ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 66, nº 3 (1997), p. 323-357
<http://www.numdam.org/item?id=AIHPA_1997__66_3_323_0>

Ann. Inst. Henri Poincaré,

Vol. 66, n° 3, 1997, p. 323-357. *Physique théorique*

Uniform Finsler Hadamard manifolds

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ABSTRACT. – The subject of this work are *reversible uniform Finsler Hadamard manifolds*, the Finsler analogues of simply connected Riemannian manifolds of nonpositive curvature. We introduce asymptotic geodesics, the geodesic ray boundary and study visibility, introduced by P. Eberlein, and δ-hyperbolicity in the sense of M. Gromov.

In Finsler geometry sharp comparison statments, such as the Aleksandrov-Toponogov comparison theorem, do not exist. Hence, the synthetic methods developed for Aleksandrov spaces of bounded curvature can not be used to study Finsler manifolds.

To apply techniques developed in Riemannian geometry we face the problem to integrate Jacobi field estimates. Unfortunately, this integration process only leads to "coarse" estimates of the Finsler distance.

However, under the hypothesis of nonpositive curvature these "coarse" distance estimates are sufficient to establish a satisfactory theory of uniform Finsler Hadamard manifolds, extending thereby many results already known in the Riemannian situation.

RÉSUMÉ. – Ce travail traite des *variétés de Finsler Hadamard, uniformes et reversibles*, qui peuvent être considerées comme les analogues des variétés Riemanniennes simplement connexes à courbure non positive. Nous

1991 Mathematics Subject Classification: 53 C 60; 53 C 22, 53 B 40.

Key words and phrases. Finsler Geometry, Uniform Finsler Hadamard Manifolds, Sphere at Infinity, Visibility, δ-Hyperbolicity.

Supported by the Swiss National Fonds, 21-36'185.92 and 2000-042054.94/1.

définissons la relation asymptotique pour des géodésiques, ainsi que le bord à l'infini. Ensuite, nous étudions la visibilité introduite par P. Eberlein et la δ-hyperbolicité de M. Gromov.

Dans le cadre Finslerien, des résultats précis de comparaison, comme le théorème d' Aleksandrov-Toponogov, n'existent pas. En particulier, les méthodes synthétique developpées pour les espaces d'Aleksandrov à courbure bornée ne peuvent pas être appliquées.

Lorsque nous voulons appliquer les techniques developpées pour des variétés Riemanniennes, nous sommes confrontés au problème de l'intégration des estimations des champs de Jacobi. Cette intégration ne nous permet de contrôler la distance de Finsler que de manière grossière.

Toutefois, sous l'hypothèse de courbure non positive, le contrôle grossier de la distance de Finsler nous suffit pour construire une théorie ayant la plupart des propriétés déjà établies dans le cadre Riemannien.

CONTENTS

1. OVERVIEW

First, in §2 we set up our notations and consider basic concepts for Finsler manifolds in general. We introduce a "connection" which is similar to the Riemannian Levi-Civita connection but lives on $\pi^*TM \to PTM$. An appropriate extension of this partial connection would lead to Chern's connection, [6].

In the next section, uniform Finsler Hadamard manifolds are studied. The main difficulty is to integrate Jacobi field estimates to distance estimates: The Finsler version of Rauch's comparison theorems estimate Jacobi fields as sections of $\pi^*TM \to PTM$ along the canonical lift of the geodesic with respect to the metric g defined in §2.2.3. As a consequence of the uniformity hypothesis we then obtain "coarse" distance estimates, for example, if $K^F \leq 0$, $exp_p : T_pM \to M$ is *quasi*-nondecreasing. For many questions such weaker comparison results are sufficient. Unfortunately this is not the case for the Flat Strip Lemma, which is used to study the geometry of parallel geodesics and is a major tool to relate abelian subgroups of the fundamental group to totally geodesic flat tori.

Because Hilbert geometries are used all over to obtain examples we give a short treatment in appendix A.

In appendix B we recall the well known fact that Finsler manifolds can not be studied as Aleksandrov spaces and make some comments on more general notions of curvature due to Busemann and Kann.

The author is thankful to E. Ruh for his interest in the work and to P. Foulon for the many illuminating discussions, which lead to new ideas. He is grateful too, for the hospitality at the IRMA (1). This article is part of the authors dissertation [26] written at Fribourg University, Switzerland.

2. FINSLER MANIFOLDS

2.1. Fundamental Differences between Finsler and Riemannian Manifolds

A *Finsler manifold* is a differentiable manifold for which a norm is prescribed on every tangent space. The unit sphere of this norm, called the *indicatrix*, is assumed to be *strictly convex* in the sense that the Hessian is positive definite.

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One fundamental difference between Finsler and Riemannian manifolds is the absence of a unique "local model": There are infinitely many affinely inequivalent normed vector spaces. Also there are many different ways to associate a scalar product to a norm. The most known ones are the inscribed or circumscirbed Loewner ellipsoids and the ellipsoid of inertia.

In Riemannian geometry bounds on the sectional curvature are as precise as "synthetic curvature bounds" defined through comparison of geodesic triangles. This is not the case in Finsler geometry. The flag curvature, being the Finsler analogon of the sectional curvature, determines only a part of the Finsler curvature. Hence it cannot control the global behaving of Finsler geodesics completely. Moreover it is not at all clear how "synthetic curvature bounds" are related to the flag curvature.

2.2. Preliminaries

In this section we fix our notations. Let M be a smooth manifold, $\tau_M : TM \to M$ the tangent bundle, and $\pi : PTM = (TM - \{0\})/\mathbb{R} \to M$ the projectivized tangent bundle, $p : TM - \{0\} \to PTM$ the canonical projection, $p(v) = \widehat{v}$. If $E = TM$ or $E = PTM$ there are short exact sequences

$$0 \to V(E) \xrightarrow{i} TE \xrightarrow{j=(d\pi,\tau_E)} TM \times_M E = \pi^* TM \to 0. \qquad (1)$$

Sections $\Gamma(E, V(E))$ are called *vertical vector fields*. Let ∇^f be the vertical derivative given by the Lie derivative along vertical vector fields.

2.2.1. *Finsler Metrics.* – A *Lagrangian (function)* is a function $F : TM \to \mathbb{R}^+$ which is C^0 on all of TM and C^∞ on $TM - \{0\}$. F is *strictly homogeneous* of degree one if $F(\lambda X) = |\lambda| F(X)$ for $\lambda \in \mathbb{R}^*$.

DEFINITION 2.1. – A *reversible Finsler metric* is a strictly homogeneous Lagrangian F with strictly convex level surfaces in the fibres, in the sense that the Hessian is positive definite.

If F is a reversible Finsler metric, the action or length $L(c) = \int_a^b F(\dot{c}(t))dt$ of a C^1 curve is independent of the parametrization of c and induces a symmetric distance

$$d(p,q) = \inf_{\substack{c \text{ piecewise } C^1 \\ \text{joining } p \text{ to } q}} L(c) \qquad (2)$$

on M. Other authors call reversible Finsler metrics also symmetric. We propose the new terminology "reversible" to avoid future naming conflicts.

2.2.2. *Potential and Reeb Field.* – Let F be a Finsler metric. Then, $A_1 \stackrel{\text{def}}{=} \nabla^f F$ is a section in $\Gamma(TM, \tau_M^* T^*M)$ and descends, by homogeneity, to a section in $\Gamma(PTM, \pi^* T^*M)$ which we again denote by A_1. The *potential* of F is defined as the horizontal 1-form $A = j^* A_1$. Then, A is a contact structure on PTM, [31].

Let X_A be the *Reeb field* of A. It is uniquely determined by the conditions $A(X_A) = 1$ and $i_{X_A} dA = 0$. Moreover, $L_{X_A} A = 0$.

The *geodesic flow* φ_t of F is the flow with infinitesimal generator X_A, which consists of contact diffeomorphisms. In particular, the Liouville measure $\mu_L = A \wedge dA^{n-1}$ is an invariant volume form on PTM.

2.2.3. *Vertical Endomorphism and Vertical Metric.* – For all $Y \in V(PTM)$, $j[X_A, Y^*]$ does not depend on the extension of Y to a vertical vector field Y^*. Moreover, $j[X_A, Y] \in \ker(A_1)$. The mapping

$$w_{X_A} : \ \ker(A_1) \to V(PTM), \quad -j[X_A, Y] \mapsto Y \tag{3}$$

yields an isomorphism, [30, Theorem II.1]. Foulon's *vertical endomorphism* is now given by $v_{X_A} = w_{X_A} \circ j$. On $V(PTM) \to PTM$

$$g(Y_1, Y_2) = i_{Y_1} i_{[X_A, Y_2]} dA$$

defines a Riemannian metric, [31]. The Reeb field X_A gives rise to the *distinguished section* $\mathcal{T}_A = j(X_A) \in \Gamma(PTM, \pi^*TM)$. On $\pi^*TM \to PTM$ we introduce a metric such that w_{X_A} is an isometry, $g(\mathcal{T}_A, \mathcal{T}_A) = 1$ and $\mathbb{R} \cdot \mathcal{T}_A \perp \ker(A_1)$.

2.2.4. *The Levi-Civita connection of Finsler Geometry.*

THEOREM 2.1 ([26], [6]). – *Let F be a Finsler metric, g the induced Riemannian metric on $\pi^*TM \to PTM$. Then there is a unique connection distribution N of $TPTM \to PTM$ and a unique partial connection ∇^h along N such that*

(i) $\nabla^h \mathcal{T}_A = 0$

(ii) $\bar{X} g(Y, Z) = g(\nabla^h_{\bar{X}} Y, Z) + g(Y, \nabla^h_{\bar{X}} Z)$

(iii) $\nabla^h_{\bar{X}} Y - \nabla^h_{\bar{Y}} X - j[\bar{X}, \bar{Y}] = 0$

with $X, Y, Z \in \Gamma(PTM, \pi^*TM)$ and $\bar{X} = h^N \circ X$. Here, h^N is the splitting of (1) *given by N.*

Proof. – The definition of g and the properties of X_A yield: For all $V, Y \in V(PTM)$

$$(\nabla^f_V g)(\mathcal{T}_A, \mathcal{T}_A) = 0, \quad (\nabla^f_V g)(\mathcal{T}_A, j[X_A, Y]) = 0. \tag{4}$$

Given any connection distribution $N_0 \subset TPTM$ there is a unique $\nabla^{h,0}$ along N_0 satisfying (ii) and (iii). The proof is similar to the derivation of the Levi-Civita in Riemannian geometry.

Parametrizing the connections in the form $N_s = N_0 + s$, s a horizontal 1-form with values in the vertical vector fields, the difference of $\nabla^{h,0}$ and $\nabla^{h,s}$ may be expressed as

$$2g(\nabla^{h,0}_{\tilde{X}}Y - \nabla^{h,s}_{\bar{X}}Y, Z) = -2g(\nabla^{f}_{s(X)}Y, Z) \\ - (\nabla^{f}_{s(X)}g)(Y,Z) - (\nabla^{f}_{s(Y)}g)(X,Z) + (\nabla^{f}_{s(Z)}g)(X,Y) \quad (5)$$

where $\tilde{X} = h^{N_0}(X)$ and $\bar{X} = h^{N_0+s}(X)$. Condition (i) imposes equations on s, which, by (4), can be solved. □

Note that $X_A = h^N(\mathcal{T}_A)$. Let $h_{X_A}(PTM) = h^N(\ker(A_1))$. We pull the metric on π^*TM back to $h_{X_A}(PTM)$ and require that $TPTM = h_{X_A}(PTM) \oplus \mathbb{R}X_A \oplus V(PTM)$ be orthogonal. In this way $TPTM \to PTM$ is equiped with a Riemannian metric g.

We extend ∇^h such that w_{X_A} and h^N become parallel. This partial connection agrees with Foulon's dynamical covariant derivative as long as one differentiates along the flow, *i.e.* in direction of X_A.

2.2.5. *Curvature.* – Let $X, Y, Z, W \in \Gamma(PTM, \pi^*TM)$, $V \in \Gamma(PTM, V(PTM))$. Let pr_N be the projection on N along $V(PTM)$, and pr_V the projection on $V(PTM)$ along N. Let $R^N(\bar{X}, \bar{Y}) = \mathrm{pr}_V[\bar{X}, \bar{Y}]$ be the curvature of N. Then, $R_{X_A}(Y) = w^{-1}_{X_A} R^N(X_A, \bar{Y})$ is a g-symmetric endomorphism of π^*TM, called *Jacobi endomorphism.* In the next section we sketch the proof of this fact. The *flag curvature*

$$K^F(Y, v) = \frac{g(R_{X_A}(Y), Y)}{g(Y,Y) - g(Y, \mathcal{T}_A)^2}\,(p(v)), \quad (6)$$

with $v \in TM - \{0\}$ and $Y \in TM$, generalizes the Riemannian sectional curvature, [6].

Let R be the curvature of the extension $\nabla = \nabla^h \oplus \nabla^f$. By Theorem 2.1, $R(\bar{Y}, X_A)\mathcal{T}_A = w^{-1}_{X_A} R^N(X_A, \bar{Y})$. Moreover, R shares a lot of algebraic properties with the curvature of the Levi-Civita connection in Riemannian geometry. First, it is a 2-form on PTM with values in $\mathrm{End}(\pi^*TM)$ and satisfies the first Bianchi identity along N

$$R(\bar{X}, \bar{Y})Z + R(\bar{Y}, \bar{Z})X + R(\bar{Z}, \bar{X})Y = 0. \quad (7)$$

In addition there is the remarkable symmetry $R(\bar{X},V)Y - R(\bar{Y},V)X = 0$. The metric g is in general not ∇^f-parallel, hence the 2-form R has not necessarily values in the skewsymmetric endomorphisms of π^*TM:

$$g(R(\bar{X},\bar{Y})Z,W) + g(Z,R(\bar{X},\bar{Y})W) = \nabla^f_{R^N(\bar{X},\bar{Y})}(g)(Z,W). \tag{8}$$

Because $\nabla^f_V g$ has a nontrivial kernel, *see* (4), the skewsymmetry of $g(R(\bar{Y},X_A)\mathcal{T}_A,Z)$ in the first as well as in the last pair together with the first algebraic Bianchi identity imply for purely algebraic reasons $g(R(\bar{Y},X_A)Z,\mathcal{T}_A) = g(R(\bar{Z},X_A)Y,\mathcal{T}_A)$, proving the symmetry of R_{X_A}.

2.2.6. *Geodesics, Jacobi Fields and Exponential Map.* – Let $c : t \in [a,b] \mapsto c(t) \in M$ be parametrized by arclength, $c_s(t)$ a variation of c with variation vector field V, $\widehat{c}_s = p(\dot{c}_s)$. By the first variation formula

$$\frac{\partial}{\partial s}L(c_s) = g(V,\dot{c}_s)(\widehat{c}_s)|_a^b - \int_a^b g(V,\nabla^h_{h^N(\dot{c})(\widehat{c}_s)}\dot{c}_s)(\widehat{c}_s)dt, \tag{9}$$

hence c is a weak local minimum, *i.e.* $L(c) \le L(\gamma)$ for all piecewise C^1 curves γ, piecewise C^1 close to c, if and only if $\nabla^h_{X_A}\dot{c} = 0$, where $\dot{c}$ is viewed as a section $\Gamma(PTM,\pi^*TM)$. This is the geodesic differential equation expressed with ∇^h. Geodesics are precisely the projections $c_v(t) = \pi\varphi_t(\widehat{v})$ of orbits of the geodesic flow φ_t.

In contrast to the Riemannian geometry we have to distinguish in Finsler geometry between *perpendicular* and *transversal.*

Definition 2.2. – Let $v,w \in T_pM$. We say that v is *perpendicular* to w and w is *transversal* to v if $g(v,w)(\widehat{v}) = 0$, *i.e.* $w \in \ker(A_1(\widehat{v}))$ or w is tangent to the indicatrix $I_p = \{F \equiv 1\} \subset T_pM$ at $\frac{v}{F(v)}$.

Jacobi fields along geodesics, defined as variation vector fields of geodesic variations, can be identified with flow invariant vector fields. In fact, if $\xi(t) = d\varphi_t(\xi_0)$ is flow invariant, $j\xi(t)$ is the corresponding Jacobi field. We set $Y'(t) = \nabla^h_{X_A}Y(t)$. The Jacobi field equation then reads $Y''(t) + R_{X_A}(Y(t)) = 0$.

Important. – *From now on we always consider Jacobi fields as sections of $\pi^*TM \to PTM$ along $p(\dot{c}(t))$.*

Conjugate points and *focal points* are defined as usual. M has no focal point if and only if for every Jacobi field $Y(t)$ along $c(t)$ with $Y(0) = 0$ and $Y'(0) \ne 0$, we have $\frac{d}{dt}g(Y(t),Y(t)) > 0$, $\forall t > 0$.

Rauch's comparison theorems extend in a straightforward manner to Finsler manifolds. The fundamental difference to the Riemannian case is that we estimate the g-length of Jacobi fields as sections of $\pi^*TM \to PTM$ along $p(\dot{c}(t))$ and not their actual Finsler length! Let

$$s_\kappa(t) = \begin{cases} \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa} t) & \text{if } \kappa < 0 \\ t & \text{if } \kappa = 0 \\ \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa} t) & \text{if } \kappa > 0 \end{cases} \tag{10}$$

and $c_\kappa(t) = s'_\kappa(t)$.

PROPOSITION 2.2. – *Let $J(t)$ be a Jacobi field along $c(t)$ with $J(0) = 0$ and $g(J, \dot{c})(\widehat{c}) = 0$. If $K^F \le \Delta$ then for $0 < t_1 \le t_2 < \frac{\pi}{\sqrt{\Delta}}$*

$$\frac{||J(t_1)||_{\widehat{c}}}{s_\Delta(t_1)} \le \frac{||J(t_2)||_{\widehat{c}}}{s_\Delta(t_2)}.$$

If $K^F \ge \delta$ and there are no conjugate points on $(0, r)$ then for $0 < t_1 \le t_2 < r$

$$\frac{||J(t_1)||_{\widehat{c}}}{s_\delta(t_1)} \ge \frac{||J(t_2)||_{\widehat{c}}}{s_\delta(t_2)},$$

where $|| \cdot ||_{\widehat{v}}$ is the norm on $(\pi^*TM)_{\widehat{v}}$ induced by g.

The proof can be done either by Riccati type inequalities or by comparison of the index form, [6]. Results in connection with the first conjugate point, such as Bonnet-Myers, Morse-Schoenberg, Synge and Cartan-Hadamard extend to the Finsler setting without any problems, we refer to [4] and [6].

Let X_F be the vector field on $TM - \{0\}$ projecting to the Reeb field. The exponential map exp is defined as the exponential map of the spray $F \cdot X_F$, *see* also [6]. As in Riemannian geometry, the differential of exp is can be expressed by Jacobi fields, [6].

2.2.7. *Uniformity.* – To obtain distance comparison results from Rauch's comparison theorem, the norms $|| \cdot ||_{\widehat{v}}$ on T_xM induced by the metric g of the bundle $\pi^*TM \to PTM$, restricted to a fibre PTM_x, have to be uniformly equivalent with a constant independent of x: A Finsler manifold is called *uniform* with *uniformity constant* C_0 if $\forall x \in M$ and $\widehat{v}, \widehat{w} \in PTM_x$

$$\frac{1}{C_0} || \cdot ||_{\widehat{v}} \le || \cdot ||_{\widehat{w}} \le C_0 || \cdot ||_{\widehat{v}}. \tag{11}$$

Note that for $v \in TM - \{0\}$, we have $F(v) = ||v||_{\widehat{v}}$. Hence, Jacobi field estimates on uniform Finsler manifolds lead to "coarse" distance control. Compact manifolds, their coverings, and more generally *compactly homogeneous* manifolds, *i.e.* manifolds which can be covered by isometric translates of a compact set, are uniform.

Remark 2.1. – It is not clear if Hilbert geometries, defined in a strictly convex domain, are uniform.

2.2.8. *Projective Change.* – Replacing X_A by mX_A, m is a nonvanishing function on PTM, changes the parametrization of the geodesics. The Jacobi endomorphism transforms under such a projective change $X_A \to mX_A$ in the following way, [26] and [30]:

$$R_{mX_A} = m^2 R_{X_A} + \left(\frac{1}{2} m L^2_{X_A}(m) - \frac{1}{4}(L_{X_A}(m))^2\right) \mathrm{id}. \qquad (12)$$

The correction term $\frac{1}{2} m L^2_{X_A}(m) - \frac{1}{4}(L_{X_A}(m))^2$ in (12) is related to the Schwarz differential

$$[l]_t = \frac{2\dddot{l}\,\dot{l} - 3\ddot{l}^2}{2\dot{l}^2} \qquad (13)$$

of a function $l(t)$ as follows: Substitute $m(l(t)) = \dot{l}(t)$ into (13). Denote the differentiation of m with respect to l by m'. Then $\frac{1}{2}[l]_t = \frac{1}{2}mm'' - \frac{1}{4}(m')^2$.

2.3. Gauss' Lemma

We give geometrical proof of the Finsler Gauss Lemma which is due to Foulon, [33], *see* also the remarks in [6, p. 167, 168].

The first variation formula shows that radial geodesics $c_v(t) = exp_p(tv)$ intersect the distance spheres $S_p(r)$ perpendicularly. Define in the total space PTM the "framing" of a ball $B_p(r)$ in M by radial unit vectors, $V = \cup_{0 \le t \le r} \varphi_t(PTM_p)$. The boundary ∂V of V consists of two components both diffeomorphic to an $(n-1)$-dimensional projective space. Because A is horizontal and invariant under the geodesic flow, $A \restriction T(\partial V) = 0$. Also $dA \restriction \Lambda^2 V = 0$, because $i_{X_A} dA = 0$. Hence, A is closed on V.

Let now $q = exp_p(v) \in S_r(p)$, $c_v(t) = exp_p(tv)$ and $\gamma : [0, a] \to M$ be any curve joining p to q. The curve $\widehat{\gamma}(t)$ will not necessarily remain in V. Note that we denoted by $\widehat{\gamma}(t)$ the point in PTM determined by $\dot{\gamma}(t)$. Its vertical projection to the manifold V yields a curve $\sigma : [0, a] \to V \subset PTM$ which is defined as $\sigma(t) = V \cap PTM_{\widehat{\gamma}(t)}$. This definition makes sense because $\pi : V - PTM_p \to B_p(r) - \{p\}$ is a diffeomorphism. $\sigma(t)$ leaves V at a boundary point, say $\widehat{w}$. Joining $\widehat{v}$ and $\widehat{\gamma}(0)$ as well as $\widehat{c}_v(\varepsilon)$ and $\widehat{w}$

with curves on ∂V we obtain a closed curve in V. By Stokes' theorem for manifolds with piecewise smooth boundary the integral over this closed curve vanishes and because of $A \restriction \partial V = 0$, the curves on the boundary ∂V do not contribute, hence

$$r = \int_{\widehat{C}_v} A = \int_\sigma A.$$

LEMMA 2.3 (Convexity Lemma). – *For all $\widehat{z} \in PTM$, $Z \in T_{\widehat{z}}PTM$*

$$A_{\widehat{z}}(Z) \leq F(d\pi(Z))$$

and equality holds if and only if $\widehat{d\pi Z} = \widehat{z}$ or equivalently if $Z \in \mathbb{R} \cdot X_A$.

Proof. – Consider the zero homogeneous section $A_1 \in \Gamma(TM - \{0\}, \tau_M^* T^* M)$. Let $z \in TM - \{0\}$ and $W \in (\tau_M^* TM)_z$. From the definition of A_1 and the convexity of F one obtains:

$$\begin{aligned} A_1(z)(W) - F(z) &= A_1(z)(W - z) \\ &= \lim_{t \to 0} \frac{1}{t}(F(z + t(W - z)) - F(z)) \leq F(W) - F(z). \end{aligned}$$

Conseqently, $A_1(z)(W) \leq F(W)$ with equality if and only if z is a multiple of W. Apply this to $W = d\pi(Z)$ and note that A_1 is zero homogeneous and that $A = j^* A_1$.

The vectors z and $d\pi(Z)$ are proportional if and only if $jZ \in \mathbb{R} \cdot \mathcal{T}_A(\widehat{z})$, where $\mathcal{T}_A$ is the distinguished section. But $\mathcal{T}_A = jX_A$. □

By definition of $\sigma(t)$ we have $\pi\sigma(t) = \gamma(t)$. The Convexity Lemma then implies

$$r = \int_\sigma A = \int A_{\sigma(t)}(\dot\sigma(t))\,dt \leq \int F(d\pi(\dot\sigma(t)))\,dt = \int F(\dot\gamma(t))\,dt = L(\gamma).$$

Equality holds if and only if the point in PTM determined by $d\pi(\dot\sigma(t)) = \dot\gamma(t)$ is equal to $\sigma(t)$. In this case, because $\sigma(t) \in V$, $\gamma(t)$ has to be a radial curve emanating from p, but not necessarily parametrized proportional to arc length. Because the enpoint of $\gamma(t)$ is q, it has to be a reparametrization of $c_v(t)$. This proves the Finsler version of Gauss' Lemma.

LEMMA 2.4 (Gauss). – *Assume that $q = exp_p(v)$ is in the image of the exponential map. Let $c_v(t) = exp_p(tv)$ be the radial geodesic joining p to q. Then $c_v(t)$ intersects the distance spheres $S_p(r)$ perpendicularly. Furthermore, every curve γ joining p to q is not shorter,* i.e. $L(c_v) \leq L(\gamma)$.

Moreover, equality holds if and only if γ agrees with c_v up to a reparametrization. In particular $||dexp_p(v)||_{\widehat{v}} = ||v||_{\widehat{v}} = F(v)$.

COROLLARY 2.5. – *Finsler geodesics satisfy locally the strict strong minimization property,* i.e. *for any geodesic $c(t)$ short enough, every curve joining their endpoints, close enough to c in the C^0 sense, is strictly longer.*

For general regular Lagrangians *see* for example [43, Theorem 4.1, p. 181].

3. UNIFORM FINSLER HADAMARD MANIFOLDS

3.1. Finsler Hadamard Manifolds

Let M be a reversible Finsler manifold. If $K^F \leq 0$ then M is free of conjugate points and without focal points. It is important to note that $K^F \leq 0$ does not imply convexity of the distance function as in Riemannian geometry. Nevertheless, a weaker property holds.

DEFINITION 3.1. – Let $I \subset \mathbb{R}$ be an intervall and $f : I \to \mathbb{R}$ a function.

$$f(t) \leq \max\{f(t_1), f(t_2)\} \quad \forall t_1 < t < t_2. \tag{P1}$$

$$\text{“} = \text{''} \text{ in (P1)} \Rightarrow f(t_1) = f(t_2), \text{ hence } f \restriction [t_1, t_2] = \text{const.} \tag{P2}$$

$$f(t) \text{ is never constant.} \tag{P3}$$

Let f be a function on a geodesic metric space. We say f is *weakly peakless, peakless* respectively *strictly peakless* if for every geodesic $c(t)$, $f \circ c(t)$ satisfies (P1), (P1) and (P2), respectively (P1) and (P3), hereby excluding (P2).

PROPOSITION 3.1 (Cartan-Hadamard). – *A simply connected complete Finsler manifold M^n with $K^F \leq 0$ is diffeomorphic to $\mathbb{R}^n$. If the Finsler metric is reversible, perpendiculars from a point onto a geodesic line or geodesic segment exist uniquely. The spheres are strictly convex and the distance function $d(p, \cdot)$ to a fixed point p is strictly peakless.*

Proof. – The first part is the extension of Cartan-Hadamard to Finsler manifolds, *see* [6]. Because there are no focal points, every point has exactly one foot on a geodesic line or geodesic segment, which is, by [11, Theorem 20.9], equivalent to the peaklessness of $t \mapsto d(p, c(t))$ and the strict convexity of spheres. Finally, a function f is weakly peakless if and only if the level sets $\{f \leq \text{const}\}$ are convex. But because of the strict convexity of spheres, $t \mapsto d(p, c(t))$ can not be constant. □

DEFINITION 3.2. – A *uniform Finsler Hadamard manifold*, abbreviated as *UFH manifold*, is a simply connected, complete, reversible, uniform Finsler manifold with $K^F \leq 0$.

Example 3.1. – *Minkowski spaces, i.e.* normed vector spaces with smooth strictly convex norm unit spheres, are Finsler manifolds of vanishing flag curvature.

Example 3.2. – Let ∂C be a smooth, strictly convex hypersurface in $\mathbb{R}^n$, in the sense of positive definite second fundamental form, with interior C. The *Hilbert geometry* (C, h) is a Finsler generalization of the hyperbolic geometry and has constant negative flag curvature. It is obviously simply connected, *see* appendix A.

3.2. Distance Distorsion of exp_p

Let M be a reversible UFH manifold, $a, b > 0$ and $c_1(t)$, $c_2(t)$ unit speed geodesics emanating form p. Proposition 2.2 yields

$$d(c_1(at), c_2(bt)) \leq C_0^2\, t\, d(c_1(a), c_2(b)) \qquad \forall 0 < t \leq 1. \tag{14}$$

DEFINITION 3.3. – A map $f : (M, d_M) \to (N, d_N)$ between metric spaces is *distance quasi-nondecreasing* if there is $C \geq 1$ such that $d_M(x, y) \leq C\ d_N(f(x), f(y))\ \forall x, y \in M$.

Remark 3.3. – By Proposition 22, $exp_p : (T_pM, F \restriction T_pM) \to M$ is distance quasi-nondecreasing.

To control the distance distorsion of exp_p under general upper curvature bounds, we associate to the Finsler norm in a fixed tangent space a Riemannian metric. The circumscribed Loewner ellipsoid centered at the origin turns to be a convenient choice. We equip the domain

$$D_p^{\kappa,c} = \left(0, \frac{\pi}{\sqrt{\kappa}}\right) \times_{s_\kappa^2(r)} S^{n-1,c} \tag{15}$$

with the wraped product metric $g_{\kappa,c}$, where $S^{n-1,c}$ is the unit ball of the circumscirbed Loewner ellipsoid ($\frac{\pi}{\sqrt{\kappa}} = \infty$ if $\kappa \leq 0$). The metric $g_{\kappa,c}$ is a Riemannian metric of constant sectional curvature. For our purpose the following coarse comparison theorem is sufficient.

PROPOSITION 3.2. – *Let (M, F) be a reversible uniform Finsler manifold. If $K^F \leq \kappa \leq 0$ then $exp_p : (D_p^{\kappa,c}, g_{\kappa,c}) \to M$ is distance quasi-nondecreasing.*

Proof. – Let again $|v| = F(v)$ be the Finsler length of a tangent vector v. Let P_E denote the projections on the tangent hyperplanes of the indicatrix along the radial rays and P_{E_c} the corresponding projections determined by the circumscribed Loewner ellipsoid centered at the origin. Let $s \mapsto c(s)$ be a curve in M, $\tilde{c}(s)$ its lift w.r.t. exp_p. We decompose

$$\tilde{c}'(s) = P_E\tilde{c}'(s) + (1 - P_E)\tilde{c}'(s)$$

relative to the Finsler indicatrix in spherical and radial part. Because the ellipsoids determined by the metrics $g^{\widehat{\tilde{c}(s)}}$ are tangent of second order to the indicatrix it coincides with the decomposition induced by the family of metrics $g^{\widehat{\tilde{c}(s)}}$. Applying the Jacobi field estimates to

$$J_s(t) = dexp_p|_{t\tilde{c}(s)}\,(tP_E\tilde{c}'(s))$$

yields

$$|J_s(1)| \geq \frac{1}{C_0}\frac{||P_E\tilde{c}'(s)||_{\widehat{\tilde{c}(s)}}}{|\tilde{c}(s)|}s_\kappa(|\tilde{c}(s)|)$$

and together with Gauss' Lemma

$$|c'(s)|^2 \geq \frac{1}{C_0^4}\left(\frac{|P_E\tilde{c}'(s)|^2}{|\tilde{c}(s)|^2}s_\kappa^2(|\tilde{c}(s)|) + |(1 - P_E)\tilde{c}'(s)|^2\right).$$

Now because $\frac{s_\kappa(r)}{r} \geq 1$ is monotonically increasing and $|\cdot| \geq |\cdot|_c$ we get

$$\begin{aligned}|c'(s)|^2 &\geq \frac{1}{C_0^4}\left(\frac{|P_{E_c}P_E\tilde{c}'(s)|_c^2}{|\tilde{c}(s)|_c^2}s_\kappa^2(|\tilde{c}(s)|_c) + |(1 - P_{E_c})P_E\tilde{c}'(s)|_c^2\right.\\ &\qquad\left. + |(1 - P_E)\tilde{c}'(s)|_c^2\right)\\ &\geq \frac{1}{2C_0^4}\left(\frac{|P_{E_c}\tilde{c}'(s)|_c^2}{|\tilde{c}(s)|_c^2}s_\kappa^2(|\tilde{c}(s)|_c) + |(1 - P_{E_c})\tilde{c}'(s)|_c^2\right).\end{aligned}$$

For the latter inequality note that $\ker(P_E) = \ker(P_{E_c})$ hence $P_{E_c}P_E = P_{E_c}$ and

$$|(1 - P_{E_c})x|_c^2 \leq 2(|(1 - P_{E_c})P_Ex|_c^2 + |(1 - P_E)x|_c^2).$$

We obtain

$$|c'(s)| \geq \frac{1}{2C_0^2}\sqrt{g_{\kappa,c}(\tilde{c}'(s), \tilde{c}'(s))}. \tag{16}$$

□

3.3. Asymptotic Geodesics

Let M be a reversible Finsler Hadamard manifold. We denote the geodesic segment joining p to q by $[p,q]$ or c_{pq}, whatever is more convenient.

LEMMA 3.3 (Divergence Property). – *Let M be a reversible UFH manifold. Let c_2, c_2 be distinct geodesic lines emanating from p. Then for any $k \geq 0$*

$$d(c_1(kt), c_2(t)) \to \infty \quad \text{for } t \to \pm\infty.$$

Proof. – Let first $k \neq 1$. We can assume $k > 1$. Then $d(c_1(kt), c_2(t)) \geq kt - t = (k-1)t \to \infty$ for $t \to \infty$. Let $k = 1$ and $s_n = (2C_0^2)^n$. Then applying (14) repeatedly with $t = \frac{1}{2C_0^2}$ yields $2^n d(c_1(1), c_2(1)) \leq d(c_1(s_n), c_2(s_n))$, hence if $s \geq s_n$ then $\frac{2^n}{C_0^2} d(c_1(1), c_2(1)) \leq d(c_1(s), c_2(s))$. □

The following important remark shows the necessity of the uniformity hypothesis to guarantee the divergence property.

Remark 3.4. – Consider planar Hilbert geometries such that the convex curve contains a line segment. The geodesics emanating from an interior point with end points on the line segment do not satisfy the divergence property. Their distance remains bounded for all $t > 0$. In particular, these Finsler manifolds cannot be uniform. The examples obviously generalize to higher dimensions.

PROPOSITION 3.4. – *Let $\partial C \subset \mathbb{R}^n$ be a smooth bounded convex hypersurface with interior C. If ∂C contains flat regions of dimension k, $1 \leq k \leq n-1$, then the Hilbert geometry (C, h) does not allow compact quotients.*

Proof. – If they would allow a compact quotients, they have to be uniform. Because $K^F \leq 0$, the divergence property has to hold, contradicting Remark 3.4.

We introduce an angle measure to be able to decide whether two directions coincide, *i.e.* enclose a zero angle, or enclose an angle bounded away from 0. On the indicatrix I_p there is a Riemannian metric given by the scalar products $g(\cdot,\cdot)(\widehat{v})$, $v \in T_pM - \{0\}$, g being the vertical metric.

DEFINITION 3.4. – The *Finsler angle* between two directions $v_0, v_1 \in T_pM - \{0\}$ is defined as

$$\measuredangle_p(v_0, v_1) = d_{I_p}\left(\frac{v_0}{F(v_0)}, \frac{v_1}{F(v_1)}\right)$$

where d_{I_p} is the distance induced by the canonical Riemannian metric on I_p.

The angle measure $\measuredangle$ can also be obtained by integrating an "infinitesimal angle". For Finsler surfaces this reduces to the Landsberg angle, [9]. If the Finsler manifold is uniform the angle measure is bounded from above.

LEMMA 3.5 (Angle Vanishing Property). – *Let M be a reversible UFH manifold. Let p, q be points in M and (r_n) be a sequence of points with $d(p, r_n) \to \infty$ for $n \to \infty$. Then*

$$\measuredangle_{r_n}(p, q) \to 0 \quad \text{for } n \to \infty.$$

where $\measuredangle$ denotes the Finsler angle measure at r_n between the geodesic segments to p resp. q.

Proof. – This follows from Remark 3.3 and the fact that for uniform Finsler manifolds, the angle measure is bounded from above. □

DEFINITION 3.5. – Let M be a reversible UFH manifold. Let $c(t)$ be a geodesic ray. The limit $a(t)$ of a converging sequence of geodesic segments $[p_n, c(t_n)]$, where $p_n \to p$ is a sequence of points converging to p and $t_n \to \infty$ a divergent sequence of real numbers, is called an *asymptotic ray* to $c(t)$ through p.

Reversible UFH manifolds are examples of *straight Busemann spaces* (*G-spaces* in the terminology of [11]). From [11, 23.2] we obtain: Any ray containing an asymptotic ray to $c(t)$ is again asymptotic to $c(t)$ and the asymptotic ray to $c(t)$ a through any given point is unique. Note that this is not true in the general case, even in the Riemannian situation.

DEFINITION 3.6. – Let M be a reversible UFH manifold. Let $c(t)$ be an oriented geodesic line, $p_n \to p$, $t_n \to \infty$. The unique oriented geodesic line determined by the limit of the geodesic segments $[p_n, c(t_n)]$ is called the *asymptote* to $c(t)$ through the point p.

If $a(t)$ is asymptotic to $c(t)$ for $t \to \infty$ and $a(-t)$ is asymptotic to $c(-t)$ for $t \to \infty$ we call $a(t)$ a *parallel* to $c(t)$.

The following proposition is adapted from [11, (37.4)] to uniform Finsler Hadamard manifolds.

PROPOSITION 3.6. – *Let M be a reversible UFH manifold. Let $a(t)$, $c(t)$ be oriented geodesics in M, c^+, a^+ their positive geodesic rays. We have the following equivalent characterisations:*

(i) $a(t)$ is an asymptote to $c(t)$.

(ii) $d(a(t), c(t))$ is bounded for $t \geq 0$.

(iii) $d(a(t), c^+)$ is bounded for $t \geq 0$.

(iv) $d(a(t), c)$ and $d(c(t), a)$ are bounded for $t \geq 0$.

In particular, the asymptote relation is an equivalence relation.

Proof. – "(i) $\Rightarrow$ (ii)": Let $a_\tau(t)$ be the geodesic joining $a(0)$ to $c(\tau)$. Let $l_\tau = d(a(0), c(\tau))$ and $l = d(a(0), c(0))$. By the triangle inequality, $\tau - l \leq l_\tau$ and therefore $l_\tau \to \infty$ for $\tau \to \infty$. Then, because $l_\tau - l \leq \tau \leq l_\tau + l$, we have

$$\frac{\tau}{l_\tau} \to 1 \quad \text{for} \qquad \tau \to \infty.$$

By Proposition 14 we have for every $t \leq l_\tau$

$$d\left(a_\tau(t), c\left(t\frac{\tau}{l_\tau}\right)\right) \leq d\left(a_\tau(t), c\left(\tau - (l_\tau - t)\frac{\tau}{l_\tau}\right)\right) \leq C\, d(c(0), a(0)) = Cl.$$

Letting $\tau \to \infty$ gives (ii).

"(ii) $\Rightarrow$ (iii)", "(ii) $\Rightarrow$ (iv)": Obvious.

"(iii) $\Rightarrow$ (i)": Note first that under our hypothesis a point has a unique foot on a geodesic ray.

Let $c(s(t))$ be the unique foot point of $a(t)$ on the geodesic ray c^+. Let $d(a(t), c^+) \leq C_1$. $s(t) \geq 0$ by definition and by the triangle inequality

$$s(t) \geq t - d(a(0), c(0)) - d(a(t), c(s(t)) \geq t - d(a(0), c(0)) - C_1 \to +\infty.$$

If $a(t)$ is not an asymptote to $c(t)$ let $b(t)$ be the asymptote to $c(t)$ through $a(0)$. Then

$$d(a(t), b(s(t))) \leq d(a(t), c(s(t))) + d(c(s(t)), b(s(t))) \leq \text{const}$$

This follows from $s(t) \to \infty$, by assumption (iii) and because we already know from "(i) $\Rightarrow$ (ii)" that if $b(t)$ is asymptotic to $c(t)$ then $d(c(t), b(t))$ is bounded for $t \to \infty$. But by the divergence property the left hand side tends to ∞, a contradiction.

"(iv) $\Rightarrow$ (i)": Similar to "(iii) $\Rightarrow$ (i)". We only have to be concerned what happens if $s(t) \to -\infty$.

Denote by $c^{-1}(t) = c(-t)$ the reversed geodesic. By the same argument as above, $a(t)$ is an asymptote to $c^{-1}(t)$.

Because by assumption also $d(c(t), a)$ is bounded for $t \geq 0$ we obtain for the foot $a(s_1(t))$ of $c(t)$ as above $s_1(t) \to \pm\infty$ and if $s_1(t) \to \infty$ then $c(t)$ is an asymptote to $a(t)$. But this leads to a contradiction: Because we already verified "(i) $\Leftrightarrow$ (ii)", we know that the asymptote relation is an equivalence relation. Hence, by transitivity, $c(t)$ would have to be asymptotic to $c^{-1}(t)$ which is absurd in a reversible UFH manifold. Therefore, $s_1(t) \to -\infty$ and $c(t)$ is asymtotic to $a^{-1}(t)$, *i.e.* by symmetry of the asymtote relation, $a(t)$ and $c(t)$ are parallel. □

3.4. The Geodesic Ray Boundary

Let M be a reversible UFH manifold. The *geodesic ray boundary* or *visual boundary* $\partial M = \partial_r M$ is defined as the set of all equivalence classes of asymptotic geodesics. Set $\bar{M} = M \cup \partial M$. For a fixed base point p let

$$\phi_p : \bar{B}_p(1) \to \bar{M} \qquad \phi_p(v) = exp_p\left(\frac{v}{1 - ||v||_v}\right),$$

where $\bar{B}_p(1) = \{v \in T_pM \,|\, F(v) \leq 1\}$ is the closed Finsler ball of radius 1. We can introduce a topology on $\bar{M}$ such that ϕ_p is continuous. The induced topology is independent of the selection of the base point. With this topology $\bar{M}$ is a compact, ∂M closed and M a dense subset of $\bar{M}$. This follows as in the Riemannian situation.

It is also possible to define the *cone topology*. For $p \in M$ and $x, y \in \bar{M}$ different from p define the angle $\measuredangle_p(x, y) = \measuredangle(\dot{c}_{px}(0), \dot{c}_{py}(0))$, as the Finsler angle between the geodesics c_{px} and c_{py} joining p to x resp. y. The topology generated by the open set of M and the set of all cones

$$C_p(\xi, \varepsilon) = \{x |\, x \neq p,\ \measuredangle_p(\xi, x) \leq \varepsilon\}$$

with $p \in M$, $\xi \in \partial M$ and $\varepsilon > 0$ is called the cone topology and the induced topology on ∂M the sphere topology. Then the cone topology coincides with the topology induced by any ϕ_p. Again the proof goes along the same lines as in Riemannian geometry.

Many of the results of Eberlein and O'Neill [25, §2] can now be extended to the Finsler setting without any substancial modifications. The law of cosine used in the proofs of [25, §2] is only of a qualitative nature.

Proposition 3.7. – *Let M be a reversible UFH manifold. The cone topology κ is admissible in the sense of* [25], i.e. *it satisfies the following properties:*

(i) *The topology induced on M by κ is the original topology, M is a dense open subset of $\bar{M}$.*

(ii) *If $\gamma : \mathbb{R} \to M$ is any geodesic then the asymptotic extension $\gamma : [-\infty, \infty] \to \bar{M}$ is continuous.*

(iii) *Every isometry extends continuously to $\bar{M}$ and is therefore a homeomorphism of $\bar{M}$.*

(iv) *Extend the metric trivially to $\bar{M}$ such that $d(\xi,\eta) = \infty\ \forall \xi \neq \eta \in \bar{M}$. Let U be a neighborhood of $\xi \in \partial M$, $r > 0$. Then there exists a neighborhood V of ξ such that $\{x \in \bar{M} | d(p,V) < r\} \subset U$.*
An isometry has always a fixed point in $\bar{M}$. Furthermore, the map

$$V : (p,\xi) \in (M \times \partial M) \mapsto \hat{c}_{p\xi}(0) \in PTM$$

is a homeomorphism and the map

$$\measuredangle : (p,\xi,\eta) \in (M \times \bar{M} \times \bar{M} \text{ with } p \neq \xi, p \neq \eta) \mapsto \measuredangle_p(\xi,\eta)$$

is continuous.

We could also compactify M by adding the horofunction boundary, *see* for example [5, p. 21]. In $CAT(0)$-spaces the horofunction boundary and the geodesic ray boundary coincide. This is in general not the case for reversible UFH manifolds.

3.5. Busemann Functions

Let M be a reversible UFH manifold. We can introduce Busemann functions and horospheres as their level sets. If $c : \mathbb{R} \to M$ is a geodesic line, $t \mapsto d(p,c(t)) - t$ is bounded from below and monotone non-increasing. Hence

$$h_c(p) = \lim_{t\to\infty} (d(p,c(t)) - t) \tag{17}$$

exists. h_c is called the Busemann function of the geodesic line c. The horospheres or limit spheres are defined as the level sets of Busemann functions.

DEFINITION 3.7. – The *horosphere* $HS(c,p)$ and the *horoball* $HB(c,p)$ through p centered at $c(\infty)$ are defined as

$$HS(c,p) = \{x|\, h_c(x) = h_c(p)\}, \quad HB(c,p) = \{x|\, h_c(x) < h_c(p)\}. \tag{18}$$

A very general result of Busemann [11] states that in straight Busemann spaces horospheres are indeed limits of spheres:

$$HS(c,p) = \lim_{t\to\infty} S_{c(t)}(d(c(t),p)).$$

The asymptotes $a(t)$ to $c(t)$ are perpendicular to all horospheres $HS(c,p)$ and a geodesic $b(t)$ is asymptotic to $c(t)$ if and only if $h_c(b(t)) - h_c(b(s)) = s - t$.

PROPOSITION 3.8. – *Let M be a reversible UFH manifold. Then the horospheres are convex,* i.e. *the Busemann functions are weakly peakless. Furthermore Busemann functions are at least C^1: For $p \in M$ let $a(t)$ be the unit speed asymptote to $c(t)$ through p. Define $X(p) = -\dot{a}(0)$.*

$$dh_c(v) = g(X, v)(\widehat{X}) \quad \forall v \in TM.$$

It can be proved that X is C^1, see [26].

Proof. – The first statement is immediate from the convexity of spheres. To show the regularity of the Busemann functions we follow [37, Proposition 3.1]. Let $f_p(x) = d(p, x)$ and X_p the outward Finsler unit normal fields of balls with centers p. f_p is smooth on $M - \{p\}$ and the first variation formula (9) shows

$$\frac{d}{dt}|_{t=0} f_p(c(t)) = g(X_p, \dot{c}(0))(\widehat{X}_p).$$

Therefore, X_p is the Legendre transform of the 1-form df_p. Uniformly on compact sets we have $h_c(x) = \lim_{n\to\infty} f_{c(n)}(x) - n$. We obtain that $X_{c(n)} \to X$, uniformly on compact sets. This follows because $\measuredangle_p(c(n), c(\infty))$ tends to zero uniformly if $p \in K$, K a compact set. Now the Legendre transforms of the 1-forms df_p converge uniformly on compact set. Therefore, h_c is C^1 and the Legendre transform of dh_c is the limit X of the $X_{c(n)}$. □

COROLLARY 3.9. – *Let M be a reversible UFH manifold. If $c(t)$ and $a(t)$ are asymptotic then $h_c(x) - h_a(x) = \text{const}$ on all of M. In particular, $HS(c, p)$ and $HB(c, p)$ depend only on the asymptotic class of c.*

Proof. – This follows from the above proposition and the fact that under our assumptions, the asymptotic relation is symmetric. □

4. VISIBILITY AND δ-HYPERBOLICITY

δ-hyperbolicity in the sense of Gromov proved to be very successfull in the investigation of geodesic metric spaces, such as the Cayley graphs of hyperbolic groups. It is a kind of a negative curvature property which does not detect infinitesimal properties of geodesics. Riemannian manifolds of negative curvature and Cayley graphs of hyperbolic groups are examples of δ-hyperbolic spaces.

A related circle of ideas is the visibility introduced by P. Eberlein. It is known that for Riemannian and even $CAT(0)$-spaces uniform visibility and δ-hyperbolicity are equivalent. Also visibility has a lot of equivalent characterizations. Again, Riemannian manifolds of negative curvature are visible, and uniformly visible if the sectional curvature is strictly negative.

We study now some of these aspects for Finsler manifolds. In the sequel let M be a reversible UFH manifold. The main difficulty is that we now have to distinguish between perpendicular and transversal.

DEFINITION 4.1. – M satisfies the *visibility axiom* if every pair of distinct points $\xi, \eta \in \partial M$ can be joined by a geodesic $c : \mathbb{R} \to M$. We then call M a visibility space.

We do not require uniqueness of the connecting geodesic.

Remark 4.1. – The geodesic ray boundary of a Hilbert geometry (C, h) is ∂C. These spaces obviously satisfy the visibility axiom.

DEFINITION 4.2. – M is *locally visible* if for $p \in M$ and $\varepsilon > 0$ there exists a constant $R(p, \varepsilon)$ such that for every geodesic segment $[x, y]$ we have the property

$$d(p, [x, y]) \geq R(p, \varepsilon) \Rightarrow \measuredangle_p(x, y) \leq \varepsilon.$$

M is *uniformly visible* if it is locally visible with constants $R(\varepsilon)$ independent of the point p.

DEFINITION 4.3. – Let (X, d) be a geodesic metric space. X is *δ-hyperbolic* in the sense of Gromov if for every geodesic triangle $\Delta(p, q, r)$ and a point x on one edge of the triangle the distance of x to the union of the other two edges is bounded by the universal constant δ. M is *hyperbolic* in the sense of Gromov if it is δ-hyperbolic for some $\delta \geq 0$.

Note that a compact geodesic metric space is always δ-hyperbolic with $\delta = \operatorname{diam}(M)$.

4.1. Equivalent Characterizations

THEOREM 4.1. – *Let M be a reversible UFH manifold. Then*

(i) *M is uniformly visible.*

(ii) *M is hyperbolic in the sense of Gromov.*

are equivalent.

Proof. – Let M be uniformly visible. Consider a geodesic triangle with vertices p, q, r. Let $x \in [q, r]$ be any point on the geodesic joining q to r. Because the Finsler angles $\measuredangle_x(p, q)$ and $\measuredangle_x(p, r)$ sum up to $\frac{1}{2} \operatorname{diam} I_x(1)$

one of these, say $\measuredangle_x(p,q)$, has to be at least $\frac{1}{4}\operatorname{diam} I_x(1)$, which is universally bounded from below by a constant $c(M)$ because the Finsler metric is uniform.

Because M is uniformly visible, there is a point y on $[p,q]$ with $d(y,x) \leq R(c(M))$, hence M is δ-hyperbolic with $\delta = c(M)$.

Conversely, let again p, q, r be the vertices of a geodesic triangle. Let $d(p,[q,r]) \geq R$ with $R \geq 2\delta$. By δ-hyperbolicity, the δ-neighborhood of the sides $[p,q]$ and $[p,r]$ cover the side $[q,r]$. Hence, we find a point $x \in [q,r]$ and points $y \in [p,q]$, $z \in [p,r]$ such that $d(x,y) \leq \delta$, $d(x,z) \leq \delta$. Then $d(y,z) \leq 2\delta$ and because $[q,r]$ lies outside a ball around p of radius R we have $d(p,y) \geq R-\delta$, $d(p,z) \geq R-\delta$. We lift the triangle (p,y,z) with exp_p to $(D_p^{0,c}, g_{0,c})$ and obtain a triangle with two sides of length bounded from below and the side opposite to 0_p bounded from above. A standard comparison argument yields an upper bound of the Finsler angle between the sides $[p,q]$ and $[p,r]$, which is independent of the point p because the Finsler metric is uniform. $\square$

THEOREM 4.2. – *Let M be a reversible UFH manifold. The following two equivalent properties* (i) *and* (ii) *imply the visibility axiom. The converse need not to hold in general.*

(i) *M is locally visible.*

(ii) *Let $p_n, q_n \in M$ be sequences of points tending to different points $\xi, \eta \in \partial M$. Then the geodesic segments $[p_n, q_n]$ meet a fixed compact set. Hence the distance $d(p,[p_n,q_n])$ to a fixed base point remains bounded by a constant $R(p, \measuredangle_p(\xi,\eta))$.*

Proof. – First we show the equivalence of the two conditions:

Assume (i) and that (ii) does not hold. Then we find sequences p_n, q_n tending to different points z, w in ∂M and a point p such that $d(p,[p_n,q_n]) \to \infty$ for $n \to \infty$. Because $z \neq w$ we have $\lim \measuredangle_p(p_n,q_n) > 0$ and therefore for n sufficiently large we find $\varepsilon > 0$ such that $\measuredangle_p(p_n,q_n) > \varepsilon$.

But this contradics (i).

The other implication is proved similarly.

We show now that (ii) implies the visibility axiom. To do this let $z \neq w \in \partial M$ two points on the boundary and $p_n \to z$, $q_n \to w$. Again we find $p \in M$ and $\varepsilon > 0$ such that $\measuredangle_p(p_n,q_n) > \varepsilon$ for n big enough. By (ii) there is $R = R(p,\varepsilon)$ with $d(p,[p_n,q_n]) \leq R$. We have to show that the segments $[p_n,q_n]$ have an accumulation geodesic: Choose $m_n \in [p_n,q_n]$ with $d(p,m_n) \leq R$. Because the sequence m_n is bounded and M is complete we can assume that $m_n \to m$ for $n \to \infty$. Let c_1 be the unique geodesic from m to z, c_2 the unique geodesic form m to w. We claim that the composition $-c_1 * c_2$ is a geodesic: Let $x \in c_1([0,\infty))$, $y \in c_1([0,\infty))$. Then $d(x,y) = d(x,m) + d(m,y)$. This equation holds because $[m_n, p_n] \to c_1$, $[m_n, q_n] \to c_2$ and $d(p_n, q_n) = d(p_n, m_n) + d(m_n, q_n)$. □

To show that the converse need not to hold we first sketch a proof for Riemannian metrics avoiding angles. Assume M is not locally visible, and $p_n \to z$, $q_n \to w$, $z \neq w \in \partial M$, $x \in M$ such that $d(x, [p_n, q_n]) \to \infty$.

Then the composition of the geodesics $-c_{zy} * c_{yw}$ is a broken geodesic at y for any $y \in M$: Let $m_n \in [p_n, q_n]$ be closest point to y. For large n we can assume that $m_n \neq p_n$ by eventually changing the role of p_n and q_n. For n big enough $d(p_n, m_n) > \varepsilon > 0$ and by the triangle inequality $d(y, [p_n, q_n]) \to \infty$. Thus, a Fermi coordiante argument implies $d(p_n, m_n) + \varepsilon_1 \leq d(p_n, y)$ for some $\varepsilon_1 > 0$. Here we use that transversal and perpendicular are equivalent. But now $d(p_n, y) + d(y, q_n) \geq d(p_n, q_n) + \varepsilon$ for n big enough, proving the claim.

The problem is that we can not control the error we make by dropping the perpendicular to $[y, m_n]$.

THEOREM 4.3. – *Let M be a reversible UFH manifold. Then the following characterisations are equivalent:*

(i) *M satisfies the visibility axiom.*

(ii) *Let h_c be a Busemann function of the geodesic c. If γ is a geodesic which is not asymptotic to c then $h_c \circ \gamma(t) \to \infty$ for $t \to \infty$, i.e. γ leaves every horoball centered at $c(\infty)$.*

(iii) *The intersections of two horoballs centered at different points at infinity is bounded.*

Proof. – "(i) $\Rightarrow$ (ii)": Choose c_1 such that $c_1(\infty) = \gamma(\infty)$, $c_1(-\infty) = c(\infty)$. Then h_c agrees with h_{-c_1} up to a constant and $h_{-c_1}(c_1(t)) = t$. Let p_{c_1} be the projection onto the geodesic c_1 which exists by the convexity of the spheres.

In Riemannian geometry we could now conclude because $h_{-c_1}(x) \geq h_{-c_1}(p_{c_1}(x))$. In Finsler geometry we have to handle the problem that perpendicularity and transversality differ in general.

Let $c_1(s(t)) = p_{c_1}(\gamma(t))$ and y_t the point on the transversal to c_1 from $c_1(s(t))$ with initial direction in the plane determined by the perpendicular from $\gamma(t)$ and c_1 such that $d(\gamma(t), c_1(s(t))) = d(y_t, c_1(s(t)))$. Then $d(c_1(-r), y_t) \geq d(c_1(-r), c_1(s(t)))$ and therefore $h_{-c_1}(y_t) \geq$

$h_{-c_1}(c_1(s(t)))$. Because Busemann functions are 1-Lipschitz we get

$$\begin{aligned} h_{-c_1}(\gamma(t)) &= h_{-c_1}(y_t) + (h_{-c_1}(\gamma(t)) - h_{-c_1}(y_t) \\ &\geq h_{-c_1}(c_1(s(t))) - d(\gamma(t), y_t) \geq s(t) - \text{const}(M). \end{aligned}$$

For the last inequality we used that $\gamma(t)$ is asymptotic to $c_1(t)$ and so $d(\gamma(t), c_1(s(t)))$ is bounded for $t \to \infty$. Then a comparison argument applied to the equilateral triangle with lateral sides of length $\leq d(\gamma(t), c_1(s(t)))$ shows that the opposite side $d(\gamma(t), y_t)$ remains bounded because the angle between transversal and perpendicular directions is universally bounded. Finally note that $s(t) \to \infty$ for $t \to \infty$ because $\gamma(\infty) \neq c(\infty)$ and γ is asymptotic to $c_1 = -c$.

"(ii) $\Rightarrow$ (iii)": Assume (iii) is wrong and p_n a sequence of points in the intersection tending to ∞. Then by the convexity of the horoballs, the segments $[p_0, p_n]$ tend to a geodesic ray c with either $c(\infty) \neq z$ or $c(\infty) \neq z$, z, w the points at ∞ of the two Busemann functions h_1 and h_2. We get a geodesic such that $h_1(c(t))$ and $h_2(c(t))$ remains bounded, a contradiction to (ii).

"(iii) $\Rightarrow$ (i)": Let h_{c_1} and h_{c_2} be the Busemann functions of two horoballs centered at z, w. Let $r_2 \in \mathbb{R}$ be fixed. Because $D_{r_1} = \overline{HB}(z, r_1) \cap \overline{HB}(w, r_2)$ is bounded we can find r such that $D_r = \emptyset$. Choose simply $r = r_1 - 2\,\text{diam}\,(D_{r_1}) > -\infty$. Let $r_0 = \inf\{r | \overline{HB}(z, r) \cap \overline{HB}(w, r_2) \neq \emptyset\}$. Then because M is complete we find $x \in \overline{HB}(z, r_0) \cap \overline{HB}(w, r_2)$ Let c_{xz} and c_{xw} be the geodesics from x to z, w. Because the Finsler metric is reversible we necessarily have $\dot{c}_{xz}(0) = -\dot{c}_{xw}(0)$ because $\dot{c}_{xz}(0)$ is the outer normal of the common tangent plane of the horoball $\overline{HB}(z, r_0)$ in x. Therefore, the composition $-c_{xz} * c_{xw}$ is a geodesic joining z to w. □

4.2. Uniform Finsler Manifolds of Strictly Negative Flag Curvature

THEOREM 4.4. – *Let M be a reversible UFH manifold. If $K^F \leq -a^2 < 0$ then:*

(i) *M is uniformly visible.*

(ii) *M is hyperbolic in the sense of Gromov.*

By Theorem 4.1 we know already that the two properties are equivalent. We prove property (i). To do this we proceed in two steps. First, we show in Proposition 4.5 that the concatenation $-c_1 * c_2$ of two geodesics c_1, c_2 is in fact a quasi-geodesic. In a second step we generalize Morse's lemma on quasi-geodesics to the Finsler case and apply it to broken geodesics of the form $-c_1 * c_2$, enclosing a Finsler angle bounded from below.

PROPOSITION 4.5. – *Let M be a reversible UFH manifold with $K^F \leq -a^2 < 0$.* Let c_1, c_2 *be geodesics emanating form a point p, enclosing a Finsler angle α. There are constants $C_1 = C_1(M)$, $C_2 = C_2(\alpha, a, C_0) \geq 0$ and $C_3 = C_3(C_1, C_2, M)$ such that for s, t big enough,* i.e. $s, t \geq C_1 \max\{0, \frac{1}{a}\ln(\frac{1}{\alpha})\}$,

$$s + t - C_3 \leq C_3 d(c_1(s), c_2(t))$$

holds, where C_0 is the uniformity constant.

This result shows that in the case of strictly negative flag curvature the divergence of geodesics can be bounded from below. By the triangle inequality we have $d(c_1(t), c_2(t)) \leq 2t$. Therefore, the estimate in the above proposition is in a certain sense optimal.

For the proof of Proposition 4.5 we need a lemma in the hyperbolic space $H^n(-a^2)$ of constant curvature $-a^2$, *see* also [1].

LEMMA 4.6. – *Let c_1, c_2 be unit speed geodesics in $H^n(-a^2)$ enclosing a fixed angle α. Then there is a constant $C(a)$ such that if s, t are big enough,* i.e. $s, t \geq \max\{0, \frac{1}{a}\ln(\frac{1}{\alpha})\}$

$$s + t + \frac{2}{a}\ln(\alpha) - C(a) \leq d(c_1(s), c_2(t)) \leq s + t + \frac{2}{a}\ln(\alpha) + C(a).$$

Proof. – This follows by applying the law of cosine to express the length of the side opposite to the angle α. We then find upper and lower bounds for $d(c_1(s), c_2(t))$ if we take the hypothesis satisfied by s, t into consideration.

Proof of Proposition 4.5. – Let $p = c_1(0) = c_2(0)$. By Proposition 3.2 we know that $exp_p : H^n(-a^2) \to M$ is a distance quasi non-decreasing map. Moreover, in radial direction exp_p is almost an isometry: There is a constant $C_1 \geq 1$ such that exp_p does not decrease length by more than a factor $\frac{1}{C_1}$. Apply now the lemma to the preimage of the geodesic triangle $p, c_1(s), c_2(t)$. □

DEFINITION 4.4. – A continuous curve $\gamma : [0, l] \to M$ is called a (λ, k)*-quasi-geodesic* if

$$L(\gamma \restriction [t_0, t_1]) \leq \lambda d(\gamma(t_0), \gamma(t_1)) + k$$

for all $t_0, t_1 \in [0, l]$.

Now, Proposition 4.5 can be rephrased in the following way: The composition of the geodesic from $c_1(s)$ to p with the geodesic from p to $c_2(t)$, which we denote by $-c_1 * c_2$, is a quasi-geodesic. The next task is to generalize Morse' famous lemma on quasi-geodesics to Finsler manifolds with $K^F \leq -a^2 < 0$, *see* also [24], [41].

THEOREM 4.7. – *Let M be a reversible UFH manifold with $K^F \leq -a^2 < 0$. Let $\gamma : [0, l] \to M$ be a (λ, k)-quasi-geodesic. Then there is a constant $d = d(\lambda, k, a, C_0)$ such that the Hausdorff distance between γ and the geodesic joining the endpoints satisfies*

$$HD(\gamma([0, l]), [\gamma(0), \gamma(1)]) \leq d.$$

LEMMA 4.8. – *Let M be a reversible UFH manifold with $K^F \leq -a^2 < 0$. Let $b(s)$ be a curve from p_1 to q_1. Let p, q be the footpoints of p_1, q_1 on a geodesic $c(s)$. If $d(b(s), c) \geq \rho > 0\ \forall s$ then $C_0^2 L(b) \geq d(p, q)\cosh(a\rho)$.*

Proof. – First note that M has no focal point and the perpendicular from a point to a geodesic line exists uniquely. Let $\beta(s) = c(\sigma(s))$ the curve of footpoints of $b(s)$ on c and $X(s)$ the perpendicular vector field along c such that $exp_{\beta(s)}(X(s)) = b(s)$. Consider the variation $\gamma_s(t) = exp_{\beta(s)}(\frac{t}{l_0}X(s))$. Take some s_0 and set $l_0 = d(\beta(s_0), b(s_0))$.

The variation vector field of γ yields a Jacobi field J along γ with $J(0) = \beta'(s_0)$ and $J(l_0) = b'(s_0)$, $\nabla^h_{\dot\gamma} J(0) = \nabla^h_{\beta'} X(s_0)(\widehat{\gamma})$.

Replacing X by the ∇^N-parallel vector field Y with $Y(s_0) = X(s_0)$ we get a perpendicular Jacobi field $J^\perp$ along γ_{s_0} with $J^\perp(l_0) = (b'(s_0))^\perp$, the transversal component of $b'(s_0)$ relative to the geodesic γ_{s_0}, and $J(0) = \beta'(s_0)$. Now Jacobi field estimates yield

$$||J^\perp(t)||_{\widehat{\gamma}} \geq \cosh(at)||J^\perp(0)||_{\widehat{\gamma}}.$$

By uniformity of our Finsler manifold we can compare the norms $||\cdot||_{\widehat{\gamma}}$ and $||\cdot||_{b'(s_0)}$ up to the constant C_0 an obtain

$$C_0^2 L(b) \geq \cosh(a\rho)L(\beta) \geq \cosh(a\rho)d(p, q). \qquad \square$$

Proof of Theorem 4.7. – Let $\rho > 0$. If $d(\gamma(t), c) \geq \rho\ \forall t \in [t_0, t_1]$ and p_0, q_0 are the footpoints of $\gamma(t_0)$ and $\gamma(t_1)$ on the geodesic c then by the above lemma we obtain

$$L(\gamma \restriction [t_0, t_1]) \geq \frac{1}{C_0^2} d(p_0, q_0)\cosh(a\rho).$$

But because γ is a quasi-geodesic we also have an upper bound on the length of $\gamma \restriction [t_0, t_1]$ hence

$$\begin{aligned}\frac{1}{C_0^2} d(p_0, p_1)\cosh(a\rho) &\leq L(\gamma \restriction [t_0, t_1]) \leq \lambda d(\gamma(t_0), \gamma(t_1)) + k \\ &\leq \lambda(2\rho + d(p_0, q_0)) + k.\end{aligned}$$

Therefore, we can find a constant ρ_0 such that $\forall\rho > \rho_0$, if p_0, q_0 are the footpoints of endpoints of a quasi-geodesic segment γ with $d(\gamma(t), c) \geq \rho$, then $d(p_0, q_0) \leq 1$.

Assume now there is a point $p = \gamma(t)$ on γ with $d(p, c) > \rho_0$. We find t_0, t_1 such that $d(\gamma(t_0), c) = d(\gamma(t_1), c) = \rho_0$. But now

$$\begin{aligned} d(p,c) &\leq L(\gamma \restriction [t_0, t]) + \rho_0 \leq L(\gamma \restriction [t_0, t_1]) + \rho_0 \\ &\leq \lambda d(\gamma(t_0), \gamma(t_1)) + k + \rho_0 \\ &\leq \lambda(2\rho_0 + d(p_0, q_0)) + k + \rho_0 = K(a, \lambda, k, C_0). \end{aligned}$$

As a result, γ is in the K-neighborhood of c.

Conversely introduce $0 = t_0 < t_1 < \ldots < t_n = l$ such that $d(\gamma(t_i), \gamma(t_{i+1})) \leq 1$. Let p_i denote the footpoints of $\gamma(t_i)$ on c. Note that $d(p_i, \gamma(t_i)) \leq K$ by the first part of the proof and that every point q on c lies between p_i, p_{i+1} for some i. But then

$$\begin{aligned} d(q,\gamma) &\leq d(q, p_i) + d(p_i, \gamma(t_i)) \\ &\leq d(p_i, p_{i+1}) + d(p_i, \gamma(t_i)) \\ &\leq 2d(p_i, \gamma(t_i)) + d(\gamma(t_i), \gamma(t_{i+1})) + d(\gamma_{i+1}, p_{i+1}) \\ &\leq 3K + 1. \end{aligned} \qquad \square$$

Because the space of all geodesics emanating form a point $p \in M$ with endpoints in a metric ball $B_p(r)$, equiped with the topology of uniform convergence, is, by Arzela-Ascoli, compact we can extend Theorem 4.7 to geodesic rays and complete geodesics. We remark that for general δ-hyperbolic spaces one has to impose properness.

COROLLARY 4.9. – *Let $\gamma : I \to M$ a (λ, k)-quasi-geodesic, $I = \mathbb{R}^+$ or $I = \mathbb{R}$. Then there is a constant $d = d(\lambda, k, a, C_0)$ and a geodesic $c : I \to M$ such that the Hausdorff distance between γ and c satisfies $HD(\gamma(I), c(I)) \leq d$.*

We do not have a Flat Strip Lemma as in Riemannian geometry. The uniqueness of c is shown in [26] by a dynamical argument using properties of the stable and unstable foliations of the geodesic flow.

Finally, we show that reversible UFH manifolds with $K^F \leq -a^2 < 0$ are uniformly visible.

Proof of Theorem 4.4. – Let c_1, c_2 be two geodesics emanating from a point p enclosing a fixed Finsler angle α. In any case, the composition of the geodesic from $c_1(s)$ to p with the geodesic from p to $c_2(t)$ is a quasi-geodesic and the result follows from Theorem 4.7. $\square$

APPENDIX A. HILBERT GEOMETRIES

In 1894 D. Hilbert discovered a generalization of the hyperbolic geometry, for which geodesics are still euclidian segments. A general reference for these geometries is [19, IV.28] and [11, §18]. *See* also [34], [8], 15 and [16].

Let A^n be the affine n-space, $C \subset A^n$ open and geometrically convex, *i.e.* the euclidian segment joining two points on the boundary ∂C is contained in C. Let $e(\cdot,\cdot)$ be any euclidian metrization of A^n and $|\cdot|_e$ the euclidian norm. Let $[a,b)$ be the euclidian ray through a,b starting at a, (a,b) the line through a,b and $[a,b]$ the segment joining a to b. Let $R(a,b,y,x)$ be the cross-ratio of the ordered colinear points $\{x,a,b,y\}$. If $a=(1-\tau_a)x+\tau_a y$ and $b=(1-\tau_b)x+\tau_b y$ then $R(a,b,y,x)=\frac{(1-\tau_a)}{\tau_a}\frac{\tau_b}{(1-\tau_b)}>1$. The cross-ratio satisfies the following elementary properties:

(i) If four lines concurrent at a point p are given and l_1,l_2 are lines passing not through p, intersecting the for lines in $\{x_i,a_i,b_i,y_i\}$, $i=1,2$, then $R(x_1,a_1,b_1,y_1)=R(x_2,a_2,b_2,y_2)$.

(ii) If $\{x,a,b,c,y\}$ are five ordered colinear points then $R(a,b,y,x)R(b,c,y,x)=R(a,c,y,x)$.

(iii) If $\{x,x_1,a,b,y_1,y\}$ are six colinear points then $R(a,b,y_1,x_1)\geq R(a,b,y,x)$, where the strict inequality holds if either $x_1\neq x$ or $y_1\neq y$.

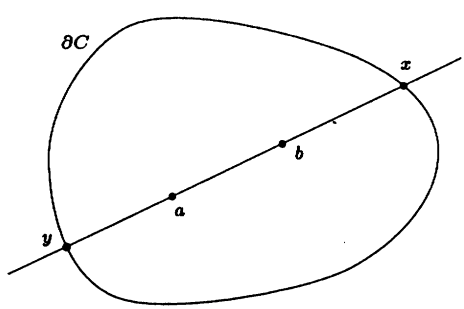

FIG. 1. - Hilbert metric

COROLLARY A.1. – *Let $a,b\in C$ and $x=[a,b)\cap\partial C$, $y=[b,a)\cap\partial C$,* see *figure* 1. *Then*

$$h(a,a)=0,$$

$$h(a,b)=\frac{1}{2\sqrt{k}}\log R(a,b,x,y)=\frac{1}{2\sqrt{k}}\log\left(\frac{e(a,x)}{e(a,y)}\,\frac{e(b,y)}{e(b,x)}\right)$$

defines a metric on C.

The metric space (C, h) is a noncompact geodesic metric space where the affine segments $(a, b) \cap C$ are isometric to $\mathbb{R}$. The triangle inequality is readily verified from elementary properties of the cross-ratio: Let $a, b, c \in C$. Consider Figure 2.

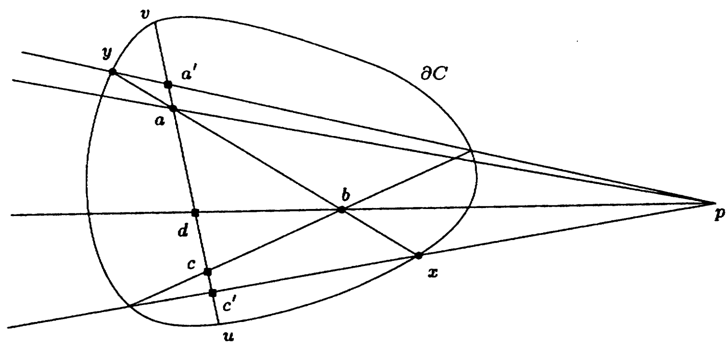

FIG. 2. - Hilbert metric, Triangle Inequality.

Then $R(a, b, x, y) = R(a, d, c', a') \geq R(a, d, u, v)$ with equality if and only if $u = c'$ and $v = a'$. Therefore, $h(a, b) \geq h(a, d)$ and similarly $h(b, c) \geq h(c, d)$ and $h(a, b) + h(b, c) \geq h(a, c)$. In case of equality, $u = c'$ and $v = a'$, hence the part of ∂C from y to $[y, a') \cap \partial C$ has to be a straight euclidian segment. The same holds for the part of ∂C from x to $\partial C \cup [x, c')$. If p is not on the line at infinity this yields two noncolinear euclidian segments on ∂C. The point p may also be at infinity. Then the two segments are parallel, but equality still holds. Therefore, such a geodesic metric space allows points with more than one geodesic joining these points. Even locally, the geodesic between two points is not uniquely determined and there are geodesic segments $[a, b]$ joining two points such that in every neighborhood of $[a, b]$ there are geodesic segments joining a to b. Also geodesics are not necessarily C^1, although they are Hölder continuous with a bounded Hölder coefficient.

LEMMA A.2. – *There is a unique geodesic joining two points if and only if there is no plane section which contains two straight euclidian segments. If there is a unique geodesic joining two points in (C, h) then (C, h) is straight.*

Proof. – Under the assumptions we always have strict inequality in the triangle inequality, hence euclidian segments are the only geodesic segments.

PROPOSITION A.3. – *If ∂C is smooth, the Lagrange function of h is given by*

$$F(p,y) = \frac{1}{2\sqrt{k}}\,|y|_e\left(\frac{1}{e(p,y_-)} + \frac{1}{e(p,y_+)}\right) \tag{19}$$

where $y_\pm = [p, p \pm y) \cap \partial C$. F is a Finsler metric if ∂C has positive definite Hessian, except for one flat region.

Proof. – Define $\alpha = e(p,y_-), \beta = e(p,y_+)$. Then

$$l(t) = h(p,p+ty) = \frac{1}{2\sqrt{k}}\log\left(\frac{\beta(\alpha + t|y|_e)}{\alpha(\beta - t|y|_e)}\right), \tag{20}$$

$$F(p,y) = \frac{d}{dt}|_{t=0}\; h(p,p+ty) = \frac{1}{2\sqrt{k}}\,|y|_e\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = |y|_e\Phi(p,y) \tag{21}$$

For simplicity we only consider the 2-dimensional case. Let $r_p(\vartheta)$ be the polar coordinate representation of ∂C from $p \in C$. Then $\Phi(p,\vartheta)^{-1} = 2\sqrt{k}(r_p(\vartheta)^{-1} + r_p(\pi+\vartheta)^{-1})^{-1}$ is the poolar coordinate representation of the indicatrix. If ∂C has positive curvature except for one flat region, $(r_p^{-1})'' + r_p^{-1} > 0$ either at ϑ or at $\pi + \vartheta$, which shows $\Phi_{\vartheta\vartheta} + \Phi > 0$. □

The isometry group of a Hilbert geometry consists of the projective transformations which map ∂C onto itself. Such projectivities map also the interior C onto itself. If the isometry group of (C,h) operates transitively on the boundary ∂C then C has to be an ellipsoid, [15, (2) p. 34]. Also, if (C,h) is homogeneous, *i.e.* the isometry group operates transitively, and ∂C has an Euler point of nonvanishing Gauss curvature, then ∂C is an ellipsoid, [15, (7) p. 38]. The assumption on the Euler point is essential because the Hilbert geometry in a triangle has a transitive abelian group of isometries.

To calculate the curvature of Hilbert geometries, we use the fact that they are projectively flat. Let X_A be the Reeb field of F and X_e the Reeb field of the flat euclidian metric. Then, there is a positive function m on HC such that $X_A = mX_e$. Let $(p,\widehat{y}) \in HC$ and y the euclidian unit vector determined by $\widehat{y}$. From the Convexity Lemma and because X is a second order differential equation, we have $m^{-1}(p,\widehat{y}) = A(X)(p,\widehat{y}) = F(p,\widehat{y}) = \Phi(p,y)$. If t is the euclidian arc length parameter and $\alpha = e(p,y_-), \beta = e(p,y_+)$, then

$$m(t) = 2\sqrt{k}\left(\frac{1}{\alpha+t} + \frac{1}{\beta - t}\right)^{-1}. \tag{22}$$

By (12), $R_{X_A} = (\frac{1}{2}m\ddot{m} - \frac{1}{4}(\dot{m})^2)\,\mathrm{id}$, where $\dot{m}(t) = \frac{dm}{dt}$, and a straightforward calculation shows that $m(t)$ of (22) satisfies $\frac{1}{2}m\ddot{m} - \frac{1}{4}(\dot{m})^2 = -k$.

PROPOSITION A.4. – *Let (C, h) be a Hilbert geometry with Hilbert metric h introduced in Corollary* A.1. *Then $R_{X_A} = -k \cdot \mathrm{id}$.*

If l denotes the Hilbert arc length, then $m(t) = \frac{dt}{dl}(l(t))$. Integration yields $t(l) = \frac{\alpha\beta}{|y|_e} \frac{e^{2\sqrt{k}l}-1}{\alpha e^{2\sqrt{k}l}+\beta}$ which satisfies $\frac{1}{2}[t]_l = \frac{1}{2}m\ddot{m} - \frac{1}{4}(\dot{m})^2 = -k$.

There is an important converse to Proposition A.4, namely, Hilbert geometries are the "Finsler models of constant curvature". A modern approach is given in [28].

THEOREM A.5 ([34], [8]). – *A simply connected, projectively flat and reversible Finsler manifold with constant negative definite Jacobi endomorphism is a Hilbert geometry.*

Note the fundamental difference to the Riemannian situation where up to isometry there is only one simply connected manifold of constant curvature.

If we give up the reversible character of the Finsler metric, we obtain a further generalization of Hilbert geometries which is due to Funk [34]. These geometries are basically constructed from two different "boundaries at infinity".

APPENDIX B. SOME REMARKS ON SYNTHETIC NOTIONS OF CURVATURE

B.1. Curvature Bounds in the Sense of Aleksandrov

Let M be a C^1 manifold and $F : TM \to \mathbb{R}^+$ a continuous positive function, strictly homogeneous of degree one. Then (2) introduces a metric $d = d_F$ on M. For a metric space (X, d) we can define the length of a continuous curve $c : [a, b] \to X$ as

$$l(c) = \sup_{\Delta} \sum_{i=0}^{n(\Delta)-1} d(c(t_i), c(t_{i+1})), \tag{23}$$

where the supremum is taken over all partitions Δ of the interval $[a, b]$. The definition of $l(c)$ is independent of the parametrization of c. If $l(c)$ is finite we can introduce the arclength measured from a fixed point of the curve as parameter. A continuous curve is called a *geodesic segment* if c is parametrized by arclength and $l(c) = d(p, q)$. In particular, a geodesic segment is an isometry $c : [0, d(p,q)] \to X$. A *geodesic* is a locally isometric map from $\mathbb{R}$ to X.

In general, without any further conditions on the function F, $L(c)$ and $l(c)$ do not agree for piecewise C^1 curves. But it is known that $L(c) = l(c)$ holds for C^1 curves if and only if the level sets $\{F \equiv \text{const}\} \cap T_pM$ are convex for all $p \in M$, [18], [11, p. 83].

Also there exists not necessarily a geodesic segment joining two points. This holds if (M, d_F) is proper, *i.e.* bounded sets are relatively compact, or equivalently closed balls are compact. The proof uses the wellknown midpoint construction. Obviously, the geodesic segment joining two points does not need to be determined uniquely. To obtain the existence of geodesic segments for general metric spaces we have to impose the additional requirement that the metric is interior, respectively intrinsic, *see* [14, (5) p. 3, Hopf-Rinow p. 4].

Note that, even locally, existence and uniqueness of prolongation of geodesic segments does in general not hold as in the case of smooth Lagrangians, which are convex in the fibres. An example is given by Hilbert geometries with two flat segments on the boundary.

We may now be tempted to approach the geometry of Finsler geodesics from the point of view of metric spaces in which two points can be joined by a geodesic and study *curvature in the sense of Aleksandrov*. Unfortunately, bounded curvature from above or below in the sense of Aleksandrov is far too strong in the setting of Finsler manifolds because it implies strong restrictions on the infinitesimal behaviour of geodesic triangles. For details on the definitions consult for example the survey article [7]. An important concept in a metric space (X, d) is the *angle in the sense of Aleksandrov* between two curves c_1 and c_2 emanating from a common point p. It is defined as follows. Let $s, t > 0$ be arbitrary and consider the triangle $\Delta(p, c_1(s), c_2(t))$. Let $\Delta_\kappa(p, q, r)$ be a comparison triangle in the Riemannian space form of constant sectional curvature κ. Let $\alpha_{s,t}$ be the angle of $\Delta_\kappa(p, q, r)$ in p. There is an angle between the curves c_1 and c_2 if

$$\alpha(c_1, c_2) = \underline{\lim}_{s,t\to 0}\alpha_{s,t} = \overline{\lim}_{s,t\to 0}\alpha_{s,t} \tag{24}$$

are equal. Note that the *lower angle* $\underline{\lim}_{s,t\to 0}\alpha_{s,t}$ and *upper angle* $\overline{\lim}_{s,t\to 0}\alpha_{s,t}$ always exist and are independent of κ. We refer to [7, §1.4] or to the original paper of Aleksandrov. For Finsler manifolds, the following theorem is due to Aleksandrov.

THEOREM B.1. – *Let M be a C^1 manifold and $F : TM \to \mathbb{R}^+$ be a continuous positive function, strictly homogeneous of degree one, convex in the fibres such that the induced metric is proper.*

If (M, d) has bounded curvature from above respectively below in the sense of Aleksandrov, then the level sets $\{F \equiv \text{const}\} \cap T_pM$ are ellipsoids.

Proof. – For Aleksandrov spaces it is known that the tangent cone is an Aleksandrov space of curvature ≤ 0 respectively ≥ 0 (in the sense of Aleksandrov). *See* [7, Theorem 8.2] and [10, §7]. Hence, strong angles in the sense of Aleksandrov exist in the tangent cone at each point. But in our case the tangent cone is exactly the tangent space which is a normed vector space. The result follows now from the following lemma. $\square$

LEMMA A.. – *If in a normed space $(\mathbb{R}^n, |\cdot|)$ a strong angle in the sense of Aleksandrov exists for any geodesic rays, then the space is Euclidian.*

B.2. Other Notions of Curvature

Busemann proposed weaker notions of nonpositive or negative curvature, *see* [11, p. 237]. These are still strong in the sense that Hilbert geometries are not of nonpositve curvature in the sense of Busemann although they have nonpositive curvature in the sense of convex capsules, *see* [11, (18.9) p. 108]. A capsule of radius r is by definition the r-neighbourhood of a geodesic segment $c_1(s)$. All these synthetic notions of bounded curvature mentioned before do not require any smoothness of the metric or that the metric comes from a Lagrangian function.

We make some remarks on the relation between these synthetic curvature notions and the flag curvature. The following claim due to Pedersen, stated as a theorem in [42] and [11, (41.8)] is not true in general.

Claim B.1 (Pedersen, Busemann). – A smooth Finsler surface with convex capsules has nonpositive flag curvature, and if the capsules are strictly convex the flag curvature is negative.

In his argument Pedersen uses Finsler parallel curves $c(s, t)$ of a geodesic $\gamma(t) = c(0, t)$, perpendicular to the geodesic segment c_1 of the capsule, with foot $c_0(0) = \gamma(t_1)$. The problem is now that the Finsler geodesic c_1 and the transversal $s \mapsto c(s, t_1)$ through $\gamma(t_1)$ are in general different already in second order, *see* [26]. The same holds for the Finsler geodesic c_0 tangent to the boundary of the capsule in $\gamma(t_0)$ and $s \mapsto c(s, t_0)$. Hence, the second variation of the length $d(c_0(s), c_1(s))$ and $L(c(s, \cdot) \restriction [t_0, t_1])$ are different. But only the latter one leads to the flag curvature through the second variation formula.

This shows that in general the relation between the convexity of capsules and the nonpositivity of the flag curvature given by Pedersen is wrong.

The relation between flag curvature bounds and nonpositive curvature in the sense of Busemann, the generalization thereof due to Kann and convex capsules is not clear.

Examples of spaces with convex capsules but not nonpositively curved in the sense of Busemann are given by Hilbert geometries. Note that they have constant negative flag curvature.

The interesting rigidity result of Kelly-Straus [40] shows that for Hilbert geometries the nonpositive curvature hypothesis in the sense of Busemann is very restrictive:

THEOREM B.3 (Kelly-Straus). – *If a Hilbert geometry with its canonical metric has at each point curvature defined in the sense of Busemann, then the metric is hyperbolic and the convex hypersurface used to define the Hilbert geometry an ellipse.*

Remark B.2. – These facts show that the different synthetic curvature notions and the flag curvature seem to capture different curvature aspects of Finsler manifolds. This may also be supported by the variational point of view: The flag curvature only represents the "horizontal" part of the Finsler curvature and can therefore never reflect the full dynamics of Finsler geodesics. This is in sharp contrast to the Riemannian case where curvature bounds in the sense of Aleksandrov and sectional curvature bounds are equivalent and the relation of the different synthetic curvature notions is determined.

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(Manuscript received on October 26 th, 1995;
Revised version received on April 26 th, 1996.)