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# A new look at Classical Mechanics of constrained systems (*)

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ABSTRACT. – A geometric formulation of Classical Analytical Mechanics, especially suited to the study of non-holonomic systems is proposed. The argument involves a preliminary study of the geometry of the space of kinetic states of the system, followed by a revisitation of Chetaev's definition of virtual work, viewed here as a cornerstone for the implementation of the principle of determinism. Applications to ideal non-holonomic systems (equivalence between d'Alembert's and Gauss' principles, equations of motion) are explicitely worked out.

*Key words:* Lagrangian Dynamics, Non-holonomic Systems.

RÉSUMÉ. – Nous présentons dans ce papier une formulation de la Mécanique Analytique qui permet de traiter des systèmes non-holonômes.

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Cette étude préliminaire concerne la géométrie de l'espace des états cinétiques du système. Cela nous a conduit à revoir la définition de Chetaev du travail virtuel, qui joue ici un rôle remarquable dans la réalisation d'un modèle mécanique compatible avec le principe de déterminisme. L'ensemble de ce travail nous a permis de l'appliquer aux systèmes non-holonômes idéaux, de comparer les formulations de D'Alembert et de Gauss et d'écrire de façon explicite les équations de la dynamique.

## 1. INTRODUCTION

A central issue in the development of Classical Mechanics is the identification of general statements – often called "principles" – characterizing the behaviour of the reactive forces for large classes of constraints of actual physical interest.

This is e.g. the role of d'Alembert's principles of virtual work ([1], [2], [3]), or of Gauss' principle of least constraint ([4], [5], [6]) – both providing equivalent characterizations of the class of ideal constraints – or of Coulomb's laws of friction [7], etc.

In general, of course, the choice of a suitable characterization of the reactive forces is a physical problem, intimately related with the structural properties of the devices involved in the implementation of the constraints. In any case, however, a basic condition to be fulfilled by any significant model is the requirement of consistency with the principle of determinism, *i.e.* the ability to give rise to a dynamical scheme in which the evolution of the system from given initial data is determined uniquely by the knowledge of the active forces, through the solution of a well-posed Cauchy problem.

In this paper, we propose a thorough discussion of this point. For generality, we shall deal with arbitrary (finite-dimensional) non-holonomic systems. The analysis will be carried on in a frame-independent language, using the standard techniques of jet-bundle theory ([8], [9], [10], [11], [12], [13]).

The main emphasis will be on the intrinsic geometry of the space of kinetic states of the system. On this basis, we shall identify a general definition of the concept of virtual work, extending the traditional one to arbitrary non-holonomic systems, along the lines proposed by Chetaev ([14], [15], [16], [17], [18]). The resulting scheme will throw new light on the interplay between determinism and constitutive characterization of the reactive forces, as well as on the relation between d'Alembert's principle and Gauss' principle in the case of ideal non-holonomic systems.

The plan of presentation is as follows:

Sections 2.1 and 2.2 are introductory in nature. In section 2.1 we review the basic concepts involved in the construction of a frame-independent model for a mechanical system with a finite number of degrees of freedom. In particular, we formalize the concept of configuration space-time, defined as the abstract manifold $t$: $\mathcal{V}_{n+1} \to \mathfrak{R}$ formed by the totality of admissible configurations of the system, fibered over the real line $\mathfrak{R}$ through the absolute time function.

In a similar way, in section 2.2 we recall some standard aspects of the geometry of the first jet space $j_1(\mathcal{V}_{n+1})$, especially relevant to the subsequent discussion ([8], [9], [10], [11], [13], [19]).

The study of the non-holonomic aspects arising from the presence of kinetic constraints begins in section 2.3. Following [12], the geometrical environment is now identified with a submanifold $i : \mathcal{A} \to j_1(\mathcal{V}_{n+1})$, fibered over $\mathcal{V}_{n+1}$, and representing the totality of admissible kinetic states of the system. In addition to the "natural" structures, either implicit in the existence of the fibration $\mathcal{A} \to \mathcal{V}_{n+1}$, or obtained by pulling back the analogous structures over $j_1(\mathcal{V}_{n+1})$, the manifold $\mathcal{A}$ is seen to carry further significant geometrical objects, depending in a more sophisticated way on the properties of the embedding $i : \mathcal{A} \to j_1(\mathcal{V}_{n+1})$.

A thorough discussion of this point is presented. Among other topics, the analysis includes the definition of the concept of *fundamental tensor* of the manifold $\mathcal{A}$, as well as the introduction of a distinguished vector bundle of differential 1-forms over $\mathcal{A}$, called the *Chetaev bundle*.

In section 3, the geometrical scheme is applied to the study of the dynamical aspects of the theory. In section 3.1 we show that, in the presence of kinetic constraints, the Poincaré-Cartan 2-form of the system splits into two terms, only one of which is effectively significant in the determination of the evolution of the system.

An analysis of this point provides a hint for a geometrical revisitation of the concepts of virtual displacement and virtual work. The argument is formalized in section 3.2. A comparison with the traditional definitions is explicitly outlined.

The discussion is completed by a factorization theorem, allowing to express the content of Newton's 2[nd] law in a form especially suited to a precise mathematical formulation of the principle of determinism. The interplay between determinism and constitutive characterization of the reactive forces is considered in section 3.3. The geometrical meaning of d'Alembert's principle, as well as the relation of the latter with Gauss' principle of least constraint are discussed in detail.

The paper is concluded by an analysis of the equations of motion for a general non-holonomic system subject to ideal constraints. Both the intrinsic formulation, in terms of local fibered coordinates on the submanifold $\mathcal{A}$, and the extrinsic formulation, based on the "cartesian" representation of the embedding $i : \mathcal{A} \to j_1(\mathcal{V}_{n+1})$ are explicitly outlined.

## 2. CLASSICAL DYNAMICS ON JET BUNDLES

### 2.1. Preliminaries

(i) In this Section we review a few basic aspects of Classical Mechanics, especially relevant to the subsequent applications. Throughout the discussion, a major role will be played by the space-time manifold $\mathcal{V}_4$, defined as the totality of *events*, and viewed as the natural environment for the frame-independent formulation of physical laws.

According to the axioms of absolute time and absolute space, $\mathcal{V}_4$ has the nature of an affine bundle over the real line $\Re$, with projection $t : \mathcal{V}_4 \to \Re$ (the "absolute-time function"), and standard fibre $E_3$ (euclidean 3-space). The totality of vectors tangent to the fibres form a vector bundle $V(\mathcal{V}_4)$, called the vertical bundle over $\mathcal{V}_4$.

Every section $\zeta : \Re \to \mathcal{V}_4$ – henceforth called a *world line* – provides the representation of a possible evolution of a point particle. In this respect, the first jet space $j_1(\mathcal{V}_4)$ has a natural interpretation as the totality of *kinetic states* of particles.

From an algebraic viewpoint, we recall that $j_1(\mathcal{V}_4)$ has the nature of an affine bundle over $\mathcal{V}_4$, modelled on the vertical bundle $V(\mathcal{V}_4)$. The first jet extension of a world-line $\zeta : \Re \to \mathcal{V}_4$ will be denoted by $\dot{\zeta} : \Re \to j_1(\mathcal{V}_4)$.

(ii) Every trivialization $\mathcal{I} : \mathcal{V}_4 \to \Re \times E_3$ mapping each fibre of $\mathcal{V}_4$ isometrically into $E_3$ identifies what is usually called a *physical frame of reference*. The trivialization $\mathcal{I}$ may be "lifted" to the first jet space $j_1(\mathcal{V}_4)$, thus giving rise to the further identification

$$j_1(\mathcal{V}_4) \simeq j_1(\Re \times E_3) \simeq \Re \times E_3 \times V_3 \simeq \mathcal{V}_4 \times V_3. \tag{2.1}$$

$V_3$ denoting the space of "free vectors" in $E_3$.

The projection $\mathbf{x} : \mathcal{V}_4 \to E_3$ associated with $\mathcal{I}$ will be called the *position map* relative to $\mathcal{I}$. The projection $\mathbf{v} : j_1(\mathcal{V}_4) \to V_3$ induced by the representation (2.1) – clearly identical to the restriction to $j_1(\mathcal{V}_4)$ of the tangent map $\mathbf{x}_* : T(\mathcal{V}_4) \to T(E_3)$, followed by the canonical projection $T(E_3) \to V_3$ – will be called the *velocity map* relative to $\mathcal{I}$.

We let the reader verify that, for each world line $\zeta$, the composite map $\mathbf{v} \cdot \dot{\zeta} : \mathfrak{R} \to V_3$ does indeed coincide with the time derivative of the map $\mathbf{x} \cdot \zeta : \mathfrak{R} \to E_3$.

(iii) Quite generally, a material system $\mathcal{B}$ may be viewed as a *measure space*, *i.e.* as a triple $(\mathfrak{B}, \mathcal{S}, m)$ in which $\mathfrak{B}$ is an abstract space (the "material space", formed by the totality of points of the system), $\mathcal{S}$ is a $\sigma$-ring of measurable subsets of $\mathfrak{B}$, and $m$ is a finite positive measure over $(\mathfrak{B}, \mathcal{S})$, assigning to each $\Omega \in \mathcal{S}$ a corresponding *inertial mass* $m(\Omega) := \int_\Omega m(d\xi)$. In the case of discrete systems the simplified notation $\mathfrak{B} = \{P_1, \ldots, P_N\}$, $m(P_i) = m_i$ (whence also $m(\Omega) = \sum_{P_i \in \Omega} m_i$), will be implicitly understood.

A *configuration* of the system is defined as a map $\mathfrak{P} : \mathfrak{B} \to \mathcal{V}_4$ satisfying the condition $t \cdot \mathfrak{P} = \text{const.}$, *i.e.* sending all points $\xi \in \mathfrak{B}$ into one and the same fiber of $\mathcal{V}_4$. In a similar way, a *kinetic state* of the system is a map $\mathfrak{V} : \mathfrak{B} \to j_1(\mathcal{V}_4)$ satisfying $t \cdot \mathfrak{V} = \text{const.}$, $t$ denoting the (pull-back of the) absolute-time function on $j_1(\mathcal{V}_4)$.

According to the stated definitions, we may attach a time label to each configuration $\mathfrak{P}$ (to each kinetic state $\mathfrak{V}$), according to the identification

$$t(\mathfrak{P}) := t \cdot \mathfrak{P}(\xi) \qquad \forall \xi \in \mathfrak{B} \tag{2.2}$$

(respectively $t(\mathfrak{V}) := t \cdot \mathfrak{V}(\xi)\ \forall \xi \in \mathfrak{B}$).

The totality of admissible configurations, or of admissible kinetic states, does not depend only on the nature of the material space $\mathfrak{B}$, but also, explicitly, on the *constraints* imposed on the system.

In general, a satisfactory insight into the situation is gained by splitting the description into two steps, focussing at first on the positional constraints, and deferring to a subsequent stage the description of the additional kinetic ones [1].

In connection with the first step, we shall restrict our analysis to the class of *holonomic constraints*, completely characterized by the following properties:

*a*) the totality of admissible configurations form an $(n+1)$-dimensional differentiable manifold $\mathcal{V}_{n+1}$, fibered over the real line $\mathfrak{R}$ through the map (2.2);

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[1] A more elastic strategy, accounting for the substantial interchangeability between positional constraints and integrable kinetic ones, would be to base step 1 on a preselected subfamily of constraints, of strictly positional nature, and to express all other constraints in kinetic terms. In this connection, see also [12].

*b*) the correspondence $x \to \mathfrak{P}_x$ between points of $\mathcal{V}_{n+1}$ and admissible configurations $\mathfrak{P}_x : \mathfrak{B} \to \mathcal{V}_4$ has the property that, for each $\xi \in \mathfrak{B}$, the image $\mathfrak{P}_x(\xi)$ depends differentiably on $x$.

As a consequence of *a*) and *b*) it is easily seen that every admissible evolution of the system determines a corresponding section $\gamma : \mathfrak{R} \to \mathcal{V}_{n+1}$. Conversely, if no further restrictions are imposed on the system, every such section provides the representation of an admissible evolution. In this respect, the first jet space $j_1(\mathcal{V}_{n+1})$ has therefore a natural interpretation as the totality of kinetic states consistent with the given holonomic constraints.

Additional (non-holonomic) constraints, when present, are then easily inserted into the scheme, in the form of restrictions on the class of admissible sections, thus shrinking the family of allowed kinetic states to a subset $\mathcal{A} \subset j_1(\mathcal{V}_{n+1})$. This point will be explicitly considered in section 2.3.

The manifold $\mathcal{V}_{n+1}$ will be called the *configuration space-time* associated with the given system; the fibration $t : \mathcal{V}_{n+1} \to \mathfrak{R}$ will be called the absolute time function over $\mathcal{V}_{n+1}$. The vertical bundle over $\mathcal{V}_{n+1}$ will be denoted by $V(\mathcal{V}_{n+1})$.

We recall that, by definition, both $V(\mathcal{V}_{n+1})$ and the first jet space $j_1(\mathcal{V}_{n+1})$ may be indentified with corresponding submanifolds of the tangent space $T(\mathcal{V}_{n+1})$, according to the identifications

$$V(\mathcal{V}_{n+1}) = \{X | X \in T(\mathcal{V}_{n+1}), \langle X, dt\rangle = 0\}$$
$$j_1(\mathcal{V}_{n+1}) = \{X | X \in T(\mathcal{V}_{n+1}), \langle X, dt\rangle = 1\}.$$

From these, it is readily seen that $j_1(\mathcal{V}_{n+1})$ has the nature of an affine bundle over $\mathcal{V}_{n+1}$, modelled on $V(\mathcal{V}_{n+1})$ ([8], [9], [11], [12], [13], [19]).

In view of the requirement *b*) stated above, each point $\xi \in \mathfrak{B}$ determines a differentiable map $\mathfrak{P}_\xi : \mathcal{V}_{n+1} \to \mathcal{V}_4$ through the prescription

$$\mathfrak{P}_\xi(x) := \mathfrak{P}_x(\xi) \qquad \forall\, x \in \mathcal{V}_{n+1}. \tag{2.3}$$

We denote by $(\mathfrak{P}_\xi)_* : T(\mathcal{V}_{n+1}) \to T(\mathcal{V}_4)$ the associated tangent map. A straightforward check shows that $(\mathfrak{P}_\xi)_*$ preserves the property of verticality, *i.e.* it satisfies $(\mathfrak{P}_\xi)_*(V(\mathcal{V}_{n+1})) \subset V(\mathcal{V}_4)$.

The restriction of $(\mathfrak{P}_\xi)_*$ to the submanifold $j_1(\mathcal{V}_{n+1}) \subset T(\mathcal{V}_{n+1})$, clearly identical to the first jet extension of the map (2.3), will be denoted by $\dot{\mathfrak{P}}_\xi$.

(iv) Given an arbitrary frame of reference $\mathcal{I}$, let $\mathbf{x} : \mathcal{V}_4 \to E_3$ and $\mathbf{v} : j_1(\mathcal{V}_4) \to V_3$ denote the corresponding position and velocity maps.

For each $\xi \in \mathfrak{B}$ the composite map $\mathbf{x}_\xi := \mathbf{x} \cdot \mathfrak{P}_\xi : \mathcal{V}_{n+1} \to E_3$ and $\dot{\mathbf{x}}_\xi := \mathbf{v} \cdot \dot{\mathfrak{P}}_\xi : j_1(\mathcal{V}_{n+1}) \to V_3$ are then easily recognized as representations of the position and velocity of $\xi$ relative to $\mathcal{I}$, respectively as functions of the configuration and of the kinetic state of the system [2].

In local fibered coordinates $t, q^1, \ldots, q^n$ over $\mathcal{V}_{n+1}$, and $t, q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n$ over $j_1(\mathcal{V}_{n+1})$, we have the familiar expressions

$$\mathbf{x}_\xi = \mathbf{x}_\xi\big(t, q^1, \ldots, q^n\big) \tag{2.4}$$

$$\dot{\mathbf{x}}_\xi = \frac{\partial \mathbf{x}_\xi}{\partial q^k}\dot{q}^k + \frac{\partial \mathbf{x}_\xi}{\partial t} \tag{2.5}$$

($\mathbf{x}_i = \mathbf{x}_i(t, q^1, \ldots, q^n), \dot{\mathbf{x}}_i = \frac{\partial \mathbf{x}_i}{\partial q^k}\dot{q}^k + \frac{\partial \mathbf{x}_i}{\partial t}$ for discrete systems).

By means of eq. (2.4), the ordinary (euclidean) scalar product in $V_3$ may be lifted to a frame-dependent scalar product in $T(\mathcal{V}_{n+1})$, according to the prescription

$$(X, Y)_{\mathcal{I}} := \int_{\mathfrak{B}} \mathbf{x}_{\xi *}(X) \cdot \mathbf{x}_{\xi *}(Y)\, m(d\xi) \qquad \forall X, Y \in T(\mathcal{V}_{n+1}). \tag{2.6}$$

A straightforward check shows that, when both vectors $X$ and $Y$ are vertical, the right-hand-side of eq. (2.6) is invariant under arbitrary transformations of the frame of reference $\mathcal{I}$. The restriction of the product (2.6) to the vertical bundle $V(\mathcal{V}_{n+1})$ is therefore a frame-independent attribute of the manifold $\mathcal{V}_{n+1}$, henceforth denoted by ( , ), and called the fiber (or vertical) metric ([8], [9], [11], [12], [13], [19]).

A system is said to be non-singular if and only if the positivity condition $(X, X) > 0$ holds for all $X \neq 0$ in $V(\mathcal{V}_{n+1})$. This condition will be implicitely assumed throughout the subsequent discussion.

In local coordinates, the representation of the fiber metric is summarized into the set of components

$$g_{hk} := \left(\frac{\partial}{\partial q^h}, \frac{\partial}{\partial q^k}\right) = \int_{\mathfrak{B}} \frac{\partial \mathbf{x}_\xi}{\partial q^h} \cdot \frac{\partial \mathbf{x}_\xi}{\partial q^k} m(d\xi) \tag{2.7}$$

($g_{hk} := \sum m_i \frac{\partial \mathbf{x}_i}{\partial q^h} \cdot \frac{\partial \mathbf{x}_i}{\partial q^k}$ for discrete systems).

Returning to the general case, it may be seen that the knowledge of the scalar product $(X, Y)_{\mathcal{I}}$ for arbitrary (not necessarily vertical) vectors is mathematically equivalent to the knowledge of the quadratic form $(X, Y)_{\mathcal{I}}$

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[2] It goes without saying that, in the presence of non-holonomic constraints, the stated interpretation applies only to the restriction of the maps $\dot{\mathbf{x}}_\xi$ to the family of admissible kinetic states.

over $T(\mathcal{V}_{n+1})$, which, in turn, is determined uniquely by the knowledge of its restriction $(z, z)_{\mathcal{I}}$ to the submanifold $j_1(\mathcal{V}_{n+1}) \subset T(\mathcal{V}_{n+1})$.

The function $\mathcal{T}$ defined globally on $j_1(\mathcal{V}_{n+1})$ by $\mathcal{T}(z) := \frac{1}{2}(z, z)_{\mathcal{I}}$ will be called the *holonomic kinetic energy* relative to the frame of reference $\mathcal{I}$. Comparison with eq. (2.6) yields the representation

$$\mathcal{T}(z) = \frac{1}{2} \int_{\mathfrak{B}} |\dot{\mathbf{x}}_\xi(z)|^2 m(d\xi) \tag{2.8}$$

showing that the restriction of $\mathcal{T}$ to the class of admissible kinetic states does indeed represent the kinetic energy of the system relative to $\mathcal{I}$.

## 2.2. Geometry of $j_1(\mathcal{V}_{n+1})$

For later use, in this Subsection we review the main aspects of the geometry of the first jet space $j_1(\mathcal{V}_{n+1})$. For an exhaustive discussion, *see* e.g. ([8], [9], [11], [12], [13]), and references therein.

Keeping the same notation as in [13], we denote by $V(j_1(\mathcal{V}_{n+1}))$ the vertical bundle over $j_1(\mathcal{V}_{n+1})$ with respect to the fibration $\pi : j_1(\mathcal{V}_{n+1}) \to \mathcal{V}_{n+1}$, and by $j_2(\mathcal{V}_{n+1})$ the second jet extension of $\mathcal{V}_{n+1}$, viewed as an affine bundle over $j_1(\mathcal{V}_{n+1})$, modelled on $V(j_1(\mathcal{V}_{n+1}))$. The definitions and elementary properties of both spaces will be regarded as known ([8], [9], [11], [12], [13]). The annihilator of the $(n+1)$-dimensional distribution spanned by $j_2(\mathcal{V}_{n+1})$ will be denoted by $\mathcal{C}(j_1(\mathcal{V}_{n+1}))$, and will be called the *contact bundle* over $j_1(\mathcal{V}_{n+1})$.

Every section $X : j_1(\mathcal{V}_{n+1}) \to V(j_1(\mathcal{V}_{n+1}))$ will be called a vertical vector field over $j_1(\mathcal{V}_{n+1})$. In a similar way, sections $\sigma : j_1(\mathcal{V}_{n+1}) \to \mathcal{C}((\mathcal{V}_{n+1}))$ will be called *contact 1-forms*, while sections $\breve{Z} : j_1(\mathcal{V}_{n+1}) \to j_2(\mathcal{V}_{n+1})$, viewed as a vector fields over $j_1(\mathcal{V}_{n+1})$, will be called *semi-sprays*.

In local fibered coordinates, introducing the notation

$$\omega^i := dq^i - \dot{q}^i dt \tag{2.9}$$

the situation is summarized into the explicit representations

$$X \text{ vertical vector} \quad \Leftrightarrow \quad X = X^i \frac{\partial}{\partial \dot{q}^i}$$

$$\breve{Z} \text{ semi-spray} \quad \Leftrightarrow \quad \breve{Z} = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + Z^i \frac{\partial}{\partial \dot{q}^i}$$

$$\sigma \text{ contact 1-form} \quad \Leftrightarrow \quad \sigma = \sigma_i \omega^i$$

with $X^i$, $Z^i$, $\sigma_i \in \mathcal{F}(j_1(\mathcal{V}_{n+1}))$.

On the basis of the stated definitions, one can prove the following general results ([8], [9], [11], [13], [19]).

(i) By regarding each $z \in j_1(\mathcal{V}_{n+1})$ as a vector in $T_{\pi(z)}(\mathcal{V}_{n+1})$, we can define a linear map $\Theta_z : T_z(j_1(\mathcal{V}_{n+1})) \to T_{\pi(z)}(\mathcal{V}_{n+1})$ according to the prescription

$$\Theta_z(X) = (\pi_z)_*(X) - \langle X, (dt)_z \rangle z. \tag{2.10}$$

By duality, this gives rise to a map $\Theta_z^*$ of the cotangent space $T^*_{\pi(z)}(\mathcal{V}_{n+1})$ into $T_z^*(j_1(\mathcal{V}_{n+1}))$, on the basis of the equation

$$\Theta_z^*(\nu) = (\pi_z)_*{}^*(\nu) - \langle z, \nu \rangle (dt)_z \qquad \forall \nu \in T^*_{\pi(z)}(\mathcal{V}_{n+1}). \tag{2.11}$$

In local coordinates, recalling the representation (2.9) for the 1-forms $\omega^i$, we have the explicit relations

$$\Theta_z(X) = \langle X, \omega^i_{|z} \rangle \left( \frac{\partial}{\partial q^i} \right)_{\pi(z)} \tag{2.12}$$

$$\Theta_z{}^*(\nu) = \left\langle \nu, \left( \frac{\partial}{\partial q^i} \right)_{\pi(z)} \right\rangle \omega^i_{|z}. \tag{2.13}$$

Accordingly, we shall call $\Theta_z$ and $\Theta_z^*$ respectively the *vertical push-forward* and the *contact pull-back* at $z$.

The map (2.13) may be extended in a standard way to *fields* of 1-forms, by requiring the condition

$$(\Theta^*(\eta))_{|z} := \Theta_z^*(\eta_{|\pi(z)}), \qquad \eta \in \mathcal{D}_1(\mathcal{V}_{n+1}). \tag{2.14}$$

For every $f \in \mathcal{F}(\mathcal{V}_{n+1})$, the contact pull-back of $df$ will be denoted by $d_c f$, and will be called the *contact differential* of $f$. In local coordinates, eqs. (2.13), (2.14) provide the explicit representations

$$\Theta^*(\eta) = \left\langle \pi^*(\eta), \frac{\partial}{\partial q^i} \right\rangle \omega^i, \qquad d_c f = \Theta^*(df) = \frac{\partial f}{\partial q^i} \omega^i. \tag{2.15}$$

(ii) According to eq. (2.13), the kernel of the map $\Theta_z^*$ coincides with the annihilator of the vertical space $V_{\pi(z)}(\mathcal{V}_{n+1})$, while the image space $\Theta_z^*\left(T^*_{\pi(z)}(\mathcal{V}_{n+1})\right)$ is identical to the space $\mathcal{C}_z(j_1(\mathcal{V}_{n+1}))$ of contact 1-forms at $z$.

This provides a canonical identification of $C_z(j_1(\mathcal{V}_{n+1}))$ with the dual of $V_{\pi(z)}(\mathcal{V}_{n+1})$, summarized into a bilinear pairing $\langle \, \| \, \rangle : \mathcal{C}_z(j_1(\mathcal{V}_{n+1})) \times V_{\pi(z)}(\mathcal{V}_{n+1}) \to \mathfrak{R}$, defined implicitly by the relation

$$\langle \Theta_z^*(\nu) \| V \rangle = \langle \nu, V \rangle \qquad \forall \nu \in T^*_{\pi(z)}(\mathcal{V}_{n+1}), \quad V \in V_{\pi(z)}(\mathcal{V}_{n+1}).$$

In local coordinates, a straigthforward comparison with eq. (2.13) yields the explicit representation

$$\left\langle \nu_i \omega^i_{|z} \,\Big\|\, V^j \left( \frac{\partial}{\partial q^j} \right)_{|\pi(z)} \right\rangle = \nu_i V^i . \tag{2.16}$$

(iii) The affine character of the fibration $j_1(\mathcal{V}_{n+1}) \to \mathcal{V}_{n+1}$ ensures that, for each $z \in j_1(\mathcal{V}_{n+1})$, the vertical spaces $V_z(j_1(\mathcal{V}_{n+1}))$ and $V_{\pi(z)}(\mathcal{V}_{n+1})$ are canonically isomorphic ([8], [9], [11], [13], [19]). This gives rise to a vertical lift of vectors from $\mathcal{V}_{n+1}$ to $j_1(\mathcal{V}_{n+1})$, denoted symbolically by $V \to V^v$, and expressed in coordinates as

$$V = V^i \frac{\partial}{\partial q^i} \to V^v = V^i \frac{\partial}{\partial \dot{q}^i} . \tag{2.17}$$

Due to this fact, the scalar product between vertical vectors may be lifted from $\mathcal{V}_{n+1}$ to $j_1(\mathcal{V}_{n+1})$, thus defining a frame-independent structure over the first jet-space, called once again the fiber (or vertical) metric. In local coordinates, the evaluation of the scalar products relies on the identifications

$$\left( \frac{\partial}{\partial \dot{q}^h}, \frac{\partial}{\partial \dot{q}^k} \right)_{|z} = \left( \frac{\partial}{\partial q^h}, \frac{\partial}{\partial q^k} \right)_{|\pi(z)} = (g_{hk})_{|\pi(z)} \tag{2.18}$$

with the $g_{hk}$'s given by eq. (2.7).

(iv) By composing the vertical push forward (2.10) with the vertical lift (2.17), one gets a linear endomorphism $J : T_z(j_1(\mathcal{V}_{n+1})) \to T_z(j_1(\mathcal{V}_{n+1}))$, mapping each vector $X$ into the vertical vector

$$J(X) := (\Theta_z(X))^v = \left\langle X, \omega^i_{|z} \right\rangle \left( \frac{\partial}{\partial \dot{q}^i} \right)_z . \tag{2.19}$$

By the quotient law, this defines a tensor field of type (1,1) over $j_1(\mathcal{V}_{n+1})$, known as the *fundamental tensor* ([8], [9], [13]). In local coordinates, eq. (2.19) provides the explicit representation

$$J = \frac{\partial}{\partial \dot{q}^i} \otimes \omega^i . \tag{2.20}$$

(v) The simultaneous presence of the fundamental tensor and of the fiber metric determines a vector bundle isomorphism

$$\begin{array}{ccc} V(j_1(\mathcal{V}_{n+1})) & \xrightarrow{g} & \mathcal{C}(j_1(\mathcal{V}_{n+1})) \\ \pi \downarrow & & \downarrow \pi \\ j_1(\mathcal{V}_{n+1}) & = & j_1(\mathcal{V}_{n+1}) \end{array} \tag{2.21}$$

(process of "lowering the indices"), assigning to each vertical vector $X$ a corresponding contact 1-form $g(X)$ on the basis of the requirement

$$\langle g(X), Y \rangle = (X, J(Y)) \qquad \forall Y \in T(j_1(\mathcal{V}_{n+1})).$$

In local coordinates, taking eq. (2.18) into account, we have the representation

$$g\left(\frac{\partial}{\partial \dot{q}^i}\right) = g_{ij}\omega^j$$

with inverse

$$g^{-1}\left(\omega^i\right) = g^{ij}\frac{\partial}{\partial \dot{q}^j}$$

$g^{ij}$ denoting the matrix inverse of $g_{ij}$.

By means of the isomorphism (2.21), the scalar product (2.18) may be extended to the bundle of contact 1-forms, by requiring the identification

$$(\sigma, \tau) := \left(g^{-1}(\sigma), g^{-1}(\tau)\right) \qquad \forall \sigma, \tau \in \mathcal{C}(j_1(\mathcal{V}_{n+1})).$$

In local coordinates, this provides the relations

$$\left(\omega^i, \omega^j\right) = g^{ir}g^{js}\left(\frac{\partial}{\partial \dot{q}^r}, \frac{\partial}{\partial \dot{q}^s}\right) = g^{ij}. \tag{2.22}$$

(vi) The linear endomorphism (2.19) may be extended to an (algebraic) derivation $v : \mathcal{D}(j_1(\mathcal{V}_{n+1})) \to \mathcal{D}(j_1(\mathcal{V}_{n+1}))$ of the entire tensor algebra over $j_1(\mathcal{V}_{n+1})$, commuting with contractions and vanishing on functions, *i.e.* satisfying the conditions

$$v(f) = 0, \quad v(X) = J(X), \quad \langle v(X), \sigma\rangle + \langle X, v(\sigma)\rangle = 0 \tag{2.23}$$

$\forall f \in \mathcal{F}(j_1(\mathcal{V}_{n+1})), X \in \mathcal{D}^1(j_1(\mathcal{V}_{n+1})), \sigma \in \mathcal{D}_1(j_1(\mathcal{V}_{n+1}))$.

By means of $v$ one can construct a special anti-derivation $d_v$ of the Grassmann algebra $\mathcal{G}(j_1(\mathcal{V}_{n+1}))$, called the *fiber differentiation* [13], defined on arbitrary $r$-forms according to the equation

$$d_v\sigma := dv(\sigma) - v(d\sigma) - r\, dt \wedge \sigma. \tag{2.24}$$

This implies, among others, the following relations

$$\begin{aligned} d_v f &= -v\, df = \frac{\partial f}{\partial \dot{q}^k}\omega^k && \forall f \in \mathcal{F}(j_1(\mathcal{V}_{n+1})) \\ d_v\omega^k &= 0 && k = 1, \ldots, n \\ d_v \cdot d_v\sigma &= 0 && \forall \sigma \in \mathcal{G}(j_1(\mathcal{V}_{n+1})). \end{aligned}$$

For further details, the reader is referred to [13].

### 2.3. Non-holonomic constraints

(i) As pointed out in section 2.1, when the set of constraints imposed on the system is larger than the holonomic subset explicitly involved in the definition of the configuration space-time $\mathcal{V}_{n+1}$, the family of admissible kinetic states does no longer fill the entire first-jet space $j_1(\mathcal{V}_{n+1})$, but only a subregion $\mathcal{A} \subset j_1(\mathcal{V}_{n+1})$. The cases explicitly accounted for in Analytical Mechanics are those in which $\mathcal{A}$ has the nature of an embedded submanifold of $j_1(\mathcal{V}_{n+1})$, fibered over $\mathcal{V}_{n+1}$ [12].

The situation is summarized into the commutative diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{i} & j_1(\mathcal{V}_{n+1}) \\ \pi \downarrow & & \downarrow \pi \\ \mathcal{V}_{n+1} & = & \mathcal{V}_{n+1}. \end{array} \tag{2.25}$$

In what follows, we shall stick to the stated assumption. Preserving the notation $t,\, q^1,\dots,q^n,\, \dot q^1,\dots,\dot q^n$ for the coordinates on $j_1(\mathcal{V}_{n+1})$, and referring $\mathcal{A}$ to local fibered coordinates $t,\, q^1,\dots,q^n,\, z^1,\dots,z^r$, the embedding $i\,:\,\mathcal{A}\to j_1(\mathcal{V}_{n+1})$ will be represented locally in either forms

$$\dot q^i = \psi^i\big(t,\, q^1,\dots,q^n,\, z^1,\dots,z^r\big) \tag{2.26}$$

with rank $\|\partial(\psi^1,\dots,\psi^n)/\partial(z^1,\dots,z^r)\| = r$ (intrinsic representation), or

$$g^\sigma\big(t,\, q^1,\dots,q^n,\, \dot q^1,\dots,\dot q^n\big) = 0 \qquad \sigma = 1,\dots,n-r \tag{2.27}$$

with rank $\|\partial(g^1,\dots,g^{n-r})/\partial(\dot q^1,\dots,\dot q^n)\| = n-r$ (cartesian representation).

For each $\xi \in \mathfrak{B}$, the map $\mathfrak{P}_\xi\,:\,\mathcal{V}_{n+1}\to\mathcal{V}_4$ and $\dot{\mathfrak{P}}_\xi\,:\,j_1(\mathcal{V}_{n+1})\to j_1(\mathcal{V}_4)$ retain the meaning already discussed in the holonomic case. In particular, given an arbitrary frame of reference $\mathcal{I}$, the representation of the position and of the velocity of the point $\xi$ as functions respectively of the configuration and of the kinetic state of the system relies on the composite maps

$$\mathbf{x}_\xi := \mathbf{x}\cdot\mathfrak{P}_\xi\,:\,\mathcal{V}_{n+1}\to E_3, \qquad \mathbf{v}_\xi\,:\,\mathbf{v}\cdot\dot{\mathfrak{P}}_\xi\cdot i = \dot{\mathbf{x}}_\xi\cdot i\,:\,\mathcal{A}\to V_3$$

expressed in coordinates as

$$\mathbf{x}_\xi = \mathbf{x}_\xi\big(t,\, q^1,\dots,q^n\big) \tag{2.28}$$

$$\mathbf{v}_\xi = \frac{\partial \mathbf{x}_\xi}{\partial q^k}\psi^k\big(t,\, q^i,\, z^A\big) + \frac{\partial \mathbf{x}_\xi}{\partial t} \tag{2.29}$$

($\mathbf{x}_i = \mathbf{x}_i(t,\, q^1,\dots,q^n), \mathbf{v}_i = \frac{\partial \mathbf{x}_i}{\partial q^k}\psi^k + \frac{\partial \mathbf{x}_i}{\partial t}$ in the discrete case).

(ii) The concepts of vertical bundle and of contact bundle are easily adapted to the submanifold $\mathcal{A}$. A vector field $Z \in \mathcal{D}^1(\mathcal{A})$, $i$-related to a semi-spray on $j_1(\mathcal{V}_{n+1})$, will be called a *dynamical flow* on $\mathcal{A}$.

In local coordinates, introducing the notation

$$\tilde\omega^i := i^*\big(\omega^i\big) = dq^i - \psi^i\big(t,\, q^k,\, z^A\big)dt \tag{2.30}$$

we have the obvious identifications

$$
\begin{aligned}
&V \text{ vertical vector} &&\Leftrightarrow && V = V^A \frac{\partial}{\partial z^A} \\
&Z \text{ dynamical flow} &&\Leftrightarrow && Z = \frac{\partial}{\partial t} + \psi^i(t,\, q^k,\, z^A)\frac{\partial}{\partial q^i} + Z^A \frac{\partial}{\partial z^A} \\
&\sigma \text{ contact 1–form} &&\Leftrightarrow && \sigma = \sigma_i \widetilde{\omega}^i.
\end{aligned}
$$

In a similar way, by composing the homomorphisms (2.10), (2.11) with the maps $(i_z)_*$ and $(i_z)_*{}^*$, we can extend the notion of vertical push-forward of vectors, or of contact pull-back of 1-forms at each point $z \in \mathcal{A}$, as well as the subsequent identification between contact 1-forms at $z$ and linear functionals on the vertical space $V_{\pi(z)}(\mathcal{V}_{n+1})$ based on the pairing (2.16).

In addition to the natural structures described above, another important class of geometrical objects is obtained through the following construction: for each $z \in \mathcal{A}$, consider the image space $i_*(T_z(\mathcal{A}))$ under the map $i_* : T(\mathcal{A}) \to T(j_1(\mathcal{V}_{n+1}))$, and denote by $\mathcal{N}_{i(z)}(\mathcal{A}) \subset T^*_{i(z)}(j_1(\mathcal{V}_{n+1}))$ the corresponding annihilator. Take the image $v(\mathcal{N}_{i(z)}(\mathcal{A}))$ under the derivation induced by the fundamental tensor $J$, and pull it back to a subspace $(i_z)_*{}^* v(\mathcal{N}_{i(z)}(\mathcal{A})) := \chi_z(\mathcal{A}) \subset T^*_z(\mathcal{A})$.

In this way, performing the same construction at each $z \in \mathcal{A}$, we end up with a vector bundle $\chi(\mathcal{A}) = \bigcup_{z \in \mathcal{A}} \chi_z(\mathcal{A})$. For reasons that will be clear later on, we shall call the latter the *Chetaev bundle* over $\mathcal{A}$. Every section $\nu : \mathcal{A} \to \chi(\mathcal{A})$ will be called a Chetaev 1-form over $\mathcal{A}$. The role of the Chetaev bundle in the study of the integrability conditions for a given set of kinetic constraints is briefly analysed in Appendix A.

A description of $\chi(\mathcal{A})$ in local coordinates is easily obtained, starting with an arbitrary cartesian representation (2.27) for the submanifold $\mathcal{A}$, and observing that, by definition, the differentials $(dg^\sigma)_{i(z)}, \sigma = 1, \ldots, n-r$ span $\mathcal{N}_{i(z)}(\mathcal{A})$. Recalling eq. (2.24), we conclude that the virtual differentials $(d_v g^\sigma)_{i(z)} = -v(dg^\sigma)_{i(z)}$ span $v(\mathcal{N}_{i(z)}(\mathcal{A}))$, which is the same as saying that the differential 1-forms

$$i^*(d_v g^\sigma)_{|z} = \left(\frac{\partial g^\sigma}{\partial \dot{q}^k}\right)_{|i(z)} \bar{\omega}^k_{|z} \tag{2.31}$$

span $\chi_z(\mathcal{A})$. Moreover, due to the assumption on the rank of the Jacobian $\|\partial(g^1, \ldots, g^{n-r})/\partial(\dot{q}^1, \ldots, \dot{q}^n)\|$, the 1-forms (2.31) are linearly independent, so that they form a basis in $\chi_z(\mathcal{A})$. To sum up, every Chetaev 1-form $\nu$ on $\mathcal{A}$ may be expressed locally as

$$\nu = \lambda_\sigma i^*(d_v g^\sigma) \tag{2.32}$$

with $\lambda_\sigma \in \mathcal{F}(\mathcal{A})$.

Switching to the intrinsic description (2.27) for the submanifold $\mathcal{A}$, it is easily seen that the characterization (2.32) is mathematically equivalent to a representation of the form $\nu = \nu_k \tilde{\omega}^k$, with the components $\nu_k(t, q^i, z^A)$ subject to the conditions

$$\nu_k \frac{\partial \psi^k}{\partial z^A} = 0 \qquad A = 1, \ldots, r. \tag{2.33}$$

(iii) At each $z \in \mathcal{A}$, the vertical space $V_z(\mathcal{A})$ is canonically isomorphic to its image $i_* V_z(\mathcal{A}) \subset V_{i(z)}(j_1(\mathcal{V}_{n+1}))$. The fiber metric on $j_1(\mathcal{V}_{n+1})$ may therefore be used to induce a scalar product on $V_z(\mathcal{A})$, as well as an "orthogonal projection" $\mathcal{P}_\mathcal{A} : V_{i(z)}(j_1(\mathcal{V}_{n+1})) \to V_z(\mathcal{A})$. Both operations are entirely straightforward. The geometric structure determined by the scalar product on $V(\mathcal{A})$ will be called the fiber metric over $\mathcal{A}$.

In local coordinates, making use of the intrinsic representation (2.26) for the embedding $i$, and omitting all unnecessary subscripts, we have the explicit relations

$$\left( \frac{\partial}{\partial z^A}, \frac{\partial}{\partial z^B} \right) := G_{AB} = g_{ij} \frac{\partial \psi^i}{\partial z^A} \frac{\partial \psi^i}{\partial z^B} \tag{2.34}$$

$$\mathcal{P}_\mathcal{A}\left( \frac{\partial}{\partial \dot{q}^i} \right) = G^{AB}\left( \frac{\partial}{\partial \dot{q}^i}, i_* \frac{\partial}{\partial z^B} \right) \frac{\partial}{\partial z^A} = G^{AB} g_{ij} \frac{\partial \psi^j}{\partial z^B} \frac{\partial}{\partial z^A} \tag{2.35}$$

with $G^{AB} G_{BC} = \delta^A_C$.

By means of $\mathcal{P}_\mathcal{A}$, recalling the definition (2.19) of the fundamental tensor of $j_1(\mathcal{V}_{n+1})$, we construct a linear map $\hat{J} : T_z(\mathcal{A}) \to T_z(\mathcal{A})$ according to the equation

$$\hat{J}(X) := \mathcal{P}_\mathcal{A}(J(i_* X)) \qquad \forall X \in T_z(\mathcal{A}). \tag{2.36}$$

By the quotient law, the latter identifies a tensor field $\hat{J}$ of type (1,1) over $\mathcal{A}$, henceforth called the *fundamental tensor* of $\mathcal{A}$. In local coordinates, recalling eqs. (2.20), (2.35), we have the explicit expression

$$\hat{J}(X) = \langle i_*(X), \omega^i \rangle \mathcal{P}_\mathcal{A}\left( \frac{\partial}{\partial \dot{q}^i} \right) = \langle X, \tilde{\omega}^i \rangle G^{AB} g_{ij} \frac{\partial \psi^i}{\partial z^B} \frac{\partial}{\partial z^A}$$

resulting in the representation

$$\hat{J} = \sigma^A \otimes \frac{\partial}{\partial z^A} \tag{2.37}$$

with

$$\sigma^A = G^{AB} \frac{\partial \psi^j}{\partial z^B} g_{ij} \tilde{\omega}^i. \tag{2.38}$$

In connection with the previous definition, it is worth noticing that, unlike the fundamental tensor $J$ of $j_1(\mathcal{V}_{n+1})$ – whose construction had a universal character, involving only the *definition* of the concept of first jet space – the tensor $\hat{J}$ depends also explicitly on the fiber metric, *i.e.*, through eq. (2.7), on the material properties of the system in study.

In this respect, rather than as a geometrical attribute of the manifold $\mathcal{A}$, the field $\hat{J}$ is therefore to be regarded as a *mechanical object* over $\mathcal{A}$ [3].

By means of $\hat{J}$, paralleling the procedure followed in $j_1(\mathcal{V}_{n+1})$, we may set up a correspondence between vertical vectors and contact 1-forms over $\mathcal{A}$, by assigning to each $X \in V_z(\mathcal{A})$ the unique element $\hat{g}(X) \in \mathcal{C}_z(\mathcal{A})$ satisfying the condition

$$\langle \hat{g}(X), Y \rangle = (X, \hat{J}(Y)) \qquad \forall Y \in T_z(\mathcal{A}). \tag{2.39}$$

A straightforward check shows that the previous definition is consistent with the identification

$$\hat{g}(X) = (i_z)_* {}^* g(i_{z*} X)$$

$g$ denoting the isomorphism between vertical vectors and contact 1-forms over $j_1(\mathcal{V}_{n+1})$ described in section 2.2.

The image of the vertical bundle $V(\mathcal{A})$ under the map $\hat{g}$ will be denoted by $V^*(\mathcal{A})$. Every element $\eta_{|z} \in V^*(\mathcal{A})$, or, more generally, every section $\eta : \mathcal{A} \to V^*(\mathcal{A})$ will be called a *virtual differential form* over $\mathcal{A}$.

In local coordinates, eqs. (2.37), (2.38), (2.39) imply the representation

$$\hat{g}\left(\frac{\partial}{\partial z^A}\right) = G_{AB}\sigma^B \quad \Leftrightarrow \quad \sigma^A = G^{AB}\hat{g}\left(\frac{\partial}{\partial z^B}\right). \tag{2.40}$$

(iv) The linear endomorphism $\hat{J} : T(\mathcal{A}) \to T(\mathcal{A})$ described by eq. (2.36) may be extended to an algebraic derivation of the tensor algebra $\mathcal{D}(\mathcal{A})$, commuting with contractions and vanishing on functions. The resulting operation, henceforth denoted by $\hat{v}$, is characterized by properties formally analogous to eqs. (2.23), namely

$$\hat{v}(f) = 0, \quad \hat{v}(X) = \hat{J}(X), \quad \langle \hat{v}(X), \eta \rangle + \langle X, \hat{v}(\eta) \rangle = 0$$

for all $f \in \mathcal{F}(\mathcal{A}), X \in \mathcal{D}^1(\mathcal{A}), \eta \in \mathcal{D}_1(\mathcal{A})$.

Exactly as we did in section 2.2, by means of $\hat{v}$ we construct an anti-

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[3] An alternative proposal for the construction of a "fundamental tensor" over the constraint submanifold $\mathcal{A}$ is outlined in Ref. [20]. The argument relies on the preliminary assignment of a fibration of the configuration space time $\mathcal{V}_{n+1}$ over an $(r+1)$-dimensional base manifold $(\mathcal{V}_{r+1})$. The resulting geometrical framework – intrinsically different from the one discussed here – is especially suited to the study of coupled systems of second-and first-order differential equations.

derivation $\hat{d}_v$ of the Grassmann algebra $\mathcal{G}(\mathcal{A})$, called once again the *fiber differentiation*, acting on arbitrary $r$-forms according to the equation

$$\hat{d}_v \eta := d\hat{v}(\eta) - \hat{v}(d\eta) - r dt \wedge \eta. \tag{2.41}$$

In local coordinates, the evaluation of $\hat{d}_v$ relies on the pair of relations

$$\hat{d}_v f = -\hat{v}\, df = \frac{\partial f}{\partial z^A} \sigma^A \tag{2.42}$$

$$\hat{d}_v df = d\,\hat{v}\, df - dt \wedge df = -d\left(\frac{\partial f}{\partial z^A}\sigma^A - f\, dt\right) \tag{2.43}$$

for all $f \in \mathcal{F}(\mathcal{A})$.

(v) At each $z \in \mathcal{A}$, the pull-back $(i_z)_*{}^*$ sets up a 1-1 correspondence between contact 1-forms in $T^*_{i(z)}(\mathcal{A})$ and contact 1-forms in $T^*_z(\mathcal{A})$. This fact, together with eq. (2.22), determines a scalar product in the vector bundle $\mathcal{C}(\mathcal{A})$, expressed locally by the relations

$$\left(\tilde{\omega}^i, \tilde{\omega}^j\right) = g^{ij} \tag{2.44}$$

with $g^{ij} = g^{ij}(t, q^1, \ldots, q^n, z^1, \ldots, z^r)$ obtained by pulling back the quantities (2.22) to the submanifold $A$. In particular, by eqs. (2.38), (2.44) we have the relations

$$\left(\tilde{\omega}^i, \sigma^A\right) = G^{AB} \frac{\partial \psi^i}{\partial z^B} \tag{2.45}$$

whence, recalling eq. (2.34)

$$\left(\sigma^A, \sigma^B\right) = G^{AM} G^{BN} \frac{\partial \psi^i}{\partial z^M} \frac{\partial \psi^j}{\partial z^N} g_{ik} g_{jl} \left(\tilde{\omega}^k, \tilde{\omega}^l\right) = G^{AB} \tag{2.46}$$

as it was to be expected, on the basis of the identifications (2.40).

The previous results are completed by the following

PROPOSITION 2.1. – *The bundle $V^*(\mathcal{A}) = \hat{g}(V(\mathcal{A}))$ of virtual 1-forms over $\mathcal{A}$ coincides with the orthogonal complement of the Chetaev bundle $\chi(\mathcal{A})$ in the vector bundle $\mathcal{C}(\mathcal{A})$ of contact 1-forms over $\mathcal{A}$.*

*Proof.* – The orthogonality between $V^*(\mathcal{A})$ and $\chi(\mathcal{A})$ under the scalar product (2.44) is easily checked by direct computation. Indeed, if $\nu$ denotes an arbitrary Chetaev 1-form, eqs. (2.33), (2.40), (2.45) imply

$$\left(\nu, \hat{g}\left(\frac{\partial}{\partial z^A}\right)\right) = \nu_i G_{AB} \left(\tilde{\omega}^i, \sigma^B\right) = \nu_i \frac{\partial \psi^i}{\partial z^A} = 0.$$

The required conclusion then follows from a dimensionality argument, based on the fact that, at each $z \in \mathcal{A}$, the space $V^*_z(\mathcal{A})$ is $r$-dimensional, while the Chetaev space $\chi_z(\mathcal{A})$ is $(n - r)$-dimensional. □

(vi) In view or Proposition 2.1, every contact 1-form $\eta$ over $\mathcal{A}$ admits a unique representation as the sum of a virtual 1-form a Chetaev 1-form. In local coordinates, setting $\eta = \eta_i \widetilde{\omega}^i$, the required splitting is accomplished through the decomposition

$$\eta = \eta_i \hat{d}_v \psi^i + \eta_i \left( \widetilde{\omega}^i - \hat{d}_v \psi^i \right). \tag{2.47}$$

Indeed, in view of eqs. (2.40), (2.42), (2.45), (2.46), it is easily seen that the fiber differentials $\hat{d}_v \psi^i$ are automatically in $V^*(\mathcal{A})$, while the differences

$$\chi^i := \widetilde{\omega}^i - \hat{d}_v \psi^i \tag{2.48}$$

satisfy the orthogonality conditions

$$\left( \chi^i, \sigma^A \right) = \left( \widetilde{\omega}^i - \frac{\partial \psi^i}{\partial z^B} \sigma^B, \sigma^A \right) = 0 \quad A = 1, \ldots, r \tag{2.49}$$

*i.e.* they are all Chetaev 1-forms.

*Remark* 2.1. – The previous discussion points out that the 1-forms $\{\chi^i, i = 1, \ldots, n\}$ generate (locally) the Chetaev bundle. Needless to say, these generators are not independent. By eqs. (2.34), (2.38) they are in fact subject to the linear relations

$$g_{ij} \frac{\partial \psi^i}{\partial z^A} \chi^j = g_{ij} \frac{\partial \psi^i}{\partial z^A} \widetilde{\omega}^j - G_{AB} \sigma^B = 0 \qquad A = 1, \ldots, r \tag{2.50}$$

mathematically equivalent to eqs. (2.49).

In a similar way, the fiber differentials $\left\{ \hat{d}_v \psi^i, i = 1, \ldots, n \right\}$ generate locally the bundle $V^*(\mathcal{A})$ of virtual 1-forms over $\mathcal{A}$. Once again, a straightforward dimensionality argument indicates that these generators are not independent, but are subject to $n - r$ linear relations. In terms of an arbitrary cartesian representation (2.27) for the submanifold $\mathcal{A}$, these are summarized into the system

$$i^* \left( \frac{\partial g^\sigma}{\partial \dot{q}^k} \right) \hat{d}_v \psi^k = \frac{\partial g^\sigma}{\partial \dot{q}^k} \frac{\partial \psi^k}{\partial z^A} \sigma^A \equiv 0 \qquad \sigma = 1, \ldots, n - r. \tag{2.51}$$

## 3. DYNAMICS

### 3.1. Mechanical quantities

In this section we shall discuss the dynamical aspects of the theory of constrained systems, within the framework introduced in section 2. As compared with the analysis proposed in [12], the present contribution is aimed at a better understanding of the role of d'Alembert's principle as a general tool for the constitutive characterization of the ideal constraints, independently of any requirement of holonomy, or of linearity.

Needless to say, the analysis is much indebted to the pioneering work of N. Chetaev on the extension of the concept of virtual displacement to the class of non-holonomic systems ([14], [15], [16], [17], [18]).

In what follows, we shall restate Chetaev's ideas in a rigorous geometrical setting, especially suited to the formal developments of Analytical Mechanics.

To keep the language as close as possible to the traditional one, we shall discuss everything in the euclidean three-space associated with an (arbitrarily chosen) frame of reference $\mathcal{I}$, sticking to eqs. (2.28), (2.29) in order to express the (relative) positions and velocities of the points of the system.

More precisely, for notational uniformity, we shall regard the functions (2.28) as representing the maps $\mathbf{x}_\xi \cdot \pi : \mathcal{A} \to E_3$, *i.e.* we shall regard them as being defined on $\mathcal{A}$, rather than on $\mathcal{V}_{n+1}$.

The (pull-back of the) contact differentials of the functions $\mathbf{x}_\xi$ will be denoted by $d_c\mathbf{x}_\xi$. Comparison with eq. (2.15), (2.28), (2.29), (2.30) yields the explicit representation

$$d_c\mathbf{x}_\xi = \frac{\partial \mathbf{x}_\xi}{\partial q^k}\tilde{\omega}^k = d\mathbf{x}_\xi - \mathbf{v}_\xi dt. \tag{3.1}$$

By eqs. (2.29), (2.42), (2.48), the latter may be splitted into the sum

$$d_c\mathbf{x}_\xi = \frac{\partial \mathbf{x}_\xi}{\partial q^k}(\hat{d}_v\psi^k + \chi^k) = \hat{d}_v\mathbf{v}_\xi + d_\chi\mathbf{x}_\xi \tag{3.2}$$

with

$$d_\chi\mathbf{x}_\xi := \frac{\partial \mathbf{x}_\xi}{\partial q^k}\chi^k. \tag{3.3}$$

The expression (3.3) will be called the *Chetaev differential* of $\mathbf{x}_\xi$.

The contact differentials (3.1), together with the position and velocity maps (2.28), (2.29), are explicitly involved in the representation of the mechanical quantities associated with the system $\mathcal{B}$ in the frame of reference $\mathcal{I}$. Among these, especially relevant in a Lagrangian context are the kinetic energy

$$T = \frac{1}{2}\int_{\mathfrak{B}} |\mathbf{v}_\xi|^2 m(d\xi) \tag{3.4}$$

and the kinetic momenta $p_k$, defined collectively by the equation

$$p_k\tilde{\omega}^k := \int_{\mathfrak{B}} \mathbf{v}_\xi \cdot d_c\mathbf{x}_\xi m(d\xi) = \left(\int_{\mathfrak{B}} \mathbf{v}_\xi \cdot \frac{\partial \mathbf{x}_\xi}{\partial q^k} m(d\xi)\right)\tilde{\omega}^k. \tag{3.5}$$

A straightforward comparison with the representation (2.8) of the holonomic kinetic energy $\mathcal{T}$ yields the identifications

$$T = i^*(\mathcal{T}), \qquad p_k = i^*\left(\frac{\partial \mathcal{T}}{\partial \dot{q}^k}\right). \tag{3.6}$$

In a similar way, by eqs. (2.7), (2.29), (3.4), (3.5) we have the identities

$$\frac{\partial T}{\partial z^A} = \int_{\mathfrak{B}} \mathbf{v}_\xi \cdot \frac{\partial \mathbf{v}_\xi}{\partial z^A} m(d\xi) = \int_{\mathfrak{B}} \mathbf{v}_\xi \cdot \frac{\partial \mathbf{x}_\xi}{\partial q^k} \frac{\partial \psi^k}{\partial z^A} m(d\xi) = p_k \frac{\partial \psi^k}{\partial z^A} \tag{3.7}$$

$$\frac{\partial p_k}{\partial z^A} = \int_{\mathfrak{B}} \frac{\partial \mathbf{v}_\xi}{\partial z^A} \cdot \frac{\partial \mathbf{x}_\xi}{\partial q^k} m(d\xi) = \int_{\mathfrak{B}} \left(\frac{\partial \mathbf{x}_\xi}{\partial q^r} \frac{\partial \psi^r}{\partial z^A}\right) \cdot \frac{\partial \mathbf{x}_\xi}{\partial q^k} m(d\xi) = g_{kr} \frac{\partial \psi^r}{\partial z^A} \tag{3.8}$$

the exterior 2-form

$$\Omega := \int_{\mathfrak{B}} d\mathbf{v}_\xi \wedge d_c \mathbf{x}_\xi m(d\xi) \tag{3.9}$$

will be called the kinetic Poincaré-Cartan 2-form of the system relative to the frame of reference $\mathcal{I}$ [4]. Eqs. (3.1), (3.4), (3.5) imply the relation

$$\Omega = d\big(p_k \tilde{\omega}^k\big) + \left(\int_{\mathfrak{B}} \mathbf{v}_\xi \cdot d\mathbf{v}_\xi m(d\xi)\right) \wedge dt = d\big(p_k \tilde{\omega}^k + T\, dt\big). \tag{3.10}$$

Together with eqs. (2.30), (3.7), the latter gives rise to the representation

$$\Omega = dp_k \wedge \tilde{\omega}^k + p_k d\tilde{\omega}^k + dT \wedge dt = \left[dp_k + \left(p_r \frac{\partial \psi^r}{\partial q^k} - \frac{\partial T}{\partial q^k}\right) dt\right] \wedge \tilde{\omega}^k. \tag{3.11}$$

A better insight into the nature of the kinetic Poincaré-Cartan 2-form is provided by the splitting (3.2). The latter allows to express eq. (3.9) in the equivalent form

$$\Omega = \hat{\Omega} + \Omega_\chi \tag{3.13}$$

with

$$\hat{\Omega} := \int_{\mathfrak{B}} d\mathbf{v}_\xi \wedge \hat{d}_v \mathbf{v}_\xi m(d\xi) = \left(\int_{\mathfrak{B}} d\mathbf{v}_\xi \cdot \frac{\partial \mathbf{x}_\xi}{\partial q^k} m(d\xi)\right) \wedge \hat{d}_v \psi^k \tag{3.14}$$

$$\Omega_\chi := \int_{\mathfrak{B}} d\mathbf{v}_\xi \wedge d_\chi \mathbf{x}_\xi m(d\xi) = \left(\int_{\mathfrak{B}} d\mathbf{v}_\xi \cdot \frac{\partial \mathbf{x}_\xi}{\partial q^k} m(d\xi)\right) \wedge \chi^k. \tag{3.15}$$

[4] The term "kinetic" is meant to point out the fact that, as compared with the standard definition ([21], [22]), eq. (3.9) leaves out all contributions coming from the *forces* acting on the system.

The meaning of the representation (3.13) is clarified by the following

THEOREM 3.1. – *For every vertical vector field $V$ over $\mathcal{A}$, the interior product $V \rfloor \Omega$ satisfies the identity*

$$V \rfloor \Omega = V \rfloor \hat{\Omega} = \hat{g}(V) \tag{3.16}$$

*$\hat{g} : V(\mathcal{A}) \to V^*(\mathcal{A})$ denoting the process of lowering the indices induced by the fundamental tensor of the manifold $\mathcal{A}$ through eq. (2.39).*

*Proof.* – By the definition (3.5) of the kinetic momenta, setting $V = V^A \frac{\partial}{\partial z^A}$, and recalling the identity (3.8), we have easily

$$\int_{\mathfrak{B}} V(\mathbf{v}_\xi) \cdot \frac{\partial \mathbf{x}_\xi}{\partial q^k} m(d\xi) = V(p_k) = V^A g_{kr} \frac{\partial \psi^r}{\partial z^A}$$

Together with eqs. (2.34), (2.40), (2.50), taking the representations (3.14), (3.15) into account, this implies the relations

$$V \rfloor \hat{\Omega} = V(p_k) \hat{d}_v \psi^k = V^A g_{kr} \frac{\partial \psi^r}{\partial z^A} \frac{\partial \psi^k}{\partial z^B} \sigma^B = V^A G_{AB} \sigma^B = \hat{g}(V)$$

$$V \rfloor \Omega_\chi = V(p_k) \chi^k = V^A g_{kr} \frac{\partial \psi^r}{\partial z^A} \chi^k = 0$$

mathematically equivalent to eq. (3.16). □

If, rather than a vertical vector field $V$, we consider a dynamical flow $Z$ over $\mathcal{A}$, eq. (3.13) provides a splitting of the interior product $Z \rfloor \Omega$ into the sum of a virtual 1-form and a Chetaev one, namely

$$Z \rfloor \Omega = Z \rfloor \hat{\Omega} + Z \rfloor \Omega_\chi. \tag{3.17}$$

In analogy with Theorem 3.1 we have then the following

COROLLARY 3.1. – *Let $Z$ denote an arbitrary dynamical flow. Then:*

*a) the Chetaev 1-form $Z \rfloor \Omega_\chi$ is independent of $Z$,* i.e. *it is determined uniquely in terms of the mechanical properties of the system;*

*b) the knowledge of the virtual 1-form $Z \rfloor \hat{\Omega}$ is mathematically equivalent to the knowledge of $Z$.*

*Proof.* – Starting with an arbitrarily chosen dynamical flow $Z_0$, set $Q_0 := Z_0 \rfloor \hat{\Omega}$, and recall that the most general dynamical flow $Z$ is obtained by adding to $Z_0$ an arbitrary vertical vector field $V$. In view of Theorem 3.1 we have then the relation

$$(Z - Z_0) \rfloor \Omega_\chi = V \rfloor \Omega_\chi = 0$$

proving assertion *a*) and

$$(Z - Z_0) \rfloor \ \hat{\Omega} = V \rfloor \ \hat{\Omega} = \hat{g}(V)$$

whence also

$$Z = Z_0 + \hat{g}^{-1}\Big[(Z - Z_0) \rfloor \ \hat{\Omega}\Big] = Z_0 + \hat{g}^{-1}\Big(Z \rfloor \ \hat{\Omega} - Q_0\Big) \qquad (3.18)$$

showing that the knowledge of $Z \rfloor \ \hat{\Omega}$ is indeed equivalent to the knowledge of $Z$. □

In local coordinates, recalling eqs. (2.48), (3.11), (3.17) (or also by direct computation, starting with eqs. (3.14), (3.15)) one can easily verify the validity of the expressions

$$Z \rfloor \ \hat{\Omega} = \left[Z(p_k) - \frac{\partial T}{\partial q^k} + p_r \frac{\partial \psi^r}{\partial q^k}\right] \hat{d}_v \psi^k \qquad (3.19)$$

$$Z \rfloor \ \Omega_\chi = \left[Z(p_k) - \frac{\partial T}{\partial q^k} + p_r \frac{\partial \psi^r}{\partial q^k}\right] \chi^k. \qquad (3.20)$$

As a concluding remark we observe that, as a consequence of Corollary 3.1, strictly associated with the kinetic Poincaré-Cartan 2-form – and thus, ultimately, with the choice of the frame of reference $\mathcal{I}$ – is a distinguished dynamical flow $Z_\mathcal{I}$, defined by the condition

$$Z_\mathcal{I} \rfloor \ \hat{\Omega} = 0. \qquad (3.21)$$

In view of eq. (3.18), every other dynamical flow $Z$ is then determined uniquely by the knowledge of the virtual 1-form $Q = Z \rfloor \ \hat{\Omega}$, according to the relation

$$Z = Z_\mathcal{I} + \hat{g}^{-1}(Q). \qquad (3.22)$$

The expression (3.22) resembles very closely the situation discussed in [12], where the term $\hat{g}^{-1}(Q)$ was taken axiomatically as a representation of the forces acting on the system in the frame of reference $\mathcal{I}$.

In local coordinates, taking eqs. (2.34), (3.8), (3.19) into account, the solution of eq. (3.21) is easily recognized to be

$$Z_\mathcal{I} = \frac{\partial}{\partial t} + \psi^k \frac{\partial}{\partial q^k} - G^{AB}\left(\frac{\partial p_k}{\partial t} + \psi^r \frac{\partial p_k}{\partial q^r} + p_r \frac{\partial \psi^r}{\partial q^k} - \frac{\partial T}{\partial q^k}\right) \frac{\partial \psi^k}{\partial z^B} \frac{\partial}{\partial z^A}. \qquad (3.23)$$

### 3.2. Virtual work

The (frame-dependent) correspondence between dynamical flows and virtual 1-forms described in Corollary 3.1, namely

$$Z \leftrightarrow Z \lrcorner \hat{\Omega} = \int_{\mathfrak{B}} Z(\mathbf{v}_\xi) \cdot \hat{d}_v \mathbf{v}_\xi m(d\xi)$$

plays a crucial role in the geometrization of Dynamics.

To ensure consistency with the traditional language, we introduce the following

DEFINITION 3.1. – *For each $z \in \mathcal{A}$, the map $\mathfrak{B} \to V^*_z(\mathcal{A})$ assigning to each $\xi \in \mathfrak{B}$ the vector valued virtual 1-form*

$$(\hat{d}_v \mathbf{v}_\xi)_{|z} = \left(\frac{\partial \mathbf{x}_\xi}{\partial q^k}\right)_{|z} (\hat{d}_v \psi^k)_{|z} \tag{3.24}$$

*will be called the virtual displacement of the system at $z$.*

Before discussing the dynamical significance of Definition 3.1, a comparison with the conventional concept of virtual displacement is in order. To this end, we make use of the fact that every contact 1-form at $z$ may be viewed as a linear functional on the vertical space $V_{\pi(z)}(\mathcal{V}_{n+1})$, through the pairing (2.16).

In particular, if we let the functionals associated with the 1-forms $(\hat{d}_v \mathbf{v}_\xi)_{|z}$ and $(\hat{d}_v \psi^k)_{|z}$ be denoted respectively by $\delta \mathbf{x}_\xi$ and $\delta q^k$, $k = 1, \ldots, n$, eq. (3.24) takes the traditional form [5]

$$\delta \mathbf{x}_\xi = \left(\frac{\partial \mathbf{x}_\xi}{\partial q^k}\right)_{|\pi(z)} \delta q^k. \tag{3.25}$$

In general, however, as pointed out in Remark 2.1, the fiber differentials $\hat{d}_v \psi^k$ – and thus also the functionals $\delta q^k$ – are not linearly independent, but are subject to the inner identities (2.51), $g^\sigma(t, q^i, \dot{q}^i) = 0$, $\sigma = 1, \ldots, n-r$ denoting any cartesian representation for the submanifold $\mathcal{A}$. When expressed in terms of the functionals $\delta q^k$ these reproduce the so-called *Chetaev conditions*

$$\left(\frac{\partial g^\sigma}{\partial \dot{q}^k}\right)_{|z} \delta q^k = 0 \qquad \sigma = 1, \ldots, n-r. \tag{3.26}$$

[5] The viewpoint of regarding both sides of eq. (3.25) as functionals on *vertical* vectors reflects the classical notion of virtual displacement as an "infinitesimal change of the configuration of the system, resulting from arbitrary infinitesimal changes in the positions of all points, consistent with the restrictions imposed by the constraints at the given instant $t$" [3].

It goes without saying that in the holonomic case (corresponding to $r = n$), the conditions (3.26) are empty, and the functional $\delta q^k$ are linearly independent.

To sum up, Definition 3.1, translated into the language of linear functionals over $V_{\pi(z)}(\mathcal{V}_{n+1})$, yields back the traditional notion of virtual displacement, with the Chetaev conditions (3.26) automatically embodied into the formalism.

To undestand the role of Definition 3.1 in a dynamical context, we now turn our attention to the description of the interactions. To this end, we rely on the usual classification of the mechanical forces into active and reactive ones, as well as on the identification – already outlined in section 2.1 – of the material system $\mathcal{B}$ with a measure space $(\mathfrak{B}, \mathcal{S}, m)$, $\mathcal{S}$ denoting the $\sigma$-ring of measurable subsets of the material space $\mathfrak{B}$. By definition, a representation of the active forces is then a map $\mathbf{F} : \mathcal{S} \times \mathcal{A} \to V_3$, assigning to each $\Omega \in \mathcal{S}$ a corresponding vector valued function $\mathbf{F}_\Omega(z) := \mathbf{F}(\Omega, z)$, expressing the total (active) force on $\Omega$ in terms of the kinetic state of the system.

With the standard notation of measure theory, we shall write

$$\mathbf{F}_\Omega = \int_\Omega \mathbf{F}(d\xi) \tag{3.27}$$

the dependence of both sides on the variable $z$ being implicitly understood. The quantities

$$Q_k := \int_{\mathfrak{B}} \frac{\partial \mathbf{x}_\xi}{\partial q^k} \cdot \mathbf{F}(d\xi) \tag{3.28}$$

($Q_k = \sum_{i=1}^N \mathbf{F}_i \cdot \frac{\partial \mathbf{x}_\xi}{\partial q^k}$ in the discrete case) will be called the Lagrangian components of $\mathbf{F}$.

Recalling the representation (3.1) for the contact differentials $d_c\mathbf{x}_\xi$, we have the obvious identification

$$\int_{\mathfrak{B}} d_c\mathbf{x}_\xi \cdot \mathbf{F}(d\xi) = \left(\int_{\mathfrak{B}} \frac{\partial \mathbf{x}_\xi}{\partial q^k} \cdot \mathbf{F}(d\xi)\right)\tilde{\omega}^k = Q_k\tilde{\omega}^k. \tag{3.29}$$

The description of the reactive forces follows a similar pattern, the only (substantial!) difference being that, in general, the associated map $\Phi : \mathcal{S} \times \mathcal{A} \to V_3$ is a-priori unknown. In place of it, one has a statement – henceforth called a *constitutive characterization of the constraints* – indicating which reactive forces are effectively allowed and which are not, the selection depending on the physical properties of the devices involved in the implementation of the constraints. We shall return on this point in section 3.3. The totality of admissible maps $\Phi : \mathcal{S} \times \mathcal{A} \to V_3$ will be indicated by $\mathcal{H}(\mathcal{S} \times \mathcal{A})$.

Exactly as we did with the active forces, we introduce the notation $\Phi_\Omega = \int_\Omega \Phi(d\xi)$, and define the Lagrangian components $\varphi_k$ of $\Phi$ collectively through the equation

$$\int_{\mathfrak{B}} d_c \mathbf{x}_\xi \cdot \Phi(d\xi) = \left( \int_{\mathfrak{B}} \frac{\partial \mathbf{x}_\xi}{\partial q^k} \cdot \Phi(d\xi) \right) \tilde{\omega}^k = \varphi_k \tilde{\omega}^k. \tag{3.30}$$

*Remark* 3.1. – In the case of *distributed* forces (defined by the condition $\mathbf{F}_\Omega = \Phi_\Omega = 0$ whenever $m(\Omega) = 0$), the measures $\mathbf{F}$, $\Phi$ may be expressed in terms of the corresponding Radon-Nikodym derivatives $\mathbf{f} = d\mathbf{F}/dm$, $\varphi = d\Phi/dm$ (*specific forces*, or forces per unit mass) in the standard form $\mathbf{F} = \mathbf{f}m, \Phi = \varphi m$ ([23], [24]). This possibility breaks down in the presence of *concentrated* forces, *i.e.* of force measures not absolutely continuous with respect to $m$.

*Remark* 3.2. – Various instances of vector-valued, $z$-dependent measures on $\mathfrak{B}$, absolutely continuous with respect to $m$, have already been met in the description of the mechanical quantities. The most significant example is provided by the "momentum measure" $\mathbf{P}$, related to the velocity map $\mathbf{v}(\xi, z) := \mathbf{v}_\xi(z)$ by the identification $\mathbf{P} = \mathbf{v}m$. For each measurable domain $\Omega \subset \mathfrak{B}$, the quantity

$$\mathbf{P}_\Omega(z) := \int_\Omega \mathbf{v}(\xi, z) m(d\xi) \tag{3.31}$$

expresses the total linear momentum of $\Omega$ as a function of the kinetic state of the system.

Let us now denote by $\mathcal{M}(\mathcal{S} \times \mathcal{A})$ ($\mathcal{M}$ for short) the totality of vector valued, $z$-dependent measures $\mathbf{U} : \mathcal{S} \times \mathcal{A} \to V_3$ such that, for each $\Omega \in \mathcal{S}$, the function $\mathbf{U}_\Omega(z) := \mathbf{U}(\Omega, z)$ depends differentiably on $z$. From an algebraic viewpoint, $\mathcal{M}$ has the nature of a *module* over the ring $\mathcal{F}(\mathcal{A})$ of differentiable functions over $\mathcal{A}$.

DEFINITION 3.2. – *For each* $\mathbf{U} \in \mathcal{M}$, *the virtual 1-form*

$$\wedge(\mathbf{U}) := \int_{\mathfrak{B}} \hat{d}_v \mathbf{v}_\xi \cdot \mathbf{U}(d\xi) = \left( \int_{\mathfrak{B}} \frac{\partial \mathbf{x}_\xi}{\partial q^k} \cdot \mathbf{U}(d\xi) \right) \hat{d}_v \psi^k \tag{3.32}$$

*(*$\Lambda(\mathbf{U}) = \sum_{i=1}^N \mathbf{U}_i \cdot \hat{d}_v \mathbf{v}_i$ *in the discrete case), will be called the virtual content of* $\mathbf{U}$. *In the special case when* $\mathbf{U}$ *is a force measure, the expression (3.32) will be called the virtual work of* $\mathbf{U}$.

Comparison with eq. (3.2) shows that the virtual work of the active forces coincides with the virtual part of the 1-form (3.29). In local coordinates, recalling eq. (2.48), we have the explicit representations

$$\Lambda(\mathbf{F}) = \left( \int_{\mathfrak{B}} \frac{\partial \mathbf{x}_\xi}{\partial q^k} \cdot \mathbf{F}(d\xi) \right) \hat{d}_v \psi^k = Q_k \frac{\partial \psi^k}{\partial z^A} \sigma^A := Q_A \sigma^A \tag{3.33}$$

$$Q_k \tilde{\omega}^k = Q_k (\hat{d}_v \psi^k + \chi^k) = \Lambda(\mathbf{F}) + Q_k \chi^k. \tag{3.34}$$

In particular at each $z \in \mathcal{A}$, keeping the same notation as in the discussion following Definition 3.1, and denoting by $\delta W$ the linear functional over $V_{\pi(z)}(\mathcal{V}_{n+1})$ associated with $\Lambda(\mathbf{F})$, we get the familiar expression

$$\delta W = \left( \int_{\mathfrak{B}} \frac{\partial \mathbf{x}_\xi}{\partial q^k} \cdot \mathbf{F}(d\xi) \right) \delta q^k = Q_k \delta q^k$$

with the Chetaev conditions (3.26) automatically embodied into the formalism. Fairly similar conclusions apply to the virtual work $\Lambda(\Phi)$ of the reactive forces.

We conclude this Subsection by proving a factorization theorem, which will play a major role in the discussion of the constitutive characterization of the reactive forces. To this end, within the module $\mathcal{M}$ of vector valued, $z$-dependent measures defined above, we single out the submodule $\mathcal{M}_\perp$ formed by the totality of measures with vanishing virtual content, namely

$$\mathcal{M}_\perp = \{\mathbf{U} | \mathbf{U} \in \mathcal{M}, \Lambda(\mathbf{U}) = 0\}. \tag{3.35}$$

Moreover, we introduce a correspondence between vertical vector fields and vector valued measures, based on the prescription

$$V = V^A \frac{\partial}{\partial z^A} \to V(\mathbf{v}) m = V^A \frac{\partial \mathbf{v}(\xi, z)}{\partial z^A} m \tag{3.36}$$

In view of eqs. (3.14), (3.32), taking Theorem 3.1 into account, this implies the identity

$$\Lambda(V(\mathbf{v})m) = \int_{\mathfrak{B}} (V \,\lrcorner\, d\mathbf{v}_\xi) \cdot \hat{d}_v \mathbf{v}_\xi m(d\xi) = V \,\lrcorner\, \hat{\Omega} = \hat{g}(V) \tag{3.37}$$

showing that the knowledge of the measure $V(\mathbf{v})m$ is mathematically equivalent to the knowledge of the field $V$.

Collecting all previous definitions, we can now state

THEOREM 3.2. – *Every measure* $\mathbf{U} \in \mathcal{M}$ *admits a unique factorization of the form*

$$\mathbf{U} := \mathbf{U}_\perp + V_{\mathbf{U}}(\mathbf{v}) m \tag{3.38}$$

*in terms of a measure* $\mathbf{U}_\perp \in \mathcal{M}_\perp$, *and of a vertical vector field* $V_{\mathbf{U}}$ *over* $\mathcal{A}$.

*Proof.* – Simply observe that, on account of eqs. (3.35), (3.37), eq. (3.38) is mathematically equivalent to the condition $V_{\mathbf{U}} = \hat{g}^{-1}(\Lambda(\mathbf{U}))$. $\square$

For dynamical purposes, the content of Theorem 3.2 is conveniently summarized into the pair of assertions

*a*) the correspondence $\mathbf{U} \to \mathbf{U}_\perp = \mathbf{U} - V_{\mathbf{U}}(\mathbf{v})m$ determines an $\mathcal{F}$-linear projection of the module $\mathcal{M}$ onto $\mathcal{M}_\perp$;

*b*) every measure $\mathbf{U} \in \mathcal{M}$ is uniquely determined by the knowledge of its projection $\mathbf{U}_\perp \in \mathcal{M}_\perp$, and of the virtual 1-form $\Lambda(\mathbf{U})$.

The proof is entirely straightforward, and is left to the reader.

### 3.3. Equations of motion

To complete our analysis, we shall now discuss the role of the previous definitions in the study of the problem of motion for the system $\mathcal{B}$.

As widely explained in the literature (*see*, e.g. [12] and references therein), the problem can be stated geometrically as follows: given the active forces, expressed by a vector-valued, $z$-dependent measure $\mathbf{F} \in \mathcal{M}$, as well as the constitutive characterization of the constraints, summarized into the assignment of a suitable class $\mathcal{H}(\mathcal{S} \times \mathcal{A})$ of admissible reactive forces, determine a dynamical flow $Z$ over $\mathcal{A}$ in such a way that, for each $z \in \mathcal{A}$, the integral curve of $Z$ through $z$ coincides with the first jet extension of the section $\gamma : \mathfrak{R} \to \mathcal{V}_{n+1}$ describing the evolution of the system from the initial kinetic state $z$.

Within the stated framework, the principle of determinism has then an obvious mathematical counterpart in the requirement that, for each assignment of $\mathbf{F}$, the constitutive characterization of the constraints be sufficient to determine a unique such $Z$.

Let us examine this point in detail: with the notation of section 3.2, Newton's 2nd law requires that, for each measurable subset $\Omega \in \mathcal{S}$, the total linear momentum $\mathbf{P}_\Omega$, viewed as a vector-valued function over $\mathcal{A}$, be related to the total (active + reactive) force $\mathbf{F}_\Omega + \Phi_\Omega$ by the evolution equation

$$Z(\mathbf{P}_\Omega) = \mathbf{F}_\Omega + \Phi_\Omega \tag{3.39}$$

$Z$ denoting the (so far unknown) dynamical flow. Recalling the representations (3.27), (3.31), as well as the analogous expression for $\Phi_\Omega$, eq. (3.39) may be written in the equivalent form

$$\int_\Omega Z(\mathbf{v})m(d\xi) = \int_\Omega (\mathbf{F}(d\xi) + \Phi(d\xi)) \qquad \forall \Omega \in \mathcal{S}$$

*i.e.* as a relation

$$\mathbf{F} + \Phi = Z(\mathbf{v})m \tag{3.40}$$

between vector valued measures in the class $\mathcal{M}$.

Keeping the same notation as in Theorem 3.2, we can then state

COROLLARY 3.2. – *Eqs. (3.40) are mathematically equivalent to the system*

$$Z \lrcorner\ \hat{\Omega} = \Lambda(\mathbf{F} + \Phi) \tag{3.41}$$

$$Z_{\mathcal{I}}(\mathbf{v})m = \mathbf{F}_{\perp} + \Phi_{\perp} \tag{3.42}$$

*$Z_{\mathcal{I}}$ denoting the dynamical flow associated with the frame of reference $\mathcal{I}$ through the prescription (3.21).*

*Proof.* – For every dynamical flow $Z$, eqs. (3.14), (3.32) yield the identity

$$\Lambda(Z(\mathbf{v})m) = \int_{\mathfrak{B}} Z(\mathbf{v}_{\xi}) \cdot \hat{d}_v \mathbf{v}_{\xi} m(d\xi) = Z \lrcorner\ \hat{\Omega}.$$

Setting $V = \hat{g}^{-1}(Z \lrcorner\ \hat{\Omega})$, and recalling eqs. (3.21), (3.22), this implies the relations

$$\Lambda(Z_{\mathcal{I}}(\mathbf{v})m) = 0; \qquad Z(\mathbf{v})m = Z_{\mathcal{I}}(\mathbf{v})m + V(\mathbf{v})m$$

which, together with eqs. (3.35), (3.38), provide the further identification

$$Z_{\mathcal{I}}(\mathbf{v})m = (Z(\mathbf{v})m)_{\perp}.$$

The system (3.41), (3.42) is therefore equivalent to the pair of conditions

$$\Lambda(Z(\mathbf{v})m) = \Lambda(\mathbf{F} + \Phi)$$
$$Z_{\mathcal{I}}(\mathbf{v})m = \mathbf{F}_{\perp} + \Phi_{\perp}.$$

The conclusion is then a direct consequence of assertion b) following Theorem 3.2. □

Corollary 3.2 allows a precise mathematical formulation of the concept of determinism. This is readily understood by recalling that, according to eq. (3.21), the vector field $Z_{\mathcal{I}}$ is a datum of the problem, completely determined by the knowledge of the frame of reference $\mathcal{I}$, and expressed locally through eq. (3.23).

Taking eqs. (3.41), (3.42) as well as Corollary 3.1 into account, and recalling the comments following Theorem 3.2, one is then led to the conclusion that the only freedom left by Newton's 2nd law in the determination of both the dynamical flow $Z$ and the reactive force measure $\Phi$ is the value of the virtual work $\Lambda(\Phi)$.

The missing information is of course to be found in the constitutive characterization of the constraints. A criterion for the latter to be consistent with the principle of determinism is therefore the requirement that, for each choice of the active force measure $\mathbf{F}$, the condition $\Phi \in \mathcal{H}(\mathcal{S} \times \mathcal{A})$, possibly completed by eq. (3.42), be sufficient to determine the virtual 1-form $\Lambda(\Phi)$ uniquely in terms of $\mathbf{F}$, and of the kinetic state of the system.

By far the simplest and most significant application of the stated criterion is provided by the celebrated d'Alembert's principle, summarized into the following

DEFINITION 3.3. – *A set of constraints is said to ideal if and only if the corresponding constitutive characterization is expressed by the condition*

$$\Phi \in \mathcal{H}(\mathcal{S} \times \mathcal{A}) \quad \Leftrightarrow \quad \Lambda(\Phi) = \int_{\mathfrak{B}} \hat{d}_v \mathbf{v}_\xi \cdot \Phi(d\xi) \equiv 0 \tag{3.43}$$

i.e. *if and only if the class of admissible reactive forces coincides with the submodule $\mathcal{M}_\perp \subset \mathcal{M}$.*

As pointed out above, the constitutive characterization (3.43) is automatically deterministic.

An important insight into the dynamical content of d'Alembert's principle is provided by an equally celabrated statement, known as the *principle of least constraint*, of K. F. Gauss. In the traditional formulation, valid for discrete systems, the latter reads:

*For a mechanical system subject to ideal constraints, the actual motion under the action of given active forces, is the one for which, at any instant $t$, the quantity*

$$\hat{C} = \frac{1}{2} \sum_{i=1}^{N} \frac{|\Phi_i|^2}{m_i} \tag{3.44}$$

*attains a minimum within the class of kinematically admissible evolutions* ([3], [5], [6], [12], [17], [25]).

In strictly logical terms, rather than as a characterization of the class of admissible reactive forces, Gauss' principle is to be seen as a prescription indicating how to handle the dynamical equations $\mathbf{F}_i + \Phi_i = m_i \mathbf{a}_i$, in order to determine the evolution of the system – and thus also, indirectly, the reactive forces $\Phi_i$ – in terms of the active forces $\mathbf{F}_i$.

To this end, one has simply to replace each vector $\Phi_i$, $i = 1, \ldots, N$ in eq. (3.44) by the difference $m_i \mathbf{a}_i - \mathbf{F}_i$, looking then, at each instant $t$, for the values $\mathbf{a}_i$ that minimize the resulting expression within the class of admissible accelerations.

In order to compare the content of d'Alembert's and Gauss' principles, we shall first extend the latter to the case of reactive forces described by vector valued *measures*.

An apparent limitation here arises from the fact that the expression (3.44) has no natural counterpart in the case of force measures not absolutely continuous with respect to the mass measure $m$. The difficulty, however, is easily overcome by observing that, as far as the dependence on the accelerations is concerned, the function (3.44) has the same extremal points as the difference

$$C := \hat{C} - \frac{1}{2}\sum_{i=1}^{N} \frac{|\mathbf{F}_i|^2}{m_i} = \frac{1}{2}\sum_{i=1}^{N} \frac{(\Phi_i + \mathbf{F}_i)}{m_i} \cdot (\Phi_i - \mathbf{F}_i).$$

Unlike the original expression (3.44), the newer one can be extended to arbitrary force measures, due to fact that, as a consequence of Newton's 2nd law, the sum $\Phi + \mathbf{F}$ is always absolutely continuous with respect to $m$. Introducing the Radon-Nikodym derivative $d(\Phi + \mathbf{F})/dm$, we can therefore restate Gauss principle as a minimality request for the functional

$$C = \frac{1}{2}\int_{\mathfrak{B}} \frac{d(\Phi + \mathbf{F})}{dm} \cdot [\Phi(d\xi) - \mathbf{F}(d\xi)] \tag{3.45}$$

at any instant $t$, within the class of admissible accelerations.

We can now prove

THEOREM 3.3. – *When formulated in terms of the functional (3.45), Gauss' characterization of the class of ideal constraints is mathematically equivalent to d'Alembert's constitutive characterization (3.43).*

*Proof.* – Taking Newton's 2nd law (3.40) into account, the functional (3.45) may be written as

$$C = \frac{1}{2}\int_{\mathfrak{B}} Z(\mathbf{v}_\xi) \cdot [Z(\mathbf{v}_\xi)m(d\xi) - 2\mathbf{F}(d\xi)].$$

Viewed as a prescription for the evaluation of the dynamical flow $Z$ in terms of $\mathbf{F}$, Gauss minimality request may therefore be stated in the form

$$\int_{\mathfrak{B}} Z(\mathbf{v}_\xi)\cdot[Z(\mathbf{v}_\xi)m(d\xi) - 2\mathbf{F}(d\xi)] \leq \int_{\mathfrak{B}} Z'(\mathbf{v}_\xi)\cdot[Z'(\mathbf{v}_\xi)m(d\xi) - 2\mathbf{F}(d\xi)]$$

for all admissible flows $Z'$. Setting $Z' - Z := V$, this amounts to the requirement

$$\int_{\mathfrak{B}} \{V(\mathbf{v}_\xi) \cdot [Z(\mathbf{v}_\xi)m(d\xi) - \mathbf{F}(d\xi)] + |V(\mathbf{v}_\xi)|^2 m(d\xi)\} \geq 0$$

$$\forall V = V^A \frac{\partial}{\partial z^A}$$

whence, by the arbitrariness of $V^A$.

$$0 = \int_{\mathfrak{B}} \frac{\partial \mathbf{v}_\xi}{\partial z^A} \cdot [Z(\mathbf{v}_\xi)m(d\xi) - \mathbf{F}(d\xi)], \qquad A = 1, \dots, r.$$

The previous relations may be summarized into the single equation

$$\begin{aligned} 0 &= \left(\int_{\mathfrak{B}} \frac{\partial \mathbf{v}_\xi}{\partial z^A} \cdot [Z(\mathbf{v}_\xi)m(d\xi) - \mathbf{F}(d\xi)]\right)\sigma^A \\ &= \int_{\mathfrak{B}} \hat{d}_v \mathbf{v}_\xi \cdot [Z(\mathbf{v}_\xi)m(d\xi) - \mathbf{F}(d\xi)] \end{aligned}$$

clearly identical to the prescription that would arise by inserting Newton's 2nd law (3.40) directly into d'Alembert's condition (3.43). □

The equations of motion for a mechanical system subject to ideal constraints are obtained by inserting d'Alembert's characterization (3.43) directly into eq. (3.41), namely

$$Z \lrcorner\ \hat{\Omega} = \Lambda(\mathbf{F}). \tag{3.46}$$

In local coordinates, making use of the representations (3.19), (3.33), this gives rise to the system

$$\left(Z(p_k) + p_r \frac{\partial \psi^r}{\partial q^k} - \frac{\partial \mathbf{T}}{\partial q^k}\right) \frac{\partial \psi^k}{\partial z^A} = Q_A \qquad A = 1, \dots, r. \tag{3.47}$$

Alternatively, on the basis of eq. (3.22), the solution of eq. (3.46) may be written in compact form as

$$Z = Z_{\mathcal{I}} + \hat{g}^{-1}\Lambda(\mathbf{F}). \tag{3.48}$$

From this, recalling eqs. (3.23), (3.33), we conclude that the evolution of the system determined by the $n + r$ ordinary differential equations

$$\begin{cases} \dfrac{dq^i}{dt} = \psi^i\left(t,\, q^k,\, z^A\right) \\ \dfrac{dz^A}{dt} = G^{AB}\left(-\dfrac{\partial p_k}{\partial t} - \psi^r \dfrac{\partial p_k}{\partial q^r} - p_r \dfrac{\partial \psi^r}{\partial q^k} + \dfrac{\partial T}{\partial q^k} + Q_k\right) \dfrac{\partial \psi^k}{\partial z^B} \end{cases} \tag{3.49}$$

for the unknowns $q^i = q^i(t)$, $z^A = z^A(t)$.

The previous arguments provide a complete solution of the problem of motion, valid whenever the embedding $i\ :\ \mathcal{A} \to j_1(\mathcal{V}_{n+1})$ is expressed in the intrinsic form (2.26).

Or course, the same problem can also be tackled starting with a *cartesian* representation for the submanifold $\mathcal{A}$, of the form

$$g^\sigma\left(t,\, q^1, \dots, q^n,\, \dot{q}^1, \dots, \dot{q}^n\right) = 0 \quad \sigma = 1, \dots, n - r$$

with rank $\|\partial(g^1, \dots, g^{n-r})/\partial(\dot{q}^1, \dots, \dot{q}^n)\| = n - r$.

Under the stated assumption, rather than looking for a direct evaluation of the dynamical flow $Z$ over $\mathcal{A}$, it is more convenient to consider the push-forward $i_*(Z)$, viewed as the restriction to the submanifold $i(\mathcal{A})$ of a semi-spray $\breve{Z} = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + Z^i \frac{\partial}{\partial \dot{q}^i}$, defined in a neighbourhood of $i(\mathcal{A})$, and $i$-related to $Z$.

By definition, this implies the identification $\breve{Z}_{|i(z)} = i_{z*}(Z_z) \qquad \forall z \in \mathcal{A}$, showing that the evaluation of $\breve{Z}$ at each point $i(z)$ is indeed equivalent to the solution of the problem of motion.

A first set of conditions on $\breve{Z}$ comes from the relations

$$i^*\left(\breve{Z}(g^\sigma)\right) = Z(i^*(g^\sigma)) = 0 \qquad \sigma = 1, \ldots, n-r \tag{3.50}$$

expressing the requirement that the field $\breve{Z}$ be everywhere tangent to the hypersurface $i(\mathcal{A})$.

In addition to this, recalling the representation (3.12) of the Poincaré-Cartan 2-form on $\mathcal{A}$, we have the identification

$$Z \lrcorner\ \Omega = Z \lrcorner\ i^*\left(d\left(\frac{\partial \mathcal{T}}{\partial \dot{q}^k}\right) - \frac{\partial \mathcal{T}}{\partial q^k} dt\right) \wedge \tilde{\omega}^k = i^*\left[\breve{Z}\left(\frac{\partial \mathcal{T}}{\partial \dot{q}^k}\right) - \frac{\partial \mathcal{T}}{\partial q^k}\right] \tilde{\omega}^k \tag{3.51}$$

$\mathcal{T}$ denoting the holonomic kinetic energy (2.8).

On the other hand, in view of eqs. (3.9), (3.29), (3.30), Newton's 2nd law (3.40) implies the relation

$$\begin{aligned} Z \lrcorner\ \Omega &= \int_{\mathfrak{B}} Z(\mathbf{v}_\xi) \cdot d_c \mathbf{x}_\xi m(d\xi) \\ &= \int_{\mathfrak{B}} d_c \mathbf{x}_\xi \cdot [\mathbf{F}(d\xi) + \Phi(d\xi)] = (Q_k + \varphi_k)\tilde{\omega}^k \end{aligned} \tag{3.52}$$

$Q_k$ and $\varphi_k$ denoting the Lagrangian components of the active and reactive forces.

All this holds independently of any constitutive assumption concerning the nature of the constraints. In particular, on account of eqs. (3.2), (3.32), the 1-form (3.30) may be splitted into

$$\varphi_k \tilde{\omega}^k = \int_{\mathfrak{B}} d_c \mathbf{x}_\xi \cdot \Phi(d\xi) = \Lambda(\Phi) + \int_{\mathfrak{B}} d_\chi \mathbf{x}_\xi \cdot \Phi(d\xi). \tag{3.53}$$

The content of d'Alembert's principle (3.43) is then that, within the class of ideal constraints, the expression at the left-hand side of eq. (3.53) is always a Chetaev 1-form over $\mathcal{A}$.

Nothing further that the differential 1-forms $i^*(d_v g^\sigma)$ span the Chetaev bundle, the constitutive characterization (3.43) may therefore be rephrased as

$$\Phi \in \mathcal{H}(\mathcal{S} \times \mathcal{A}) \Leftrightarrow \varphi_k \tilde{\omega}^k = \lambda_\sigma i^*(d_v g^\sigma) = \lambda_\sigma i^* \left( \frac{\partial g^\sigma}{\partial \dot{q}^k} \right) \tilde{\omega}^k \tag{3.54}$$

the coefficients $\lambda_\sigma$ denoting $n - r$ a-priori unspecified functions over $\mathcal{A}$.

The rest is now entirely straightforward: by eqs. (3.51), (3.52), taking the characterization (3.54) into account, we get the $n$ independent relations

$$i^* \left[ \breve{Z} \left( \frac{\partial \mathcal{T}}{\partial \dot{q}^k} \right) - \frac{\partial \mathcal{T}}{\partial q^k} \right] = Q_k + \lambda_\sigma i^* \left( \frac{\partial g^\sigma}{\partial \dot{q}^k} \right). \tag{3.55}$$

These, together with eqs. (3.50), form a system of $2n - r$ equations for the $n$ components $Z^i$ of the semi-spray $\breve{Z}$ along $i(\mathcal{A})$, and for the $n - r$ coefficients $\lambda_\sigma$, thus providing once again a complete solution for the problem of motion.

*Example* 3.1. – As an illustration of the methods discussed above, we shall sketch the familiar problem of a rigid disc, rolling without sliding on a horizontal plane, under the action of given active forces. For simplicity, the positional constraints will be assumed to include the requirement that the plane of the disk remains vertical throughout the evolution.

Denoting by $m$ and $R$ the mass and radius of the disk, and introducing Lagrangian coordinates $x$, $y$, $\varphi$, $\theta$, expressing respectively the cartesian coordinates of the point of contact of the disk with the plane, the angle between the normal to the disk and the $x$ axis, and the angle of rotation of the disk around its axis, the holonomic kinetic energy is given by the equation

$$\mathcal{T} = \frac{1}{2} m \left( \dot{x}^2 + \dot{y}^2 \right) + \frac{1}{2} \left( \frac{mR^2}{4} \dot{\varphi}^2 + \frac{mR^2}{2} \dot{\theta}^2 \right) \tag{3.56}$$

while the rolling condition is summarized into the pair of relations

$$\dot{x} + R\dot{\theta} \sin \varphi = 0 \qquad \dot{y} - R\dot{\theta} \cos \varphi = 0.$$

Choosing the generalized velocities $z^1 := \dot{\varphi}$, $z^2 := \dot{\theta}$ as independent variables, one can easily write down the explicit representation of the constraint manifold $\mathcal{A}$

$$\begin{aligned} &\dot{x} = \psi^1(t, q, z) = -Rz^2 \sin \varphi \qquad &&\dot{y} = \psi^2(t, q, z) = Rz^2 \cos \varphi \\ &\dot{\varphi} = \psi^3(t, q, z) = z^1 &&\dot{\theta} = \psi^4(t, q, z) = z^2 \end{aligned}$$

as well as the (pull-back of the) contact 1-forms

$$\tilde{\omega}^1 = dx + Rz^2 \sin\varphi dt \quad \tilde{\omega}^2 = dy - Rz^2 \cos\varphi dt$$
$$\tilde{\omega}^3 = d\varphi - z^1 dt \qquad \tilde{\omega}^4 = d\theta - z^2 dt.$$

The Chetaev bundle is generated by the pair of independent 1-forms

$$\nu^1 = \tilde{\omega}^1 + R \sin\varphi \quad \tilde{\omega}^4 = dx + R \sin\varphi d\theta$$
$$\nu^2 = \tilde{\omega}^2 - R \cos\varphi \quad \tilde{\omega}^4 = dy - R \cos\varphi d\theta.$$

Taking the expression (3.56) for the holonomic kinetic energy into account, and denoting by $Q_x$, $Q_y$, $Q_\varphi$, $Q_\theta$ the Lagrangian components of the active forces, a straightforward comparison with eqs. (3.55) yields the equations of motion

$$m\frac{d^2x}{dt^2} = Q_x + \lambda_1 \quad m\frac{d^2y}{dt^2} = Q_y + \lambda_2$$
$$\frac{mR^2}{4}\frac{d^2\varphi}{dt^2} = Q_\varphi \qquad \frac{mR^2}{2}\frac{d^2\theta}{dt^2} = Q_\theta + \lambda_1 R \sin\varphi - \lambda_2 R \cos\varphi.$$

These determine the evolution of the system, as well as the Lagrangian components of the reactive forces, summarized into the Chetaev 1-form

$$\varphi_k\tilde{\omega}^k = \lambda_1\nu^1 + \lambda_2\nu^2 = \lambda_1\tilde{\omega}^1 + \lambda_2\tilde{\omega}^2 + (\lambda_1 \sin\varphi - \lambda_2 \cos\varphi)R\tilde{\omega}^4.$$

In order to write down the equations of motion in the intrinsic form, we notice that, on account of eqs. (3.6), (3.56), the (pull-back of the) kinetic energy and of the kinetic momenta $p_k$ over $\mathcal{A}$ are given by the expressions

$$T = i^*(\mathcal{T}) = \frac{1}{2}mR^2\left(z^2\right)^2 + \frac{1}{2}\left[\frac{mR^2}{4}\left(z^1\right)^2 + \frac{mR^2}{2}\left(z^2\right)^2\right]$$

and

$$p_1 = -mRz^2\, sin\varphi,\ p_2 = mRz^2 \cos\varphi,\ p_3 = \frac{mR^2}{4}z^1,\ p_4 = \frac{mR^2}{2}z^2.$$

In view of eq. (3.49) the required equations are then summarized into the system

$$\frac{dx}{dt} = -Rz^2 \sin\varphi \quad \frac{dy}{dt} = Rz^2 \cos\varphi$$
$$\frac{d\varphi}{dt} = z^1 \qquad \frac{d\theta}{dt} = z^2$$
$$\frac{dz^1}{dt} = \frac{4}{mR^2}Q_\varphi \quad \frac{dz^2}{dt} = \frac{2}{3mR^2}[-R \sin\varphi Q_x + R \cos\varphi Q_y + Q_\theta].$$

## APPENDIX A

Keeping the same notation as in section 2.3, the fibration

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{i} & j_1(\mathcal{V}_{n+1}) \\ \pi \downarrow & & \downarrow \pi \\ \mathcal{V}_{n+1} & = & \mathcal{V}_{n+1} \end{array} \tag{A.1}$$

will be said to be *integrable* (or more precisely, to identify a set of integrable kinetic constraints) if and only if, in a neighbourhood of each point $i(z)$, $z \in \mathcal{A}$, the submanifold $i(A) \subset j_1(\mathcal{V}_{n+1})$ admits at least one cartesian representation (2.27) of the special form

$$g^\sigma = \frac{\partial f^\sigma}{\partial q^k}\dot{q}^k + \frac{\partial f^\sigma}{\partial t} = 0 \quad \sigma = 1, \ldots, n-r \tag{A.2}$$

involving (the pull-back of) $n - r$ differentiable functions $f^\sigma(t,\, q^1, \ldots, q^n)$ defined on $\mathcal{V}_{n+1}$. Recalling the definition of the Chetaev bundle over $\mathcal{A}$, we prove

THEOREM A.1. – *The fibration (A.1) is integrable if and only if the ideal $\mathcal{J}$ generated by the module of Chetaev 1-forms over $\mathcal{A}$ is a differential ideal.*

*Proof.* – Necessity: the existence of a local representation of the form (A.2) for the submanifold $\mathcal{A}$ implies that the Chetaev bundle is generated locally by the family of 1-forms

$$\nu^\sigma = i^*(d_v g^\sigma) = \frac{\partial f^\sigma}{\partial q^k}\tilde{\omega}^k \qquad \sigma = 1, \ldots, n-r. \tag{A.3}$$

Comparison with eqs. (2.30), (A.2) provides the identifications

$$\nu^\sigma = df^\sigma - i^*\left(\frac{\partial f^\sigma}{\partial q^k}\dot{q}^k + \frac{\partial f^\sigma}{\partial t}\right)dt = df^\sigma$$

showing that, under the stated assumption, $\mathcal{J}$ is a differential ideal.

Sufficiency: according to Frobenius theorem, the assumption that $\mathcal{J}$ is differential ideal is mathematically equivalent to the assertion that the Chetaev bundle is generated locally by $n - r$ exact 1-forms $\nu^\sigma = df^\sigma$. Since – by definition – the latter are automatically contact 1-forms, the condition $Z \lrcorner\ df^\sigma = 0$ for all dynamical flows $Z$ implies the validity of the relations

$$\frac{\partial f^\sigma}{\partial z^A} = 0 \quad \Rightarrow \quad f^\sigma = f^\sigma(t,\, q^1, \ldots, q^n) \tag{A.4}$$

$$\left(\frac{\partial}{\partial t} + \psi^k\frac{\partial}{\partial q^k}\right) \lrcorner\ df^\sigma = 0 \quad \Rightarrow \quad i^*\left(\frac{\partial f^\sigma}{\partial q^k}\dot{q}^k + \frac{\partial f^\sigma}{\partial t}\right) = 0. \tag{A.5}$$

On the basis of eqs. (A.4), (A.5), a straightforward dimensionality argument shows that, under the stated assumption, the submanifold $\mathcal{A}$ is represented locally in the form

$$\frac{\partial f^\sigma}{\partial q^k}\dot{q}^k + \frac{\partial f^\sigma}{\partial t} = 0 \qquad \sigma = 1, \ldots, n - r$$

*i.e.* it satisfies the integrability requirement (A.2). □

## REFERENCES


[1] J. L. Lagrange, *Mécanique Analytique*, (Veuve Desaint, Paris, 1788), in *Œuvres de Lagrange*, Gauthier-Villars, Paris, 1867-1892.

[2] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, MIR, Moscow, 1975.

[3] T. Levi-Civita, U. Amaldi, *Lezioni di Meccanica Razionale*, Zanichelli, Bologna, 1984, first edition 1922, second edition 1929.

[4] K. Gauss, Uber ein neues allgemeines Grundgesetz der Mechanik, *Crelle J. reine Math.*, Vol. **4**, 1829, pp. 232.

[5] E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, (4th ed.), Cambridge University Press, Cambridge, 1959.

[6] G. Hamel, *Theoretische Mechanik. Eine einheitliche Einfuhrung in die gesamte Mechanik*, Springer-Verlag, Berlin, 1949.

[7] A. Sommerfeld, *Mechanics, Lectures on Theoretical Physics*, Vol. **1**, Academic Press, New York, San Francisco, London, 1942.

[8] D. J. Saunders, *The Geometry of Jet Bundles*, London Mathematical Society, Lecture Note Series 142, Cambridge University Press, 1989.

[9] M. Crampin, G. E. Prince and G. Thompson, A Geometrical Version of the Helmholtz Conditions in Time-Dependent Lagrangian Dynamics, *J. Phys. A: Math. Gen.*, Vol. **17**, 1984, pp. 1437-1447.

[10] F. Cantrijn, J. F. Cariñena, M. Crampin and L. A. Ibort, Reduction of Degenerate Lagrangian Systems, *J. G. P.*, Vol. **3**, 1986, pp. 353-400.

[11] W. Sarlet, F. Cantrijn and M. Crampin, A New Look at Second Order Equations and Lagrangian Mechanics, *J. Phys. A: Math. Gen.*, Vol. **17**, 1984, pp. 1999-2009.

[12] E. Massa and E. Pagani, Classical Dynamics of non-holonomic systems: a geometric approach, *Ann. Inst. Henri Poicaré*, Vol. **55**, 1991, pp. 511-544.

[13] E. Massa and E. Pagani, Jet bundle geometry, dynamical connections, and the inverse problem of Lagrangian Mechanics, *Ann. Inst. Henri Poicarré*, Vol. **61**, 1994, pp. 17-62.

[14] N. G. Chetaev, O printsipe Gaussa (On the Gauss Principle), *Izd. Fiz.-Matem. Obshch. pri Kazansk. Univ.*, Ser. 3, Vol. **6**, 1932-1933, pp. 323-326, *Izvestia of the Physico-Mathematical Society of Kazan University*, Series 3, Vol. **6**, 1932-1933, pp. 62-67.

[15] N. G. Chetaev, Papers on Analytical Mechanics. On the Gauss Principles, Moscow, *Izd. Akad. Nauk SSSR*, 1962.

[16] V. V. Rumiantsev, Certain Variational Principles of Mechanics, in *Adv. Theor. Appl. Mech.*, A. Ishlinsky and F. Chernousko Eds., English translation, MIR Publishers, Moscow, 1981.

[17] I. I. Neimark and N. A. Fufaev, Dynamics of Non-holonomic Systems, *Translations of Mathematical Monographs*, Vol. **33**, American Mathematical Society, Providence, Rhode Island, 1972.

[18] Y. Pironneau, Sur les liaisons non holonômes, non linéaires déplacements virtuels à travail nul, conditions de Chetaev, Proceedings of the IUTAM-ISIMM Symposium on Modern Developments in Analytical Mechanics, Academy of Sciences of Turin, Turin, June 7-11, 1982, *Atti della Accademia delle Scienze di Torino*, Supplemento al Vol. **117**, (1983), p. 671-686.

[19] M. de LEON and P. R. RODRIGUES, *Methods of Differential Geometry in Analytical Mechanics*, North Holland, Amsterdam, 1989.
[20] W. SARLET, F. CANTRIJN and D. J. SAUNDERS, A Geometrical Framework for the Study of Non-holonomic Lagrangian Systems, *J. Phys. A.: Math. Gen.*, Vol. **28**, 1995, pp. 3253-3268.
[21] C. GODBILLON, *Géométrie Différentielle et Mécanique Analytique*, Hermann, Paris, 1969.
[22] R. ABRAHAM and J. E. MARSDEN, *Foundations of Mechanics*, The Benjamin/Cumming Publishing Company, Reading (Massachussets), 1978.
[23] R. B. ASH, *Measure, Integration and Functional Analysis*, Academic Press, New York and London, 1972.
[24] K. YOSIDA, *Functional Analysis*, Springer-Verlag, Berlin and New York, 1968.
[25] F. CARDIN and G. ZANZOTTO, On Constrained Mechanical Systems: D'Alembert and Gauss' Principles, *J. Math. Phys.*, Vol. **30**, 1989, pp. 1473-1479.

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