ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 66, nº 1 (1997), p. 109-136
<http://www.numdam.org/item?id=AIHPA_1997__66_1_109_0>

Ann. Inst. Henri Poincaré,

Vol. 66, n° 1, 1997, p. 109-136. *Physique théorique*

Gauge symmetries of an extended phase space for Yang-Mills and Dirac fields

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ABSTRACT. – The extended phase space $\mathbf{P}$ for Yang-Mills and Dirac fields in the Minkowski space is a Sobolev space of Cauchy data. We prove in $\mathbf{P}$ the existence and uniqueness theorem for the evolution equations. We show that the Lie algebra $gs(\mathbf{P})$ of infinitesimal gauge symmetries of $\mathbf{P}$ is a Hilbert-Lie algebra carrying a Beppo Levi topology. The connected group $GS(\mathbf{P})$ of gauge symmetries of $\mathbf{P}$ with the Lie algebra $gs(\mathbf{P})$ is a Hilbert-Lie group acting properly in $\mathbf{P}$. We construct a closed connected subgroup $GS(\mathbf{P})_0$ of $GS(\mathbf{P})$, acting in $\mathbf{P}$ with a momentum map $\mathcal{J}_0$, such that the constraint equations are given by $\mathcal{J}_0 = 0$. The action of $GS(\mathbf{P})_0$ in $\mathbf{P}$ is free and proper.

Key words: Gauge group, Hilbert Lie group, Yang-Mills theory, Cauchy problem, gauge fixing.

RÉSUMÉ. – L'espace de phases prolongé P pour le champ de Yang-Mills couplé avec le champ de Dirac sur l'espace-temps de Minkowski est un espace Sobolev des données de Cauchy. Dans P, nous démontrons un théorème d'existence et d'unicité pour les équations d'évolution.

On démontre que l'algèbre $gs(P)$ des symétries infinitésimales de jauge de P est une algèbre de Hilbert-Lie avec une topologie de Beppo Levi. Le groupe connexe $GS(P)$ de symétries de jauge de P, avec l'algèbre de Lie $gs(P)$, est un groupe de Hilbert-Lie et son opération sur P est propre.

Nous construisons un sous-groupe fermé $GS(P)_0$ de $GS(P)$, opérant sur P avec le moment $\mathcal{J}_0$, tel que les équations de contrainte soient données par $\mathcal{J}_0 = 0$. L'opération de $GS(P)_0$ sur P est libre et propre.

1. INTRODUCTION

The aim of this paper is to study the group of gauge symmetries for minimally interacting Yang-Mills and Dirac fields. Since we want to cast the gauge theory in a Hamiltonian form, we have to specify a gauge condition. This gauge condition is needed to determine the evolution of the scalar potential of the Yang-Mills fields. The extended phase space **P** of the theory consists of all Cauchy data which admit a (finite time) existence and uniqueness theorem for the evolution equations. The gauge symmetries considered here are the gauge transformations which preserve the gauge condition and the extended phase space. They give rise to the conservation laws and the constraints. Moreover, the space of solutions of the field equations with Cauchy data in the extended phase space **P** is uniquely determined by the constraints. Other gauge transformations, which are not gauge symmetries considered here, intertwine equivalent Hamiltonian descriptions of the theory.

We study the Yang-Mills-Dirac system in the Minkowski space $M^4 = \mathbb{R} \times \mathbb{R}^3$. The the structure group G is assumed to be a compact classical group, that is it is a compact subgroup of the $Gl(k, \mathbb{R})$. This implies that G is a closed submanifold of the space $gl(k, \mathbb{R})$ of $k \times k$ matrices. Let $\{T^a\}$ be a basis of the structure algebra $\mathfrak{g}$, and $[T^a, T^b] = f_c^{ab} T^c$ the Lie bracket. The usual $(3+1)$ splitting of space-time yields a splitting of the Yang-Mills field $A_\mu = (A_0, A)$ into the scalar potential A_0 and the vector potential $A = A_i dx^i$. It leads to a representation of the field strength $F_{\mu\nu}$ in terms of the "electric" field E and the "magnetic" field

$$B = \operatorname{curl} A + [A \times, A]. \tag{1.1}$$

We use the Euclidean metric in $\mathbb{R}^3$ to identify vector fields and forms, and $\times$ to denote the cross product. The field equations split into the evolution equations

$$\partial_t A = E + \operatorname{grad} A_0 - [A_0, A], \tag{1.2}$$
$$\partial_t E = -\operatorname{curl} B - [A\times, B] - [A_0, E] + J, \tag{1.3}$$
$$\partial_t \Psi = -\gamma^0(\gamma^j \partial_j + im + \gamma^j A_j + \gamma^0 A_0)\Psi, \tag{1.4}$$

and the constraint equation

$$\operatorname{div} E + [A; E] = J^0. \tag{1.5}$$

Here A, E, and B are treated as time dependent $\mathfrak{g}$-valued vector fields on $\mathbb{R}^3$, and Ψ is a time dependent spinor field with values in the space $\mathbb{R}^k$ of the fundamental representation of G. Moreover $[A; E]$ means the Lie bracket contracted over the vector indices, and

$$J^0 = \Psi^\dagger (I \otimes T^a)\Psi T_a, \qquad J^k = \Psi^\dagger (\gamma^0\gamma^k \otimes T^a)\Psi T_a \,. \tag{1.6}$$

Gauge transformations act on the fields (A_0, A, E, Ψ) via

$$\begin{aligned} &A_0 \mapsto \varphi A_0 \varphi^{-1} + \varphi \partial_t \varphi^{-1}, \\ &A \mapsto \varphi A \varphi^{-1} + \varphi \operatorname{grad} \varphi^{-1}\,, \quad E \mapsto \varphi E \varphi^{-1}\,, \quad \Psi \mapsto \varphi \Psi, \end{aligned} \tag{1.7}$$

where $\varphi(t)$ is a curve of maps from $\mathbb{R}^3$ to the structure group G.

Since the scalar potential A_0 does not appear as an independent degree of freedom in (1.2) through (1.4), it can be fixed by a choice of an appropriate gauge transformation. The most common gauge fixing for studies of Yang-Mills fields as a dynamical system are the temporal gauge $A_0 = 0$, *cf.* [1-3], the Lorentz gauge $\partial^\mu A_\mu = 0$, [4], or the Coulomb gauge $\operatorname{div} A = 0$, [5]. Here we use a different gauge condition

$$\Delta A_0 \;=\; -\operatorname{div} E \quad \text{and} \quad \int_{\mathbb{R}^3} A_0 (1 + |x|^2)^{-2} d_3 x = 0 \,. \tag{1.8}$$

It enables us to deal with the evolution equations also off the constraint set and allows for static solutions with $A = 0$ and $E = -\operatorname{grad} A_0$, where A_0 is proportional to a fixed direction in the structure algebra $\mathfrak{g}$, and behaves as $|x|^{-1}$ when $|x| \to \infty$. We show that this gauge fixing can be achieved by an appropriate gauge transformation.

In order to specify the extended phase space of the theory we first prove the finite time existence and uniqueness theorem for the evolution equations (1.2) through (1.4) in

$$\mathbf{P} = \{(A, E, \Psi) \in H^2(\mathbb{R}^3, \mathfrak{g}) \times H^1(\mathbb{R}^3, \mathfrak{g}) \times H^2(\mathbb{R}^3, \mathbb{R}^k)\} \,. \quad (1.9)$$

Here $H^s(\mathbb{R}^3, \mathfrak{g})$ and $H^s(\mathbb{R}^3, \mathbb{R}^k)$ are Sobolev spaces of the $\mathfrak{g}$-valued forms and $\mathbb{R}^k$-valued spinors, respectively, which are square integrable over $\mathbb{R}^3$ together with their partial derivatives up to order s, [6]. Moreover we show that the constraint (1.5) is preserved under the time evolution in $\mathbf{P}$. For the sake of completeness we show also the following regularity result. If the initial data for the Yang-Mills-Dirac system are in

$$\begin{aligned}\mathbf{P}^s = \{(A, E, \Psi) | A \in H^{s+1}(\mathbb{R}^3, \mathfrak{g}),\\ E \in H^s(\mathbb{R}^3, \mathfrak{g}), \Psi \in H^{s+1}(\mathbb{R}^3, \mathbb{R}^k)\},\end{aligned} \quad (1.10)$$

then the solution curve is in $\mathbf{P}^s$. In order to take account into the largest class of classical solutions, we choose $\mathbf{P}$ as the extended phase space of the system. The mathematically important problem of the infinite time existence of solutions, *cf.* [3-5], is beyond the scope of this paper.

Having established an admissible extended phase space $\mathbf{P}$, we can turn to the study of the group of gauge transformations, acting via (1.7), which preserve $\mathbf{P}$. We denote by $gs(\mathbf{P})$ the Lie algebra of all time independent infinitesimal gauge transformations which preserve $\mathbf{P}$. We prove that it is a Hilbert-Lie algebra which carries a Beppo Levi topology with the norm

$$\|\xi\|^2_{B^3} := \int_{D_1} |\xi|^2 d_3x + \|\text{grad}\,\xi\|^2_{H^2} \,, \quad (1.11)$$

where D_1 denotes the unit ball in $\mathbb{R}^3$ centered at 0. This algebra admits a splitting

$$gs(\mathbf{P}) = gs(\mathbf{P})_0 \oplus \mathfrak{g}, \quad (1.12)$$

where the subalgebra $gs(\mathbf{P})_0$ is the completion of the space of smooth compactly supported maps $\xi : \mathbb{R}^3 \to \mathfrak{g}$ in the topology given by (1.11), *cf.* [7].

We construct a connected Hilbert-Lie group $GS(\mathbf{P})$ of gauge symmetries for the Yang-Mills and Dirac fields in the phase space $\mathbf{P}$ with Lie algebra $gs(\mathbf{P})$. It carries the uniform topology induced by topology of $gs(\mathbf{P})$. We prove that the action of $GS(\mathbf{P})$ in $\mathbf{P}$ is continuous and proper.

We prove that for each curve $(A(t), E(t), \Psi(t)) \in \mathbf{P}$ satisfying the constraint equation (1.5) and each $A_0(t) \in \mathcal{B}^3(I\!R^3, \mathfrak{g})$ there exists a gauge transformation such that the transformed fields satisfy the gauge condition (1.8) on a finite time interval.

The extended phase space $\mathbf{P}$ is weakly symplectic. The action of $GS(\mathbf{P})$ on $\mathbf{P}$ is Hamiltonian with an equivariant momentum map $\mathcal{J}$. The vanishing of the restriction of $\mathcal{J}$ to the subalgebra $gs(\mathbf{P})_0$ gives rise to the constraints of the theory. More precisely, if $\mathbf{C}$ is the constraint set of the theory, i.e. the set of all $(A, E, \Psi) \in \mathbf{P}$ satisfying Eq. (1.5), then subalgebra $gs(\mathbf{P})_0$ can be given a geometric interpretation as

$$gs(\mathbf{P})_0 = \{\xi \in gs(\mathbf{P}) \mid \langle \mathcal{J}(\mathbf{A}, \mathbf{E}, \Psi) | \xi \rangle = 0 \quad \forall (\mathbf{A}, \mathbf{E}, \Psi) \in \mathbf{C}\}. \quad (1.13)$$

We construct a connected Banach-Lie subgroup $GS(\mathbf{P})_0$ of $GS(\mathbf{P})$ with Lie algebra $gs(\mathbf{P})_0$ and prove that it acts freely and properly in $\mathbf{P}$. Conversely, the constraint set is shown to be the zero level of the momentum map $\mathcal{J}_0$ for the action of $GS(\mathbf{P})_0$ in $\mathbf{P}$,

$$\mathbf{C} = \mathcal{J}_0^{-1}(0). \quad (1.14)$$

It follows from Eq. (1.14) that the natural choice for the reduced phase space is the space $\check{\mathbf{P}} = \mathbf{C}/GS(\mathbf{P})_0$ of $GS(\mathbf{P})_0$ orbits in $\mathbf{C}$. If $\mathbf{C}$ were a submanifold of $\mathbf{P}$, the reduced phase space $\check{\mathbf{P}}$ would be a symplectic (Hausdorff) manifold with an exact symplectic form. The structure of the constraint set and of the reduced phase space will be studied elsewhere.

This paper is organized as follows. In Section 2 we analyse our gauge condition, and in Section 3 we prove the finite time existence and uniqueness theorem for the evolution equations with Cauchy data in $\mathbf{P}$. Section 4 is devoted to the study of the gauge symmetry group of $\mathbf{P}$. In Section 5 we prove that the gauge condition (1.8) can be achieved by a gauge transformation. Constraints and their reduction are discussed in Section 6. In Appendix A we consider some decomposition results and estimates for Beppo Levi spaces.

The authors would like to thank L. Bos and U. Wernick for useful comments on decompositions of Beppo Levi spaces, and to the referee for valuable suggestions.

2. THE GAUGE CONDITION

Let $H^s(I\!R^3, \mathfrak{g})$ be the Sobolev space of $\mathfrak{g}$-valued vector fields on $I\!R^3$. Each each $X \in H^s(I\!R^3, \mathfrak{g})$ allows for a Helmholtz decomposition

$$X = X^L + X^T \text{ such that } \operatorname{curl} X^L = 0 \text{ and } \operatorname{div} X^T = 0. \quad (2.1)$$

The components $X^L \in H^s(I\!R^3, \mathfrak{g})$ and $X^T \in H^s(I\!R^3, \mathfrak{g})$ are uniquely determined by $\operatorname{div} X$ and $\operatorname{curl} X$, and called the longitudinal and transverse components of X, respectively. For details see the Appendix. Splitting the gauge fields via (2.1) we obtain

$$\begin{aligned}\partial_t A^L &= E^L + \operatorname{grad} A_0 - [A_0, A]^L, \\ \partial_t E^L &= -[A\times, B]^L - [A_0, E]^L + J^L, \\ \partial_t A^T &= E^T - [A_0, A]^T, \\ \partial_t E^T &= -\operatorname{curl} B - [A\times, B]^T - [A_0, E]^T + J^T .\end{aligned} \tag{2.2}$$

The gauge condition (1.8) is chosen in such a way, that the longitudinal component E^L and the gradient of the scalar potential $\operatorname{grad} A_0$ cancel each other. In order to prove that this gauge can be achieved we need the Beppo Levi spaces $BL_m(L^2(I\!R^3, \mathfrak{g}))$, which are defined as the spaces of $\mathfrak{g}$-valued distributions on $I\!R^3$ with square integrable partial derivatives of order m. For the intersection of s Beppo Levi spaces we write

$$\mathcal{B}^s(I\!R^3, \mathfrak{g}) := \bigcap_{m=1}^{s} BL_m(L^2(I\!R^3, \mathfrak{g})) . \tag{2.3}$$

These are Banach spaces with respect to the norm

$$\int_{D_1} |\xi| d_3x + \|\operatorname{grad} \xi\|_{H^{s-1}} , \tag{2.4}$$

cf. [8]. The integral over the unit ball D_1 is essential for (2.4) to define a norm. However, each L^p norm on D_1 yields an equivalent norm. In particular we may choose as in (1.11) the integral of $|\xi|^2$ to define the norm $\|\xi\|_{\mathcal{B}^3}$.

THEOREM 2.1. – *For each $E \in H^1(I\!R^3, \mathfrak{g})$ there exists a unique scalar potential $A_0 \in \mathcal{B}^2(I\!R^3, \mathfrak{g})$ obeying the gauge condition*

$$\operatorname{grad} A_0 = -E^L \quad \textit{and} \quad \int_{I\!R^3} \rho^{-4} A_0 \, d_3x = 0 \quad \textit{where} \quad \rho = \sqrt{1 + |x|^2} . \tag{2.5}$$

The potential $A_0 \in \mathcal{B}^2(I\!R^3, \mathfrak{g})$ satisfies the estimate

$$\|A_0\|_{\mathcal{B}^2} \leq C \|E\|_{H^1}. \tag{2.6}$$

If $(A, E, \Psi) \in \mathbf{C}$ satisfy the constraint equation (1.5), *then $A_0 \in \mathcal{B}^3(I\!R^3, \mathfrak{g})$, and*

$$\|A_0\|_{\mathcal{B}^3} \leq C\big(\|E\|_{H^1} \|A\|_{H^2} + \|\Psi\|^2_{H^1}\big) . \tag{2.7}$$

Proof. – Let $H^2_{-2}(I\!R^3, \mathfrak{g})$ and $H^1_{-1}(I\!R^3, \mathfrak{g})$ denote the ρ-weighted Sobolev spaces, *cf.* (A.6). It is shown in [9], that the Laplace operator $\Delta : H^2_{-2}(I\!R^3, \mathfrak{g}) \to L^2(I\!R^3, \mathfrak{g})$ is onto, Fredholm and has kernel $\ker(\Delta) = \mathfrak{g}$. Therefore, for each $\chi \in L^2(I\!R^3, \mathfrak{g})$, there exists a unique $\Phi \in H^2_{-2}(I\!R^3, \mathfrak{g})$ such that

$$\Delta\Phi = \chi \qquad \text{and} \qquad \int_{I\!R^3} \rho^{-4}(\Phi \cdot \xi) d_3x = 0 \qquad \forall \xi \in \mathfrak{g}. \tag{2.8}$$

By Fredholmness of Δ there exists a constant C independent of χ such that

$$\|\Phi\|^2_{H^2_{-2}} \leq C\|\chi\|^2_{L^2}. \tag{2.9}$$

Given $E \in H^1(I\!R^3, \mathfrak{g})$ we consider $\chi_E = -\operatorname{div} E \in L^2(I\!R^3, \mathfrak{g})$. The corresponding solution of (2.8) we denote by Φ_E. Then the vector field $Y_E := \operatorname{grad} \Phi_E + E^L$ is harmonic, that is $\operatorname{curl} Y_E = 0$ and $\operatorname{div} Y_E = 0$. By the estimates (A.4) and (A.5),

$$\sum_{j=1}^{3} \|\partial_j Y_E\|^2_{L^2} \leq \|\operatorname{curl} Y_E\|^2_{L^2} + \|\operatorname{div} Y_E\|^2_{L^2} , \tag{2.10}$$

see also [10]. Therefore Y_E is constant. Since, by construction, Y_E has a finite norm in $H^1_{-1}(I\!R^3, \mathfrak{g})$, this implies that $Y_E = 0$, *cf.* (A.7). Therefore we may set $A_0 = \Phi_E$ and have constructed the unique solution of the problem (2.5).

From the a-priori estimate (2.9) and (A.6) we conclude that

$$\frac{1}{4}\Big(\int_{D_1} |A_0|\, d_3x\Big)^2 \leq \Big(\int_{I\!R^3} \rho^{-2} |A_0|\, d_3x\Big)^2 \leq \|A_0\|^2_{H^2_{-2}} \leq C\|\operatorname{div} E\|^2_{L^2} . \tag{2.11}$$

Since $\|\operatorname{grad} A_0\|^2_{H^1} \leq \|E\|^2_{H^1}$, this proves (2.6). Now assume $(A, E, \Psi) \in$ **C**, that is

$$\begin{gathered} \operatorname{div} E + [A; E] = J^0 \\ \text{with} \\ (A, E, \Psi) \in H^2(I\!R^3, \mathfrak{g}) \times H^1(I\!R^3, \mathfrak{g}) \times H^2(I\!R^3, I\!R^k) \end{gathered} \tag{2.12}$$

and J^0 is determined from Ψ by (1.6). By standard Sobolev estimates

$$\|\operatorname{div} E\|_{H^1} \leq C\big(\|A\|_{H^2}\|E\|_{H^1} + \|\Psi\|^2_{H^2}\big) . \tag{2.13}$$

On the other hand we infer from (A.4) that

$$\|\operatorname{grad} A_0\|^2_{H^2} = \|E^L\|^2_{H^2} \leq \|\operatorname{div} E\|^2_{H^1} + \|E\|^2_{H^1} . \tag{2.14}$$

This proves the estimate (2.7). Q.E.D.

This particular gauge fixing allows for static solutions of the Yang-Mills equations (2.2) with $A = 0$ and $E = -\mathrm{grad}\, A_0$, where A_0 is proportional to a fixed direction in the structure algebra $\mathfrak{g}$, and behaves as $|x|^{-1}$ when $|x| \to \infty$. On the constraint set **C** this gauge can be achieved by a gauge transformation

$$A_0 \mapsto \varphi A_0 \varphi^{-1} + \varphi \partial_t \varphi^{-1} \tag{2.15}$$

where $\varphi(t)$ is a curve of maps from $\mathbb{R}^3$ to the structure group G. This will be considered in detail in Section 5.

3. Existence and uniqueness results

Using the gauge fixing of Theorem 2.1 and linearizing the Yang-Mills and Dirac equations (2.2) we obtain

$$\frac{d}{dt}\begin{bmatrix} A^L \\ E^L \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{3.1}$$

$$\frac{d}{dt}\begin{bmatrix} A^T \\ E^T \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \Delta & 0 \end{bmatrix}\begin{bmatrix} A^T \\ E^T \end{bmatrix} =: \mathcal{T}(A^T, E^T), \tag{3.2}$$

$$\frac{d}{dt}\Psi = -\gamma^0(\gamma^j \partial_j + im)\Psi =: \mathcal{D}\Psi . \tag{3.3}$$

We shall study these linear equations in the Hilbert spaces

$$\mathbf{H}_L = \{(A^L, E^L) \in H^2(\mathbb{R}^3, \mathfrak{g}) \times H^1(\mathbb{R}^3, \mathfrak{g})\}, \tag{3.4}$$

$$\mathbf{H}_T = \{(A^T, E^T) \in H^1(\mathbb{R}^3, \mathfrak{g}) \times L^2(\mathbb{R}^3, \mathfrak{g})\}, \tag{3.5}$$

$$\mathbf{H}_D = \{\Psi \in L^2(\mathbb{R}^3, \mathbb{R}^k)\}. \tag{3.6}$$

PROPOSITION 3.1. – *The operator $\mathcal{T}$, defined by (3.2), with domain*

$$\mathbf{D}_T = \{(A^T, E^T) \in H^2(\mathbb{R}^3, \mathfrak{g}) \times H^1(\mathbb{R}^3, \mathfrak{g})\} \tag{3.7}$$

is the generator of a continuous group $\exp(t\mathcal{T})$ *of transformations in* $\mathbf{H}_T$.

Proof. – By standard arguments, the operator

$$\widetilde{\mathcal{T}} = \begin{bmatrix} 0 & 1 \\ \Delta & 0 \end{bmatrix} \quad \text{with domain}\, H^2(\mathbb{R}^3, \mathfrak{g}) \times H^1(\mathbb{R}^3, \mathfrak{g}) \tag{3.8}$$

is dissipative, and satisfies

$$\mathrm{range}(\widetilde{\mathcal{T}} - \lambda I) = H^1(\mathbb{R}^3, \mathfrak{g}) \times L^2(\mathbb{R}^3, \mathfrak{g}) \quad \text{and} \quad \ker(\widetilde{\mathcal{T}} - \lambda I) = \{0\} \tag{3.9}$$

for $\lambda > 0$. In fact, $\widetilde{\mathcal{T}}$ is the infinitesimal generator corresponding to the wave equation, [11]. We have to show that $\exp(t\widetilde{\mathcal{T}})$ preserves the Hilbert space $\mathbf{H}_T$ of transverse fields. Given $(X^T, Y^T) \in \mathbf{H}_T$ we consider $(\widetilde{A}, \widetilde{E})$, satisfying the equation

$$(\widetilde{\mathcal{T}} - \lambda I)(\widetilde{A}, \widetilde{E}) = (X^T, Y^T) . \tag{3.10}$$

Since Δ maintains the Helmholtz decomposition $\widetilde{A} = \widetilde{A}^T + \widetilde{A}^L$, this implies that

$$(\widetilde{A}^L, \widetilde{E}^L) \in \ker(\widetilde{\mathcal{T}} - \lambda I) = \{0\}. \tag{3.11}$$

Therefore, since $\mathcal{T} = \widetilde{\mathcal{T}}|_{\mathbf{D}_T}$, we have

$$\operatorname{range}(\mathcal{T} - \lambda I) = \operatorname{range}(\widetilde{\mathcal{T}} - \lambda I)|_{\mathbf{D}_T} = \mathbf{H}_T. \tag{3.12}$$

The Lumer-Phillips theorem implies that $\mathcal{T}$ generates a one parameter group of continuous transformations $\exp(t\mathcal{T})$ in $\mathbf{H}_T$. Q.E.D.

PROPOSITION 3.2.
(i) *The operator* $\mathcal{D}$, *with domain*

$$\mathbf{D}_D = \{\Psi \in H^1(\mathbb{R}^3, \mathbb{R}^k)\} \tag{3.13}$$

is the generator of a continuous group of (unitary) transformations $\exp(t\mathcal{D})$ *in* $\mathbf{H}_D$.

(ii) $\exp(t\mathcal{D})$ *restricts to a group of continuous transformations in* $H^2(\mathbb{R}^3, \mathbb{R}^k)$.

Proof. – (i) It is known, [12], that the operator $\mathcal{D}$ with domain $\mathbf{D}_D$ is skew-adjoint in $\mathbf{H}_D$. Thus, $\mathcal{D}$ generates a group $\exp(t\mathcal{D})$ of unitary transformations in $\mathbf{H}_D$.

(ii) The operator $\mathcal{D} : H^1(\mathbb{R}^3, \mathbb{R}^k) \to L^2(\mathbb{R}^3, \mathbb{R}^k)$ is continuous, and its square

$$\mathcal{D}^2 = \Delta - m^2 : H^2(\mathbb{R}^3, \mathbb{R}^k) \longrightarrow L^2(\mathbb{R}^3, \mathbb{R}^k) \tag{3.14}$$

is continuous and elliptic. With the elliptic a-priori estimate this implies that

$$C_1 ||\mathcal{D}^2\Psi||_{L^2} \le ||\Psi||_{H^2} \le C_2\big(||\mathcal{D}^2\Psi||_{L^2} + ||\Psi||_{H^1}\big) . \tag{3.15}$$

Moreover, from the identity $\gamma^i\gamma^k = -\delta^{ik} + \frac{1}{2}[\gamma^i, \gamma^k]$, we obtain

$$||\mathcal{D}\Psi||^2_{L^2} = \sum_{j=1}^{3} ||\partial_j\Psi||^2_{L^2} - A(\Psi) + m^2||\Psi||^2_{L^2} , \tag{3.16}$$

where

$$A(\Psi) = \frac{1}{2} \sum_{j,k=1}^{3} \langle [\gamma^j, \gamma^k] \partial_k \Psi, \partial_j \Psi \rangle_{L^2}. \tag{3.17}$$

Integration by parts shows that $A(\Psi)$ vanishes for all Ψ in $C^\infty(I\!R^3, I\!R^k) \cap H^1(I\!R^3, I\!R^k)$. Thus, by a density argument, $A(\Psi) = 0$ for all $\Psi \in H^1(I\!R^3, I\!R^k)$. Therefore

$$C_3 \|\mathcal{D}\Psi\|_{L^2} \leq \|\Psi\|_{H^1} \leq C_4 \|\mathcal{D}\Psi\|_{L^2} \tag{3.18}$$

and

$$\|\Psi\|_{H^2} \leq C_5 \left(\|\mathcal{D}^2\Psi\|_{L^2} + \|\mathcal{D}\Psi\|_{L^2} \right) . \tag{3.19}$$

Since $\exp(t\mathcal{D})$ is a unitary operator, which commutes on the domain $\mathbf{D}_D$ with its generator $\mathcal{D}$, *cf.* [13], we can estimate for all $\Psi \in H^2(I\!R^3, I\!R^k)$:

$$\begin{aligned} \|\exp(t\mathcal{D})\Psi\|_{H^2} &\leq C_5 \left(\|\mathcal{D}^2 \exp(t\mathcal{D})\Psi\|_{L^2} + \|\mathcal{D} \exp(t\mathcal{D})\Psi\|_{L^2} \right) \\ &= C_5 \left(\|\mathcal{D}^2\Psi\|_{L^2} + \|\mathcal{D}\Psi\|_{L^2} \right) \leq C_6 \|\Psi\|_{H^2}. \end{aligned} \tag{3.20}$$

Hence $\exp(t\mathcal{D})$ acts continuously in the Hilbert space $H^2(I\!R^3, I\!R^k)$. Q.E.D.

COROLLARY 3.3. – *The linear operator*

$$\mathcal{S} = 0 \oplus \mathcal{T} \oplus \mathcal{D} \tag{3.21}$$

with domain $\mathbf{D} = \mathbf{H}_L \times \mathbf{D}_T \times \mathbf{D}_D$, *corresponding to the dynamical system* (3.1), (3.2) *and* (3.3), *generates a one parameter group* $\exp(t\mathcal{S})$ *of continuous transformations in* $\mathbf{H} = \mathbf{H}_L \times \mathbf{H}_T \times \mathbf{H}_D$. *The space*

$$\mathbf{P} = \{(A, E, \Psi) | A \in H^2(I\!R^3, \mathfrak{g}),\ E \in H^1(I\!R^3, \mathfrak{g}),\ \Psi \in H^2(I\!R^3, I\!R^k)\} \tag{3.22}$$

is preserved by the action of $\exp(t\mathcal{S})$ *in* $\mathbf{H}$. *The restriction of* $\exp(t\mathcal{S})$ *to* $\mathbf{P}$ *is a continuous one parameter group* $\mathcal{U}(t)$ *of continuous transformations in* $\mathbf{P}$,

$$\mathcal{U}(t) = \exp(t\mathcal{S})|_{\mathbf{P}} : \mathbf{P} \longrightarrow \mathbf{P} \ \ \textit{such that} \ \ (A, E, \Psi) \mapsto \mathcal{U}(t)(A, E, \Psi) \tag{3.23}$$

and $\mathcal{U}(t)(A, E, \Psi)$ *is the unique solution of the linear evolution equations* (3.1), (3.2) *and* (3.3) *with initial condition* (A, E, Ψ).

Having solved the linearized problem, we can rewrite the coupled nonlinear equations (1.2), (1.3) and (1.4) in an abstract form as

$$\frac{d}{dt}(A, E, \Psi)_t = \mathcal{S}(A, E, \Psi)_t - \mathcal{F}((A, E, \Psi)_t) . \tag{3.24}$$

Here $\mathcal{F}$ describes the nonlinearity of the theory and is given by

$$\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3 \text{ where}$$
$$\begin{cases} \mathcal{F}_1(A, E, \Psi) &= (0;\ [A\times, B] + \mathrm{curl}\,[A\times, A];\ 0) \\ \mathcal{F}_2(A, E, \Psi) &= (0;\ -J;\ \gamma^0\gamma^j A_j \Psi) \\ \mathcal{F}_3(A, E, \Psi) &= ([A_0, A];\ [A_0, E];\ [A_0, \Psi])\ . \end{cases} \tag{3.25}$$

In order to solve the system (3.24) we apply the method of nonlinear semigroups. It requires the knowledge of some analytic properties of the nonlinearity.

PROPOSITION 3.4. – *The nonlinear part of the Yang-Mills-Dirac system, given by Eq. (3.25), is a map $\mathcal{F} : \mathbf{P} \longrightarrow \mathbf{P}$. It is continuous and smooth with respect to the norm*

$$\|(A, E, \Psi)\|_{\mathbf{P}}^2 = \|A\|_{H^2}^2 + \|E\|_{H^1}^2 + \|\Psi\|_{H^2}^2\ . \tag{3.26}$$

Proof. – The continuity and smoothness was proved for the component $\mathcal{F}_1$ in [3], and for the minimal coupling component $\mathcal{F}_2$ in [14]. The proof given there under the bag boundary conditions literally generalizes to $I\!R^3$. For the component $\mathcal{F}_3$ we get with the estimates of Lemma A.3 and (2.6)

$$\begin{aligned} \|\mathcal{F}_3\|_{\mathbf{P}}^2 &= \|[A_0, A]\|_{H^2}^2 + \|[A_0, E]\|_{H^1}^2 + \|[A_0, \Psi]\|_{H^2}^2 \\ &\le C\|A_0\|_{B^2}^2 \left(\|A\|_{H^2} + \|E\|_{H^1} + \|\Psi\|_{H^2}\right)^2 \\ &\le C'\|E\|_{H^1}^2 \|(A, E, \Psi)\|_{\mathbf{P}}^2\ . \end{aligned} \tag{3.27}$$

This proves the continuity of $\mathcal{F}_3 : \mathbf{P} \to \mathbf{P}$. To show differentiability we write (a, e, ψ) for an arbitrary infinitesimal variation and evaluate

$$\begin{aligned} &D\mathcal{F}_3(A, E, \Psi)(a, e, \psi) \\ &\quad = ([a_0, A] + [A_0, a];\ [a_0, E] + [A_0, e];\ [a_0, \Psi] + [A_0, \psi]) \end{aligned} \tag{3.28}$$

where $\Delta a_0 = -\mathrm{div}\, e$. Since (a, e, ψ) are of the same Sobolev class as (A, E, Ψ) we can estimate similarly as in (3.27)

$$\|D\mathcal{F}_3(A, E, \Psi)(a, e, \psi)\|_{\mathbf{P}}^2 \le C(\|a\|_{H^2} + \|e\|_{H^1} + \|\psi\|_{H^2})^2 \|(A, E, \Psi)\|_{\mathbf{P}}^2. \tag{3.29}$$

This proves that $\mathcal{F}_3 : \mathbf{P} \to \mathbf{P}$ is differentiable. Higher order differentiability is shown accordingly.

Q.E.D.

The result of Proposition 3.4 enables us to infer the existence and uniqueness of solutions of minimally coupled Yang-Mills and Dirac equations from the corresponding results for nonlinear semigroups, *cf.* [16]:

THEOREM 3.5. – *For every* $(A, E, \Psi) \in \mathbf{P}$ *there exists a unique maximal* $T \in (0, \infty]$ *and a unique curve* $(A(t), E(t), \Psi(t))$ *in* $C^1([0,T), \mathbf{P})$ *satisfying the Yang-Mills and Dirac equations (1.2), (1.3) and (1.4), and the initial condition* $(A(0), E(0), \Psi(0)) = (A, E, \Psi)$. *If* $T < \infty$, *then*

$$\lim_{t \to T} \|(A, E, \Psi)_t\|_{\mathbf{P}} = \infty . \tag{3.30}$$

Observe that the time evolution of the Yang-Mills-Dirac system discussed here gives rise to local diffeomorphisms of the phase space $\mathbf{P}$. To see this, we consider the map

$$(A, E, \Psi) \longmapsto (A, E, \Psi)_t = \mathcal{U}(t)(A, E, \Psi) + \int_0^t \mathcal{U}(t-s)\mathcal{F}((A, E, \Psi)_s)ds . \tag{3.31}$$

By differentiation of this map in the direction of a vector (a, e, ψ) in $\mathbf{P}$ we obtain

$$\begin{aligned}&((A, E, \Psi), (a, e, \psi)) \\ &\quad\longmapsto \mathcal{U}(t)(a, e, \psi) + \int_0^t \mathcal{U}(t-s)D\mathcal{F}((A, E, \Psi)_s)(a, e, \psi)ds,\end{aligned} \tag{3.32}$$

which is continuous, since $\mathcal{F}$ is smooth. A corresponding argument for the higher derivatives implies that the time evolution (3.31) is smooth. Since the dynamics is reversible, this shows that it is a local diffeomorphism. It should be emphasized that this diffeomorphism is not a symplectomorphism. To obtain a Hamiltonian evolution one has to modify the gauge condition of Theorem 2.1, *cf.* [17].

If the initial conditions for the Yang-Mills-Dirac system are more regular, say in

$$\begin{aligned}\mathbf{P}^s = \{(A, E, \Psi) | A \in H^{s+1}(\mathbb{R}^3, \mathfrak{g}), \\ E \in H^s(\mathbb{R}^3, \mathfrak{g}),\ \Psi \in H^{s+1}(\mathbb{R}^3, \mathbb{R}^k)\end{aligned} \tag{3.33}$$

with $s \geq 1$, then the time evolution maintains this regularity. To see this, note that

$$\mathbf{D}_T^s = \{(A^T, E^T) \in H^{s+1}(\mathbb{R}^3, \mathfrak{g}) \times H^s(\mathbb{R}^3, \mathfrak{g})\} \tag{3.34}$$

is the domain of the s-th power of the operator $\mathcal{T}$. Moreover, by repeating the arguments of Proposition 3.2(ii), it follows that the domain of $\mathcal{D}^s$ is

$$\mathbf{D}_D^s = \{\Psi \in H^s(I\!R^3, I\!R^k)\} . \tag{3.35}$$

It is straightforward to show that $\mathcal{F} : \mathbf{P}^s \to \mathbf{P}^s$ is continuous and smooth. Therefore we can conclude with [18]:

COROLLARY 3.6. – *For every initial condition* $(A, E, \Psi) \in \mathbf{P}^s$ *the solution of Eqs. (1.2), (1.3) and (1.4) is a curve* $(A(t), E(t), \Psi(t))$ *in* $C^1([0, T), \mathbf{P}^s)$.

4. GAUGE SYMMETRIES

The group of maps φ from $I\!R^3$ to the structure group G acts on configurations $(A, E, \Psi) \in \mathbf{P}$ via the transformation law

$$A \mapsto \varphi A \varphi^{-1} + \varphi \operatorname{grad} \varphi^{-1}, \qquad E \mapsto \varphi E \varphi^{-1} \qquad \text{and} \qquad \Psi \mapsto \varphi \Psi. \tag{4.1}$$

The group $GS(\mathbf{P})$ of gauge symmetries of $\mathbf{P}$ is the connected group of gauge transformations which preserve the space $\mathbf{P}$. The infinitesimal action of the elements ξ of the Lie algebra $gs(\mathbf{P})$ of $GS(\mathbf{P})$ is given by

$$A \mapsto A - D_A \xi, \qquad E \mapsto E - [E, \xi] \qquad \text{and} \qquad \Psi \mapsto \Psi + \xi \Psi, \tag{4.2}$$

where

$$D_A \xi = \operatorname{grad} \xi + [A, \xi] \tag{4.3}$$

is the covariant differential of ξ with respect to the connection defined by A. Since the Yang-Mills potential A in $\mathbf{P}$ is of Sobolev class $H^2(I\!R^3, \mathfrak{g})$, it follows that $\xi \in gs(\mathbf{P})$ only if $\operatorname{grad} \xi \in H^2(I\!R^3, \mathfrak{g})$. This suggests the following:

PROPOSITION 4.1. – *The set of infinitesimal gauge symmetries of* $\mathbf{P}$ *is the Hilbert-Lie algebra*

$$gs(\mathbf{P}) = \mathcal{B}^3(I\!R^3, \mathfrak{g}) . \tag{4.4}$$

The action of $gs(\mathbf{P})$ *in* $\mathbf{P}$ *is continuous.*

Proof. – The estimates of Lemma A.3 imply that

$$||[A, \xi]||_{H^2} \le C \, ||\xi||_{\mathcal{B}^3} ||A||_{H^2}, \qquad ||[E, \xi]||_{H^1} \le C \, ||\xi||_{\mathcal{B}^3} ||E||_{H^1}$$

and

$$||\xi \Psi||_{H^2} \le C \, ||\xi||_{\mathcal{B}^3} \, ||\Psi||_{H^2} \tag{4.5}$$

for $\xi \in \mathcal{B}^3(\mathbb{R}^3, \mathfrak{g})$. Therefore the infinitesimal action (4.2) of each $\xi \in \mathcal{B}^3(\mathbb{R}^3, \mathfrak{g})$ preserves $\mathbf{P}$. This implies that $\mathcal{B}^3(\mathbb{R}^3, \mathfrak{g}) \subseteq gs(\mathbf{P})$. By the argument above $(\operatorname{grad} \xi)$ has to be in $H^2(\mathbb{R}^3, \mathfrak{g})$ in order to have $\xi \in gs(\mathbf{P})$. With the definition of $\mathcal{B}^3(\mathbb{R}^3, \mathfrak{g})$, *cf.* (2.4), this proves (4.4). Moreover,

$$||[\xi, \eta]||_{\mathcal{B}^3} \leq ||\xi||_{\mathcal{B}^3}\, ||\eta||_{\mathcal{B}^3} \tag{4.6}$$

which proves that $gs(\mathbf{P})$ is a Banach-Lie algebra. Since $\mathcal{B}^3(\mathbb{R}^3, \mathfrak{g})$ is a Hilbert space, *cf.* Theorem A.2, $gs(\mathbf{P})$ is a Hilbert-Lie algebra. Finally, the continuity of the action of $gs(\mathbf{P})$ in $\mathbf{P}$ follows from the estimates (4.5). Q.E.D.

Let $C_c^\infty(\mathbb{R}^3, \mathfrak{g})$ denote the space of all smooth maps $\xi : \mathbb{R}^3 \to \mathfrak{g}$ which are constant outside a compact set, and let $C_0^\infty(\mathbb{R}^3, \mathfrak{g})$ be the subspace of compactly supported maps. From the decomposition results of [7] we infer that

$$gs(\mathbf{P}) = gs(\mathbf{P})_0 \oplus \mathfrak{g}, \tag{4.7}$$

where $gs(\mathbf{P})_0$ is the closure of $C_0^\infty(\mathbb{R}^3, \mathfrak{g})$ in the topology given by the norm (1.11). From Theorem A.2 it follows that $gs(\mathbf{P})_0 \subset C_0^1(\mathbb{R}^3, \mathfrak{g})$. Therefore infinitesimal gauge transformations in $gs(\mathbf{P})$ are C^1 maps from $\mathbb{R}^3$ the structure Lie algebra $\mathfrak{g}$, and $C_c^\infty(\mathbb{R}^3, \mathfrak{g})$ is dense in $gs(\mathbf{P})$, *cf.* also Lemma A.1.

The topology of the gauge group on non-compact manifolds with a Sobolev-Lie algebra has been studied in [1] and [19]. Here we adapt the approach of [1] to our case of a $\mathcal{B}^3$ Hilbert-Lie algebra. Let $C_c^\infty(\mathbb{R}^3, G)$ denote the space of all smooth maps $\varphi : \mathbb{R}^3 \to G$ which are constant outside a compact set. It forms a group under pointwise multiplication with the identity denoted by e. By assumption, $G \subset gl(k, \mathbb{R})$ so that $C_c^\infty(\mathbb{R}^3, G) \subset C_c^\infty(\mathbb{R}^3, gl(k, \mathbb{R}))$ can be topologized by the norm $||\cdot||_{\mathcal{B}^3}$.

One parameter subgroups of $C_c^\infty(\mathbb{R}^3, G)$ are of the form $\exp(t\xi)$, where ξ is in the dense subalgebra $C_c^\infty(\mathbb{R}^3, \mathfrak{g})$ of $gs(\mathbf{P})$. The topology of $gs(\mathbf{P})$ induces a uniform structure in $C_c^\infty(\mathbb{R}^3, G)$, with a neighbourhood basis at e consisting of the sets

$$N_\epsilon = \{\exp(\xi) \mid \xi \in C_c^\infty(\mathbb{R}^3, \mathfrak{g}),\ ||\xi||_{\mathcal{B}^3} < \epsilon\} \qquad \text{with } \epsilon > 0. \tag{4.8}$$

In order to show that the completion of $C_c^\infty(\mathbb{R}^3, G)$ in this uniform structure is a topological group, relatively to the canonically extended multiplication, we need to show:

PROPOSITION 4.2. – *The mapping* $\exp(\xi) \mapsto \exp(\xi)^{-1}$ *is uniformly continuous relative to* N_1. *That is, for every* $\epsilon > 0$, *there exists* $\delta > 0$

such that, for every $\exp(\xi) \in N_1$,

$$\exp(\xi)^{-1} N_\delta \exp(\xi) \subseteq N_\epsilon. \tag{4.9}$$

Proof. – Let $\varphi \in N_\epsilon \subset C^\infty(\mathbb{R}^3, gl(k, \mathbb{R}))$ then

$$\varphi = \exp(\xi) = \sum_{n=0}^{\infty} \frac{1}{n!} \xi^n, \tag{4.10}$$

and

$$\operatorname{grad} \varphi = \operatorname{grad} \exp(\xi) = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{1}{n!} \xi^k (\operatorname{grad} \xi) \xi^{n-k-1}. \tag{4.11}$$

Using the estimates of Lemma A.3 this implies that

$$\|\operatorname{grad} \varphi\|_{H^2} \leq \exp(C\|\xi\|_{\mathcal{B}^3}) \|\operatorname{grad} \xi\|_{H^2} \tag{4.12}$$

and

$$\|\varphi\|_{\mathcal{B}^3} \leq \sum_{n=0}^{\infty} \frac{1}{n!} \left(C\|\xi\|_{\mathcal{B}^3}\right)^n < e^{C\epsilon}. \tag{4.13}$$

For each $\zeta \in gs(\mathbf{P})$ we then obtain by using Lemma A.3 once more:

$$\begin{aligned} \| \exp(\xi)^{-1} \zeta \exp(\xi)\|_{\mathcal{B}^3} &\leq C^2 \| \exp(-\xi)\|_{\mathcal{B}^3} \| \zeta \|_{\mathcal{B}^3} \| \exp(\xi)\|_{\mathcal{B}^3} \\ &< C^2 e^{2C\epsilon} \| \zeta \|_{\mathcal{B}^3} . \end{aligned} \tag{4.14}$$

This proves (4.9) with $\delta = \epsilon(Ce^{C\epsilon})^{-2}$. Q.E.D.

By a result of [20], Proposition 4.2 implies that the completion of $C_c^\infty(\mathbb{R}^3, G)$ in this uniform structure is a topological group, relatively to the canonically extended multiplication. It is a Hilbert-Lie group, whose Lie algebra is canonically isomorphic to the Hilbert-Lie algebra $gs(\mathbf{P})$. In view of this we set:

DEFINITION 4.3. – *The Hilbert-Lie group $GS(\mathbf{P})$ of gauge symmetries is the completion of the group $C_c^\infty(\mathbb{R}^3, G)$ in the uniform structure defined by the topology of its Lie algebra $gs(\mathbf{P})$.*

The exponential map $\exp : gs(\mathbf{P}) \to GS(\mathbf{P})$ maps the unit ball in $gs(\mathbf{P})$ onto the neighbourhood of identity in $GS(\mathbf{P})$ given by the completion $\overline{N}_1$ of N_1. Since G is connected, it follows that $C_c^\infty(\mathbb{R}^3, G)$ is connected, and $GS(\mathbf{P})$ is connected. Therefore, $GS(\mathbf{P})$ is the union of the sets

$$N_1^m = \{\varphi_1 \cdot \varphi_2 \cdot \ldots \cdot \varphi_m | \varphi_i \in N_1\} . \tag{4.15}$$

The inequality (4.12) together with (4.15) implies that, for each $\varphi \in GS(\mathbf{P})$,

$$\operatorname{grad} \varphi \in H^2(\mathbb{R}^3, \mathfrak{g}). \tag{4.16}$$

Moreover, since G is compact, it is bounded in $gl(k, \mathbb{R})$, and the Sobolev embedding theorem implies that each $\varphi \in GS(\mathbf{P})$ is a bounded continuous map. Hence, $||\varphi||_{\mathcal{B}^3}$ is finite for every φ in $GS(\mathbf{P})$. We can give an alternative characterization of the topology of $GS(\mathbf{P})$.

PROPOSITION 4.4. – *A sequence $\varphi_k \in GS(\mathbf{P})$ converges to φ in $GS(\mathbf{P})$ if and only if the sequence of maps $\varphi_k : \mathbb{R}^3 \to G$ converges to φ in the topology defined by the norm $|| \cdot ||_{\mathcal{B}^3}$.*

Proof. – Suppose that φ_k converges to φ in the uniform topology of $GS(\mathbf{P})$. For sufficiently large k,

$$\varphi_k = \varphi \exp(\xi_k) \ , \tag{4.17}$$

where the sequence ξ_k converges to zero in the topology of $gs(\mathbf{P})$. The estimate (A.26) implies that

$$||\varphi_k - \varphi||_{\mathcal{B}^3} \leq C||\varphi||_{\mathcal{B}^3} \left|\left| \sum_{n=1}^{\infty} \frac{1}{n!} (\xi_k)^n \right|\right|_{\mathcal{B}^3} \leq C||\varphi||_{\mathcal{B}^3} \left| 1 - e^{C||\xi_k||_{\mathcal{B}^3}} \right| . \tag{4.18}$$

Since $\xi_k \to 0$ in the norm topology of $gs(\mathbf{P})$ the right hand side converges to zero. Therefore $\varphi_k \to \varphi$ in the topology defined by $|| \cdot ||_{\mathcal{B}^3}$.

Conversely, suppose that $||\varphi_k - \varphi||_{\mathcal{B}^3} \to 0$. Then

$$||e - \varphi^{-1}\varphi_k||_{\mathcal{B}^3} \leq \epsilon_k \tag{4.19}$$

with $\epsilon_k \to 0$ as k goes to infinity. Eq. (4.17) yields

$$\xi_k = \log(\varphi^{-1}\varphi_k) = -\sum_{n=1}^{\infty} \frac{(e - \varphi^{-1}\varphi_k)^n}{n} \tag{4.20}$$

for k sufficiently large. Therefore, by (A.26),

$$||\xi_k||_{\mathcal{B}^3} \leq \sum_{n=1}^{\infty} \frac{\left(C||e - \varphi^{-1}\varphi_k||_{\mathcal{B}^3}\right)^n}{C\,n} \leq -\frac{1}{C} \log(1 - C\epsilon_k). \tag{4.21}$$

This implies that $\xi_k \to 0$ in the topology of $gs(\mathbf{P})$, and hence $\varphi_k \to \varphi$ in the uniform topology of $GS(\mathbf{P})$. Q.E.D.

COROLLARY 4.5. – *The Hilbert-Lie group $GS(\mathbf{P})$ forms a submanifold of the space $\mathcal{B}^3(\mathbb{R}^3, gl(k,\mathbb{R}))$.*

Proof. – By construction $C_c^\infty(\mathbb{R}^3, G) \subset C_c^\infty(\mathbb{R}^3, gl(k,\mathbb{R}))$. Therefore Proposition 4.4 implies that $GS(\mathbf{P})$ can be identified with the subset of $\mathcal{B}^3(\mathbb{R}^3, gl(k,\mathbb{R}))$ given by the $\mathcal{B}^3$-maps from $\mathbb{R}^3$ with values in G. Since G is a closed submanifold of $gl(k,\mathbb{R})$, it follows that $GS(\mathbf{P})$ is a submanifold of $\mathcal{B}^3(\mathbb{R}^3, gl(k,\mathbb{R}))$, *cf.* [21]. Q.E.D.

THEOREM 4.6. – *The action of $GS(\mathbf{P})$ in $\mathbf{P}$, given by (4.2), is continuous and proper.*

Proof. – Let φ_n be a sequence in $GS(\mathbf{P})$ converging to φ, and $p_n = (A_n, E_n, \Psi_n)$ a sequence in $\mathbf{P}$ converging to $p = (A, E, \Psi)$. From (4.2) we obtain by using the estimate (A.25) and the fact that the inversion $\varphi \mapsto \varphi^{-1}$ in $GS(\mathbf{P})$ is continuous:

$$\begin{aligned}
&\|(\varphi_n A_n \varphi_n^{-1} + \varphi_n \operatorname{grad} \varphi_n^{-1}) - (\varphi A \varphi^{-1} - \varphi \operatorname{grad} \varphi^{-1})\|_{H^2} \qquad (4.22)\\
&\quad \leq \|\varphi_n A_n \varphi_n^{-1} - \varphi_n A \varphi_n^{-1}\|_{H^2} + \|\varphi_n A \varphi_n^{-1} - \varphi_n A \varphi^{-1}\|_{H^2}\\
&\qquad + \|\varphi_n A \varphi^{-1} - \varphi A \varphi^{-1}\|_{H^2} + |\varphi_n \operatorname{grad} \varphi_n^{-1} - \varphi_n \operatorname{grad} \varphi^{-1}\|_{H^2}\\
&\qquad + \|\varphi_n \operatorname{grad} \varphi^{-1} - \varphi \operatorname{grad} \varphi^{-1}\|_{H^2}\\
&\quad \leq C(\|\varphi_n\|_{\mathcal{B}^3}^2 \|A_n - A\|_{H^2} + \big(\|\varphi_n\|_{\mathcal{B}^3} + \|\varphi\|_{\mathcal{B}^3}\big)\|A\|_{H^2}\|\varphi_n - \varphi\|_{\mathcal{B}^3}\\
&\qquad + \|\varphi_n\|_{\mathcal{B}^3}\|\operatorname{grad} \varphi_n - \operatorname{grad} \varphi\|_{H^2} + \|\varphi_n - \varphi\|_{\mathcal{B}^3}\|\operatorname{grad} \varphi\|_{H^2}).
\end{aligned}$$

Writing symbolically φp for the action of $\varphi \in gs(\mathbf{P})$ on $p \in \mathbf{P}$, and $(\varphi p)_A$ for its A component, this implies that

$$\|(\varphi_n p_n)_A - (\varphi p)_A\|_{H^2} \leq C'\big(\|A_n - A\|_{H^2} + \|\varphi_n - \varphi^{-1}\|_{\mathcal{B}^3}\big) , \quad (4.23)$$

since $\|\varphi_n\|_{\mathcal{B}^3}$ is bounded. Correspondingly we estimate with (A.24) and (A.25),

$$\begin{aligned}
\|(\varphi_n p_n)_E - (\varphi p)_E\|_{H^1} &\leq C'\big(\|E_n - E\|_{H^1} + |\varphi_n - \varphi^{-1}\|_{\mathcal{B}^3}\big)\\
\|(\varphi_n p_n)_\Psi - (\varphi p)_\Psi\|_{H^2} &\leq C'\big(\|\Psi_n - \Psi\|_{H^2} + \|\varphi_n - \varphi^{-1}\|_{\mathcal{B}^3}\big) .
\end{aligned} \quad (4.24)$$

Therefore $\|\varphi_n p_n - \varphi p\|_{\mathbf{P}} \to 0$ as $n \to \infty$, which proves the continuity of the action.

Let $p_n = (A_n, E_n, \Psi_n)$ converge in $\mathbf{P}$ to $p = (A, E, \Psi)$, and φ_n be a sequence in $GS(\mathbf{P})$ such that $\varphi_n p_n$ converges to $\widetilde{p} \in \mathbf{P}$. To prove properness of the action it is to show that φ_n converges to $\varphi \in GS(\mathbf{P})$ and $\widetilde{p} = \varphi p$. The argument used in [15] for compact domains implies that, for every compact domain $M \subset \mathbb{R}^3$, the restrictions $\varphi_n|M$ converge in $H^2(M)$ to a map $\varphi_M \in H^2(M)$. Since, $M \subseteq \widetilde{M}$ implies that $\varphi|\widetilde{M}$ restricted to M coincides with φ_M, it follows that there exists a continuous map $\varphi : \mathbb{R}^3 \to G$ such that φ_M is the restriction of φ to M. The proof that grad φ_n converges to grad φ in the H^2 topology is the same as in the compact case, [15]. Hence, Proposition 4.4 implies that φ_n converges to φ in the uniform topology. Q.E.D.

Let $C_0^\infty(\mathbb{R}^3, G)$ be the subgroup of $C_c^\infty(\mathbb{R}^3, G)$ consisting of maps $\varphi : \mathbb{R}^3 \to G$ which are the identity in G outside a compact set. Its closure in the uniform topology discussed above defines the closed subgroup $GS(\mathbf{P})_0$ of $GS(\mathbf{P})$. The subalgebra $gs(\mathbf{P})_0$ of $gs(\mathbf{P})$, defined by (4.7), is an ideal and hence $GS(\mathbf{P})_0$ is a normal subgroup of $GS(\mathbf{P})$.

Proposition 4.7. – *$GS(\mathbf{P})_0$ is a Hilbert-Lie group with Lie algebra $gs(\mathbf{P})_0$. The action of $GS(\mathbf{P})_0$ in $\mathbf{P}$ is free and proper.*

Proof. – To show that the infinitesimal action is also free suppose that $\xi_0 \in gs(\mathbf{P})_0$ has a fixed point (A, E, Ψ). By (4.2)

$$D_A \xi_0 = 0, \tag{4.25}$$

that is, ξ_0 is covariantly constant with respect to the connection given by A. Since the scalar product in $\mathfrak{g}$ is ad-invariant, this implies that $|\xi_0|$ is constant. Since $gs(\mathbf{P})_0 \subset C_0^1(\mathbb{R}^3, \mathfrak{g})$, this implies that $\xi_0 = 0$. This proves that the action of $gs(\mathbf{P})_0$ is free. Since $GS(\mathbf{P})_0$ is connected, every $\varphi \in GS(P)_0$ is of the form

$$\varphi = (\exp \xi_1) \cdot (\exp \xi_2) \dots (\exp \xi_n) \tag{4.26}$$

for some $\xi_1, \dots \xi_n$ in $gs(\mathbf{P})_0$. Therefore the action of $GS(\mathbf{P})_0$ is free.

The result of Proposition 4.4 implies that the Lie algebra of $GS(\mathbf{P})_0$ is the closure of $C_0^\infty(\mathbb{R}^3, \mathfrak{g})$ in the $\mathcal{B}^3$ topology. By the decomposition (4.7) this coincides with $gs(\mathbf{P})_0$. Since $GS(\mathbf{P})_0$ is a closed subgroup of $GS(\mathbf{P})$ which acts properly in $\mathbf{P}$, it follows that the action of $GS(\mathbf{P})_0$ in $\mathbf{P}$ is proper. Q.E.D.

5. TIME DEPENDENT GAUGE TRANSFORMATIONS

Let ϕ be a Minkowski space gauge transformation, given by a map from the space $M^4 = \mathbb{R} \times \mathbb{R}^3$ to the structure group G. It acts on the Yang-Mills and Dirac fields $(A_0, A, E, \Psi) \in \mathcal{B}^3(\mathbb{R}^3, \mathfrak{g}) \times \mathbf{P}$ via

$$A_0 \mapsto \widetilde{A}^0 = \phi A_0 \phi^{-1} + \phi \partial_t \phi^{-1}, \qquad A \mapsto \widetilde{A} = \phi A \phi^{-1} + \phi \operatorname{grad} \phi^{-1},$$
$$\widetilde{E} \mapsto E = \phi E \phi^{-1}, \qquad \widetilde{\Psi} \mapsto \Psi = \phi \Psi. \tag{5.1}$$

In order to preserve $\mathbf{P}$ the map ϕ has to be chosen such that

$$\phi(t,.) =: \varphi(t) \in GS(\mathbf{P}) \tag{5.2}$$

at each instant of time t. If also $\partial_t\phi \in \mathcal{B}^3(I\!R^3, \mathfrak{g})$, then Lemma A.3 implies that the gauge transformation (5.1) preserves the Beppo Levi class of the scalar potential. Gauge transformations of this type are essential in view of the attainability of the gauge fixing given by Theorem 2.1 :

THEOREM 5.1. – *Let* $(A(t), E(t), \Psi(t))$ *be a* C^1 *curve in the constraint set* $\mathbf{C}$, *and* $A_0(t)$ *a* C^1 *curve in* $\mathcal{B}^3(I\!R^3, \mathfrak{g})$. *Then there exists a maximal* $\widetilde{T} > 0$ *and a* C^1 *curve of gauge transformations with*

$$\varphi(t) \in \mathcal{G}S(\mathbf{P}) \ \ \forall\, t \in (-\widetilde{T}, \widetilde{T}) \qquad \textit{and} \qquad \varphi(0) = e, \tag{5.3}$$

such that the gauge transformed scalar potential $\widetilde{A}_0(t)$ *satisfying the gauge condition*

$$\operatorname{grad} \widetilde{A}_0 \;=\; -\widetilde{E}^L \qquad \textit{and} \qquad \int_{I\!R^3} \rho^{-4}\, \widetilde{A}_0\, d_3x = 0\;. \tag{5.4}$$

Proof. – The aim is to find a curve of gauge transformations $\varphi : (-\widetilde{T}, \widetilde{T}) \to GS(\mathbf{P})$ which satisfies the initial value problem

$$\varphi(0) = e \qquad \text{and} \qquad \partial_t\varphi = -A_0\varphi + \varphi\,\Phi_\varphi =: \mathcal{G}\big(\varphi, A_0, E\big), \tag{5.5}$$

where

$$\Delta\Phi_\varphi = -\operatorname{div}\left(\varphi^{-1}E\varphi\right) \qquad \text{and} \qquad \int_{I\!R^3} \rho^{-4}\Phi_\varphi d_3x = 0. \tag{5.6}$$

For fixed $(A_0(t), E(t))$ in $\mathcal{B}^3(I\!R^3, \mathfrak{g}) \times H^1(I\!R^3, \mathfrak{g})$ we consider the map

$$\mathcal{G}(., A_0(t), E(t)) \;:\; \mathcal{B}^3(I\!R^3, G) \to \mathcal{B}^3(I\!R^3, \mathfrak{g}) \tag{5.7}$$

as a vector field on the Hilbert submanifold $GS(\mathbf{P}) \subset \mathcal{B}^3(I\!R^3, gl(k, I\!R))$.

First we show that $\mathcal{G}$ is (locally) Lipschitz in $\mathcal{B}^3(I\!R^3, gl(k, I\!R))$. In view of (5.6) we have to estimate

$$\begin{aligned} &\left\|\operatorname{div}\left(\varphi^{-1}E\varphi - \hat{\varphi}^{-1}E\hat{\varphi}\right)\right\|_{H^1} \\ &\quad\le \|\operatorname{div}\left(\varphi^{-1}E(\varphi - \hat{\varphi})\right)\|_{H^1} + \|\operatorname{div}\left((\hat{\varphi}^{-1} - \varphi^{-1})E\hat{\varphi}\right)\|_{H^1}. \end{aligned} \tag{5.8}$$

With Lemma A.3 and the estimate (2.13) we get

$$\begin{aligned}\|\mathrm{div}\,(\varphi^{-1}E(\varphi-\hat\varphi))\|_{H^1} &\le \|\mathrm{grad}\,\varphi^{-1}\|_{H^2}\,\|E\|_{H^1}\,\|\varphi-\hat\varphi\|_{\mathcal{B}^3}\\ &+\|\varphi^{-1}\|_{\mathcal{B}^3}\,\|\mathrm{div}\,E\|_{H^1}\,\|\varphi-\hat\varphi\|_{\mathcal{B}^3}\\ &+\|\varphi^{-1}\|_{\mathcal{B}^3}\,\|E\|_{H^1}\,\|\mathrm{grad}\,(\varphi-\hat\varphi)\|_{H^2}\\ &\le C\|\varphi-\hat\varphi\|_{\mathcal{B}^3}(\|E\|_{H^1}+\|A\|_{H^2}\|E\|_{H^1}+\|\Psi\|^2_{H^2}),\end{aligned} \tag{5.9}$$

where the constant C depends on $\|\varphi^{-1}\|_{\mathcal{B}^3}$. The second term of (5.8) can be estimated accordingly by observing that Lemma A.3 implies that

$$\|\varphi^{-1}-\hat\varphi^{-1}\|_{\mathcal{B}^3}\le C\,\|\hat\varphi-\varphi\|_{\mathcal{B}^3}, \tag{5.10}$$

with C depending on $\|\varphi^{-1}\|_{\mathcal{B}^3}$ and $\|\hat\varphi^{-1}\|_{\mathcal{B}^3}$. Taking into account (5.6) and the estimate (2.7) it then follows from Lemma A.3 that

$$\|\varphi\,\Phi_\varphi-\hat\varphi\,\Phi_{\hat\varphi}\|_{\mathcal{B}^3}\le C'\|\varphi-\hat\varphi\|_{\mathcal{B}^3}(\|E\|_{H^1}+\|A\|_{H^2}\|E\|_{H^1}+\|\Psi\|^2_{H^2}), \tag{5.11}$$

where C' depends on $\mathcal{B}^3$ norms of φ, $\hat\varphi$, φ^{-1} and $\hat\varphi^{-1}$. Applying Lemma A.3 once more to estimate the term $\|A_0\varphi-A_0\hat\varphi\|_{\mathcal{B}^3}$ we end up with

$$\|\mathcal{G}(\varphi,A_0,E)-\mathcal{G}(\hat\varphi,A_0,E)\|_{\mathcal{B}^3}\le C\|\varphi-\hat\varphi\|_{\mathcal{B}^3}$$

where (5.12)

$$C=C(\|\varphi\|_{\mathcal{B}^3},\|\varphi^{-1}\|_{\mathcal{B}^3},\|\hat\varphi\|_{\mathcal{B}^3},\|\hat\varphi^{-1}\|_{\mathcal{B}^3},\|A\|_{H^2},\|E\|_{H^1},\|\Psi\|_{H^2}).$$

This proves that $\mathcal{G}(.,A_0,E)$ is (locally) Lipschitz. Therefore the Picard proof on the existence and uniqueness of a flow applies to the case under consideration, *cf.* [21].

Since $GS(\mathbf{P})\subset\mathcal{B}^3(\mathbb{R}^3,gl(k,\mathbb{R}))$ is a closed submainfold, and the map $\mathcal{G}(.,A_0,E)$ is a vector field tangential to $GS(\mathbf{P})$, this implies that the curve $\varphi(t)$ stays in $GS(\mathbf{P})$. Thus we have shown that there exists $\widetilde{T}>0$ such that the gauge fixing (5.4) can be achieved by a gauge transformation

$$\varphi(t)\in C^1\big(-\widetilde{T},\widetilde{T}),\mathcal{G}S(\mathbf{P})\big), \tag{5.13}$$

provided that the fields $\big(A_0(t),A(t),E(t),\Psi(t)\big)$ are of class $\mathcal{B}^3(\mathbb{R}^3,\mathfrak{g})\times H^2(\mathbb{R}^3,\mathfrak{g})\times H^1(\mathbb{R}^3,\mathfrak{g})\times H^2(\mathbb{R}^3,\mathbb{R}^k)$ and satisfy the constraint equation (1.5). Q.E.D.

A time dependent gauge transformation ϕ is a gauge symmetry of our system, if for each $t \in I\!R$ it preserves the phase space $\mathbf{P}$ of Cauchy data and it preserves the gauge condition of Theorem 2.1, which implies that

$$\partial_t \phi = \Big(\Delta^{-1}(\operatorname{div} E)\Big)\phi - \phi^{-1}\Delta^{-1}\Big((\operatorname{div}(\phi^{-1}E\phi)\Big) \qquad (5.14)$$

As in Theorem 5.1 we can prove that, given $\phi_0 \in GS(\mathbf{P})$, Eq. (5.14) defines a C^1 curve $\phi(t)$ in $GS(\mathbf{P})$, such that $\phi(t_0) = \phi_0$. This implies that the Minkowski space gauge symmetries of our system are C^1 curves in $GS(\mathbf{P})$, which is uniquely determined by their initial data.

6. CONSTRAINTS AND REDUCTION

The extended phase space $\mathbf{P}$ is endowed with a 1-form θ given by

$$\langle \theta(A, E, \Psi) | (a, e, \psi) \rangle = \int_{I\!R^3} (E \cdot a + \Psi^\dagger \psi) d_3 x \ , \qquad (6.1)$$

for $(a, e, \psi) \in T\mathbf{P}$, where $E \cdot a = -\mathrm{tr}(Ea)$. The exterior differential $\omega = d\theta$ of θ is a weakly symplectic form on $\mathbf{P}$, that is ω is non-degenerate and closed, but the induced mapping $\flat : T\mathbf{P} \to T^*\mathbf{P}$ defined by $u^\flat(v) = \omega(u, v)$ is not onto. Here $T^*\mathbf{P}$, the cotangent bundle of $\mathbf{P}$, is the topological dual of the tangent bundle $T\mathbf{P}$.

The action of $gs(\mathbf{P})$ in $\mathbf{P}$ is Hamiltonian with the momentum map $\mathcal{J}$ given by

$$\langle \mathcal{J}(A, E, \Psi) | \xi \rangle = \langle \theta | \xi_{\mathbf{P}}(A, E, \Psi) \rangle = \int_{I\!R^3} (-E \cdot D_A \xi + \Psi^\dagger \xi \Psi) d_3 x. \quad (6.2)$$

Each ξ in $gs(\mathbf{P})_0$ is the limit of a sequence ξ_n of smooth and compactly supported elements of $gs(\mathbf{P})_0$. The continuity of the momentum map $\mathcal{J}$ implies that

$$\begin{aligned} \langle \mathcal{J}(A, E, \Psi) | \xi \rangle &= \lim_{n \to \infty} \langle \mathcal{J}(A, E, \Psi) | \xi_n \rangle \\ &= -\lim_{n \to \infty} \int_{I\!R^3} \big(\operatorname{div} E + [A; E] - J^0\big) \cdot \xi_n d_3 x, \qquad (6.3) \end{aligned}$$

which follows by integration by parts. Therefore, for every ξ in $gs(\mathbf{P})_0$, the momentum $\langle \mathcal{J}(A, E, \Psi) | \xi \rangle$ vanishes for all (A, E, Ψ) satisfying the constraint equation (1.5). On the other hand, if $\xi : I\!R^3 \to \mathfrak{g}$ is a constant map, then there exists $(A, E, \Psi) \in \mathbf{C}$ such that $\langle \mathcal{J}(A, E, \Psi)|\xi\rangle$ does not vanish. Hence, we have obtained a geometric characterization of $gs(\mathbf{P})_0$ as

$$gs(\mathbf{P})_0 = \{\xi \in gs(\mathbf{P})|\langle \mathcal{J}(\mathbf{A}, \mathbf{E}, \Psi)|\xi\rangle = 0 \qquad \forall(\mathbf{A}, \mathbf{E}, \Psi) \in \mathbf{C}\}. \quad (6.4)$$

Let $\mathcal{J}_0$ be the restriction of the momentum mapping $\mathcal{J}$ to the subalgebra $gs(\mathbf{P})_0$. That is, $\mathcal{J}$ is the map from $\mathbf{P}$ to $gs(\mathbf{P})_0^*$ such that

$$\langle \mathcal{J}_0|\xi\rangle = \langle \mathcal{J}|\xi\rangle \quad (6.5)$$

for all $\xi \in gs(\mathbf{P})_0$. It follows from Eq. (6.4) that the constraint set $\mathbf{C}$ is contained in the zero level of $\mathcal{J}_0$. Conversely, the vanishing of $\langle \mathcal{J}_0|\xi\rangle$ for all smooth compactly supported maps ξ from $\mathbb{R}^3$ to the Lie algebra $\mathfrak{g}$ implies the constraint equations. This follows from the Fundamental Theorem of the Calculus of Variations and Eq. (6.3). Since the momentum mapping $\mathcal{J}_0$ is continuous and every $\xi \in gs(\mathbf{P})_0$ is the limit of a sequence of smooth and compactly supported elements ξ_n, it follows that the zero level of $\mathcal{J}_0$ is contained in $\mathbf{C}$. Hence, we have proved that

$$\mathbf{C} = \mathcal{J}_0^{-1}(0). \quad (6.6)$$

We define the reduced phase space to be the space $\check{\mathbf{P}}$ of the $GS(\mathbf{P})_0$ orbits in $\mathbf{C}$,

$$\check{\mathbf{P}} = \mathbf{C}/GS(\mathbf{P})_0, \quad (6.7)$$

and denote by ρ the canonical projection from $\mathbf{C}$ to $\check{\mathbf{P}}$. Since $\mathbf{C}$ is a closed subset of $\mathbf{P}$ and the action of $GS(\mathbf{P})_0$ in $\mathbf{P}$ is proper and preserves $\mathbf{C}$, it follows that the quotient topology in $\check{\mathbf{P}}$ is Hausdorff. The differentiable structure of $\check{\mathbf{P}}$ will be analysed in another paper, [22].

It follows from Eq. (4.7) that $gs(\mathbf{P})_0$ is an ideal in $gs(\mathbf{P})$ and that the quotient algebra

$$colour(\mathbf{P}) = gs(\mathbf{P})/gs(\mathbf{P})_0 \quad (6.8)$$

is isomorphic to $\mathfrak{g}$. For $\xi \in gs(\mathbf{P})$ and $(A, E, \Psi) \in \mathbf{C}$, the momentum $\langle \mathcal{J}(A, E, \Psi)|\xi\rangle$ depends only on the class $[\xi]$ in $colour(\mathbf{P})$ and on the $GS(\mathbf{P})_0$ orbit through (A, E, Ψ). It is interpreted as the colour charge in the physical state $\rho(A, E, \Psi)$ in the direction of $[\xi] \in colour(\mathbf{P})$. It should be noted that in the decomposition (4.7) of $gs(\mathbf{P})$ the second term $\mathfrak{g}$ is not an ideal. Hence, the notion of the "constant infinitesimal gauge transformations" makes invariant sense only as an element of the quotient algebra $colour(\mathbf{P})$, [23].

Appendix : Decompositions and estimates for Beppo Levi spaces

Let $\mathcal{S}$ denote the Schwartz space of smooth fast falling test functions on $\mathbb{R}^3$. The Fourier transformation $X \mapsto \mathcal{F}(X)$ is a homeomorphism from $\mathcal{S}$ to $\mathcal{S}$ which extends to a unitary map from $L^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$. Given a vector field $X \in \mathcal{S}$, one has

$$\begin{aligned} &\mathcal{F}(\operatorname{div} X)(p) = \sum_{i=1}^{3} p_i \mathcal{F}(X_i)(p) \\ &\text{and} \\ &\mathcal{F}(\operatorname{curl} X)_l = \sum_{i,j=1}^{3} \epsilon_{ijl} p_i \mathcal{F}(X_j)(p). \end{aligned} \tag{A.1}$$

This implies a splitting of $\mathcal{F}(X) = \mathcal{F}(X)^L + \mathcal{F}(X)^T$ with the components given as

$$\begin{aligned} &\mathcal{F}(X)^L_j(p) = \frac{p_j}{|p|^2} \mathcal{F}(\operatorname{div} X)(p) \\ &\text{and} \\ &\mathcal{F}(X)^T_j(p) = \Big(\frac{p}{|p|^2} \times \mathcal{F}(\operatorname{curl} X)(p)\Big)_j. \end{aligned} \tag{A.2}$$

The Helmholtz decomposition $X = X^L + X^T$ is defined by the inverse Fourier transformation

$$X^L = \mathcal{F}^{-1}\big(\mathcal{F}(X)^L\big) \qquad \text{and} \qquad X^T = \mathcal{F}^{-1}\big(\mathcal{F}(X)^T\big) \tag{A.3}$$

on $\mathcal{S}$. It extends to a decomposition for vector fields in $L^2(\mathbb{R}^3)$. Moreover (A.2) implies that

$$\begin{aligned} \|X^L\|^2_{H^k} &= \int \big(1 + |p|^2\big)^k |\mathcal{F}(X)^L(p)|^2 d_3p \\ &\le \int \big(1 + |p|^2\big)^{k-1} |\mathcal{F}(X)^L(p)|^2 d_3p \\ &\quad + \int \big(1 + |p|^2\big)^{k-1} |\mathcal{F}(\operatorname{div} X)(p)|^2 d_3p \\ &\le \|X^L\|^2_{H^{k-1}} + \|\operatorname{div} X\|^2_{H^{k-1}} \end{aligned} \tag{A.4}$$

for $k \geq 1$. Similarly

$$\|X^T\|^2_{H^k} \leq \|X^T\|^2_{H^{k-1}} + \|\operatorname{curl} X\|^2_{H^{k-1}} . \qquad (A.5)$$

In order to solve the Laplace equation on $\mathbb{R}^3$ one needs to introduce the weighted Sobolev space $H^1_{-1}(\mathbb{R}^3, V)$ and $H^2_{-2}(\mathbb{R}^3, V)$, where V is a finite dimensional vector space. With the weight function $\rho = \sqrt{1+|x|^2}$ these spaces are defined as the respective completions of $C_0^\infty(\mathbb{R}^3, V)$ in the norms

$$\|g\|^2_{H^1_{-1}} := \int_{\mathbb{R}^3} |\rho^{-1} g|^2 d_3x + \sum_{j=1}^{3} \int_{\mathbb{R}^3} |\partial_j g|^2 d_3x$$

and $\qquad (A.6)$

$$\|g\|^2_{H^2_{-2}} := \int_{\mathbb{R}^3} |\rho^{-2} g|^2 d_3x + \sum_{j=1}^{3} \int_{\mathbb{R}^3} |\rho^{-1}\partial_j g|^2 d_3x + \sum_{j,k=1}^{3} \int_{\mathbb{R}^3} |\partial_k \partial_j g|^2 d_3x.$$

By this definition, the derivatives are continuous as maps $\partial_j : H^2_{-2}(\mathbb{R}^3, V) \to H^1_{-1}(\mathbb{R}^3, V)$. The space $H^1_{-1}(\mathbb{R}^3, V)$ does not contain the constants, since for all $c \in V$

$$\|c\|^2_{H^1_{-1}} = |c|^2 \int_{\mathbb{R}^3} \rho^{-2} d_3x = \infty \qquad \text{if} \quad c \neq 0. \qquad (A.7)$$

Let $BL_1(L^2(\mathbb{R}^3, V))$ be the first Beppo Levi space of V-valued distributions which have a square integrable gradient, [8]. The following result can be found in a paper of Aikawa [7]:

LEMMA A.1. – *The space $BL_1(L^2(\mathbb{R}^3, V))$ can be topologized by the norm*

$$\|g\|_{B^1} = \int_{D_1} |g| d_3x + \|\operatorname{grad} g\|_{L^2} . \qquad (A.8)$$

It has a direct sum decomposition

$$BL_1(L^2(\mathbb{R}^3, V)) = \overline{\mathcal{D}_1} \oplus V \qquad (A.9)$$

where V is considered as the space of constant functions from $\mathbb{R}^3$ to V and $\overline{\mathcal{D}_1}$ is the closure of the space $C_0^\infty(\mathbb{R}^3, V)$ of smooth compactly supported functions in the topology of $BL_1(L^2(\mathbb{R}^3, V))$ given by the norm (A.8).

The intersection of k Beppo Levi spaces we denote by

$$\mathcal{B}^k(I\!R^3, V) := \bigcap_{m=1}^{k} BL_k(L^2(I\!R^3, V)). \qquad (A.10)$$

This space is topologized by the norm

$$\int_{D_1} |g| d_3x + \|\mathrm{grad}\, g\|_{H^{k-1}} \ . \qquad (A.11)$$

It is not so difficult to see that this norm is equivalent to

$$\|g\|_{\mathcal{B}^k} = \left(\int_{D_1} |g|^2 d_3x + \|\mathrm{grad}\, g\|^2_{H^{k-1}}\right)^{1/2}. \qquad (A.12)$$

Let $\langle , \rangle$ and $\ll , \gg_{H^k}$ denote the scalar products in V and $H^k(I\!R^3, V)$. Then $\mathcal{B}^k(I\!R^3, V)$ is a Hilbert space with a norm corresponding to the scalar product

$$\ll g, h \gg_{\mathcal{B}^k} = \int_{D_1} \langle g, h\rangle d_3x + \ll \mathrm{grad}\, f, \mathrm{grad}\, g \gg_{H^{k-1}} \ . \qquad (A.13)$$

THEOREM A.2.

(i) *The space* $\mathcal{B}^k(I\!R^3, V)$ *splits into*

$$\mathcal{B}^k(I\!R^3, V) = \overline{\mathcal{D}_k} \oplus V \qquad (A.14)$$

where $\overline{\mathcal{D}_k}$ *is the closure of the space* $C_0^\infty(I\!R^3, V)$ *of smooth compactly supported functions in the topology given by the norm* (A.12). *Each* $g \in \mathcal{B}^k(I\!R^3, V)$ *uniquely decomposes into*

$$g = g_0 + c_g \quad where \quad g_0 \in \overline{\mathcal{D}^k} \ \ and \ \ c_g \in V. \qquad (A.15)$$

(ii) *The scalar product* (A.13) *on* $\mathcal{B}^k(I\!R^3, V)$ *is equivalent to the scalar product*

$$\ll f, g \gg_{\mathcal{B}^k} = \langle c_f, c_g\rangle + \ll \rho^{-1} f_0, \rho^{-1} g_0 \gg_{L^2} + \ll \mathrm{grad}\, f, \mathrm{grad}\, g \gg_{H^{k-1}} \qquad (A.16)$$

(iii) *For* $k \geq 2$, *each* $f \in \mathcal{B}^k(I\!R^3, V)$ *is continuous and* C^{k-2}-*differentiable. Let* $g^{(\alpha)}$ *denotes the partial derivative corresponding to a multi-index* α, *then*

$$\sum_{|\alpha| \leq k-2} \|g^{(\alpha)}\|^2_{L^\infty} \leq C \|g\|^2_{\mathcal{B}^k} \ . \qquad (A.17)$$

Proof.

(i) The decomposition (A.14) is obvious by intersecting (A.9) with $\mathcal{B}^k(I\!R^3, V)$.

(ii) On the space $C_0^\infty(I\!R^3, V)$ the $\mathcal{B}^1$-norm (A.8) is equivalent to the weighted Sobolev norm induced by the scalar product

$$\ll f, g \gg_{\mathcal{H}^1_{-1}} = \ll \rho^{-1} f_0, \rho^{-1} g_0 \gg_{L^2} + \ll \operatorname{grad} f, \operatorname{grad} g \gg_{L^2} . \tag{A.18}$$

This follows from the weighted Poincaré inequality for the weight function ρ, *cf.* [24], which states that there is a constant $C_p > 0$ such that

$$||f_0||^2_{L^2_{-1}} \le C_p ||\operatorname{grad} f_0||^2_{L^2_0} \le C_p ||\operatorname{grad} f_0||^2_{\mathcal{B}^1} \qquad \forall f_0 \in C^\infty(I\!R^3, V) . \tag{A.19}$$

Conversely

$$\frac{1}{4}\Big(\int_{D_1} |f_0|\, d_3x\Big)^2 \le \Big(\int_{I\!R^3} \rho^{-2}\, |f_0|\, d_3x\Big)^2 \le ||f_0||^2_{H^1_{-1}} , \tag{A.20}$$

which implies that $\overline{\mathcal{D}_1} = H^1_{-1}(I\!R^3, V)$.

The finite dimensional subspace $V \subset \mathcal{B}^1(I\!R^3, V)$ is split. Therefore the scalar product on $\mathcal{B}^1(I\!R^3, V)$ given by (A.18) induces a norm which is equivalent to the norm given by (A.8). The result for $\mathcal{B}^k(I\!R^3, V)$ then is obvious.

(iii) To prove the embedding result of the Sobolev type consider the Fourier transform $\mathcal{F}(g)$ of $g \in C_0^\infty(I\!R^3, V)$. Then

$$g(x) = \int e^{ipx}\, \mathcal{F}(g)(p) \left(|p|^2(1+|p|^2)^{k-1}\right)^{1/2} \left(|p|^2(1+|p|^2)^{k-1}\right)^{-1/2} d_3p . \tag{A.21}$$

Using the Cauchy-Schwarz inequality we estimate

$$|g(x)|^2 \le \Big(\int |p\,\mathcal{F}(g)(p)|^2 (1+|p|^2)^{k-1}\, d_3p\Big)\Big(\int \frac{(1+|p|^2)^{1-k}}{|p|^2}\, d_3p\Big) . \tag{A.22}$$

Since $\int_{I\!R} (1+|p|^2)^{1-k} dp < \infty$ for $k > \frac{3}{2}$ and $|p\,\mathcal{F}(g)(p)|^2 = |\mathcal{F}(\operatorname{grad} g)(p)|^2$ this implies that

$$|g(x)|^2 \le C ||\operatorname{grad} g||^2_{H^{k-1}} \le C ||\operatorname{grad} g||^2_{\mathcal{B}^k} . \tag{A.23}$$

This shows that each $f \in B^k(I\!R^3, V)$ is continuous and uniformly bounded. For the higher order derivatives the argument applies correspondingly. Q.E.D.

LEMMA A.3. – *Let f and g be maps from $\mathbb{R}^3$ to normed vector spaces, and $f \cdot g$ any pointwise multiplication with values in a normed vector space. If $f \in \mathcal{B}^k(\mathbb{R}^3, V)$ and $k \geq 2$, the following estimates hold:*

$$\|f \cdot g\|_{H^1} \leq C_1 \|f\|_{\mathcal{B}^k} \|g\|_{H^1} \qquad \forall g \in H^1(\mathbb{R}^3, W), \qquad (A.24)$$

$$\|f \cdot g\|_{H^2} \leq C_2 \|f\|_{\mathcal{B}^k} \|g\|_{H^2} \qquad \forall g \in H^2(\mathbb{R}^3, W), \qquad (A.25)$$

$$\|f \cdot g\|_{\mathcal{B}^k} \leq C_3 \|f\|_{\mathcal{B}^k} \|g\|_{\mathcal{B}^k} \qquad \forall g \in \mathcal{B}^k(\mathbb{R}^3, W). \qquad (A.26)$$

Proof. – By Theorem A.2, $f \in \mathcal{B}^k(\mathbb{R}^3, V)$ implies that $\|f\|_{L^\infty}$ is finite, and hence

$$\|f \cdot g\|_{L^2} \leq \|f\|_{L^\infty} \|g\|_{L^2} \qquad \forall g \in H^1(\mathbb{R}^3, W) . \qquad (A.27)$$

With an appropriately defined pointwise product $\cdot$ on the right hand side we have

$$\mathrm{grad}\,(f \cdot g) = \mathrm{grad}\,(f) \cdot g + f \cdot \mathrm{grad}\,(g) . \qquad (A.28)$$

If $f \in \mathcal{B}^k(\mathbb{R}^3, V)$ then $\mathrm{grad}\,(f) \in H^1(\mathbb{R}^3, V)$ and

$$\|\mathrm{grad}\,(f) \cdot g\|_{L^2} \leq \|\mathrm{grad}\, f\|_{H^1} \|g\|_{H^1} . \qquad (A.29)$$

Together with (A.17) and (A.27) this implies that

$$\begin{aligned} \|f \cdot g\|_{H^1} &\leq \|f\|_{L^\infty} \big(\|g\|_{L^2} + \|\mathrm{grad}\, g\|_{L^2} \big) + \|\mathrm{grad}\, f\|_{H^1} \|g\|_{H^1} \\ &\leq C_1 \|f\|_{\mathcal{B}^k} \|g\|_{H^1}, \end{aligned} \qquad (A.30)$$

which proves (A.24). Differentiating (A.28), we get

$$D\,\mathrm{grad}\,(f \cdot g) = D\,\mathrm{grad}\,(f) \cdot g + 2\,\mathrm{grad}\,(f) \cdot \mathrm{grad}\,(g) + f \cdot D\,\mathrm{grad}\,(g) . \qquad (A.31)$$

Therefore, for $g \in H^2(\mathbb{R}^3, \mathfrak{g})$,

$$\begin{aligned} \|D\,\mathrm{grad}\,(f \cdot g)\|_{L^2} \leq & \|D\,\mathrm{grad}\, f\|_{L^2} \|g\|_{H^2} + 2\,\|\mathrm{grad}\, f\|_{H^1} \|g\|_{H^2} \\ & + \|f\|_{L^\infty} \|g\|_{H^2}. \end{aligned} \qquad (A.32)$$

With (A.24) and (A.17) this implies that

$$\|f \cdot g\|_{H^2} \leq C_1 \|f\|_{\mathcal{B}^k} \|g\|_{H^1} + 3\,\|\mathrm{grad}\, f\|_{H^1} \|g\|_{H^2} + \|f\|_{L^\infty} \|g\|_{H^2} , \qquad (A.33)$$

which proves (A.25). Finally the estimates above yield

$$\begin{aligned} \|f \cdot g\|_{\mathcal{B}^2} \leq & \|f\|_{L^\infty} \left(\int_{D_1} |g| d_3 x + \|\mathrm{grad}\, g\|_{L^2} + \|D\mathrm{grad}\, g\|_{L^2} \right) \\ & + 2 \|\mathrm{grad}\, f\|_{H^1} \|\mathrm{grad}\, g\|_{H^1} \\ & + \|g\|_{L^\infty} \big(\|\mathrm{grad}\, g\|_{L^2} + \|D\mathrm{grad}\, g\|_{L^2} \big). \end{aligned} \qquad (A.34)$$

Since $\|f\|_{L^\infty} \leq C \|f\|_{\mathcal{B}^2}$ this proves

$$\|f \cdot g\|_{\mathcal{B}^2} \leq C_3 \|f\|_{\mathcal{B}^2} \|g\|_{\mathcal{B}^2} . \qquad (A.35)$$

For $k > 2$ the estimate (A.26) is shown accordingly. Q.E.D.

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(Manuscript received November 30, 1995;
Revised version received June 18, 1996.)