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Inverse scattering theory for Dirac operators

by

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ABSTRACT. – We derive a reconstruction procedure of the potential from the scattering matrix of the 3-dimensional Dirac operator by using the method of Faddeev for the multi-dimensional inverse scattering theory for Schrödinger operators. We consider two problems in almost the same frameworks: the reconstruction of slowly decreasing potentials from the scattering matrices for all high energies and that of exponentially decreasing potentials from the scattering matrix for a fixed energy.

RÉSUMÉ. – On déduit un procédé de la reconstitution du potentiel à partir de la matrice S de l'opérateur de Dirac en 3 dimensions en utilisant la méthode de Faddeev pour la théorie de diffusion inverse de l'opérateur de Schrödinger. On considère deux problèmes similaires : la reconstitution d'un potentiel décroissant lentement à partir des matrices S pour l'énergie assez haute et celle d'un potentiel décroissant exponentiellement à partir de la matrice S à énergie fixée.

1. INTRODUCTION

This paper deals with the inverse problem of scattering associated with the Dirac operator in $\mathbf{R}^3$:

$$H_0 = \sum_{j=1}^{3} A_j D_j + A_4, \quad H = H_0 + V, \tag{1.1}$$

where $D_j = -i\partial/\partial x_j$, A_j's are 4×4 Hermitian matrices satisfying the anti-commutation relations

$$A_j A_k + A_k A_j = 2\delta_{jk} I_4, \quad 1 \leq j, k \leq 4, \tag{1.2}$$

V is a Hermitian matrix valued function on $\mathbf{R}^3$, and I_4 is the 4×4 identity matrix. Under suitable assumptions on the potential, the existence and completeness of wave operators were proved by the time-dependent method by Thaller-Enss [21], or by the stationary method (*see* e.g. Yamada [24] and the references therein), which ensure the unitarity of the scattering matrix. Meromorphic extensions and resonances of the scattering matrix were studied by Balslev-Helffer [2], and a great deal of mathematical achievements for the Dirac equation is expounded in the monograph of Thaller [20].

Not so much is known, however, about the inverse scattering theory, which is the attempt to construct the potential from the scattering matrix. In the case of the Schrödinger operator in $\mathbf{R}^n$ with $n \geq 2$, the uniqueness of the potential with the given scattering amplitude was proved in an early age of the study of scattering theory by using the high energy Born approximation, which is due to the decay property of the resolvent of the Schrödinger operator at high energies. However, this is not the case for the Dirac operator nor in general for the 1st order systems of classical physics. Therefore even the uniqueness problem remained open in the inverse scattering theory for these operators.

The breakthrough for this difficulity comes from the recent progress of the inverse scattering theory for multi-dimensional Schrödinger operators, where an essential role is played by the Green's function of Faddeev and the high energy Born approximation is replaced by the complex Born approximation (*see* Faddeev [6] and Newton [15]). The Green's function of Faddeev has been rediscovered from various view points. It appeared as a tool for the complex geometrical optics and was used to derive the uniqueness in the inverse boundary value problems by Sylvester and Uhlmann (*see* [19], [14]). It also appeared as a key-stone in the $\overline{\partial}$-method developed by Beals-Coifmann [3] Nachman-Ablowitz [13] and was used by Nachman in an elegant way in the reconstruction of coefficients from boundary data ([11], [12]). Ola-Päivärinta-Sommersalo applied his method to the inverse boundary value problem of the Maxwell equation [18]. The $\overline{\partial}$-method was also used by Khenkin-Novikov [16] and Novikov [17] in the inverse scattering theory for the Schrödinger operator. Hachem [7] constructed a genaral frame work of the $\overline{\partial}$-theory for the Dirac equation under a smallness assumption on the potential. As for the original work of

Faddeev, we introduced in [9] a modified radiation condition of pseudo-differential form and the Rellich type uniqueness theorem to characterize it. Broader and more complete bird's-eye views of the multi-dimensional inverse problem are described in the survey articles of Uhlmann [22] and Isakov [8].

One of the motivations of this paper is inspired by the recent work of Eskin-Ralston [5] on the inverse scattering problem of a magnetic Schrödinger operator. They introduced a new Green's function different from the Faddeev's one, which is extremely useful for the inverse scattering at a fixed energy for exponentially decreasing potentials. A little attention should be paied, however, when comparing their approach with that of Faddeev, since Eskin-Ralston worked out mainly in the momentum space.

The first aim of this paper is to study these Green's functions in more detail. In §2, we investigate relations between the Green functions of Faddeev and Eskin-Ralston. The modified radiation condition introduced in [9] is an appropriate tool to study them.

The next aim of this paper is to accomodate the methods of Faddeev and Eskin-Ralston to the Dirac operators. Since the theory of Faddeev and that of Eskin-Ralston are basically the same, we consider two problems simultaneously : the reconstruction of slowly decreasing potentials from the scattering matrix for all high energies and that of exponentially decreasing potentials from a fixed energy. As a first step, following the idea of Faddeev, for any $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in S^2$, we introduce a direction dependent Green's operator for H formally defined by

$$\lim_{\epsilon \to 0}(H - E + i\epsilon \sum_j A_j \gamma_j)^{-1},$$

E being an energy parameter. The Faddeev scattering amplitude is then constructed with the usual resolvent of the Dirac operator replaced by the Green's operator of Faddeev. We then have an equation between the physical scattering amplitude and the Faddeev scattering amplitude. Regarding the physical scattering amplitude as input and the Faddeev scattering amplitude as the unknown quantity, this equation is solvable for generic values of an auxiliary parameter. The Faddeev scattering amplitude thus obtained has a meromorphic extension to the complex upper half plane.

Our reconstruction then proceeds as follows. First we take $0 \neq \xi \in \mathbf{R}^3$ arbitrarily. Next we take $\gamma, \eta \in S^2$ such that $\xi \cdot \gamma = \gamma \cdot \eta = \eta \cdot \xi = 0$. Let $S^1_\gamma = \{\omega \in S^2; \omega \cdot \gamma = 0\}$, and we restrict the Faddeev scattering amplitude on S^1_γ. Let $\mathcal{B}_\gamma(\lambda, z; \omega, \omega')$ be the associated kernel. For sufficiently large $\lambda > 0$, we take $\omega(\lambda), \omega'(\lambda) \in S^1_\gamma$ such that $\lambda(\omega(\lambda) - \omega'(\lambda)) = \xi$ and

$\omega(\lambda) \to \eta$, $\omega'(\lambda) \to \eta$. In order to guarantee the unique solvability of the Faddeev equation for complex paremeters, we assume that the potential V has the following form :

$$V = \begin{pmatrix} V_+(x)I_2 & 0 \\ 0 & V_-(x)I_2 \end{pmatrix}, \tag{1.3}$$

where $V_\pm(x)$ are real-valued. Let

$$P_+ = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_- = \begin{pmatrix} 0 & 0 \\ 0 & I_2 \end{pmatrix},$$

and let $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ be the Pauli spin matrix. Then letting

$$K_\gamma(\lambda) = \frac{1}{\lambda}\mathcal{B}_\gamma(\lambda, i\lambda; \omega(\lambda), \omega'(\lambda)),$$

we have as $\lambda \to \infty$

$$\begin{pmatrix} \sigma \cdot (\xi \times \eta) & 0 \\ 0 & \sigma \cdot (\xi \times \eta) \end{pmatrix} P_\pm (K_\gamma(\lambda) + K_{-\gamma}(\lambda))P_\pm \to \frac{i}{2}\xi^2 P_\pm \widehat{V}(\xi)^I,$$

where

$$\widehat{V}(\xi)^I = \begin{pmatrix} \widehat{V}_-(\xi)I_2 & 0 \\ 0 & \widehat{V}_+(\xi)I_2 \end{pmatrix},$$

and $\hat{f}(\xi)$ denotes the Fourier transform of f.

We impose either of the following assumptions (A-I) or (A-II) on V.

(A-I). V is of the form (1.3), and there exist constants $C, \delta_0 > 0$ such that

$$|\partial_x^\alpha V(x)| \leq C(1 + |x|)^{-2-\delta_0-|\alpha|}, \quad |\alpha| \leq 1.$$

(A-II). *V is of the form* (1.3), *and there exist constants $C, \delta_0 > 0$ such that*

$$|\partial_x^\alpha V(x)| \leq Ce^{-\delta_0|x|}, \quad |\alpha| \leq 1.$$

The main results of this paper are as follows :

THEOREM 1.1. – *Let $E_0 > 1$ be arbitrarily given. Suppose that the assumption* (A-I) *is satisfied and that we are given the scattering matrix for all energy $E > E_0$. Then we can reconstruct the potential $V(x)$ uniquely from the scattering matrix.*

THEOREM 1.2. – *Suppose that the assumption* (A-II) *is satisfied and that we are given the scattering matrix for an arbitrarily fixed energy $E > 1$. Then we can reconstruct the potential $V(x)$ uniquely from the scattering matrix.*

The contents of this paper are as follows. In §2, we study the relationships between three Green operators of the Laplacian. In §3, we summarize the basic facts on the direct problem. In §4, we solve the Faddeev equation for the Dirac operator, which is easily reduced to that of the Schrödinger operator. Further properties of Faddeev resolvents and scattering amplitudes are investigated in sections 5 and 6, respectively. §7 and §8 are devoted to the reconstruction procedures.

We shall use the standard notation in this paper. In particular, for $x \in \mathbf{R}^n$, $< x >= (1+|x|^2)^{1/2}$. For $s \in \mathbf{R}$, $L^{2,s}$ denotes the set of functions $u(x)$ such that

$$\|u\|_s = \| < x >^s u(x)\|_{L^2(\mathbf{R}^n)} < \infty.$$

For two Banach spaces X and Y, $\mathbf{B}(X;Y)$ denotes the set of all bonded operators from X to Y, and $\mathbf{B}(X) = \mathbf{B}(X;X)$.

2. GREEN OPERATORS FOR THE LAPLACIAN

In this section we introduce three kinds of Green operators for the Laplacian in $\mathbf{R}^n, n \geq 2$.

2.1. *The usual Green operator.* – The usual Green operator for the Laplacian is defined by

$$R_0(E \pm i0) = (-\Delta - (E \pm i0))^{-1}. \tag{2.1}$$

Let $s > 1/2$. For $E > 0$, this is a bounded operator from $L^{2,s}$ to $L^{2,-s}$. For $f \in L^{2,s}$, $u_\pm = R_0(E \pm i0)f$ is a unique solution of the equation

$$(-\Delta - E)u_\pm = f \tag{2.2}$$

satisfying the outgoing (for u_+) or the incoming (for u_-) radiation condition

$$u_\pm \in L^{2,-s}, \quad \left(\frac{\partial}{\partial r} \mp i\sqrt{E}\right)u_\pm \in L^{2,-\alpha},$$

for some $0 < \alpha < 1/2$.

2.2. *The Green operator of Faddeev.* – The Green operator introduced by Faddeev is written as

$$G_{\gamma,0}(\lambda, z)f = (2\pi)^{-n} \int \frac{e^{ix\cdot\xi}}{\xi^2 + 2z\gamma\cdot\xi - \lambda^2}\hat{f}(\xi)d\xi, \tag{2.3}$$

where $\gamma \in S^{n-1}, \lambda \geq 0, z \in \mathbf{C}_+ = \{z \in \mathbf{C}; \mathrm{Im}\ z > 0\}$. If $\mathrm{Im} z \neq 0$, $(\xi^2 + 2z\gamma \cdot \xi - \lambda^2)^{-1} \in L^1_{loc}(\mathbf{R}^n)$. Therefore the above integral is absolutely convergent for $f \in \mathcal{S}$. For $t \in \mathbf{R}$, $G_{\gamma,0}(\lambda, t)$ is defined as the boundary value $G_{\gamma,0}(\lambda, t + i0)$. $G_{\gamma,0}(\lambda, z)$ thus defined has the following properties. See [23].

THEOREM 2.1. – *Let $s > 1/2$.*

(1) *As a $\mathbf{B}(L^{2,s}; H^{2,-s})$-valued function, $G_{\gamma,0}(\lambda, z)$ is continuous with respect to $\lambda \geq 0, \gamma \in S^{n-1}, z \in \overline{\mathbf{C}_+}$ except for $(\lambda, z) = (0, 0)$*

(2) *$G_{\gamma,0}(\lambda, z)$ is analytic in $z \in \mathbf{C}_+$.*

(3) *For any $\epsilon_0 > 0$, there exists a constant $C > 0$ such that*

$$\|G_{\gamma,0}(\lambda, z)\|_{\mathbf{B}(L^{2,s}; H^{\alpha,-s})} \leq C(\lambda + |z|)^{\alpha - 1},$$

if $\lambda + |z| \geq \epsilon_0, 0 \leq \alpha \leq 2$.

$u = G_{\gamma,0}(\lambda, z)f$ solves the equation

$$(-\Delta - 2iz\gamma \cdot \nabla - \lambda^2)u = f. \tag{2.4}$$

If $\mathrm{Im} z \neq 0$, $u = G_{\gamma,0}(\lambda, z)f$ is characterized as the unique solution of the above equation satisfying $u \in L^{2,-s}, \quad 1/2 < s < 1$.

To characterize $u = G_{\gamma,0}(\lambda, t)f$ for $t \in \mathbf{R}$, we need a variant of the radiation condition. As is inferred from (2.3), u should be *outgoing* if $\gamma \cdot \xi < 0$, and *incoming* if $\gamma \cdot \xi > 0$. It is in fact this property that characterizes $G_{\gamma,0}(\lambda, t)$. To state it more precisely, we need a series of notations.

Let $M_\pm$ be the operator defined by

$$M_\pm f = \mathcal{F}^{-1}(F(\pm\gamma \cdot \xi \geq 0)\hat{f}(\xi)). \tag{2.5}$$

Here and in the sequel, $F(\cdots)$ denotes the characteristic function of the set $\{\cdots\}$.

For $\xi \in \mathbf{R}^n$, let $\xi_\gamma = \gamma \cdot \xi$.

DEFINITION 2.2. – *For $\epsilon > 0$, $\mathcal{L}^{(\pm)}_{1,\epsilon}$ is the set of operators of the form $L^{(\pm)}_1 = \rho_1(\gamma \cdot D_x)M_\pm$, where*

$$\rho_1(\xi_\gamma) = 1 \quad \text{if} \quad |\xi_\gamma| > 2\epsilon,$$

$$\rho_1(\xi_\gamma) = 0 \quad \text{if} \quad |\xi_\gamma| < \epsilon.$$

For $\epsilon > 0$, $\mathcal{L}^{(\pm)}_{0,\epsilon}$ is the set of operators of the form $L^{(\pm)}_0 = \rho_0(\gamma \cdot D_x)M_\pm$, where

$$\rho_0(\xi_\gamma) = 1 \quad \text{if} \quad |\xi_\gamma| < \epsilon,$$

$$\rho_0(\xi_\gamma) = 0 \quad \text{if} \quad |\xi_\gamma| > 2\epsilon.$$

For $\lambda > 0$ and $z \in \overline{\mathbf{C}_+}$, we take $\epsilon_0 > 0$ small enough so that

$$\lambda^2 - \xi_\gamma^2 - 2\mathrm{Re} z\xi_\gamma > 0 \quad \text{for} \quad |\xi_\gamma| \leq \epsilon_0. \tag{2.6}$$

We define

$$\mathcal{L}_1^{(\pm)} = \cup_{0<\epsilon<\epsilon_0} \mathcal{L}_{1,\epsilon}^{(\pm)}, \quad \mathcal{L}_0^{(\pm)} = \cup_{0<\epsilon<\epsilon_0} \mathcal{L}_{0,\epsilon}^{(\pm)}.$$

We introduce two classes of Ps.D.Op.'s. Let σ be a constant such that $0 < \sigma < 1$. Let $\zeta = z\gamma, \zeta_R = \mathrm{Re}\ \zeta$ and

$$\theta(x, \xi) = \frac{x}{< x >} \cdot \frac{\xi}{|\xi|}. \tag{2.7}$$

DEFINITION 2.3. – *$S^{(\pm)}(\zeta, \sigma)$ is the set of functions $p^{(\pm)}(x, \xi) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ having the following properties.*

(1) $|\partial_x^\alpha \partial_\xi^\beta\ p^{(\pm)}(x, \xi)| \leq C_{\alpha\beta} < x >^{-|\alpha|}, \quad \forall \alpha, \beta.$

(2) $\mathrm{supp}_\xi\ p^{(\pm)}(x, \xi) \subset \{\xi; \frac{1}{2}(\lambda^2 + (\mathrm{Re}\ z)^2) \leq (\xi + \zeta_R)^2 \leq \frac{3}{2}(\lambda^2 + (\mathrm{Re}\ z)^2)\}$.

(3) $-\sigma < \inf_{x,\xi} \pm \theta(x, \xi + \zeta_R)$ on supp $p^{(\pm)}(x, \xi)$.

For $x \in \mathbf{R}^n$, let $x_\perp = x - (x \cdot \gamma)\gamma$.

DEFINITION 2.4. – *$S_\perp^{(\pm)}(\zeta, \sigma)$ is the set of functions $p_\perp^{(\pm)}(x_\perp, \xi_\perp) \in C^\infty(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1})$ having the following properties.*

(1) $|\partial_{x_\perp}^\alpha \partial_{\xi_\perp}^\beta\ p_\perp^{(\pm)}(x_\perp, \xi_\perp)| \leq C_{\alpha\beta} < x_\perp >^{-|\alpha|}, \quad \forall \alpha, \beta.$

(2) $\mathrm{supp}_{\xi_\perp}\ p_\perp^{(\pm)}(x_\perp, \xi_\perp) \subset \{\xi_\perp; \frac{1}{2}\lambda \leq |\xi_\perp| \leq \frac{3}{2}\lambda\}$.

(3) $-\sigma < \inf_{x_\perp,\xi_\perp} \pm \theta(x_\perp, \xi_\perp)$ on supp $p_\perp^{(\pm)}(x_\perp, \xi_\perp)$.

For $s \in \mathbf{R}$, we define

$$u \in L^{2,s} \iff \|u\|_s^2 = \int_{\mathbf{R}^n} (1 + |x|)^{2s} |u(x)|^2 dx < \infty,$$

$$u \in L_\perp^{2,s} \iff \|u\|_{\perp,s}^2 = \int_{\mathbf{R}^n} (1 + |x_\perp|)^{2s} |u(x)|^2 dx < \infty.$$

DEFINITION 2.5. – *Let $1/2 < s \leq 3/4$ and $0 < \sigma < 1$. $\mathcal{R}_{\gamma,\sigma}^s$ is the set of functions $u \in H_{loc}^2 \cap \mathcal{S}'$ such that*

(1) $L_1^{(\pm)} u \in L^{2,-s}, \quad \forall L_1^{(\pm)} \in \mathcal{L}_1^{(\pm)}$,

(2) $L_0^{(\pm)} u \in L_\perp^{2,-s}, \quad \forall L_0^{(\pm)} \in \mathcal{L}_0^{(\pm)}$,

(3) $P^{(\pm)} L_1^{(\pm)} u \in L^{2,s-1}, \quad \forall L_1^{(\pm)} \in \mathcal{L}_1^{(\pm)}, \quad \forall P^{(\pm)} \in S^{(\pm)}(\zeta, \sigma)$,

(4) $P_\perp^{(\pm)} L_0^{(\pm)} u \in L_\perp^{2,s-1}, \quad \forall L_0^{(\pm)} \in \mathcal{L}_0^{(\pm)}, \quad \forall P_\perp^{(\pm)} \in S_\perp^{(\pm)}(\zeta, \sigma)$.

The following two theorems are proved in [9].

THEOREM 2.6. – *Let* $1/2 < s \leq 3/4$ *and* $\lambda > 0$. *Then if* $f \in L^{2,s}$, $G_{\gamma,0}(\lambda, z)f \in \mathcal{R}^s_{\gamma,\sigma}$ *for* $z \in \overline{\mathbf{C}_+}$ *and for any* $0 < \sigma < 1$.

THEOREM 2.7. – *Suppose* $u \in L^{2,-s}$ *satisfies* $(H_0(\zeta) - \lambda^2)u = 0$. *Suppose there exist* $0 < \epsilon < \epsilon_0$, ϵ_0 *being defined by (2.6), and* $0 < \sigma < 1$ *such that* u *satisfies the conditions in Definition 2.5 with* $\mathcal{L}_0^{(\pm)}, \mathcal{L}_1^{(\pm)}$ *replaced by* $\mathcal{L}_{0,\epsilon}^{(\pm)}, \mathcal{L}_{1,\epsilon}^{(\pm)}$. *Then* $u = 0$.

2.3 *The Green operator of Eskin-Ralston.* – Letting $\lambda^2 + t^2 = E$ for $\lambda, t \in \mathbf{R}$, one can rewrite the Green function of Faddeev as

$$(2\pi)^{-n} \int \frac{e^{i(x-y)\xi}}{\xi^2 + 2(t + i0)\gamma \cdot \xi + t^2 - E} d\xi. \tag{2.8}$$

In the application to the inverse scattering, E corresponds to the energy. In the inverse scattering at a fixed energy, it is convenient to consider an analytic continuation with respect to t of (2.8). One might imagine that one has only to take

$$(2\pi)^{-n} \int \frac{e^{i(x-y)\xi}}{\xi^2 + 2z\gamma \cdot \xi + z^2 - E} d\xi, \tag{2.9}$$

which is not analytic in z, however. This is one of the subtle points of the Green function of Faddeev. Eskin and Ralston [5] found the analytic continuation of (2.8) by passing to the momentum space. We shall rewrite it here in the configuration space and fill the details by using Theorems 2.6 and 2.7.

For small $\epsilon > 0$, let

$$D_\epsilon = \{z \in \mathbf{C}_+; |\mathrm{Re}\, z| < \epsilon/2\}. \tag{2.10}$$

Let $\varphi_1(t) \in C^\infty(\mathbf{R})$ be such that $\varphi_1(t) = 1$ for $|t| > 2\epsilon$, $\varphi_1(t) = 0$ for $|t| < \epsilon$.

The Green operator of Eskin-Ralston consists of two parts : $V_\gamma(E, z) + W_\gamma(E, z)$, where $V_\gamma(E, z)$ is defined to be

$$V_\gamma(E, z)f(x) = (2\pi)^{-n} \int \frac{e^{ix\cdot\xi}\varphi_1(\gamma \cdot \xi)}{\xi^2 + 2z\gamma \cdot \xi + z^2 - E} \hat{f}(\xi) d\xi, \tag{2.11}$$

for $E > 0, z \in D_\epsilon$. We define $W_\gamma(E, z)$ later. Letting $\gamma = (1, 0, \cdots, 0)$ and $\xi = (\xi_1, \xi')$, we have $\mathrm{Im}(\xi^2 + 2z\gamma \cdot \xi + z^2 - E) = 2\mathrm{Im}z(\xi_1 + \mathrm{Re}z)$. On the integrand of (2.11), $|\xi_1 + \mathrm{Re}z| > \epsilon/2$. Therefore $V_\gamma(E, z)$ is $\mathbf{B}(L^2(\mathbf{R}^n))$-valued analytic with respect to $z \in D_\epsilon$ and satisfies

$$(-\Delta - 2iz\gamma \cdot \nabla + z^2 - E)V_\gamma(E, z) = \varphi_1(\gamma \cdot D_x). \tag{2.12}$$

A simple computation shows that

$$V_\gamma(E,z) = e^{-i\mathrm{Re}z\gamma\cdot x} G_{\gamma,0}(\sqrt{E+(\mathrm{Im}z)^2}, i\mathrm{Im}z)\varphi_1(\gamma\cdot D_x - \mathrm{Re}z)e^{i\mathrm{Re}z\gamma\cdot x}.$$

Therefore as $\mathrm{Im}z \to 0$, $\mathrm{Re}z \to t$, $|t| < \epsilon/2$, $V_\gamma(E,z)$ converges to

$$e^{-it\gamma\cdot x} G_{\gamma,0}(\sqrt{E},0)\varphi_1(\gamma\cdot D_x - t)e^{it\gamma\cdot x}$$

in $\mathbf{B}(L^{2,s}; L^{2,-s})$.

LEMMA 2.8. – *Let* $0 < \sigma < 1$. *Then by choosing* ϵ *and* ϵ_1 *small enough we have for any* $f \in L^{2,s}$ *and* $-\epsilon/2 < t < \epsilon/2$,

$$P^{(\pm)}L_1^{(\pm)}V_\gamma(E,t+i0)f \in L^{2,s-1}, \quad \forall P^{(\pm)} \in S^{(\pm)}(\zeta,\sigma), \quad \forall L_1^{(\pm)} \in \mathcal{L}_{1,\epsilon_1}^{(\pm)}.$$

Proof. – We assume that $\gamma = (1,0,\cdots,0)$. Let $L_1^{(\pm)} = \rho_1(D_{x_1})M_\pm \in \mathcal{L}_{1,\epsilon_1}^{(\pm)}$, $P^{(\pm)} \in S^{(\pm)}(\zeta,\sigma)$ and let $p^{(\pm)}(x,\xi)$ be the symbol of $P^{(\pm)}$. Then we have

$$\begin{aligned} &P^{(\pm)}L_1^{(\pm)}e^{-itx_1}G_{\gamma,0}(\sqrt{E},0)\varphi_1(D_{x_1}-t)e^{itx_1} \\ &\quad = e^{-itx_1}P_t^{(\pm)}L_{1,t}^{(\pm)}\varphi_1(D_{x_1}-t)G_{\gamma,0}(\sqrt{E},0)e^{itx_1}, \end{aligned}$$

where the symbol of $P_t^{(\pm)}$ is $p^{(\pm)}(x,\xi_1-t,\xi')$ and the symbol of $L_{1,t}^{(\pm)}$ is $\rho_1(\xi_1-t)F(\pm(\xi_1-t)\geq 0)$. We then have

$$L_{1,t}^{(\pm)}\varphi_1(D_{x_1}-t) = \rho_1(D_{x_1}-t)\varphi_1(D_{x_1}-t)M_\pm \in \mathcal{L}_{1,\epsilon_1-\epsilon/2}^{(\pm)},$$

if $\epsilon < 2\epsilon_1$. We take $0 < \sigma < \sigma_1 < 1$. Then

$$P_t^{(\pm)} \in S^{(\pm)}(\zeta,\sigma_1)$$

if ϵ_1 is sufficiently small. Therefore the lemma follows from Theorem 2.6. □

Next let us explain $W_\gamma(E,z)$. As above we let $\gamma = (1,0,\cdots,0)$. Let $\Delta' = \sum_{j=2}^n(\partial/\partial x_j)^2$. Then for any $\lambda > 0$, $(-\Delta'-z)^{-1}$ has continuous boundary values $(-\Delta'-\lambda\mp i0)^{-1}$ in $\mathbf{B}(L^{2,s};L^{2,-s})$. Let for $a \in \mathbf{R}$

$$\mathcal{H}'_a = \left\{f : \|f\|^2_{\mathcal{H}'_a} = \int_{\mathbf{R}^{n-1}} e^{2a|x'|}|f(x')|^2dx' < \infty\right\}.$$

Then for any $\delta > 0$, $(-\Delta'-z)^{-1}$ defined on $\mathbf{C}_\pm$ has analytic continuations across the positive real axis $(0,\infty)$ into the regions $\{z; \pm\mathrm{Im}\sqrt{z} > -\delta\}$ as $\mathbf{B}(\mathcal{H}'_\delta;\mathcal{H}'_{-\delta})$-valued functions. We denote these operators by $r_\pm(z)$.

Let $\varphi_0(t) = 1-\varphi_1(t)$, where $\varphi_1(t)$ is the one appearing in the expression of $V_\gamma(E,z)$ in (2.11). Let $\mathcal{F}_{x_1\to\xi_1}$ be the Fourier transformation with respect to x_1. We define

$$\begin{aligned} W_\gamma(E,z) =&(\mathcal{F}_{x_1\to\xi_1})^{-1}[r_+(E-(\xi_1+z)^2)F(\xi_1<0)\varphi_0(\xi_1) \\ &+ r_-(E-(\xi_1+z)^2)F(\xi_1>0)\varphi_0(\xi_1)]\mathcal{F}_{x_1\to\xi_1}. \end{aligned} \tag{2.13}$$

Note that for ϵ small enough, $\mathrm{Re}(E-(\xi_1+z)^2)>0$. $W_\gamma(E,z)$ satisfies

$$(-\Delta - 2iz\gamma\cdot\nabla_x + z^2 - E)W_\gamma(E,z) = \varphi_0(\gamma\cdot D_x). \tag{2.14}$$

Let for $a \in \mathbf{R}$

$$\mathcal{H}_a = \left\{ f : \|f\|^2_{\mathcal{H}_a} = \int_{\mathbf{R}^n} e^{2a|x|}|f(x)|^2dx < \infty \right\}.$$

LEMMA 2.9. – *For any $\delta>0$, there exists $\epsilon>0$ such that $W_\gamma(E,z)$ is a $\mathbf{B}(\mathcal{H}_\delta;\mathcal{H}_{-\delta})$-valued analytic function of $z\in D_\epsilon$. Moreover it has a continuous boundary value for $z\in\overline{D_\epsilon}\cap\mathbf{R}$. For any $f\in L^{2,s}$ and $\epsilon_2>0$, we have*

$$P_\perp^{(\pm)}L_0^{(\pm)}W_\gamma(E,t)f \in L_\perp^{2,s-1},$$
$$\forall P_\perp^{(\pm)} \in S_\perp^{(\pm)}(\zeta,\sigma), \forall L_0^{(\pm)} \in \mathcal{L}_{0,\epsilon_2}^{(\pm)}, 0<\forall\sigma<1, -\epsilon/2<\forall t<\epsilon/2.$$

Proof. – By Theorem 2.3 of [9], we have for any $g\in L^{2,s}(\mathbf{R}^{n-1}_{x_\perp})$

$$P_\perp^{(\pm)}r_\mp(k)g \in L^{2,s}(\mathbf{R}^{n-1}_{x_\perp})$$

for any $k>0$ and $P_\perp^{(\pm)}\in S_\perp^{(\pm)}(\zeta,\sigma)$, from which the lemma follows immediately. □

We finally define

$$U_{\gamma,0}(E,z) \doteq V_\gamma(E,z) + W_\gamma(E,z). \tag{2.15}$$

This is the Green operator of Eskin-Ralston.

THEOREM 2.10. – *For any $\delta>0$, there exists $\epsilon>0$ such that $U_{\gamma,0}(E,z)$ is a $\mathbf{B}(\mathcal{H}_\delta;\mathcal{H}_{-\delta})$-valued analytic function of $z\in D_\epsilon$. It satisfies*

$$(-\Delta - 2iz\gamma\cdot\nabla + z^2 - E)U_{\gamma,0}(E,z)f = f \tag{2.16}$$

for any $f\in\mathcal{H}_\delta$. It has a continuous boundary value for $z\in\overline{D_\epsilon}\cap\mathbf{R}$ and

$$U_{\gamma,0}(E,t) = G_{\gamma,0}(\sqrt{E-t^2},t), \quad -\epsilon/2<t<\epsilon/2. \tag{2.17}$$

Moreover for $z=i\tau, \tau>0$

$$U_{\gamma,0}(E,i\tau) = G_{\gamma,0}(\sqrt{E+\tau^2},i\tau). \tag{2.18}$$

Proof. – (2.16) follows from (2.12) and (2.14). By Lemmas 2.8 and 2.9, for any $f\in L^{2,s}$, $u = U_{\gamma,0}(E,t)f - G_{\gamma,0}(\sqrt{E-t^2},t)f$ satisfies the assumption in Theorem 2.7, hence follows (2.17). (2.18) follows directly from (2.15). □

3. PRELIMINARIES FOR THE DIRAC OPERATOR

We summarize fundamental results for Dirac operators. For the details, see e.g. [2], [24] or [4].

3.1 *Dirac operators.* – The unperturbed Dirac operator on $L^2(\mathbf{R}^3)^4$, from now on we often omit the superscript 4, is defined by

$$H_0 = \sum_{j=1}^{3} A_j D_j + A_4, \tag{3.1}$$

where $D_j = -i\partial/\partial x_j$, and A_j's are 4×4 Hermitian matrices satisfying

$$A_j A_k + A_k A_j = 2\delta_{jk} I_4, \quad j, k = 1, \cdots, 4. \tag{3.2}$$

A standard choice of these matrices is

$$A_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad 1 \le j \le 3, \quad A_4 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},$$

where σ_j's are the Pauli spin matrices :

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let for $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbf{C}^3$,

$$h_0(\zeta) = \sum_{j=1}^{3} A_j \zeta_j + A_4. \tag{3.3}$$

Using the anti-commutation relations (3.2), we have

$$(h_0(\zeta))^2 = (1 + \zeta^2) I_4. \tag{3.4}$$

For $\xi \in \mathbf{C}^3$, let

$$\sigma \cdot \xi = \sum_{i=1}^{3} \sigma_i \xi_i, \quad A \cdot \xi = \sum_{i=1}^{3} A_i \xi_i. \tag{3.5}$$

By using the relations :

$$\sigma_j \sigma_k + \sigma_k \sigma_j = 2\delta_{jk} I_2,$$

$$\sigma_1\sigma_2 = i\sigma_3, \qquad \sigma_2\sigma_3 = i\sigma_1, \qquad \sigma_3\sigma_1 = i\sigma_2,$$

we have the following formulas for the product

$$(\sigma \cdot \xi)(\sigma \cdot \eta) = \xi \cdot \eta I_2 + i\sigma \cdot (\xi \times \eta), \tag{3.6}$$

$$(A \cdot \xi)(A \cdot \eta) = \xi \cdot \eta I_4 + \begin{pmatrix} i\sigma \cdot (\xi \times \eta) & 0 \\ 0 & i\sigma \cdot (\xi \times \eta) \end{pmatrix}. \tag{3.7}$$

We also define

$$P_{\pm} = \frac{1}{2}(I_4 \pm A_4). \tag{3.8}$$

This coincides with the one given in §1. Let $\mathcal{W}$ be the set defined by

$$\mathcal{W} = \{aI_4 + bA_4; a, b \in \mathbf{R}\}. \tag{3.9}$$

For $W = aI_4 + bA_4 \in \mathcal{W}$, we define

$$W^I = aI_4 - bA_4. \tag{3.10}$$

More explicitly

$$\begin{pmatrix} \alpha I_2 & 0 \\ 0 & \beta I_2 \end{pmatrix}^I = \begin{pmatrix} \beta I_2 & 0 \\ 0 & \alpha I_2 \end{pmatrix}.$$

The map $W \to W^I$ is an involution on $\mathcal{W}$. Since

$$A_4(A \cdot \xi) + (A \cdot \xi)A_4 = 0$$

by the anti-commutation relations, we easily have

$$W(A \cdot \xi) = (A \cdot \xi)W^I, \quad \forall W \in \mathcal{W}, \tag{3.11}$$

$$P_+(A \cdot \xi) = (A \cdot \xi)P_-. \tag{3.12}$$

Therefore by induction, we can show

$$P_+(A \cdot \zeta^{(1)}) \cdots (A \cdot \zeta^{(2n+1)})P_+ = 0, \quad \forall \zeta^{(i)} \in \mathbf{C}^3, \quad \forall n \geq 0, \tag{3.13}$$

$$P_-(A \cdot \zeta^{(1)}) \cdots (A \cdot \zeta^{(2n+1)})P_- = 0, \quad \forall \zeta^{(i)} \in \mathbf{C}^3, \quad \forall n \geq 0. \tag{3.14}$$

The perturbed Dirac operator is defined by

$$H = H_0 + V, \tag{3.15}$$

where $V(x)$ is a 4×4 Hermitian-matrix valued function on $\mathbf{R}^3$. We assume that $V(x)$ is a $\mathcal{W}$-valued function. More precisely

(A-1) V has the following form

$$V = \begin{pmatrix} V_+(x)I_2 & 0 \\ 0 & V_-(x)I_2 \end{pmatrix},$$

where $V_\pm(x)$ are real-valued.

In sections 3 and 4, we shall assume that

(A-2) $$|\partial_x^\alpha V(x)| \leq C(1+|x|)^{-1-\delta_0}, \quad \delta_0 > 0, \quad |\alpha| \leq 1.$$

Under this assumption, we have

$$\sigma_{\mathrm{cont}}(H) = (-\infty, -1] \cup [1, \infty), \quad \sigma_d(H) \subset (-1, 1).$$

It is well-known that $\sigma_{\mathrm{cont}}(H)$ is absolutely continuous and $\sigma_p(H) \cap ((-\infty, -1) \cup (1, \infty))$ is empty. Let $R(z) = (H-z)^{-1}$. Then for $|E| > 1$, we have

$$R(E \pm i0) \in \mathbf{B}(L^{2,s}; L^{2,-s}), \quad s > 1/2.$$

3.2 *Spectral representation for* H_0. – Let for $\xi \in \mathbf{R}^3$

$$\Pi_\pm(\xi) = \frac{1}{2}\left(I \pm \frac{h_0(\xi)}{<\xi>}\right), \quad <\xi>= (1+\xi^2)^{1/2}. \tag{3.16}$$

They are the eigenprojections of $h_0(\xi)$: $\Pi_\pm(\xi)^2 = \Pi_\pm(\xi)$ and

$$h_0(\xi)\Pi_\pm(\xi) = \pm <\xi> \Pi_\pm(\xi). \tag{3.17}$$

We define

$$\mathcal{F}_0^{(\pm)}(E)f(\theta) = \Pi_\pm(\sqrt{E^2-1}\theta)\int e^{-i\sqrt{E^2-1}\theta\cdot x} f(x)dx, \quad \pm E > 1, \quad \theta \in S^2. \tag{3.18}$$

Then $\mathcal{F}_0^{(\pm)}(E) \in \mathbf{B}(L^{2,s}; L^2(S^2))$, $s > 1/2$, and $\mathcal{F}_0^{(\pm)}(E)^*$ are the eigenoperators of H_0 in the sense that

$$H_0\mathcal{F}_0^{(\pm)}(E)^*\varphi = E\mathcal{F}_0^{(\pm)}(E)^*\varphi, \quad \forall\varphi \in L^2(S^2).$$

Let $\rho(E) = \sqrt{E^2-1}|E|$ for $|E| > 1$. Then we have for $f \in L^{2,s}$, $s > 1/2$,

$$\begin{aligned}(2\pi)^3\|f\|^2_{L^2(\mathbf{R}^3)} = &\int_1^\infty \|\mathcal{F}_0^{(+)}(E)f\|^2_{L^2(S^2)}\rho(E)dE \\ &+ \int_{-\infty}^{-1} \|\mathcal{F}_0^{(-)}(E)f\|^2_{L^2(S^2)}\rho(E)dE.\end{aligned}$$

Let $K_\pm(E)$ be the set of all functions $\varphi(\theta) \in L^2(S^2)^4$ such that

$$\Pi_\pm(\sqrt{E^2-1}\theta)\varphi(\theta) = \varphi(\theta) \quad a.e. \quad \theta \in S^2.$$

Let $\mathcal{H}_\pm$ be the Hilbert spaces of $K_\pm(E)$-valued functions $f_\pm$ defined for $\pm E \in (1,\infty)$ such that

$$\int_1^\infty \|f_\pm(\pm E,\cdot)\|^2_{L^2(S^2)^4}\rho(E)dE < \infty,$$

$$f_\pm(E,\cdot) \in K_\pm(E) \quad a.e. \quad \pm E > 1.$$

Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. We define for $f \in L^{2,s}, s > 1/2$,

$$(\mathcal{F}_0 f)(E) = \mathcal{F}_0^{(\pm)}(E)f, \quad \pm E > 1. \tag{3.19}$$

Then $\mathcal{F}_0$ is uniquely extended to a unitary operator from $L^2(\mathbf{R}^3)$ to $\mathcal{H}$.

3.3 *Spectral representation for H.* – Let $R(z) = (H-z)^{-1}$, and define

$$\mathcal{F}^{(\pm)}(E) = \mathcal{F}_0^{(\pm)}(E)(1-VR(E+i0)^*), \quad \pm E > 1 \quad E \notin \sigma_p(H). \tag{3.20}$$

Then $\mathcal{F}^{(\pm)}(E) \in \mathbf{B}(L^{2,s}; L^2(S^2)), s > 1/2$,

$$H\mathcal{F}^{(\pm)}(E)^*\varphi = E\mathcal{F}^{(\pm)}(E)^*\varphi, \quad \forall\varphi \in L^2(S^2),$$

and for $f \in L^{2,s}, s > 1/2$,

$$(2\pi)^3\|P_{ac}(H)f\|^2_{L^2(\mathbf{R}^3)} = \int_1^\infty \|\mathcal{F}^{(+)}(E)f\|^2_{L^2(S^2)}\rho(E)dE + \int_{-\infty}^{-1} \|\mathcal{F}^{(-)}(E)f\|^2_{L^2(S^2)}\rho(E)dE.$$

Let for $f \in L^{2,s}, s > 1/2$,

$$(\mathcal{F}f)(E) = \mathcal{F}^{(\pm)}(E)f, \quad \pm E > 1, \quad E \notin \sigma_p(H). \tag{3.21}$$

Then $\mathcal{F}$ is uniquely extended to a partial isometry on $L^2(\mathbf{R}^3)$ with initial set $\mathcal{H}_{ac}(H)$ = the absolutely continuous subspace for H and final set $\mathcal{H}$.

3.4 *S-matrix.* – The wave operators are defined by

$$W_\pm = \text{s-}\lim_{t\to\pm\infty} e^{itH}e^{-itH_0}. \tag{3.22}$$

They are partial isometries with initial set $L^2(\mathbf{R}^3)$ and final set $\mathcal{H}$. The scattering operator S is defined by

$$S = W_+^* W_-. \tag{3.23}$$

Let $\widehat{S} = \mathcal{F}_0 S \mathcal{F}_0^*$. Then as is well-known

$$\widehat{S} = \int_1^\infty \oplus \widehat{S}_+(E)dE + \int_{-\infty}^{-1} \oplus \widehat{S}_-(E)dE, \tag{3.24}$$

where $\widehat{S}_\pm(E)$ are unitary operators on $K_\pm(E)$ and are written as

$$\widehat{S}_\pm(E) = I - \frac{i\rho(E)}{4\pi^2} A_\pm(E), \tag{3.25}$$

$$A_\pm(E) = \mathcal{F}_0^{(\pm)}(E) V \mathcal{F}^{(\pm)}(E)^*, \quad \pm E > 1, \quad E \notin \sigma_p(H). \tag{3.26}$$

Another representation, which seems to be more transparent, was utilized in Balslev-Helffer [2]. Let $\widetilde{\mathcal{F}}_0$ be the usual Fourier transformation

$$\widetilde{\mathcal{F}}_0 f(\xi) = (2\pi)^{-3/2} \int e^{-ix\cdot\xi} f(x) dx,$$

and define the Foldy-Wouthuysen transformation G by

$$G = \widetilde{\mathcal{F}}_0^{-1} \exp\left\{ A_4 \sum_{j=1}^3 A_j \xi_j g(|\xi|) \right\} \widetilde{\mathcal{F}}_0,$$

where

$$g(s) = \frac{1}{2s} \arctan s.$$

We then have the following diagonalization :

$$GH_0G^{-1} = \begin{pmatrix} (1-\triangle)^{1/2} & 0 \\ 0 & -(1-\triangle)^{1/2} \end{pmatrix}.$$

Using this F-W transformation, Balslev-Helffer derived a representation of the S-matrices $\widetilde{S}_\pm(E)$ as unitary operators on $L^2(S^2)^2$. $\widehat{S}_\pm(E)$ defined above and $\widetilde{S}_\pm(E)$ are unitarily equivalent.

4. THE FADDEEV EQUATION FOR THE DIRAC OPERATOR

4.1 *Derivation of the Faddeev equation.* – The generalized eigenfunction $\Phi_{\pm}(x, k)$ of the Dirac operator is a solution to the following equation:

$$H\Phi_{\pm}(x,k) = \pm < k > \Phi_{\pm}(x,k), \quad k \in \mathbf{R}^3.$$

Letting $\Phi_{\pm}(x,k) = \Pi_{\pm}(k)e^{ix\cdot k} + u_{\pm}$, we have

$$(H \mp < k >)u_{\pm} = -V(x)\Pi_{\pm}(k)e^{ix\cdot k}.$$

We introduce an arbitrary direction $\gamma \in S^2$ and decompose k as

$$k = \eta + t\gamma, \quad \eta \cdot \gamma = 0.$$

Letting $u_{\pm} = e^{it\gamma\cdot x}v_{\pm}$ and $D_x = -i\nabla_x$, we get

$$(h(D_x; t\gamma) + V \mp \sqrt{1 + \eta^2 + t^2})v_{\pm} = -V(x)\Pi_{\pm}(k)e^{i\eta\cdot x},$$

where

$$h(\xi;\zeta) = \sum_{i=1}^{3} A_j(\xi_j + \zeta_j) + A_4. \tag{4.1}$$

The basic idea of Faddeev in the inverse scattering theory is to complexify t into $z \in \mathbf{C}$. Following this idea, we consider the equation

$$(h(D_x; z\gamma) + V \mp E(\lambda, z))v_{\pm} = f, \tag{4.2}$$

$$E(\lambda, z) = (1 + \lambda^2 + z^2)^{1/2}, \tag{4.3}$$

where $\lambda \in \mathbf{R}, \gamma \in S^2, z \in \overline{\mathbf{C}_+}$. This is the Faddeev equation for the Dirac operator.

4.2 *Unperturbed equation.* – Let us begin with the unperturbed equation

$$(h(D_x; z\gamma) \mp E(\lambda, z))v_{\pm} = f. \tag{4.4}$$

The properties of this equation is easily reduced to those of the Schrödinger equation. Multiplying $h(D_x; z\gamma) \pm E(\lambda, z)$ and noting that $h(D_x; z\gamma)^2 = -\triangle - 2iz\gamma\cdot\nabla + z^2 + 1$, we have

$$(-\triangle - 2iz\gamma\cdot\nabla - \lambda^2)v_{\pm} = (h(D_x; z\gamma) \pm E(\lambda, z))f. \tag{4.5}$$

Therefore $v_\pm$ is formally given by

$$v_\pm = (h(D_x; z\gamma) \pm E(\lambda, z))(-\Delta - 2iz\gamma \cdot \nabla - \lambda^2)^{-1} f. \tag{4.6}$$

Now let $g_{\gamma,0}(\lambda, z)$ be the Green operator of Faddeev introduced in §2. Namely

$$g_{\gamma,0}(\lambda, z) f = (2\pi)^{-3} \int_{\mathbf{R}^3} \frac{e^{ix\cdot\xi}}{\xi^2 + 2z\gamma \cdot \xi - \lambda^2} \hat{f}(\xi) d\xi. \tag{4.7}$$

We define the operator $G^{(\pm)}_{\gamma,0}(\lambda, z)$ by

$$G^{(\pm)}_{\gamma,0}(\lambda, z) = (h(D_x; z\gamma) \pm E(\lambda, z)) g_{\gamma,0}(\lambda, z). \tag{4.8}$$

Then for $f \in L^{2,s}, s > 1/2$, $v_\pm = G^{(\pm)}_{\gamma,0}(\lambda, z) f$ solves the equation (4.4). Theorem 2.1 immediately implies the following theorem.

THEOREM 4.1. – *Let $s > 1/2$.*

(1) *As a $\mathbf{B}(L^{2,s}; H^{1,-s})$-valued function, $G^{(\pm)}_{\gamma,0}(\lambda, z)$ is continuous with respect to $\lambda \in \mathbf{R}, \gamma \in S^2, z \in \overline{\mathbf{C}_+}$ except for $(\lambda, z) = (0, 0)$.*

(2) *$G^{(\pm)}_{\gamma,0}(\lambda, z)$ is analytic in $z \in \mathbf{C}_+$.*

(3) *For any $\epsilon_0 > 0, 0 \leq \alpha \leq 1$, there exists a constant $C > 0$ such that*

$$\|G^{(\pm)}_{\gamma,0}(\lambda, z)\|_{\mathbf{B}(L^{2,s}; H^{\alpha,-s})} \leq C(|\lambda| + |z|)^\alpha,$$

if $|\lambda| + |z| \geq \epsilon_0$.

4.3 *Perturbed equation.* – We next consider

$$(h(D_x; z\gamma) + V \mp E(\lambda, z)) v_\pm = f. \tag{4.9}$$

Let

$$s_0 = \min\left(\frac{3}{4}, \frac{1 + \delta_0}{2}\right)$$

and fix $1/2 < s \leq s_0$. By Theorem 4.1, $G^{(\pm)}_{\gamma,0}(\lambda, z)V$ is a compact operator on $L^{2,-s}$.

DEFINITION 4.2. – *(Exceptional points). For $\lambda \in \mathbf{R}$,* let *$\mathcal{E}_\pm(\lambda, \gamma)$ be the set of $z \in \overline{\mathbf{C}_+}$ such that $-1 \in \sigma_p(G^{(\pm)}_{\gamma,0}(\lambda, z)V)$.*

LEMMA 4.3. – *There exists a constant $C_0 > 0$ such that $i\lambda \notin \mathcal{E}_\pm(\lambda, \gamma)$ if $\lambda > C_0$.*

Proof. – We show the lemma for the + case. Suppose $u \in L^{2,-s}$ satisfies $u + G^{(+)}_{\gamma,0}(\lambda, z)Vu = 0$. Then we have

$$h(D_x; z\gamma)u = (E - V)u.$$

By virtue of (3.11), we have the following commutation relation

$$h(D_x; z\gamma)Vu = \sum_{j=1}^{3} A_j(D_jV)u + V^I h(D_x; z\gamma)u.$$

We have, therefore, by using (4.8)

$$u = -g_{\gamma,0}(\lambda, z)\Big(\sum_{j=1}^{3} A_j(D_jV)u + E(V + V^I)u - V^I Vu\Big).$$

We now let $z = i\lambda$. Then since $E(\lambda, i\lambda) = 1$, we have

$$\|u\|_{-s} \leq C\|u\|_{-s}/\lambda.$$

From this, the lemma follows immediately. □.

Since $G^{(\pm)}_{\gamma,0}(\lambda, z)V$ is analytic in $z \in \mathbf{C}_+$ and is continuous for $z \in \overline{\mathbf{C}_+}$, we have

THEOREM 4.4. – *There exists a constant $C_0 > 0$ such that for $\lambda > C_0$, $\mathcal{E}_\pm(\lambda, \gamma) \cap \mathbf{C}_+$ is discrete and $\mathcal{E}_\pm(\lambda, \gamma) \cap \mathbf{R}$ is a closed null set.*

For $\lambda > C_0$ and $z \notin \mathcal{E}_\pm(\lambda, \gamma)$, we define

$$G^{(\pm)}_{\gamma}(\lambda, z) = (I + G^{(\pm)}_{\gamma,0}(\lambda, z)V)^{-1} G^{(\pm)}_{\gamma,0}(\lambda, z). \tag{4.10}$$

The following theorem is easily proved by Theorem 4.1.

THEOREM 4.5.

(1) *As a $\mathbf{B}(L^{2,s}; H^{1,-s})$-valued function, $G^{(\pm)}_{\gamma}(\lambda, z)$ is continuous for $\lambda > C_0$ and $z \in \overline{\mathbf{C}_+} \setminus \mathcal{E}_\pm(\lambda, \gamma)$.*

(2) *$G^{(\pm)}_{\gamma}(\lambda, z)$ is analytic for $z \in \mathbf{C}_+ \setminus \mathcal{E}_\pm(\lambda, \gamma)$.*

(3) *$G^{(\pm)}_{\gamma}(\lambda, z)^* = G^{(\pm)}_{-\gamma}(\lambda, -\overline{z})$.*

(4) (*Resolvent equations*).

$$G^{(\pm)}_{\gamma}(\lambda, z) = G^{(\pm)}_{\gamma,0}(\lambda, z) - G^{(\pm)}_{\gamma,0}(\lambda, z)VG^{(\pm)}_{\gamma}(\lambda, z), \tag{4.11}$$

$$G^{(\pm)}_{\gamma}(\lambda, z) = G^{(\pm)}_{\gamma,0}(\lambda, z) - G^{(\pm)}_{\gamma}(\lambda, z)VG^{(\pm)}_{\gamma,0}(\lambda, z). \tag{4.12}$$

In the sequel, we call $G^{(\pm)}_{\gamma,0}(\lambda, z)$ and $G^{(\pm)}_{\gamma}(\lambda, z)$ as Faddeev resolvents for the Dirac operator.

4.4 *A singular limit of the Green's function of Faddeev.* – We study a limit of $g_{\gamma,0}(\lambda, z)$ which is utilized in §7. For $\omega \in S^2$ such that $\omega \cdot \gamma = 0$, let

$$M_\gamma(\lambda) = \lambda e^{-i\lambda\omega\cdot x} g_{\gamma,0}(\lambda, i\lambda) e^{i\lambda\omega\cdot x}. \tag{4.13}$$

Then we have

$$M_\gamma(\lambda) f(x) = (2\pi)^{-3} \int \frac{e^{ix\cdot\xi}}{2\xi\cdot(\omega + i\gamma) + \xi^2/\lambda} \widehat{f}(\xi) d\xi. \tag{4.14}$$

The limit of $M_\gamma(\lambda)$ as $\lambda \to \infty$ should be called the singular limit since the principal term ξ^2/λ tend to 0. Since $M_\gamma(\lambda)/\lambda$ is a unitary transform of $g_{\gamma,0}(\lambda, i\lambda)$, by Theorem 2.1 for any $m \geq 0, s > 1/2$, there exists a constant $C > 0$ such that

$$\|M_\gamma(\lambda)\|_{\mathbf{B}(H^{m,s};H^{m,-s})} \leq C, \quad \forall \lambda \geq 1. \tag{4.15}$$

We let for $f \in \mathcal{S}$

$$N_\gamma f = (2\pi)^{-3} \int \frac{e^{ix\cdot\xi}}{2\xi\cdot(\omega + i\gamma)} \widehat{f}(\xi) d\xi. \tag{4.16}$$

It is known that $N_\gamma \in \mathbf{B}(L^{2,s}; L^{2,-s}), s > 1/2$. See [19] Lemma 3.1.

THEOREM 4.6. – *For $f \in L^{2,s}, s > 1/2$,*

$$M_\gamma(\lambda) f \to N_\gamma f \quad \text{in} \quad L^{2,-s},$$

as $\lambda \to \infty$.

Proof. – In view of (4.15), we have only to consider the case that $f \in C_0^\infty(\mathbf{R}^3)$. We first show that $M_\gamma(\lambda) f \to N_\gamma f$ weakly in $L^{2,-s}$. Take $\varphi \in L^{2,s}$ such that $\widehat{\varphi}(\xi) \in C_0^\infty(\mathbf{R}^3)$. Then

$$(M_\gamma(\lambda) f, \varphi) = (2\pi)^{-3} \int \frac{\widehat{f}(\xi)\overline{\widehat{\varphi}(\xi)}}{2\xi\cdot(\omega + i\gamma) + \xi^2/\lambda} d\xi. \tag{4.17}$$

Without loss of generality, we assume that $\omega = (1, 0, 0), \gamma = (0, 1, 0)$. We make the change of variables : $2\eta_1 = 2\xi_1 + \xi^2/\lambda, \eta_2 = \xi_2, \eta_3 = \xi_3$, and let $\lambda \to \infty$ to see that

$$(M_\gamma(\lambda) f, \varphi) \to (N_\gamma f, \varphi).$$

For $s > 1/2$, we take $s > s' > 1/2$. If $f \in C_0^\infty(\mathbf{R}^3)$, by (4.15)

$$\sup_{\lambda>1} \|M_\gamma(\lambda) f\|_{H^{1,-s'}} < \infty.$$

By the theorem of Rellich, one can select a subsequence $\lambda_1 < \lambda_2 < \cdots \to \infty$ such that $M_\gamma(\lambda_n)f$ is convergent in $L^{2,-s'}_{loc}$. Since $s > s'$, $M_\gamma(\lambda_n)f$ converges in $L^{2,-s}$. (4.17) shows that $M_\gamma(\lambda_n)f \to N_\gamma f$. Consequently, for any sequence $\lambda_1 < \lambda_2 < \cdots \to \infty$, there exists a subsequence $\lambda'_1 < \lambda'_2 < \cdots \to \infty$ such that $M_\gamma(\lambda'_n)f$ converges in $L^{2,-s}$ to one and the same limit. Therefore $M_\gamma(\lambda)f$ converges to $N_\gamma f$. □

As is clear from the above proof, Theorem 4.6 also holds when ω depends on λ, $\omega = \omega(\lambda)$, and $\omega(\lambda) \to \eta \in S^2$ as $\lambda \to \infty$. Here we must replace ω in (4.16) by η.

5. PROPERTIES OF FADDEEV RESOLVENTS

The aim of this section is to derive an equation between the Faddeev resolvent and $(H_0 - z)^{-1}$, which is easily proved by using the corresponding result for the Schrödinger case. We introduce several notations.

For $|E| > 1$, let $\mathcal{F}^0(E) \in \mathbf{B}(L^{2,s}; L^2(S^2)), s > 1/2$, be the operator defined by

$$\mathcal{F}^0(E)f(\theta) = \int e^{-i\sqrt{E^2-1}\theta\cdot x} f(x)dx, \quad \theta \in S^2. \tag{5.1}$$

For $k \in \mathbf{R}^3$, $F_\gamma(k)$ denotes the operator

$$F_\gamma(k)\psi(\theta) = F(\gamma \cdot (\theta - k) \geq 0)\psi(\theta), \quad \forall\psi \in L^2(S^2). \tag{5.2}$$

Let $r_0(z)$ be the resolvent of the Laplacian :

$$r_0(z) = (-\Delta - z)^{-1}. \tag{5.3}$$

Finally for $\lambda \neq 0$ and $t \in \mathbf{R}$, let $E = E(\lambda, t) = (1 + \lambda^2 + t^2)^{1/2}$ and we introduce the operator $\widetilde{T_\gamma}(\lambda, t) \in \mathbf{B}(L^{2,s}; L^{2,-s}), s > 1/2$, by

$$\widetilde{T_\gamma}(\lambda, t) = C_0(\lambda, t)\mathcal{F}^0(E)^* F_\gamma\left(\frac{t}{\sqrt{\lambda^2 + t^2}}\gamma\right)\mathcal{F}^0(E), \tag{5.4}$$

$$C_0(\lambda, t) = \frac{i(\lambda^2 + t^2)^{1/2}}{8\pi^2}. \tag{5.5}$$

The proof of the following relation discovered by Faddeev [6] can be seen in [15], p. 119, [23], p. 552 or [9], Lemma 6.2.

LEMMA 5.1. – *For* $\lambda \neq 0$ *and* $t \in \mathbf{R}$, *we have*

$$e^{itx\cdot\gamma} g_{\gamma,0}(\lambda,t) e^{-itx\cdot\gamma} = r_0(\lambda^2+t^2+i0) - \widetilde{T_\gamma}(\lambda,t). \tag{5.6}$$

We define for $\lambda \neq 0$ and $z \in \overline{\mathbf{C}_+}$

$$G^{(\pm)}_{\gamma,0}(\lambda,z) = (h(D_x; z\gamma) \pm E(\lambda,z)) g_{\gamma,0}(\lambda,z) \tag{5.7}$$

and for $\lambda \neq 0$ and $t \in \mathbf{R}$

$$R^{(\pm)}_{\gamma,0}(\lambda,t) = e^{itx\cdot\gamma} G^{(\pm)}_{\gamma,0}(\lambda,t) e^{-itx\cdot\gamma}. \tag{5.8}$$

Let

$$R_0(z) = (H_0 - z)^{-1}, \tag{5.9}$$

$$T^{(\pm)}_\gamma(\lambda,t) = (H_0 \pm E(\lambda,t))\widetilde{T_\gamma}(\lambda,t). \tag{5.10}$$

Since $\mathcal{F}^0(E(\lambda,t))^* = \mathcal{F}^{(+)}_0(E(\lambda,t))^* + \mathcal{F}^{(-)}_0(-E(\lambda,t))^*$, we have

$$H_0\mathcal{F}^0(E(\lambda,t))^* = E(\lambda,t)(\mathcal{F}^{(+)}_0(E(\lambda,t))^* - \mathcal{F}^{(-)}_0(-E(\lambda,t))^*).$$

Therefore

$$(H_0 \pm E(\lambda,t))\mathcal{F}^0(E(\lambda,t))^* = \pm 2E(\lambda,t)\mathcal{F}^{(\pm)}_0(\pm E(\lambda,t))^*,$$

from which we can show

$$T^{(\pm)}_\gamma(\lambda,t) = C_\pm(E)\mathcal{F}^{(\pm)}_0(\pm E)^* F_\gamma\left(\frac{t}{\sqrt{\lambda^2+t^2}}\gamma\right)\mathcal{F}^{(\pm)}_0(\pm E), \tag{5.11}$$

$$C_\pm(E) = \pm\frac{i(E^2-1)^{1/2}E}{4\pi^2}, \quad E = E(\lambda,t). \tag{5.12}$$

THEOREM 5.2. – *For* $\lambda \neq 0$ *and* $t \in \mathbf{R}$ *we have*

$$R^{(\pm)}_{\gamma,0}(\lambda,t) = R_0(\pm E(\lambda,t)+i0) - T^{(\pm)}_\gamma(\lambda,t). \tag{5.13}$$

Proof. – Multiply (5.6) by $H_0 \pm E$. Then the left-hand side turns out to be $R^{(\pm)}_{\gamma,0}(\lambda,t)$. By the anti-commutation relations, we have

$$(H_0 \pm E)r_0(\lambda^2+t^2+i0) = R_0(\pm E+i0).$$

This together with (5.10) prove the theorem. □

We define for $\lambda > C_0$, C_0 being given in Lemma 4.3, and $t \in \mathbf{R}\backslash\mathcal{E}_\pm(\lambda,\gamma)$

$$R_\gamma^{(\pm)}(\lambda,t) = e^{itx\cdot\gamma}G_\gamma^{(\pm)}(\lambda,t)e^{-itx\cdot\gamma}. \tag{5.14}$$

It follows from the resolvent equation (4.12) that

$$R_\gamma^{(\pm)}(\lambda,t) = R_{\gamma,0}^{(\pm)}(\lambda,t) - R_\gamma^{(\pm)}(\lambda,t)VR_{\gamma,0}^{(\pm)}(\lambda,t). \tag{5.15}$$

We derive an equation between $R_\gamma^{(\pm)}(\lambda,t)$ and the resolvent of H :

$$R(z) = (H-z)^{-1}.$$

LEMMA 5.3. – *Let* $R_\gamma = R_\gamma^{(\pm)}(\lambda,t)$, $T = T_\gamma^{(\pm)}(\lambda,t)$ *and* $R = R(\pm E + i0)$ *with* $E = E(\lambda,t)$. *Then we have*

$$R_\gamma = R - (1 - R_\gamma V)T(1 - VR). \tag{5.16}$$

Proof. – Theorem 5.2 and the resolvent equation (5.15) yields

$$\begin{aligned} &-(1 - R_\gamma V)T(1 - VR) \\ &= (1 - R_\gamma V)(R_{\gamma,0} - R_0)(1 - VR) \\ &= (1 - R_\gamma V)R_{\gamma,0}(1 - VR) - (1 - R_\gamma V)R_0(1 - VR) \\ &= R_\gamma(1 - VR) - (1 - R_\gamma V)R \\ &= R_\gamma - R. \quad \square \end{aligned}$$

6. EIGENOPERATORS AND SCATTERING AMPLITUDES

In this section we assume that

(A-3) $\quad |\partial_x^\alpha V(x)| \leq C(1 + |x|)^{-3/2-\epsilon}, \quad \epsilon > 0, \quad |\alpha| \leq 1.$

We define a circle S_γ^1 by

$$S_\gamma^1 = \{\omega \in S^2; \omega \cdot \gamma = 0\}. \tag{6.1}$$

For $\lambda \neq 0$ and $t \in \mathbf{R}$, let $E = E(\lambda,t)$ be as in (4.3) and let $Tr^{(\pm)}(\lambda,t)$ be the trace operator defined by

$$(Tr^{(\pm)}(\lambda,t)\psi)(\omega) = \psi\left(\pm\frac{t\gamma + \lambda\omega}{\sqrt{E^2-1}}\right), \quad \omega \in S_\gamma^1, \quad \psi \in H^{1/2}(S^2). \tag{6.2}$$

We define

$$\Phi_{\gamma,0}^{(\pm)}(\lambda,t)f(\omega) = (Tr^{(\pm)}(\lambda,t)\mathcal{F}_0^{(\pm)}(\pm E(\lambda,t))f)(\omega)$$
$$= \Pi_{\pm}(t\gamma+\lambda\omega)\int e^{-i\lambda\omega\cdot x}e^{-it\gamma\cdot x}f(x)dx. \quad (6.3)$$

Then as is well-known, $\Phi_{\gamma,0}^{(\pm)}(\lambda,t) \in \mathbf{B}(L^{2,s};L^2(S_\gamma^1))$, $s > 1$. In the following we always assume that $\lambda > C_0$, where C_0 is the constant in Lemma 4.3. Let

$$\Phi^{(\pm)}(\lambda,t) = \Phi_{\gamma,0}^{(\pm)}(\lambda,t)(1 - VR(\pm E(\lambda,t)+i0)^*), \quad (6.4)$$

$$\Phi_\gamma^{(\pm)}(\lambda,t) = \Phi_{\gamma,0}^{(\pm)}(\lambda,t)(1 - VR_\gamma^{(\pm)}(\lambda,t)^*), \quad (6.5)$$

where we assume that $\pm E(\lambda,t) \notin \sigma_p(H)$ and $t \notin \mathcal{E}_\pm(\lambda,\gamma)$ in order that they are well-defined. Using these operators, we introduce the following scattering amplitudes:

$$A_\gamma^{(\pm)}(\lambda,t) = \mathcal{F}_0^{(\pm)}(\pm E(\lambda,t))V\Phi^{(\pm)}(\lambda,t)^*, \quad (6.6)$$

$$B_\gamma^{(\pm)}(\lambda,t) = \mathcal{F}_0^{(\pm)}(\pm E(\lambda,t))V\Phi_\gamma^{(\pm)}(\lambda,t)^*. \quad (6.7)$$

We then have the following important relations between these operators.

THEOREM 6.1. – *Let $E = E(\lambda,t) = (1+\lambda^2+t^2)^{1/2}$. Suppose $\pm E \notin \sigma_p(H)$ and $\lambda > C_0$, $t \notin \mathcal{E}_\pm(\lambda,\gamma)$. Then we have the following formulas:*

$$\Phi_\gamma^{(\pm)}(\lambda,t)^* = \Phi^{(\pm)}(\lambda,t)^* + C_\pm(E)\mathcal{F}^{(\pm)}(\pm E)^* F_\gamma^{(\pm)}\left(\frac{t}{\sqrt{\lambda^2+t^2}}\gamma\right)B_\gamma^{(\pm)}(\lambda,t), \quad (6.8)$$

$$B_\gamma^{(\pm)}(\lambda,t) = A_\gamma^{(\pm)}(\lambda,t) + C_\pm(E)A^{(\pm)}(\pm E)F_\gamma^{(\pm)}\left(\frac{t}{\sqrt{\lambda^2+t^2}}\gamma\right)B_\gamma^{(\pm)}(\lambda,t). \quad (6.9)$$

Proof. – We use (5.16). Then we have by (6.5)

$$\Phi_\gamma(\lambda,t)^* = \Phi_{\gamma,0}(\lambda,t)^* - R_\gamma V\Phi_{\gamma,0}(\lambda,t)^*$$
$$= \Phi(\lambda,t)^* + (1-RV)TV\Phi_\gamma(\lambda,t)^*,$$

where we have dropped the superscript $(\pm)$. Noting that $T = C_0(E)\mathcal{F}_0(E)^*F_\gamma(\frac{t}{\sqrt{E}}\gamma)\mathcal{F}_0(E)$, we obtain (6.8). By multiplying (6.8) by $\mathcal{F}_0(E)V$, we obtain (6.9). □

We define two operators $\widetilde{K}^{(\pm)}$ and $K^{(\pm)}$ by

$$\widetilde{K}^{(\pm)} = (1 - R(\pm E + i0)V)T_\gamma^{\pm)}(\lambda, t)V, \tag{6.10}$$

$$K^{(\pm)} = C_\pm(E)A^{(\pm)}(E)F_\gamma^{(\pm)}\left(\frac{t}{\sqrt{\lambda^2 + t^2}}\gamma\right). \tag{6.11}$$

LEMMA 6.2

$$1 \in \sigma_p(\widetilde{K}^{(\pm)}) \Longleftrightarrow 1 \in \sigma_p(K^{(\pm)}).$$

Proof. – Let

$$S_1 = (1 - R(\pm E + i0)V)C_\pm(E)\mathcal{F}_0^{(\pm)}(\pm E)^* F_\gamma\left(\frac{t}{\sqrt{\lambda^2 + t^2}}\gamma\right),$$

$$S_2 = \mathcal{F}_0^{(\pm)}(E)V.$$

Then we have

$$\widetilde{K}^{(\pm)} = S_1 S_2, \quad K^{(\pm)} = S_2 S_1.$$

Since $\sigma_p(S_1 S_2) \setminus \{0\} = \sigma_p(S_2 S_1) \setminus \{0\}$, the lemma follows immediately. $\square$

THEOREM 6.3. – *Let* $\lambda > C_0$ *and* $t \in \mathbf{R}$. *Suppose* $\pm(1 + \lambda^2 + t^2)^{1/2} \notin \sigma_p(H)$. *Then*

$$t \in \mathcal{E}_\pm(\lambda, \gamma) \Longleftrightarrow 1 \in \sigma_p(K^{(\pm)}).$$

Proof. – Since $\pm E = \pm(1 + \lambda^2 + t^2)^{1/2} \notin \sigma_p(H)$, we have

$$-1 \notin \sigma_p(R_0(\pm E + i0)V). \tag{6.12}$$

By a direct calculation we have

$$1 + R_{\gamma,0}^{(\pm)}(\lambda, t)V = (1 + R_0^{(\pm)}(\pm E + i0)V)(1 - \widetilde{K}^{(\pm)}). \tag{6.13}$$

Lemma 6.2 and (6.13) then imply the theorem. $\square$

By (6.9) and Theorem 6.3, for $\lambda \neq 0$ and $t \in \mathbf{R} \setminus \mathcal{E}_\pm(\lambda, \gamma)$ such that $\pm(1 + \lambda^2 + t^2)^{1/2} \notin \sigma_p(H)$, we have

$$B_\gamma^{(\pm)}(\lambda, t) = (1 - K^{(\pm)})^{-1} A_\gamma^{(\pm)}(\lambda, t). \tag{6.14}$$

We have now entered into the first step of the reconstruction procedure of the potential. Suppose we are given $E_0 > 1$ and the scattering matrix $\widehat{S}_+(E)$ for all $E > E_0$. Take any $\lambda > \max\{C_0, \sqrt{E_0^2 - 1}\}$. Then for $t \in \mathbf{R} \setminus \mathcal{E}_+(\lambda, \gamma)$ such that $(1 + \lambda^2 + t^2)^{1/2} \notin \sigma_p(H)$, one can construct $A_\gamma^{(+)}(\lambda, t)$ from $\widehat{S}_+(E)$ with $E = (1 + \lambda^2 + t^2)^{1/2}$. By (6.14), one then gets the Faddeev scattering amplitude $B_\gamma^{(+)}(\lambda, t)$. This operator has a unique analytic continuation as a function of $z \in \overline{\mathcal{C}_+} \setminus \mathcal{E}_+(\lambda, \gamma)$. In the next section, we shall discuss the reconstruction procedure of V from $B_\gamma^{(+)}(\lambda, t)$.

7. RECONSTRUCTION PROCEDURES

In this section, we shall assume that V satisfies the assumption (A-I) in §1. In order to simplify the explanation, however, we first proceed under the stronger assumption that for some constants $C, \epsilon_0 > 0$,

$$|\partial_x^\alpha V(x)| \leq C(1+|x|)^{-3-\epsilon_0}, \quad |\alpha| \leq 1, \tag{7.1}$$

which we remove later.

Let $B_\gamma^{(+)}(\lambda, t; \xi, \omega')$ be the integral kernel of $B_\gamma^{(+)}(\lambda, t)$, which is written as

$$\begin{aligned} &B_\gamma^{(+)}(\lambda, t; \xi, \omega') \\ &\quad = \int e^{-i\xi\cdot x}\Pi_+(\xi)V(x)\Pi_+(\lambda\omega' + t\gamma)e^{i(\lambda\omega'+t\gamma)x}dx \\ &\quad\quad - \int e^{-i\xi\cdot x}\Pi_+(\xi)V(x)R_\gamma^{(+)}(\lambda, t)(V\Pi_+(\lambda\omega' + t\gamma)e^{i(\lambda\omega'+t\gamma)\cdot})dx, \end{aligned}$$

where $|\xi|^2 = E^2 - 1 = \lambda^2 + t^2$. The assumption (7.1) is used only to guarantee the absolute convergence of the integral of the first term of the right-hand side. We now define for $\omega, \omega' \in S_\gamma^1$

$$\mathcal{B}_\gamma^{(+)}(\lambda, t; \omega, \omega') = B_\gamma^{(+)}(\lambda, t; \lambda\omega + t\gamma, \omega'). \tag{7.2}$$

Using (5.14), we have the following expression

$$\begin{aligned} &\mathcal{B}_\gamma^{(+)}(\lambda, t; \omega, \omega') \\ &\quad = \int e^{-i\lambda(\omega-\omega')x}\Pi_+(\lambda\omega + t\gamma)V(x)\Pi_+(\lambda\omega' + t\gamma)dx \\ &\quad\quad - \int e^{-i\lambda\omega\cdot x}\Pi_+(\lambda\omega + t\gamma)V(x)G_\gamma^{(+)}(\lambda, t)(V\Pi_+(\lambda\omega' + t\gamma)e^{i\lambda\omega'\cdot})dx. \end{aligned}$$

By virtue of Theorem 4.5, $\mathcal{B}_\gamma^{(+)}(\lambda, t; \omega, \omega')$ has a meromorphic continuation into $\mathbf{C}_+$ as a function of t. Our aim is to calculate the limit of $\mathcal{B}_\gamma^{(+)}(\lambda, i\lambda; \omega, \omega')$ as $\lambda \to \infty$. Let

$$\zeta = \lambda\omega' + z\gamma, \quad \omega' \in S_\gamma^1, \tag{7.3}$$

$$L^{(+)}(\zeta) = e^{-i\lambda\omega'\cdot x}G_{\gamma,0}^{(+)}(\lambda, z)e^{i\lambda\omega'\cdot x}, \tag{7.4}$$

$$\widetilde{G}^{(+)}(\zeta) = e^{-i\lambda\omega'\cdot x}G_\gamma^{(+)}(\lambda, z)e^{i\lambda\omega'\cdot x}, \tag{7.5}$$

and we define

$$u_+ = u_+(\zeta) = \widetilde{G}^{(+)}(\zeta)V\Pi_+(\zeta). \tag{7.6}$$

This is analytic for $z \in \Omega_\lambda$, where

$$\Omega_\lambda = \mathbf{C} \setminus (\mathcal{E}_+(\lambda, \gamma) \cup \{i\sqrt{\lambda^2 + 1}\}). \tag{7.7}$$

Then we have

$$\int e^{-i\lambda\omega\cdot x}\Pi_+(\lambda\omega + z\gamma)V(x)G_\gamma^{(+)}(\lambda, z)(V\Pi_+(\lambda\omega' + z\gamma)e^{i\lambda\omega'\cdot})dx$$

$$= \int e^{-i\lambda(\omega-\omega')\cdot x}\Pi_+(\lambda\omega + z\gamma)Vu_+(\zeta)dx. \tag{7.8}$$

As a function of z, u_+ is meromorphic in $\mathbf{C}_+$. By Lemma 4.3 and (4.10) for large $\lambda > 0$, $i\lambda$ is not the pole of u_+. Let

$$\widetilde{g}(\zeta) = e^{-i\lambda\omega'\cdot x}g_{\gamma,0}(\lambda, z)e^{i\lambda\omega'\cdot x}, \tag{7.9}$$

$g_{\gamma,0}(\lambda, z)$ being defined by (4.7). $\widetilde{g}(\zeta)$ is the Green's operator introduced by Faddeev having the following expression :

$$\widetilde{g}(\zeta)f = (2\pi)^{-3}\int \frac{e^{ix\cdot\xi}}{\xi^2 + 2\zeta\cdot\xi}\widehat{f}(\xi)d\xi.$$

LEMMA 7.1. – *Let*

$$W_+(\zeta) = \sum_{j=1}^{3} A_j(D_jV) + E(\lambda, z)(V + V^I) + A_4(V - V^I) - V^IV, \tag{7.10}$$

$$f_+ = f_+(\zeta) = W_+(\zeta)\Pi_+(\zeta). \tag{7.11}$$

Then we have for $z \in \Omega_\lambda$

$$u_+ = \widetilde{g}(\zeta)f_+ - \widetilde{g}(\zeta)W_+(\zeta)u_+. \tag{7.12}$$

Proof. – By the resolvent equation (4.11), we have

$$u_+ = L^{(+)}(\zeta)V\Pi_+(\zeta) - L^{(+)}(\zeta)Vu_+.$$

Noting that

$$(h(D_x;\zeta) - E(\lambda, z))L^{(+)}(\zeta) = I,$$

$$h(D_x;\zeta)^2 - E(\lambda, z)^2 = -\triangle + 2\zeta \cdot D_x,$$

we have

$$\begin{aligned}(-\triangle + 2\zeta \cdot D_x)u_+ = (h(D_x;\zeta) + E(\lambda, z))V\Pi_+(\zeta) \\ - (h(D_x;\zeta) + E(\lambda, z))Vu_+.\end{aligned}$$

Using (3.11), (3.17), (7.13) and (7.14), we have

$$(h(D_x;\zeta) + E(\lambda, z))V\Pi_+(\zeta) = (W_+(\zeta) + V^I V)\Pi_+(\zeta),$$

$$(h(D_x;\zeta) + E(\lambda, z))Vu_+ = W_+(\zeta)u_+ + V^I V\Pi_+(\zeta).$$

We have therefore

$$(-\triangle + 2\zeta \cdot D_x)u_+ = f_+ - W_+(\zeta)u_+.$$

Multiplying both sides by $\widetilde{g}(\zeta)$, we obtain (7.12). Here we must take notice of the following fact which follows from [1] Theorem 2.2 : Let $1/2 < s < 1$ and suppose $w \in L^{2,-s}$ satisfies $(-\triangle + 2\zeta \cdot D_x)w \in L^{2,s}$. Then $w = \widetilde{g}(\zeta)(-\triangle + 2\zeta \cdot D_x)w$. $\square$

Since $\widetilde{g}(\zeta)$ is a unitary transform of $g_{\gamma,0}(\lambda, z)$, the inequality in Theorem 2.1 (3) also holds for $\widetilde{g}(\zeta)$. We now let $z = i\lambda$. Then since $E(\lambda, i\lambda) = 1$, we have

$$\|\widetilde{g}(\lambda\omega' + i\lambda\gamma)W_+(\lambda\omega' + i\lambda\gamma)\|_{\mathbf{B}(L^{2,-s};L^{2,-s})} \leq C/\lambda,$$

where $s > 1/2$ is chosen sufficiently close to 1/2. Therefore by taking λ large enough, we get the following lemma. Note that

$$W_+(\lambda\omega' + i\lambda\gamma) = \sum_{j=1}^{3} A_j(D_j V) + (2V_+ - V_+V_-)I_4.$$

LEMMA 7.2. – *There exists $C_0 > 0$ such that if $\lambda > C_0$, we have the following expression :*

$$u_+(\lambda\omega' + i\lambda\gamma) = (1 + \widetilde{g}(\lambda\omega' + i\lambda\gamma)\widetilde{V}_+)^{-1}\widetilde{g}(\lambda\omega' + i\lambda\gamma)f_+(\lambda), \quad (7.13)$$

$$f_+(\lambda) = \widetilde{V}_+\Pi_+(\lambda\omega' + i\lambda\gamma), \quad (7.14)$$

$$\widetilde{V}_+ = \sum_{j=1}^{3} A_j(D_j V) + (2V_+ - V_+V_-)I_4. \quad (7.15)$$

We fix a non zero vector $k \in \mathbf{R}^3$. We take $\gamma, \eta \in S^2$ such that $k \cdot \gamma = k \cdot \eta = \gamma \cdot \eta = 0$. For $\lambda > |k|/2$, we let

$$\omega = \omega(\lambda) = \left(1 - \frac{k^2}{4\lambda^2}\right)^{1/2} \eta + \frac{k}{2\lambda},$$

$$\omega' = \omega'(\lambda) = \left(1 - \frac{k^2}{4\lambda^2}\right)^{1/2} \eta - \frac{k}{2\lambda}.$$

Then we have $\omega, \omega' \in S^1_\gamma$ and

$$\lambda(\omega - \omega') = k. \tag{7.16}$$

Note that as $\lambda \to \infty$,

$$\omega \to \eta, \quad \omega' \to \eta, \quad k \cdot \omega \to 0, \quad k \cdot \omega' \to 0. \tag{7.17}$$

We now define

$$K^{(+)}_\gamma(\lambda) = \frac{1}{\lambda} \mathcal{B}^{(+)}_\gamma(\lambda, i\lambda; \omega(\lambda), \omega'(\lambda)) \tag{7.18}$$

and compute the limit as $\lambda \to \infty$.

Theorem 1.1 is a consequence of the following theorem.

THEOREM 7.3. – *As $\lambda \to \infty$, we have*

$$\begin{pmatrix} \sigma \cdot (k \times \eta) & 0 \\ 0 & \sigma \cdot (k \times \eta) \end{pmatrix} P_\pm (K^{(+)}_\gamma(\lambda) + K^{(+)}_{-\gamma}(\lambda)) P_\pm \to \frac{i}{2} k^2 P_\pm \widehat{V}(k)^I.$$

In order to prove Theorem 7.3, we split $K^{(+)}_\gamma(\lambda)$ into two parts :

$$K^{(+)}_\gamma(\lambda) = K^{(+)}_{\gamma,1}(\lambda) - K^{(+)}_{\gamma,2}(\lambda), \tag{7.19}$$

where

$$K^{(+)}_{\gamma,1}(\lambda) = \frac{1}{\lambda} \int e^{-ik\cdot x} \Pi_+(\lambda\omega + i\lambda\gamma) V(x) \Pi_+(\lambda\omega' + i\lambda\gamma) dx, \tag{7.20}$$

$$K^{(+)}_{\gamma,2}(\lambda) = \frac{1}{\lambda} \int e^{-ik\cdot x} \Pi_+(\lambda\omega + i\lambda\gamma) V(x) u_+(\lambda\omega' + i\lambda\gamma) dx. \tag{7.21}$$

We first show that $K^{(+)}_{\gamma,1}(\lambda)$ gives rise to the potential.

LEMMA 7.4

$$\begin{pmatrix} \sigma\cdot(k\times\eta) & 0 \\ 0 & \sigma\cdot(k\times\eta) \end{pmatrix} P_\pm(K_{\gamma,1}^{(+)}(\lambda)+K_{-\gamma,1}^{(+)}(\lambda))P_\pm \to \frac{i}{2}k^2 P_\pm \widehat{V}(k)^I.$$

Proof. – If $\zeta^2 = 0$, $\Pi_+(\zeta) = P_+ + \frac{1}{2}A\cdot\zeta$. Therefore

$$\Pi_+(\lambda\omega + i\lambda\gamma) = \Pi_+(\lambda\omega' + i\lambda\gamma) + \frac{1}{2}A\cdot k,$$

$$V\Pi_+(\lambda\omega' + i\lambda\gamma) = \Pi_+(\lambda\omega' + i\lambda\gamma)V^I + P_+(V - V^I),$$

which implies that

$$\begin{aligned} K_{\gamma,1}^{(+)}(\lambda) =& \left(1 + \frac{1}{2}A\cdot k\right)\left(\frac{1}{\lambda}P_+ + \frac{1}{2}A\cdot(\omega' + i\gamma)\right)\widehat{V}(k)^I \\ &+ \left(\frac{1}{\lambda}P_+ + \frac{1}{2\lambda}A\cdot k + \frac{1}{2}A\cdot(\omega'+i\gamma)\right)P_+(\widehat{V}(k) - \widehat{V}(k)^I). \end{aligned}$$

Therefore we have

$$\begin{aligned} K_{\gamma,1}^{(+)} &\equiv \lim_{\lambda\to\infty} K_{\gamma,1}^{(+)}(\lambda) \\ &= \frac{1}{4}(2 + A\cdot k)A\cdot(\eta + i\gamma)\widehat{V}(k)^I \\ &+ \frac{1}{2}A\cdot(\eta + i\gamma)P_+(\widehat{V}(k) - \widehat{V}(k)^I). \end{aligned}$$

Therefore by using (3.13), (3.14), we have

$$P_\pm K_{\gamma,1}^{(+)} P_\pm = \frac{1}{4}P_\pm(A\cdot k)A\cdot(\eta + i\gamma)P_\pm\widehat{V}(k)^I.$$

Since $k\cdot\eta = k\cdot\gamma = 0$, we have by (3.7)

$$(A\cdot k)A\cdot(\eta+i\gamma) = \begin{pmatrix} i\sigma\cdot(k\times(\eta+i\gamma)) & 0 \\ 0 & i\sigma\cdot(k\times(\eta+i\gamma)) \end{pmatrix}.$$

We have, therefore

$$P_\pm(K_{\gamma,1}^{(+)} + K_{-\gamma,1}^{(+)})P_\pm = \frac{i}{2}\begin{pmatrix} \sigma\cdot(k\times\eta) & 0 \\ 0 & \sigma\cdot(k\times\eta) \end{pmatrix} P_\pm\widehat{V}(k)^I.$$

Noting that $(\sigma\cdot(k\times\eta))^2 = k^2$, we get Lemma 7.4. □

The proof of Theorem 7.3 is thus reduced to show

LEMMA 7.5. – $P_\pm K^{(+)}_{\gamma,2}(\lambda)P_\pm \to 0$ as $\lambda \to \infty$.

Proof. – We introduce operators $M_\gamma(\lambda), N_\gamma$ by

$$M_\gamma(\lambda)f = (2\pi)^{-3}\int \frac{e^{ix\cdot\xi}}{2\xi\cdot(\omega' + i\gamma) + \xi^2/\lambda}\widehat{f}(\xi)d\xi, \tag{7.22}$$

$$N_\gamma f = (2\pi)^{-3}\int \frac{e^{ix\cdot\xi}}{2\xi\cdot(\eta + i\gamma)}\widehat{f}(\xi)d\xi. \tag{7.23}$$

Then we have

$$\widetilde{g}(\lambda\omega' + i\lambda\gamma)f_+(\lambda) = M_\gamma(\lambda)\widetilde{V}_+\Pi_+(\lambda\omega' + i\lambda\gamma)/\lambda.$$

Therefore by virtue of Theorem 4.6

$$\widetilde{g}(\lambda\omega' + i\lambda\gamma)f_+(\lambda) \to N_\gamma\widetilde{V}_+A\cdot(\eta + i\gamma)/2$$

in $L^{2,-s}$ as $\lambda \to \infty$. This, combined with (7.13), shows that

$$u_+(\lambda\omega' + i\lambda\gamma) \to N_\gamma\widetilde{V}_+A\cdot(\eta + i\gamma)/2.$$

Therefore neglecting the terms convergent to 0 in $L^1(\mathbf{R}^3)$, we have

$$\begin{aligned}
&4P_\pm\frac{1}{\lambda}\Pi_+(\lambda\omega + i\lambda\gamma)Vu_+(\lambda\omega' + i\lambda\gamma)P_\pm \\
&\sim V^I N_\gamma P_\pm A\cdot(\eta + i\gamma)\widetilde{V}_+A\cdot(\eta + i\gamma)P_\pm \\
&= V^I N_\gamma P_\pm A\cdot(\eta + i\gamma)\sum_{j=1}^{3}A_j(D_jV)A\cdot(\eta + i\gamma)P_\pm
\end{aligned}$$

where we have used $(A\cdot(\eta + i\gamma))^2 = 0$. This last term vanishes by the following computations. Let $V_j = -i\partial V/\partial x_j$. Then since $A\cdot(\eta + i\gamma)A_j = 2(\eta_j + i\gamma_j) - A_jA\cdot(\eta + i\gamma)$ for $1 \le j \le 3$, we have

$$\begin{aligned}
&P_\pm A\cdot(\eta + i\gamma)A_jV_jA\cdot(\eta + i\gamma)P_\pm \\
&= P_\pm A\cdot(\eta + i\gamma)A_jA\cdot(\eta + i\gamma)P_\pm V_j^I \\
&= 2(\eta_j + i\gamma_j)P_\pm A\cdot(\eta + i\gamma)P_\pm V_j^I \\
&= 0.
\end{aligned}$$

In view of (7.21), we have thus shown that $P_\pm K^{(+)}_{\gamma,2}(\lambda)P_\pm \to 0$. □

We next remove the assumption (7.1). In the above proof, this assumption was used only to guarantee the absolute convergence of the integral

$$\int e^{-i\lambda(\omega-\omega')x} V(x)dx.$$

As is well-known the assumption (A-I) endows the above integral with the meaning of the oscillatory integral. Namely letting

$$L = i|\lambda(\omega-\omega')|^{-2}\lambda(\omega-\omega')\cdot\nabla_x,$$

we have

$$\widehat{V}(\lambda(\omega-\omega')) = \int e^{-i\lambda(\omega-\omega')x} L^* V(x)dx.$$

Using this expression, one can repeat the above proof in the same way.

8. THE FIXED ENERGY PROBLEM

In this section we shall assume that the potential V satisfies the assumption (A-II) in §1. We fix an energy $E > 0$ and reconstruct the potential V from the scattering matix $\widehat{S}_+(E)$. Let $u_{\gamma,0}(E,z)$ be the Green operator of Eskin-Ralston for the Laplacian defined by (2.15). We define $U^{(\pm)}_{\gamma,0}(E,z)$ by

$$U^{(\pm)}_{\gamma,0}(E,z) = (h(D_x; z\gamma) \pm E)u_{\gamma,0}(E^2-1,z). \tag{8.1}$$

Let $0 < \delta < \delta_0/2, \delta_0$ being the constant in (A-II). Then there exists an $\epsilon > 0$ such that $U^{(\pm)}_{\gamma,0}(E,z)$ is a $\mathbf{B}(\mathcal{H}_\delta; \mathcal{H}_{-\delta})$-valued analytic function of $z \in D_\epsilon = \{z \in \mathbf{C}_+; |\text{Re}z| < \epsilon/2\}$ and is continuous on $\overline{D_\epsilon}$. For $t \in (-\epsilon/2, \epsilon/2)$, we have by (2.17) and (4.8)

$$U^{(\pm)}_{\gamma,0}(E,t) = G^{(\pm)}_{\gamma,0}(\lambda(t),t), \tag{8.2}$$

$$\lambda(t) = (E^2-1-t^2)^{1/2}. \tag{8.3}$$

$U^{(\pm)}_{\gamma,0}(E,z)V$ is compact on $\mathcal{H}_{-\delta}$ for $z \in \overline{D_\epsilon}$.

DEFINITION 8.1. – *Let $\mathcal{E}^{(\pm)}(E,\gamma)$ be the set of $z \in \overline{D_\epsilon}$ such that $-1 \in \sigma_p(U^{(\pm)}_{\gamma,0}(E,z)V)$.*

LEMMA 8.2. – (1) *There exists a constant* $C > 0$ *such that*

$$\mathcal{E}^{(\pm)}(E,\gamma) \cap \{i\tau; \tau > C\} = \emptyset.$$

(2) $\mathcal{E}^{(\pm)}(E,\gamma) \cap D_\epsilon$ *is discrete and* $\mathcal{E}^{(\pm)}(E,\gamma) \cap \mathbf{R}$ *is a null set.*

(3) *Let* $t \in (-\epsilon/2, \epsilon/2)$. *Then there exists* $0 \neq u \in \mathcal{H}_{-\delta}$ *such that* $u + U^{(\pm)}_{\gamma,0}(E,t)Vu = 0$ *if and only if there exists* $0 \neq u \in L^{2,-s}$ *such that* $u + G^{(\pm)}_{\gamma,0}(\lambda(t),t)Vu = 0$.

(4) *If* $t \in (-\epsilon/2, \epsilon/2) \setminus \mathcal{E}^{(\pm)}(E,\gamma)$, *then* $t \notin \mathcal{E}_\pm(\lambda(t),\gamma)$.

Proof. – (1) is proved in the same way as Lemma 4.3. Here we must use $U^{(\pm)}_{\gamma,0}(E,i\tau) = G^{(\pm)}_{\gamma,0}(\lambda(i\tau),i\tau)$, which follows from (2.18). (2) follows from the analytic Fredholm theorem. (3) easily follows from (8.2). (4) is a direct consequence of (3) and Definition 4.2. □

For $z \notin \mathcal{E}^{(\pm)}(E,\gamma)$, we define

$$U^{(\pm)}_{\gamma}(E,z) = (1 + U^{(\pm)}_{\gamma,0}(E,z)V)^{-1} U^{(\pm)}_{\gamma,0}(E,z). \tag{8.4}$$

By (4.10), (8.2) and (8.4), we have for $t \in (-\epsilon/2, \epsilon/2) \setminus \mathcal{E}^{(\pm)}(E,\gamma)$,

$$U^{(\pm)}_{\gamma}(E,t) = G^{(\pm)}_{\gamma}(\lambda(t),t). \tag{8.5}$$

Therefore by virtue of Theorem 6.3 and (6.14), one can get

$$B^{(+)}_{\gamma}(\lambda(t),t) = (1-K)^{-1} A^{(+)}_{\gamma}(\lambda(t),t) \tag{8.6}$$

for $t \in (-\epsilon/2, \epsilon/2) \setminus \mathcal{E}^{(\pm)}(E,\gamma)$ from the given scattering matrix $\widehat{S}_+(E)$. We now define $\mathcal{B}^{(+)}_{\gamma}(\lambda(t),t;\omega,\omega')$ by (7.2). Then $\mathcal{B}^{(+)}_{\gamma}(\lambda(t),t;\omega,\omega')$ has a meromorphic continuation with respect to $z \in D_\epsilon$ and has the following expression

$$\begin{aligned}
&\mathcal{B}^{(+)}_{\gamma}(\lambda(z),z;\omega,\omega') \\
&\quad = \int e^{-i\lambda(z)(\omega-\omega')x} \Pi_+(\lambda(z)\omega + z\gamma)V(x)\Pi_+(\lambda(z)\omega' + z\gamma)dx \\
&\qquad - \int e^{-i\lambda(z)\omega\cdot x} \Pi_+(\lambda(z)\omega + z\gamma)V(x)U^{(+)}_{\gamma}(E,z) \\
&\qquad \times (V\Pi_+(\lambda(z)\omega' + z\gamma)e^{i\lambda(z)\omega'\cdot})dx.
\end{aligned}$$

Now let $z = i\tau$ and $\tau \to \infty$. Since $U^{(\pm)}_{\gamma}(E,i\tau) = G^{(\pm)}_{\gamma}(\lambda(i\tau),i\tau)$ and $\lambda(i\tau) = \tau + O(1/\tau)$, the remaining arguments are essentially the same as those in §7. We have only to replace $G^{(+)}_{\gamma}(\lambda,z)$ by $U^{(+)}_{\gamma}(E,z)$. We fix a non zero vector $\xi \in \mathbf{R}^3$. We take $\gamma, \eta \in S^2$ such that $\xi \cdot \gamma = \xi \cdot \eta = \gamma \cdot \eta = 0$. We let

$$\omega = \omega(\tau) = \left(1 - \frac{\xi^2}{4\tau^2}\right)^{1/2} \eta + \frac{\xi}{2\tau},$$

$$\omega' = \omega'(\tau) = \left(1 - \frac{\xi^2}{4\tau^2}\right)^{1/2} \eta - \frac{\xi}{2\tau}.$$

We also let

$$L_\gamma^{(+)}(\tau) = \frac{1}{\tau} \mathcal{B}_\gamma^{(+)}(\lambda(i\tau), i\tau; \omega(\tau), \omega'(\tau)).$$

Then as in Theorem 7.3, we have the following theorem.

THEOREM 8.3. – *As $\tau \to \infty$, we have*

$$\begin{pmatrix} \sigma \cdot (\xi \times \eta) & 0 \\ 0 & \sigma \cdot (\xi \times \eta) \end{pmatrix} P_\pm (L_\gamma^{(+)}(\tau) + L_{-\gamma}^{(+)}(\tau)) P_\pm \to \frac{i}{2} \xi^2 P_\pm \widehat{V}(\xi)^I.$$

Theorem 1.2 now follows from Theorem 8.3.

REFERENCES

[1] S. AGMON and L. HÖRMANDER, Asymptotic properties of solutions of differential equations with simple characteristics, *J. d'Anal. Math.*, Vol. **30**, 1976, pp. 1-38.

[2] E. BALSLEV, B. HELFFER, Limiting absorption principle and resonances for the Dirac operator, *Adv. in Appl. Math.*, Vol. **13**, 1992, pp. 186-215.

[3] R. BEALS and R. COIFMAN, Multidimensional inverse scattering and nonlinear P.D.E., *Proc. Sympo. Pure Math.*, Vol. **43**, 1985, pp. 45-70.

[4] A. BOUTET DE MONVEL-BERTHIER, D. MANDA and R.PURICE, Limiting absorption principle for the Dirac operator, *Ann. Inst. Henri Poincaré, Physique Théorique*, Vol. **58**, 1993, pp. 413-431.

[5] G. ESKIŃ and J. RALSTON, Inverse scattering problem for the Schrödinger equation with magnetic potential at a fixed energy, *Commun. Math. Phys*, Vol. **173**, 1995, pp. 199-224.

[6] L. D. FADDEEV, Inverse problem of quantum scattering theory, *J. Sov. Math.*, Vol. **5**, 1976, pp. 334-396.

[7] G. HACHEM, The $\bar{\partial}$-approach to inverse scattering for Dirac operators, *Inverse Problems*, Vol. **11**, 1995, pp. 123-146.

[8] V. ISAKOV, Uniqueness and stability in multi-dimensional inverse problems, *Inverse Problems*, Vol. **9**, 1993, pp. 579-621

[9] H. ISOZAKI, Multi-dimensional inverse scattering theory for Schrödinger operators, to appear in Reviews in Math. Phys..

[10] S. T. KURODA, Scattering theory for differential operators, II, *J. Math. Soc. Japan*, Vol. **25**, 1972, pp. 222-234.

[11] A. NACHMAN, Reconstruction from boundary measurements, *Ann. Math.*, Vol. **128**, 1988, pp. 531-576.

[12] A. NACHMAN, Inverse scattering at fixed energy, Proceedings of the 10th International Congress on Math. Phys., Leipzig 1991, edited by K. Schmügen, Springer Verlag, 1992, pp. 434-441.
[13] A. I. NACHMAN and M. J. ABLOWITZ, A multi-dimensional inverse scattering method, *Stud. Appl. Math.*, Vol. **71**, 1984, pp. 243-250.
[14] G. NAKAMURA and G. UHLMAN, Global uniqueness for an inverse bounday problem arising in elasticity, *Invent. math.*, Vol. **118**, 1994, pp. 457-474.
[15] R. G. NEWTON, *Inverse Schrödinger Scattering in Three Dimensions*, Springer-Verlag, Berlin-Heidelberg-New York, 1989.
[16] R. G. NOVIKOV and G. M. KHENKIN, The $\overline{\partial}$-equation in the multi-dimensional inverse scattering problem, *Russian Math. Surveys*, Vol. **42**, 1987, pp. 109-180.
[17] R. G. NOVIKOV, The inverse scattering problem at fixed energy for the three-dimensional Schrödinger equation with an exponentially decreasing potential, *Commun. Math. Phys.*, Vol. **161**, 1994, pp. 569-595.
[18] P. OLA, L. PÄIVÄRINTA and E. SOMERSALO, An inverse boundary value problem in electrodynamics, *Duke Math. J.*, Vol. **70**, 1993, pp. 617-653.
[19] J. SYLVESTER and G. UHLMAN, A global uniqueness theorem for an inverse boundary value problem, *Ann. Math.*, Vol. **125**, 1987, pp. 153-169.
[20] B. THALLER, *Dirac Equations, Texts and Monographs in Physics*, Springer, Berlin 1992.
[21] B. THALLER, V. Enss, Asympotic observables and Coulomb scattering for the Dirac equation, *Ann. Inst. H. Poincaré*, Vol. **45**, 1986, pp. 147-171.
[22] G. UHLMANN, Inverse boundary value problems and applications, *Astérisque,* Vol. **207**, 1992, pp. 153-211.
[23] R. WEDER, Generalized limiting absorption method and multidimensional inverse scattering theory, *Math. Meth. in Appl. Sci.*, Vol. **14**, 1991, pp. 509-524.
[24] O. YAMADA, Eigenfunction expansions and scattering theory for Dirac operators, *Publ. RIMS. Kyoto Univ.*, Vol. **11**, 1976, pp. 651-689.

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