 + S_1(w_1,q,\epsilon) + \lambda\epsilon^\mu(I-\mathcal{P})(a(x,y)(\epsilon^{1/4}\sum_l p_l\phi_l + w_2)) - \lambda^2\epsilon^{\mu+1/2}w_1, \tag{4.6}$$

$$\frac{\partial w_2}{\partial t} = A_2^\epsilon w_2 + S_2(w_2,p,\epsilon) + \lambda\epsilon^\mu(I-\tilde{\mathcal{P}})g(x,y,\epsilon^{1/4}\sum_k q_k\psi_k + w_1,\epsilon) - \epsilon^{\mu+1/2}w_2. \tag{4.7}$$

In equations (4.4) -(4.7)

$$R_i = < A_1(q_i\psi_i + \epsilon^{1/4}w_1),\psi_i >, \quad \tilde{R}_j = < A_2(\epsilon^{1/4}p_j\phi_j + w_2),\phi_j >, \tag{4.8}$$

the operators $\mathcal{P}$ and $\tilde{\mathcal{P}}$ are orthogonal projections on subspaces H^m and $\tilde{H}^n$ respectively

$$\mathcal{P}f = f - \sum_i < f,\psi_i > \psi_i, \quad \tilde{\mathcal{P}}f = f - \sum_j < f,\phi_j > \phi_j \tag{4.9}$$

and the functions S_1, S_2 are defined by

$$S_1 = \epsilon^{1/4}(\sum_i q_i[A_1\psi_i - < A_1\psi_i,\psi_i > \psi_i]) - \sum_k < A_1w_1,\psi_k > \psi_k, \tag{4.10}$$

$$S_2 = \epsilon^{1/4}(\sum_j p_j[A_2\phi_i - < A_2\phi_j,\phi_j > \phi_j]) - \sum_l < A_2w_2,\phi_l > \phi_l. \tag{4.11}$$

Notice that forms (4.8)-(4.11) hold due to definitions of ψ_i and ϕ_j and orthogonality properties (2.5) and (2.18).

Original system (1.4), (1.5) and the transformed one are equivalent. New equations (4.4)-(4.7) can be rewritten as

$$\frac{dz}{dt} = Bz + Z(z,w,\epsilon,\lambda), \quad z = (q,p)^t, \tag{4.12}$$

$$\frac{dw}{dt} = A_\epsilon w + G(r,w,\epsilon,\lambda), \quad w = (w_1,w_2)^t. \tag{4.13}$$

In (4.12) and (4.13) one sets

$$A_\epsilon = (A_1 - 2\lambda\epsilon^{\mu+1/2}\mathbf{I},\ A_2 - \epsilon^{\mu+1/2}\mathbf{I})^t = (\tilde{A}_1, \tilde{A}_2)^t,\quad B = (B_1, B_2)^t,$$

$$Z = (Z_1, Z_2)^t,\quad G = (G_1, G_2)^t,\quad B_1 = -2\lambda^2\epsilon^{\mu+1/2}\mathbf{I},\quad B_2 = -\epsilon^{\mu+1/2}\mathbf{I} \tag{4.14}$$

where $\mathbf{I}$ is the identity and

$$G_1 = S_1(w_1, q, \epsilon) + \lambda\epsilon^\mu(I - \mathcal{P})(a(x,y)(\epsilon^{1/4}\sum_l p_l\phi_l + w_2), \tag{4.15}$$

$$G_2 = S_2(w_2, p, \epsilon) + \lambda\epsilon^\mu(I - \tilde{\mathcal{P}})g(x, y, \epsilon, \epsilon^{1/4}\sum_k q_k\psi_k + w_1). \tag{4.16}$$

For Z_k one has the explicit forms

$$(Z_1)_i = \lambda\epsilon^{\mu-1/4} < a(\cdot,\cdot)(\epsilon^{1/4}\sum_l p_l\phi_l + w_2), \psi_i > +R_i, \tag{4.17}$$

$$(Z_2)_j = \lambda\epsilon^{\mu-1/4} < g(\cdot,\cdot,\epsilon^{1/4}\sum_k q_k\psi_k + w_1, \phi_i > +\tilde{R}_j. \tag{4.18}$$

Let us investigate equations (4.12) and (4.13) in the ambient phase space $X = Y \times E$, where $Y = \{w = (w_1, w_2) \in H : \mathcal{P}w_1 \equiv 0, \tilde{\mathcal{P}}w_2 \equiv 0\}$ and $E = \mathbf{R}^{m+n}$.

Denote $Y_\alpha = H_\alpha \bigcap Y$. Due to results of previous section, the right-hands R and G are the maps at least from $C^{1+\kappa}$-Holder class as maps from $Y_\alpha \times E$ to E and Y respectively.

Due to Proposition 2.2 in $Y \bigcap DomA$ one has for small ϵ

$$< A_\epsilon w,\ w > \leq -\beta||w||^2 \tag{4.19}$$

where β does not depend on ϵ as ϵ tends to 0. This barrier for A_ϵ yields that w is the fast mode and r is the slow one. Below we shall sometimes omit the index ϵ in the denotation of the operator A_ϵ.

To conclude this section, let us obtains some auxiliary technical estimates which are useful in the following section and help us to show that this system is dissipative.

Let us use estimate (2.6) and analogous relation for B_n. These inequalities imply

$$|A_1\psi_i|,\quad |A_2\phi_i| < c_N\epsilon^N \tag{4.20}$$

that holds for arbitrary integer $N > 0$ and sufficiently small ϵ.

These estimates are useful in the following sections.

5. DISSIPATIVE PROPERTIES OF SYSTEM

Let us define some special class $\mathcal{G}_{n,m,\epsilon}$ of functions g. *All the following results hold for* $g \in \mathcal{G}_{n,m,\epsilon}$.

Definition of class $\mathcal{G}_{n,m,\epsilon}$.

Functions from this class are defined by

$$g = \sum_{s=1}^{3} \left[\sum_{k=1}^{m} \sum_{j=1}^{n} b_s(x,y)\theta_s((x-x_k)/\sqrt{\epsilon})\eta_s((y-y_j)/\sqrt{\epsilon}) \right] K(u), \quad (5.1)$$

where $\theta_s(z), \eta_s(z), b_s$ *and* K *satisfy conditions*

$$\theta_s, \eta_s, b_s, K \in C^2, |K|_2 < C, \quad supp\ \eta_s, \quad supp\ \theta_s \subset (-c_0, c_0). \quad (5.2)$$

Suppositions (5.2) entail in particular that the perturbation g is concentrated in neighborhoods of size $O(\sqrt{\epsilon})$ at points (x_i, y_j).

Before estimation, let us remind some standard facts about linear operators [34] and obtain some auxiliary inequalities. For operator $A_\epsilon = (\tilde{A}_1, \tilde{A}_2)$ (restricted to Y) one has

$$|| \exp(\tilde{A}_k t) u ||_\alpha \leq C_0 ||u||_\alpha \exp(-\beta t), \quad (5.3)$$

$$||(-\tilde{A}_k)^\alpha \exp(\tilde{A}_k t) u|| = || \exp(\tilde{A}_k t) u||_\alpha \leq C_1 b_\alpha(t) \exp(-\beta t) ||u|| \quad (5.4)$$

where β does not depend on ϵ, λ for small ϵ, λ, $b_\alpha = \min\{1, t^{-\alpha}\}$ and $k = 1, 2$.

In fact, the operators $\tilde{A}_k$ are self-adjoint, with a dense domain and negatively defined due to (4.19) [34].

Dissipative properties of system (4.4)-(4.7) can be described by the following assertion.

PROPOSITION 5.1. – *For solutions of equations* (4.4)-(4.7) *the following a priori estimates hold:*

$$||w_2(t)||_\alpha \leq C_1 \exp(-\beta t) + C_1^* \lambda \epsilon^{\mu+1/2}, \quad (5.5)$$

$$|p(t)| < C_2 \exp(-\epsilon^{\mu+1g/2} t) + C_2^* \lambda, \quad (5.6)$$

$$||w_1(t)||_\alpha \leq C_3 \exp(-\kappa(\epsilon, \lambda) t) + C_3^* \lambda \epsilon^{\mu+1/4} \quad (5.7)$$

and

$$|q(t)| < C_4 \exp(-\kappa(\epsilon, \lambda) t) + C_4^* \quad (5.8)$$

where $\kappa(\epsilon, \lambda) > 0$. Constants C_i continuously depend on initial data and do not depend on small parameters ϵ, λ as $\epsilon, \lambda \to 0$. Constants C_i^* do not depend on small parameters and initial data.

The proof uses estimates (5.4) and can be found in Appendix 1.

Inequalities (5.5)-(5.8) allow to conclude that the set $\mathcal{A}$ defined by

$$\mathcal{A} = \{(q, p, w_1, w_2) : |q| < C_1^*, \ |p| < C_2^* \lambda, ||w||_\alpha < C\lambda\epsilon^\mu\} \tag{5.9}$$

(where C_i^* are some positive constants) is absorbing.

Finally the following assertion is proved.

LEMMA 5.1. – *If the function g from class $\mathcal{G}_{n,m,\epsilon}$ then dynamics S^t induced by* (1.4), (1.5) *is dissipative and the set $\mathcal{A}$ is absorbing.*

Let us turn into the construction of the invariant manifold.

6. EXISTENCE OF LOCALLY INVARIANT AND GLOBALLY ATTRACTING MANIFOLD $\mathcal{M}_1$

Suppose m, n, a, g, λ are fixed. Let us prove that for small ϵ system (4.4)-(4.7) has an inertial manifold. First let us define the subset $\mathcal{A}_0$ in $\mathbf{R}^{n+m}$ by

$$\mathcal{A}_0 = \{(q, p) : \ |q| < C_1^*, \ |p| < C_2^* \lambda\}. \tag{6.1}$$

PROPOSITION 6.1. – *Suppose m, n and $a, g, V_m, \tilde{V}_n, \lambda$ are fixed and satisfy assumptions* (2.1), (2.2), (5.1) *and* (3.1), (3.2). *Then there exists a positive number $\epsilon_0(a, g, m, n, \lambda)$ (depending on these parameters) such that for $0 < \epsilon < \epsilon_0$ one has:*

I. *There exists a manifold $\mathcal{M}_1$ defined by equations*

$$w_1 = W_1(p, q, \epsilon, \lambda), \quad w_2 = W_2(p, q, \epsilon, \lambda). \tag{6.2}$$

The maps $W_k : (p, q) \to w_k$ lie in Holder class $C^{1+\nu}$ of the mappings from E to H_α where $\nu \in (0, 1)$. These maps W_k satisfy estimates

$$||W_k(\cdot, \cdot, \epsilon)||_{C^1(\mathcal{A})} < C\epsilon. \tag{6.3}$$

II. *The manifold $\mathcal{M}_1$ is locally invariant in the absorbing set $\mathcal{A}$ and globally attracting in H_α.*

This means that if any initial data $w^0 = (u^0, v^0)$ lie in $\mathcal{M}_1$ and a piece of the trajectory $w(t) = S^t w^0$ for $t \in [0, t_0]$ lies in the absorbing set $\mathcal{A}$, then *all this piece lies on the manifold $\mathcal{M}_1$. Moreover, every trajectory with an exponential rate tends to the manifold $\mathcal{M}_1$*

$$\text{dist}\{w(t), \mathcal{M}_1\} < C\exp(-\beta_0 t), \quad t > 0 \tag{6.4}$$

where β_0 does not depend on ϵ and initial data but the constant C can depend on initial data.

The proof is standard and can be found in Appendix 2.

7. INVESTIGATION OF THE FINITE DIMENSIONAL DYNAMICS INDUCED BY GLOBAL SEMIFLOW IN THE MANIFOLD $\mathcal{M}_1$

Let us consider the inertial form of equations (1.4) and (1.5) in manifold $\mathcal{M}_1$.

In this case it is system of $m+n$ equations that arises if one substitutes in (4.4)-(4.7) equations (6.2) for w_k.

As a result one finds using (6.3) and the rescaling time $T = \epsilon^{\mu+1/2}t$ that equations (1.13) give the inertial form in $\mathcal{M}_1$. Variables q, p lie in $\mathcal{A}_0 \subset E = R^m \times R^n$. The vector fields $\tilde{Q}$ and $\tilde{P}$ satisfy

$$|\tilde{Q}|_1, \quad |\tilde{P}|_1 < c\epsilon. \tag{7.1}$$

In fact these contributions connected with w_k in (4.4) and (4.5) have orders $<< \epsilon^{\mu+1/2}$. For instance one has (let us remind that $\mu > 1$)

$$\lambda\epsilon^{\mu-1/4}| < aw_2, \phi_j > | < C\lambda\epsilon^{\mu-1/4}||w_2|| < C\lambda\epsilon^{\mu+3/4}.$$

Expressions Q and P in (1.13) are defined by

$$Q_{ij} = \alpha_{ij}p_j, \quad P_{ij} = \sum_{s=1}^{3} b_{ij}^s \Theta_s(q_i). \tag{7.2}$$

In (7.2) the matrix α and b_s are defined by the simple relations

$$\alpha_{ij} = a(x_i, y_j), \quad b_{ij}^{(s)} = b_s(x_i, y_j) \tag{7.3}$$

where x_i, y_j are the points from the sets X_m, Y_n defined by in sec. 2.

The functions $\Theta_s(q_i)$ are defined by more complicated expressions

$$\Theta_s(q_i) = \gamma_s \int_{-c_0}^{c_0} \theta_s(z)\exp(-z^2/2)K(cq_i\exp(-z^2/2))dz \tag{7.4}$$

where

$$\gamma_s = \int_{-\infty}^{\infty} \eta_s(y) \exp(-y^2/2) dy, \quad c > 0. \tag{7.5}$$

These forms hold for small ϵ as a consequence of hypothesis (5.2).

First let us prove an auxiliary assertion which shows that Θ_s can have sufficiently general form.

LEMMA 7.2. – *Suppose C_f is some positive number and moreover let $f_s(x)$ (where $x \in \mathbf{R}$ and $s = 1, 2, 3$) be given functions $\in C^2$ and δ be some small positive number.*

Then there exist functions $K(u)$ and θ_s, η_s satisfying (5.2) and such that

$$|f_s - \Theta_s|_{C^1(-C_f, C_f)} < \delta \tag{7.6}$$

where Θ_s are defined by (7.4).

Proof. – By choosing η_s one has $\gamma_s = 1$. Let us set $\theta_s(z) = |z|\tilde{\theta}_s(y)$ where $y = \exp(-z^2/2)$. Integral (7.4) now gives

$$\Theta_s(x) = 2 \int_{r_0}^{1} K(cxy)\tilde{\theta}_s(y) dy \tag{7.7}$$

where $r_0 = \exp(-c_0^2/2)$.

We can approximate given functions f_s in interval $(-C_f, C_f)$ by polynomials of degree N uniformly in C^1-norm and with accuracy $\delta/2$. Thus one can suppose that $f_s = \sum_{l=0}^{N} f_{ls} x^l$. Let us define the bounded kernel $K(cxy)$ such that $K(cxy) = \sum_{l=0}^{N} K_l (xy)^l$ for any $|x| < C_f$ where all coefficients $K_l \neq 0$. Furthermore one takes $\tilde{\theta}_s(y)$ such that $\int_{r_0}^{1} y^l \tilde{\theta}_s(y) dy = f_{ls} K_l^{-1}$ completing the proof.

Throughout below one supposes that θ_s and K are chosen so that relations (7.6) hold with

$$f_1(x) = \cos x, \quad f_2(x) = \sin x, \quad f_3(x) = x h_\nu(x) \tag{7.8}$$

where h_ν is a special cut-off function having properties

$$h_\nu \in C^2, \quad h_\nu(x) = 1 \quad \textit{if } |x| < 1, \tag{7.9a}$$

$$h_\nu(x) \leq 1 \ (x \in \mathbf{R}), \quad h_\nu \equiv 0 \quad \textit{for any } |x| > 1 + \nu \tag{7.9b}$$

where positive parameter ν will be chosen below.

Such choice is possible due to Lemma 7.2 and allows (as it will be shown in the following section) to simplify system (1.13).

8. SIMPLIFICATION OF EQUATIONS (1.13)

System (1.13) is complicated and it is natural to try to simplify equations. To do it let us suppose in addition that λ is small.

Throughout one assumes that ϵ lies in $I(\lambda) = \{0 < \epsilon < \epsilon_1(\lambda)\}$, where $\epsilon_1(\lambda) < \epsilon_0(\lambda)$ and ϵ_0 is the number from Proposition 5.1. Clearly for any λ one can take ϵ_1 so that the terms $\tilde{P}_{ij}, \tilde{Q}_{ij}$ are small corrections $\epsilon < \epsilon_1(\lambda)$.

Let us recall that we study system (1.13) in the set $\mathcal{A}_0$ defined by (6.1). It is clear that without any loss of generality one can increase constants C_i^* from definition 6.1. It is easy to see that (under assumptions (7.8) and (7.9)) these constants can be taken so (for small ϵ and large C_f) that the set $\mathcal{A}_0$ be invariant under dynamics (1.13). In fact, due to these assumptions terms P_{ij} are uniformly bounded and thus one can take C_i^* satisfying the inequalities $m \max P_{ij}(\cdot) < C_2^*$ and $\max |a_{ij}| C_2^* < 2C_1^*$ that implies the invariance of $\mathcal{A}_0$.

Finally (if a such set $\mathcal{A}_0$ is taken) one can restrict our consideration of dynamics (1.13) to trajectories starting in this set. All estimates that shall appear below hold in the absorbing set $\mathcal{A}_0$.

For small λ it is useful to substitute

$$p_j = \lambda \sum_{i=1}^{m} P_{ij}(q_i) + \rho_j \tag{8.1}$$

where ρ_j are new unknown functions.

Then equation (1.13b) takes the form

$$\frac{d\rho_j}{dT} = -\rho_j + \lambda^2 Y_j(q, \rho, \lambda, \epsilon) \tag{8.2}$$

where $|Y(q, \rho, \lambda, \epsilon)|_1 < C$.

Clearly the variables ρ are fast and q are slow.

For $\rho(T)$ one has

$$|\rho(T)| < c\lambda^2 + c_1 \exp(-T). \tag{8.3}$$

Using estimates (7.1) and (8.3) and repeating standard arguments one can obtain the following proposition which allows to simplify inertial form (1.13). In fact it is clear that since for large T the quantities ρ_j are very small one can substitute in (1.13a) the form $p_j = \lambda \sum_i P_{ij}(q_i)$.

PROPOSITION 8.1. – **I.** *The finite dimensional semiflow defined by equations* (1.13) *has in* $\mathcal{A}_0$ *for* $\lambda < \lambda_0$ *and* $\epsilon \in I(\lambda)$ *a new inertial manifold* $\mathcal{M}_2$ *of the dimension* n. *It is defined by equations*

$$p_i = \lambda \sum_i P_{ij}(q_i) + \sigma_i(q, \lambda, \epsilon), \quad \sigma \in C^1, \quad |\sigma(\cdot, \lambda, \epsilon)|_1 < c(\lambda^2 + \epsilon^{\delta_0}),$$

where $\delta_0 > 0$. *Moreover*

$$\operatorname{dist}\{p(T), \mathcal{M}_2\} < C \exp(-\beta_1 T), \quad T > 0 \tag{8.4}$$

and the constant β_1 *does not depend on* ϵ *and* λ.

II. *The inertial form of (1.13) (which holds on* $\mathcal{M}_2$*) has the form*

$$\frac{dq_i}{d\tau} = -q_i + \sum_{j=1}^{n} \alpha_{ij} \Phi_j(q) + S_i(q, \epsilon, \lambda), \quad i = 1, \dots, m \tag{8.5}$$

where $\tau = \lambda^2 T$ *and*

$$\Phi_j(q) = \sum_{l=1}^{m} P_{lj}(q_l). \tag{8.6}$$

The corrections S_i *satisfy estimates*

$$|S_i(\cdot, \epsilon, \lambda)|_{C^1} < C\sigma(\lambda, \epsilon), \quad \sigma \to 0 \quad \text{as} \quad \lambda, \epsilon \to 0. \tag{8.7}$$

We omit the proof since it is absolutely standard. It can be found in the Appendix 2. The new inertial form can be studied in a simple way. This analysis shows that in order to prove the Theorem we should use some special matrices a_{ij} and $b_{ij}^{(3)}$. We define it in the next section.

9. PRELIMINARIES: SPECIAL CHOICE OF α_{ij} AND SIMPLIFICATION OF EQUATIONS (8.5)

Step 1. Special choice of matrix a *and* $b^{(3)}$.

Consider the grid $\mathbf{Z}^n$ consisting of integer n-dimensional vectors $\mathbf{k}$. Every vector $\mathbf{k}$ defines some linear form in $\mathbf{R}^n$ by relation $q \to \mathbf{k} \cdot q = \sum_{s=1}^{n} k_s q_s$.

Let us consider infinite matrix $\bar{\alpha}_{ij}$ (where $i \in \mathbf{Z}_+$ and $j = 1, ..., n$). Each matrix α defines some family of linear forms by

$$E_l(q) = \sum_{j=1}^{n} \bar{\alpha}_{lj} q_j. \tag{9.1}$$

Let us choose a such $\bar{\alpha}$ that the following conditions hold:

I. relation

$$\bar{\alpha}_{ij} = \delta_{ij}, \quad i = 1, ..., n, \quad j = 1, ..., m \tag{9.2}$$

is fulfilled (where δ_{lj} is the Kronecker symbol).

II. the infinite set of linear forms E_l (which are defined by (9.1)), where $l = 1, 2, ...$ contains all forms $\mathbf{k} \cdot q$.

Now let us set

$$\alpha_{ij} = \bar{\alpha}_{ij}, \quad b_{ij}^{(3)} = 2\delta_{ij} \quad i = 1, ..., m, \quad j = 1, ..., n. \tag{9.3}$$

Step 2. Simplification of equations (8.5)

Under supposition (9.2), equations (8.5) can be reduced to a system consisting of only n equations.

Let us define new variables Q and z

$$z_i = q_i - \sum_{j=1}^{n} \alpha_{ij} q_j, \quad i = n+1, ..., m, \quad Q_j = q_j, \quad j = 1, ..., n. \tag{9.4}$$

Using property (9.2) one finds

$$\frac{dQ_j}{d\tau} = \Phi_j(q(Q, z)) - 2Q_j + S_j(Q, z, \epsilon, \lambda). \tag{9.5}$$

By simple calculations one obtains

$$\frac{dz_i}{d\tau} = \frac{dq_i}{d\tau} - \sum_j \alpha_{ij} \frac{dQ_j}{d\tau} = -2z_i + \tilde{S}_i, \quad i > n, \tag{9.6}$$

where

$$\tilde{S}_i = S_i - \alpha_{ij} S_j, \quad |\tilde{S}_j|_1 = \delta(\lambda, \epsilon) \to 0 \quad \text{if} \quad \lambda, \epsilon \to 0. \tag{9.7}$$

Finally we see (setting $\Psi_j(Q, z) = \Phi_j(q(Q, z)) - 2Q_j$) that system (8.5) can be transformed to the form

$$\frac{dQ_j}{d\tau} = \Psi_j(Q, z) + S_j(Q, z, \epsilon, \lambda), \quad j = 1, ..., n, \tag{9.8}$$

$$\frac{dz_i}{d\tau} = -2z_i + \tilde{S}_i(Q, z, \epsilon, \lambda), \quad i = n+1, ..., m. \tag{9.9}$$

Under our choice θ, η, K one has for Ψ_j the following forms

$$\Psi_j = \tilde{F}_j(Q, z) + 2h_\nu(Q_j)Q_j - 2Q_j, \tag{9.10}$$

where

$$\tilde{F}_j(Q, z) = \sum_{l=1}^{m} b_{lj}^{(1)} \cos(E_l(Q) + z_l) + b_{lj}^{(2)} \sin(E_l(Q) + z_l), \tag{9.11}$$

where forms E_l are defined by (9.1).

The main result of this section (simplification of equations (8.5)) allows to obtain (in the next section) a description of the large time behavior of trajectories. It finds out (under some additional assumptions) that there are possible a new reduction of these simplified equations.

Step 3. The large time behaviour of reduced system (9.8), (9.9).

Since $\tilde{S}$ is bounded one has

$$|z(\tau)| < c\delta(\lambda, \epsilon) + C|z(0)| \exp(-\tau). \tag{9.12}$$

Taking into account this estimate naturally one sets

$$F_j(Q) = \Psi_j(Q, 0) \tag{9.13}$$

i.e. field F is the value of $\Psi(Q, z)$ at $z = 0$.

The following analysis of (9.10), (9.11) holds under the additional assumptions

$$\max_i \sum_j |\frac{\partial F_i(Q)}{\partial Q_j}| < 1 \quad \text{for any} \quad Q \in \mathbf{R}^n \tag{9.14a}$$

and for each i, $i = 1, 2, ..., n$

$$Q_i F_i(Q) < 0 \text{ for any } Q \text{ such that } Q \in \mathcal{Q}_1 \quad \text{and} \quad |Q_i| = 1 \tag{9.14b}$$

that means that the field F is directed inside a cube $\mathcal{Q}_1$ at the cube boundary.

Let us prove the simple but important lemma.

LEMMA 9.1. – *Denote $\mathcal{Q}_1$ the cube $\{Q : |Q_j| \leq 1\}$. Then one can choose a function h_ν and a number ν from* (7.9) *so that the set*

$$W = \mathcal{Q}_1 \times \mathcal{Z}, \quad \mathcal{Z} = \{z : |z| < \sigma_1(\epsilon, \delta)\} \tag{9.15}$$

where σ_1 tends to 0 as $\epsilon, \lambda \to 0$ is absorbing.

Proof. – Clearly $z(\tau) < \sigma_1$ holds for large τ that follows from estimate (9.12).

Let us define the norm $|Q|$ in Euclidian space as $\max_i |Q_i|$. This norm define some distance between points in $\mathbf{R}^n$. Let $r(\tau)$ the distance between unit cube and $Q(\tau)$. Consider the time evolution of $r(\tau)$. If ν is sufficiently small and $r > \nu$ one has due to (7.9) and (9.14)

$$\frac{dr}{d\tau} < r - 2(r+1) + c\sigma < -\delta_2 < 0. \tag{9.16}$$

On the other hand, if $r < \nu$, one obtains

$$\frac{dr}{d\tau} < -\delta_3 < 0 \tag{9.17}$$

as a consequence of (7.9) and condition (9.14b). Thus $\frac{dr}{d\tau} < -\delta_4 < 0$ holds anywhere out of the unit cube that proves the lemma.

Finally it is sufficient to investigate the dynamics (9.8), (9.9) in the set W. Inside W one has $h_\nu \equiv 1$ and equations (9.8) takes the very simple form

$$\frac{dQ_j}{d\tau} = F_j(Q) + \bar{S}_j(Q, z, \epsilon, \lambda), \quad j = 1, ..., n \tag{9.18}$$

where corrections $\bar{S}_j$ are small in the norm $C^1(\mathcal{Q})$. Now again using standard arguments it is easily to prove that system (9.8), (9.9) has an invariant n-dimensional manifold $\mathcal{M}$. One has

PROPOSITION 9.2. – *Under assumptions* (9.13) *and* (9.14) *the system* (9.8), (9.9) *has the inertial manifold $\mathcal{M}$ from C^1-class which is locally invariant in the absorbing set W. This surface is defined by equations*

$$z_k = Z_k(Q, \lambda, \epsilon), \quad |Z|_{C^1(\mathcal{Q})} < \sigma(\lambda, \epsilon) \tag{9.19}$$

where $\sigma \to 0$ as ϵ, $\lambda \to 0$. Here

$$\mathrm{dist}((\mathrm{Q}, \mathrm{z}(\tau)), \ \mathcal{M}) < \mathrm{C}\exp(-\mathrm{c}\tau), \quad \tau > 0 \tag{9.20}$$

The inertial form of (9.8), (9.9) in this manifold can be written as

$$\frac{dQ_j}{d\tau} = F_j(Q) + \Sigma_j(Q, \lambda, \epsilon), \quad |\Sigma|_{C^1(\mathcal{Q})} \to 0 \ \ (\epsilon, \lambda \to 0). \tag{9.21}$$

One omits the trivial proof (It can be proved analogously to prop. 8.1, see Appendix 2, part 2).

In the next section using results of sections 5-9 one will prove main Theorem 1.1.

10. MAIN ASSERTION ABOUT EMBEDDING OF ARBITRARY PRESCRIBED DYNAMICS IN THE DYNAMICS IN THE INERTIAL MANIFOLD $\mathcal{M}$

Let us consider the cube $\mathcal{Q} = \{Q : |Q_i| \leq 1\} \subset R^n$ and the system of ordinary differential equations of the class C^1 in $\mathcal{Q}$

$$\frac{dQ_i}{d\tau} = F_i^{pr}(Q), \quad F^{pr} \in C^1, \quad |F^{pr}|_{C^1(\mathcal{Q})} < 1. \tag{10.1}$$

Suppose that the cube $\mathcal{Q}$ is invariant under prescribed dynamics (10.1) *i.e.* for the field F^{pr} condition (9.14b) holds.

The proof falls into some steps. We are going to present an explicit algorithm of solution of the following inverse problem: having given dynamics one will find the number m and the coefficients $a, g, V_m, \tilde{V}_n$ in (1.4) and (1.5) so that the inertial manifold $\mathcal{M}$ exists and the inertial dynamics "almost" coincides with the prescribed one *i.e.* inequality (1.12) holds.

Step 0. First for any number $m > n$ let us choose matrix a_{ij} satisfying conditions **I** and **II** from section 9 and matrix $b_{ij}^{(3)}$ satisfying (9.3). Let us continue the field F^{pr} in $\mathbf{R}^n$ so that a new field (one denotes it again by F^{pr}) will 2π-periodic over all arguments and in addition will satisfy the condition $|F^{pr}|_{C^1(\mathbf{R}^n)} < 1$.

Step 1. Choice of m and matrices $b^{(1)}$ and $b^{(2)}$.

Let us approximate the prescribed field F^{pr} uniformly in the unit cub $\mathcal{Q}_\pi$ in the norm C^1 with accuracy $\delta/4$ by a new field from $C^{n+2}(\mathcal{Q}_\pi)$. Such approximation for instance can be obtained by the averaging

$$F_{av}^{pr} = \int_{\mathcal{Q}} F^{pr}(Q - Q')\omega(Q')d^nQ'. \tag{10.2}$$

The field F_{av}^{pr} can be approximated uniformly in the norm C^1 in $\mathcal{Q}_1$ by the Fourier sums. For sake of the choice of the matrix a_{ij}, the field F_{av}^{pr} can be approximated by the field F. In fact due to (9.13) this field is Ψ from (9.11) at $z = 0$, *i.e.* one has

$$F_j(Q) = \sum_{l=1}^{m} b_{lj}^{(1)} \cos E_l(Q) + b_{lj}^{(2)} \sin E_l(Q). \tag{10.3}$$

The sums (10.3) contain in particular all finite Fourier sums as follows from supposition **II** from section 9.

Thus if m is large enough one obtains for appropriate coefficients $b_{ij}^{(s)}$

$$|F_j^{pr}(\cdot) - F(\cdot, b)|_{C^1(\mathbf{R}_n)} < \delta/4. \tag{10.4}$$

Step 2. Choice of h_ν.

Having field F constructed by steps 1 and 2 (which is close to prescribed and satisfies conditions (9.14)) let us take h_ν satisfying Lemma 9.1.

Step 3. Choice of sets X_n, Y_m, coefficients $V_n, \tilde{V}_m$ and nonlinearity g.

Let us define arbitrary sets X_n and Y_m as it was explained above in section 2. Furthermore one constructs V_m and $\tilde{V}_n$ according to sec. 2.

Having the points x_i, y_j from sets X_n and Y_m one constructs coefficients $\alpha, b_s(x, y)$ so that $a(x_i, y_j) = \alpha_{ij}, b_s(x_i, y_j) = b_{ij}^{(s)}$ where matrices α, b^s in the right hand have been defined above (steps 0 and 1).

Let us define g with the help of Lemma 7.2 with a sufficiently small δ_f in (7.6). In (7.6) the functions f_s are defined by (7.8) and the interval $(-C_f, C_f)$ is sufficiently large.

Last Step. Let us take sufficiently small $\lambda_0(\nu, a, b, n, ...)$ such that as $\epsilon < \epsilon_1(\lambda)$ one has for $\lambda < \lambda_0$:

1. Lemma 9.1 and Proposition 9.2 hold and thus system (9.8), (9.9) (or, that is the same, system (8.5)) asymptotically reduces to (9.18) as $t \to \infty$.

2. If it is necessary, let us decrease λ so that Proposition 8.1 holds and thus system (1.13) asymptotically reduces to (8.5);

3. If it is necessary, let us choose (for any $\lambda < \lambda_0$) a new critical $\epsilon_1(\lambda)$ so that Proposition 6.1 holds that allows to asymptotically reduce (1.4), (1.5) to (1.13).

Finally one sees that the combination of results of sections 2-10 proves the Theorem 1.1 and moreover one has the constructive algorithm allowing to find system (1.4), (1.5) with the prescribed dynamics.

11. CONCLUSION

The construction of a rigorous mathematical theory of developed hydrodynamic turbulence and the proof of the strange attractor existence for the Navier-Stokes equations is a very difficult problem. In fact even the Kolmogorov lattice is not resolved yet [35].

However one can modify the Kolmogorov system so that the problem of *chemical turbulence* (*i.e.* diffusion chaos in reaction-diffusion systems) can be solved at least from the formal mathematical point of view.

Prof. D. Ruelle have supposed that for reaction-diffusion systems the strange attractors can appear which can generate a chemical turbulence (diffusion chaos). Prof. Y. Kuramoto have formulated the problem [38] on a description of a connection between the diffusion chaos and the determined chaos in reaction-diffusion systems.

This paper shows that Y. Kuramoto's and D. Ruelle's foreseens were correct and the strange attractor existence in the reaction-diffusion systems can be found. Moreover, the existence of strange attractors of arbitrary dimensions is an analytical fact that holds even for reaction-diffusion systems with the minimal number of component (*i.e.* with two components) where such attractors are possible.

The author is thankful to Prof E. L. Aero, Prof. V. S. Buslaev, Prof. Yu. S. Il'yashenko, Prof. D. Ruelle and Prof. J. L. Lions for an interest in previous author works that stimulated and encouraged this investigation.

The author is grateful to Prof. P. Polacik and Prof. I. A. Molotkov for the help and Dr. V. M. Shelkovich for a discussion. Also the author is very grateful to Dr. J. Denzler (Munich), Dr. R. Pines (New York) and the Soros Fund for financial support.

This work is supported by Russian Fund of Basic Research (RFBR), grant No. 95-01-01084a.

APPENDIX 1

Proof of proposition 5.1

Our aim is to obtain estimates (5.5-5.8).

Proof splits into two steps.

1. *Preliminary step: rough* a priori *estimate of* $||u||$ *and* $||v||$.

Consider equation (1.5). Notice that as follows from spectral analysis in sec. 2 one has $< A_2(\epsilon)v, v > \leq c_N\epsilon^N||v||^2$ for small ϵ. Thus by multiplying (1.5) through v one sees that equation (1.5) yields

$$\frac{d||v||}{dt} < -c\epsilon^{\mu+1/2}||v|| + c_1\lambda\epsilon^{\mu}$$

and thus for sufficiently large t one has

$$||v(\cdot,t)||, |p(t)| < c\epsilon^{-k} \tag{A1.1}$$

for some k. In a similar way, using (A1.1) one can obtain *a priori* estimate for $||u||$ and $|q|$. These preliminaries allow to avoid diffuculties connected with estimates of the terms h_ϵ below.

2. *Exact estimates for* w, q *and* p.

First let us notice that due to assumptions (5.2) on the supports of θ_s, η_s one has

$$| < g(\cdot,\cdot,u), \phi_j > | \leq c\epsilon^{3/4} \tag{A1.2}$$

and

$$||g(\cdot,\cdot,u,\epsilon)|| \leq c\epsilon^{1/2}. \tag{A1.3}$$

It gives (together with (A1.1)) the following inequalities

$$|Z_2| \leq c\lambda\epsilon^{\mu+1/2} + h_\epsilon, \tag{A1.4}$$

$$||G_2|| \leq c\lambda\epsilon^{\mu+1/2} + h_\epsilon \tag{A1.5}$$

where the expressions h_ϵ denote the quantities which are less than $C_N\epsilon^N(1 + ||w_2|| + |p|)$ for any $N > 0$ and sufficiently small ϵ. Due to preliminary estimate (A1.1) one has that $h_\epsilon < c_N\epsilon^N$ where N is any integer. All such contributions are negligible in estimates that follow below and we omit these terms.

One finds using definitions of ψ_i, ϕ_j that

$$| < \psi_i, \phi_j > | = \sqrt{\epsilon}(1 + O(\exp(-c\epsilon^{-1/2})) \tag{A1.6}$$

that yields

$$|Z_1| < C[\lambda\epsilon^{\mu+1/2}|p| + \lambda\epsilon^{\mu-1/4}||w_2||] \tag{A1.7}$$

and

$$||G_1|| < C[\ \lambda\epsilon^{\mu+1/4}|p| + \lambda\epsilon^{\mu}||w_2||\ ]. \tag{A1.8}$$

Let us begin with the estimate of $||w_2(t)||_\alpha$. Rewriting (4.7) in the integral form

$$w_2(t) = \exp(-\tilde{A}_2 t)w_2(0) + \int_0^t \exp(-\tilde{A}_2(t-t_1))G_2(u(t_1))dt_1 \tag{A1.9}$$

one finds from (5.3), (5.4) and (A1.5) that

$$||w_2(t)||_\alpha \leq C(||w_2(0)||_\alpha \exp(-\beta t) + \lambda\epsilon^{\mu+1/2}). \tag{A1.10}$$

Thus inequality (5.5) is proved.

There is an important detail mentioned by the Referee of this work. In fact, this estimate (A1.10) involves $|p|$ through h_ϵ. Thus the first impression that these arguments (giving (A1.10) are non quite correct since the estimate of $|p|$, that follows below, is based on (A1.10).

However one can use here the preliminary step *i.e.* (A1.1). Due to (A1.1) one has that h_ϵ *a priori* bounded. In fact, $h_\epsilon < C_N\epsilon^N(1+|p|+|w_2|) < \tilde{C}_N\epsilon^{N-k}$ where k could be large but fixed and N is any positive number. It shows that here and below h_ϵ are *a priori neglectable* quantities and, thus, they can be omitted in estimates.

Let us turn to estimates of $|p(t)|$. One obtains from (4.5)

$$|p(t)| < |p(0)|\exp(-\epsilon^{\mu+1/2}t) + c\int_0^t \exp(-\epsilon^{\mu+1/2}(t-t_1))\sup|Z_2(t_1)|dt_1$$

that immediately gives (5.6) by (A1.4).

Inequalities for norms of p and w_2 in turn can be used to obtain the estimates for q and w_1.

Consider (4.6) and let us rewrite this equation by an integral form. Using (A1.8), (A1.10) and (5.6) one obtains

$$\begin{aligned}||w_1(t)||_\alpha &\leq \exp(-\beta t)||w_1(0)|| \\ &\quad + C\int_0^t b_\alpha(t-t_1)\exp(-\beta(t-t_1))||G_1(v(t_1))||dt_1 \\ &\leq \exp(-\beta t)||w_1(0)||_\alpha + C\lambda\epsilon^\mu[\epsilon^{1/4}\exp(-\epsilon^{\mu+1/2}t) \\ &\quad + c\lambda\epsilon^{\mu+1/4} + \lambda||w_2(0)||\exp(-\beta t)]. \end{aligned} \tag{A1.11}$$

Substituting into (A1.11) estimates (5.5) and (5.6) one has (5.7). In a similar way, one finds from equation (4.7) that

$$|q(t)| \leq \exp(-\lambda^2\epsilon^{\mu+1/2}t)|q(0)| + C\int_0^t \exp(-\lambda^2\epsilon^{\mu+1/2}(t-t_1))|Z_1(t_1)|dt_1.$$

Let us use estimate (A1.7) for Z_1 where one substitutes results (5.5) and (5.6) for $|p|$ and $||w_1||$. After some calculations one has (5.8) that completes the proof.

APPENDIX 2

Proof of propositions 6.1 and 8.1

1. *Proof of Proposition* 6.1.

Due to Proposition 5.1 one can suppose that the initial data lie in domain $\mathcal{A}$ defined by (5.9). Let us prove that this domain contains locally invariant for (1.4), (1.5) surface which attracts (with exponential rate) all solutions starting from $\mathcal{A}$. Again the proof splits into two steps. The first preliminary step shows that the terms $R, \tilde{R}, S_1$ and S_2 are very small ($O(\epsilon^N)$) and they can be thrown away from equations (4.4-4.7).

1.1 *Step 1. Estimates* $S, R, \tilde{R}$.

Using explicit forms (4.8)-(4.11) one concludes by (4.20) that for any $N > 0$

$$||S_1||, |R_i| < c_N \epsilon^N (1 + ||w_1|| + |q|), \tag{A2.1}$$

$$||S_2||, |\tilde{R}_i| < c_N \epsilon^N (1 + ||w_2|| + |p|)) \tag{A2.2}$$

More simple estimates hold for the Frechet derivative of these terms S, R. Here and below $|||\ \ |||$ denotes standard norms in corresponding spaces of linear bounded operators. For instance $|||S'_{1w}||| \equiv ||S'_{1w}||_{Lin(Y,Y)}$.

For the Frechet derivatives one has

$$|||S'_{1r}|||, |||S'_{1w}|||, |||R_i||| < c_N \epsilon^N, \tag{A2.3}$$

$$|||S'_{2r}|||, |||S'_{2w}|||, |||\tilde{R}_i||| < c_N \epsilon^N. \tag{A2.4}$$

Let us remind that in Appendix 1 estimate (A1.1) has been obtained. Thus the terms $S, R, \tilde{R}$ are uniformly bounded together with the Frechet derivatives by $O(\epsilon^N)$ for any N and for sufficiently small ϵ. It shows that these contributions are negligible and we omit it below.

1.2 *Step* 2. *Local considerations in* $\mathcal{A}$.

Let us apply Lyapunov-Perron approach using the classical Theorems 6.1.2, 6.1.4 and 6.1.7 from [34]. Let us remind that one denotes $z = (q, p)$ and $w = (w_1, w_2)$ and (4.4)-(4.7) can be rewritten as

$$\frac{dz}{dt} = Z(z, w, \epsilon, \lambda), \tag{A2.5}$$

$$\frac{dw}{dt} = A_\epsilon w + G(z, w, \epsilon, \lambda). \tag{A2.6}$$

Recall also that the discrete spectrum of the operator A_ϵ is contained in the interval $(-\infty, -\beta)$, where $\beta > 0$ and does not depend on ϵ, λ for small ϵ, λ (that shows the existence of the spectral barrier).

Let positive numbers ν, K, L, M_2 be defined by

$$\nu = \beta/4, \quad L = \sup |||G'_z||| + \sup |||G'_w||| \tag{A2.7}$$

$$M_2 = \sup |||Z'_w|||, \quad K = \sup ||G||, \tag{A2.8}$$

where $|||\ \ |||$ are the norms of bounded linear operators from the operator spaces $\mathcal{L}(Y, Y)$. These operators are Frechet derivatives and sup in (A2.7), (A2.8) should be taken over all $p, q, \tilde{u}, \tilde{v}$ from the absorbing set $\mathcal{A}$ that has been obtained in Prop. 5.1. Recall that $\alpha \in [3/4, 1)$.

Using explicit forms (4.15)-(4.18) for Z, G one can obtain

$$\nu > \sup_B |||Z'_z|||, \quad K < c\lambda\epsilon^{\mu+1/4}, \tag{A2.9a}$$

$$L < c\lambda\epsilon^{\mu}, \quad M_2 < C\epsilon^{\mu-1/4}. \tag{A2.9b}$$

In fact, relations (A2.9a) is a immediate consequence of estimates (5.7) and (5.9) if one takes into account that $(z, w) \in \mathcal{A}$.

Estimates for Frechet derivatives giving (A2.9b) can be obtained similarly. For instance, by the Cauchy-Schwartz inequality one obtains

$$|||{Z_1}'_w||| \leq \sup_{||h||=1} \lambda\epsilon^{\mu-1/4}| < ah, \psi_i > | \leq c\lambda\epsilon^{\mu-1/4},$$

and similarly for other derivatives. Finally, inequalities (A2.9) hold that concludes the first step of the proof.

For any $\Delta_1 > 0$ let us define

$$\Theta(\Delta_1) = cL \int_0^\infty t^{-\alpha} \exp((-\beta + \nu + \Delta_1 M_2)t)dt$$

Let us define the neighborhood Ω_D in Y_α by

$$\Omega_D = \{||w||_\alpha^2 < c_0\epsilon^\mu = D(\epsilon)\}$$

where the constant c_0 does not depend on ϵ, λ.

To prove the existence of an attracting with an exponential rate invariant manifold in $\mathcal{A}$, it is sufficient show that some estimates hold [34]. For some $\Delta_1 > 0$ one checks the following estimates

$$\Theta < \Delta_1(1+\Delta_1)^{-1}, \quad \Theta \max[1, (1+\Delta_1)M_2/(\nu+\Delta_1 M_2)] < 1, \tag{A2.10}$$

$$C_1 K \int_0^\infty t^\alpha \exp(-\beta t)dt < D(\epsilon), \tag{A2.11}$$

and

$$r = \Theta(1 + M_2(1 + \Delta_1))/(\beta - \nu') < [\beta/(\beta - \nu']^{1-\alpha}. \tag{A2.12}$$

Here $\nu' = \nu + \Delta_1 M_2$, C_1 does not depend on ϵ and λ and the quantity

$$\gamma = \beta - (\beta - \nu')r$$

defines the exponent γ in the function $\exp(-\gamma t)$ which gives the upper estimate for the rate of the trajectory convergence to $\mathcal{M}_1$.

The number Δ_1 gives the upper estimate for the Lipschitz constants of the functions W_k.

Let us take $\Delta_1 = 2L$.

It is easy to see due to (A2.9) estimates (A2.10)-(A2.12) are fulfilled for sufficiently small ϵ.

Finally the manifold $\mathcal{M}_1$ from the Lipschitz class exists and the maps W_1, W_2 satisfy

$$||W_k|| < c\epsilon, \quad Lip W_k < c_1\epsilon, \quad k = 1, 2. \tag{A2.13}$$

Due to theorem 6.1.7 [34] the functions W_k are really from $C^{1+\nu}$-Holder class and inequalities (A2.13) immediately yields (6.3).

Estimate of the convergence rate can be in a standard way following the theorem 6.1.7 [34]. In fact, $\gamma > \beta/2$ for small ϵ. The proof completes.

Remark. – Another approach is the Hadamard Graph Transform method with the using of the Cone Condition (for example, see [11], [32]). This method also allows easily to prove the same assertion.

2. *Proof of Proposition* 8.1.

We use the same scheme as above in subsec. 1 for Prop. 6.1. However technical details are more simple since now we are dealing only with the bounded operators in finite dimensional spaces.

Let us use the change $p \to \rho$ defined by (8.1) and rewrite (1.13) as

$$\frac{dq}{dT} = Z(q, \rho, \epsilon, \lambda), \tag{A2.14}$$

$$\frac{d\rho}{dT} = -\rho + \lambda^2 Y(q, \rho, \epsilon, \lambda) \tag{A2.15}$$

where $q \in \mathbf{R}^m$, $p \in \mathbf{R}^n$, the right hand Z has the form

$$Z(q, \rho, \epsilon, \lambda) = \sum \lambda^2 \alpha_{ij} \tilde{p}_j(q) - 2\lambda^2 q_i + \lambda^2 \tilde{Q}(q, \rho, \epsilon, \lambda) \tag{A2.16}$$

where $\tilde{p}_j = \sum_i P_{ij}(q_i)$ and where $|Q|_{C^1(\mathcal{A}_0)} \to 0$ as $\epsilon \to 0$ and Y is uniformly bounded in the norm $C^1(\mathcal{A}_0)$.

Let us apply Lyapunov-Perron approach again using the Theorems 6.1.2, 6.1.4 and 6.1.7 from [34]. According to estimate (8.2) one can investigate (A2.15) and (A2.16) in region $B(\lambda) = \{|q| < C_1^*, |\rho| < c\lambda^{1.99} = D(\lambda)\}$.

Let positive numbers ν, K, L, M_2 be defined by

$$\nu = 1/4, \quad L = \sup |||Y_q'||| + \sup |||Y_\rho'||| \tag{A2.17}$$

$$M_2 = \sup |||Z_\rho'|||, \quad K = \sup ||Y|| \tag{A2.18}$$

where $|||\ |||$ are the norms of bounded linear operators in the corresponding Euclidian spaces. The suprema in (A2.17), (A2.18) should be taken over all ρ, q from the absorbing set $\mathcal{B} = \mathcal{A}_0 \bigcap B(\lambda)$. Since all operators are bounded one can set $\alpha = 0$. Using (A2.14) immediately one obtains for sufficiently small λ

$$\nu > \sup |||Z_q'||| \leq c\lambda^2, \quad K < c\lambda^2, \tag{A2.19a}$$

$$L < c\lambda^2, \quad M_2 < C\lambda. \tag{A2.19b}$$

In fact, relations (A2.9a) is a immediate consequence of estimates (5.7) and (5.9) if one takes into account that $(\rho, q) \in \mathcal{B}$.

For any $\Delta_1 > 0$ let us define

$$\Theta(\Delta_1) = cL \int_0^\infty \exp((-1/2 + \nu + \Delta_1 M_2)t)dt.$$

For some $\Delta_1 > 0$ one checks the following estimates:

$$\Theta < \Delta_1(1+\Delta_1)^{-1}, \quad \Theta \max[1, (1+\Delta_1)M_2/(\nu+\Delta_1 M_2)] < 1, \tag{A2.20}$$

$$C_1 K \int_0^\infty \exp(-\beta t)dt < D(\epsilon), \quad \beta = 1/2, \tag{A2.21}$$

and

$$r = \Theta(1 + M_2(1 + \Delta_1))/(\beta - \nu') < \beta/(\beta - \nu') \tag{A2.22}$$

where $\nu' = \nu + \Delta_1 M_2$.

Let us take $\Delta_1 = 2L$. It is clear that for sufficiently small λ these inequalities (A2.20-A2.22) hold thus the invariant manifold exists. The corresponding Lipschitz constant does not increase $2L$. Following subsec.1 one establishes the attracting properties and that the manifold lies in C^1-class. The proof completes.

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(Manuscript received on February 5th, 1996;
Revised version received on May 13rd, 1996.)